Topology of complete manifolds with radial curvature bounded from below*†

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Abstract

We investigate the topology of a complete Riemannian manifold whose radial curvature at the base point is bounded from below by that of a von Mangoldt surface of revolution. Sphere theorem is generalized to a wide class of metrics, and it is proven that such a manifold of a noncompact type has finitely many ends.

1 Introduction

The notion of radial curvature is first introduced by Klingenberg [Kl] to investigate compact Riemannian manifolds whose radial curvatures are pinched in between $(1/4, 1]$. Here the standard sphere is employed as a reference space in comparison theorems. After the works of Berger [B] and Klingenberg [Kl] for the classical sphere theorem initiated by Rauch [R], Grove and Shiohama have proved the following.

Theorem 1.1 (Diameter sphere theorem, [GS]) *Let X be a compact Riemannian n-manifold whose sectional curvature is bounded from below by 1. If the diameter is larger than $\pi/2$, then X is homeomorphic to the sphere $\mathbb{S}^n$.*

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Here the reference space is the standard sphere again.

One purpose of this article is to extend the sphere theorem to a wider class of metrics than those described in [GS]. Before stating our results, we mention the history of comparison theorems for radial curvature to clarify the validity of our theorems. Elerath first discusses in [E] a reference space which does not have constant sectional curvature. There a von Mangoldt surface of revolution $\widetilde{Y} \subset \mathbb{R}^3$ with nonnegative Gaussian curvature is employed as a reference space. He has proved the generalized Toponogov comparison theorem (abbreviated to *GTCT*) for complete open Riemannian manifolds whose radial curvatures are bounded from below by that of $\widetilde{Y}$.

The notion of radial curvature is then employed by Greene and Wu [GW] to investigate a function theoretic approach of complete open Riemannian manifolds. The Hessian comparison theorem is the key tool for their investigations. Gap theorems and other important results have been obtained by using special reference surfaces, called models, whose underlying manifolds are $\mathbb{R}^2$ and whose metrics are expressed in terms of the geodesic polar coordinates $d\tilde{r}^2 = dt^2 + g(t)^2 d\theta^2, (t, \theta) \in (0, \infty) \times \mathbb{S}^1_{\tilde{o}}$, around a fixed base point $\tilde{o} \in \widetilde{Y}$. Here $g : [0, \infty) \longrightarrow [0, \infty)$ is a nonnegative smooth function satisfying the Jacobi equation $g'' + Gg = 0$ with $g(0) = 0$ and $g'(0) = 1$. Further restriction is imposed on G as follows

$$G < 0, \quad G' \geq 0, \quad \int_0^\infty -tG(t)\, dt < \infty. \tag{1.1}$$

Thus, the model surface $(\widetilde{Y}, \tilde{o})$ is an Hadamard surface with finite total curvature.

Abresch [A] has proved the GTCT for complete open Riemannian manifolds whose radial curvatures are bounded from below by G with (1.1), so-called *asymptotically nonnegatively curved* manifolds. On the other hand, Machigashira [Ma2] has also proved the GTCT for complete open Riemannian manifolds whose radial curvatures are bounded from below by G without (1.1) of an Hadamard surface with finite total curvature. Recently, the GTCT is extended by Itokawa, Machigashira, and Shiohama [IMS] to more general class of models and also von Mangoldt surfaces (see Section 2 in this article). Our investigation is based upon this fact, for the Toponogov comparison theorem plays an important role in the study of curvature and topology of Riemannian manifolds.

Let (M, p) be a complete Riemannian n-manifold with a base point $p \in M$. We say that (M, p) has *radial curvature bounded from below by* $K : [0, \ell) \longrightarrow \mathbb{R}$ if, along every unit speed minimal geodesic $\gamma : [0, a) \longrightarrow M$ with $\gamma(0) = p$, its sectional curvature K_M satisfies

$$K_M(\gamma'(t), X) \geq K(t)$$

for all $t \in [0, a)$ and $X \in T_{\gamma(t)}M$ with $X \perp \gamma'(t)$. Here $0 < \ell \leq \infty$ and $0 < a \leq \infty$ are constant. The function K is called the *radial curvature function* of a model surface $(\widetilde{M}, \tilde{p})$ such that its metric $d\tilde{s}^2$ is expressed by, in terms of the geodesic polar coordinates around a base point $\tilde{p} \in \widetilde{M}$,

$$d\tilde{s}^2 = dt^2 + f(t)^2 d\theta^2, \quad (t, \theta) \in (0, \ell) \times \mathbb{S}^1_{\tilde{p}}.$$

Here $f : (0, \ell) \longrightarrow \mathbb{R}$ is a positive smooth function satisfying the Jacobi equation

$$f'' + Kf = 0, \quad f(0) = 0, \quad f'(0) = 1.$$

Throughout this article, let $(\widetilde{M}, \tilde{p})$ be a von Mangoldt surface of revolution (cf. [SST, Chapter 7]). Namely, the radial curvature function $K : [0, \ell) \longrightarrow \mathbb{R}$ of $(\widetilde{M}, \tilde{p})$ is assumed to be monotone non-increasing on $(0, \ell)$, instead of (1.1). A round sphere is the only compact 'smooth' (i.e., $\lim_{t \to \ell} f'(t) = -1$) von Mangoldt surface of revolution. If a von Mangoldt surface of revolution $(\widetilde{M}, \tilde{p})$ has the property $\ell < \infty$ and if it is not a round sphere, then $\lim_{t \to \ell} f(t) = 0$ and $\lim_{t \to \ell} f'(t) > -1$. Therefore $(\widetilde{M}, \tilde{p})$ has a singular point, say $\tilde{q} \in \widetilde{M}$, at the maximal distance from $\tilde{p} \in \widetilde{M}$ such that $d(\tilde{p}, \tilde{q}) = \ell$. Its shape can be understood as a 'balloon'. Define

$$\mathrm{rad}_p := \sup_{x \in M} d(p, x)$$

and, from now on, fix a point $p^* \in M$ satisfying $d(p, p^*) = \mathrm{rad}_p$. (We will prove that such a point is unique, see Proposition 3.3.) We denote by $\mathrm{vol}(M)$ the volume of M. Our results are the following.

Theorem A *Let (M, p) be a compact Riemannian n-manifold whose radial curvature is bounded from below by $K : [0, \ell) \longrightarrow \mathbb{R}$ for $\ell < \infty$, and let $\rho \in (0, \ell)$ be the zero of f' on $(0, \ell)$. If $\mathrm{rad}_p > \rho$ and if p is a critical point for some point $z \in M \setminus \overline{B_\rho(p)}$, then M is homeomorphic to a sphere $\mathbb{S}^n$.*

Theorem A provides a sphere theorem for a new class of metrics, for the radial curvature may change signs. Furthermore, it containes Theorem 1.1 as a special case, that is, p and $p^* = z$ are points satisfying $d(p, p^*) = \operatorname{diam} M$, $f(t) = \sin t$, $\rho = \pi/2$, and, moreover, all sectional curvatures are bounded. Related results have been obtained in [Ma1] ($1/4 < K \leq 1$) and [MM].

Theorem B *Let (M, p) be a compact Riemannian n-manifold whose radial curvature is bounded from below by $K : [0, \ell) \longrightarrow \mathbb{R}$ for $\ell < \infty$, and let $\rho \in (0, \ell)$ be the zero of f'. If we have*

$$\mathrm{vol}(M) > \frac{1}{2}\left\{ \mathrm{vol}\left(B^n_\rho(\tilde{p})\right) + \mathrm{vol}(\widetilde{M}^n)\right\},$$

then (M, p) is homeomorphic to a sphere $\mathbb{S}^n$. Here $\widetilde{M}^n$ is an n-model of a Mangoldt type, and $B^n_\rho(\tilde{p}) \subset \widetilde{M}^n$ is the distance ρ-ball around the base point $\tilde{p} \in \widetilde{M}^n$.

Theorem B also provides a sphere theorem for a new class of metrics. Related results have been proved in [MaS] and [Mh]. Our last theorem concerns the case of $\ell = \infty$.

Theorem C *Let (M, p) be a complete, noncompact Riemannian n-manifold whose radial curvature is bounded from below by $K : [0, \infty) \longrightarrow \mathbb{R}$, and denote by $c(\widetilde{M})$ the total curvature of $\widetilde{M}$.*

(C-i) *If $c(\widetilde{M}) > 0$, then (M, p) has exactly one end;*

(C-ii) *If $c(\widetilde{M}) \le 0$, then (M,p) has at most $\mathcal{N}$ ends, where we set*

$$\mathcal{N} = \mathcal{N}\big(n, c(\widetilde{M})\big) := 2\left(1 - \frac{c(\widetilde{M})}{2\pi}\right)^{n-1}.$$

In most of the previous investigations, (M,p) is assumed to have asymptotically nonnegative curvature, that is, (1.1) is imposed on K (see [A], [Ka], [O], [Z], and [Ma2]).

Theorem A is proved jointly by both authors, and Theorems B and C are proved by the first author. The article contains a part of the first author's dissertation [Ko].

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2 Preliminaries

The basic tool used in this article is the Alexandrov-Toponogov comparison theorem for geodesic triangles of the form $\triangle(pxy) \subset M$ whose reference surface is a von Mangoldt surface of revolution. Elerath [E] first discusses such a type of the Toponogov comparison theorem for a von Mangoldt surface of revolution in $\mathbb{R}^3$ which is a flat cone with a convex cap near the vertex. Tanaka [T] proves that the *cut locus* $\mathrm{Cut}(\tilde{z})$ to each point $\tilde{z} \in \widetilde{M} \setminus \{\tilde{p}\}$ of a von Mangoldt surface of revolution is either an empty set, or a geodesic ray, or a segment properly contained in the meridian $\theta^{-1}(\theta(\tilde{z})+\pi)$ laying opposite to $\tilde{z}$, and that the end point of $\mathrm{Cut}(\tilde{z})$ is the first conjugate point to $\tilde{z}$ along the unique minimizing geodesic from $\tilde{z}$ sitting in $\theta^{-1}(\theta(\tilde{z})) \cup \theta^{-1}(\theta(\tilde{z}) + \pi)$. This special property makes it possible to find a corresponding geodesic triangle $\tilde{\triangle}(pxy) := \triangle(\tilde{p}\tilde{x}\tilde{y}) \subset \widetilde{M}$ to an arbitrarily given geodesic triangle $\triangle(pxy) \subset M$. This property is automatically satisfied if the model is an Hadamard surface of revolution (see [Ma2]) as well as the one satisfing (1.1).

Theorem 2.1 (GTCT-II, [IMS, Theorem 1.3]) *Let (M,p) be a complete Riemannian n-manifold whose radial curvature is bounded from below by $K : [0,\ell) \longrightarrow \mathbb{R}$. Then, for every geodesic triangle $\triangle(pxy) \subset M$, there exists a geodesic triangle $\tilde{\triangle}(pxy) = \triangle(\tilde{p}\tilde{x}\tilde{y}) \subset \widetilde{M}$ such that*

$$d(\tilde{p},\tilde{x}) = d(p,x), \quad d(\tilde{p},\tilde{y}) = d(p,y), \quad d(\tilde{x},\tilde{y}) = d(x,y)$$

and that

$$\angle(pxy) \ge \angle(\tilde{p}\tilde{x}\tilde{y}), \quad \angle(pyx) \ge \angle(\tilde{p}\tilde{y}\tilde{x}), \quad \angle(xpy) \ge \angle(\tilde{x}\tilde{p}\tilde{y}).$$

Here we denote by $\angle(pxy)$ the angle between the geodesics from x to p and y forming the triangle $\triangle(pxy)$.

Theorem 2.2 (ACT-II, [IMS, Remark 2]) *Under the same assumption as in Theorem* 2.1*, let $\gamma : [0,a] \longrightarrow M$ and $\tilde{\gamma} : [0,a] \longrightarrow \widetilde{M}$ be the edges of $\triangle(pxy)$ and $\tilde{\triangle}(pxy)$ from x and $\tilde{x}$ to y and $\tilde{y}$, respectively. Then we have, for all $s \in [0,a]$,*

$$d\big(p,\gamma(s)\big) \ge d\big(\tilde{p},\tilde{\gamma}(s)\big).$$

3 Proof of Theorem A

We first prove a lemma on a von Mangoldt surface of revolution $(\widetilde{M}, \tilde{p})$. Let $K : [0, \ell) \longrightarrow \mathbb{R}$ and $f : [0, \ell) \longrightarrow \mathbb{R}$ be as in Section 1 with $\ell < \infty$, and let $\rho \in (0, \ell)$ be the zero of f'.

Lemma 3.1 *The set $\widetilde{M} \setminus B_{\rho'}(\tilde{p})$ is strictly convex for all $\rho' \in (\rho, \ell)$. Namely, for any distinct two points $x, y \in \partial B_{\rho'}(\tilde{p})$ and minimal geodesic $\tilde{\eta} : [0, 1] \longrightarrow \widetilde{M}$ between them, we have*

$$\tilde{\eta}\big((0,1)\big) \subset \widetilde{M} \setminus \overline{B_{\rho'}(\tilde{p})}.$$

Proof. Fix $\rho' \in (\rho, \ell)$ and $\tilde{x} \in \partial B_{\rho'}(\tilde{p})$, and let $\tilde{\gamma} : [0, \rho'] \longrightarrow \widetilde{M}$ be a minimal geodesic from $\tilde{p}$ to $\tilde{x}$. If we denote by $\widetilde{J}$ a Jacobi field along $\tilde{\gamma}$ with

$$\widetilde{J}(0) = 0, \qquad \|\widetilde{J}'(0)\| = 1, \qquad \widetilde{J}(t) \perp \tilde{\gamma}'(t)$$

for all $t \in [0, \rho']$, then it can be represented by $\widetilde{J} = f\widetilde{E}$ for some unit parallel vector field $\widetilde{E}$ along $\tilde{\gamma}$ which is perpendicular to $\tilde{\gamma}'$. As $(\widetilde{M}, \tilde{p})$ is a von Mangoldt surface of revolution, we know that $f' < 0$ holds on (ρ, ℓ). It yields

$$\begin{aligned}
I_{\tilde{\gamma}}(\widetilde{J}, \widetilde{J}) &= \int_0^{\rho'} \{\langle \widetilde{J}', \widetilde{J}' \rangle - \langle R(\tilde{\gamma}', \widetilde{J})\tilde{\gamma}', \widetilde{J} \rangle\}\, dt \\
&= \int_0^{\rho'} \{\langle \widetilde{J}', \widetilde{J}' \rangle + \langle \widetilde{J}'', \widetilde{J} \rangle\}\, dt \\
&= \int_0^{\rho'} \frac{d}{dt} \langle \widetilde{J}', \widetilde{J} \rangle\, dt = \langle \widetilde{J}'(\rho'), \widetilde{J}(\rho') \rangle - \langle \widetilde{J}'(0), \widetilde{J}(0) \rangle \\
&= f'(\rho')f(\rho') < 0,
\end{aligned}$$

where $I_{\tilde{\gamma}}$ denotes the index form of $\tilde{\gamma}$. Define the variation $\nu : (-\varepsilon, \varepsilon) \times [0, \rho'] \longrightarrow \widetilde{M}$ by $\nu(s, t) := \theta_{s\|\widetilde{J}(t)\|}(\tilde{\gamma}(t))$, where we denote by $\theta_{s\|\widetilde{J}(t)\|}$ the rotation with the length $s\|\widetilde{J}(t)\|$ to the direction $\widetilde{J}(t)$. By the second variation formula, we have

$$\begin{aligned}
0 &= \frac{d^2}{ds^2}\Big|_{s=0} \int_0^{\rho'} \left\| \frac{\partial \nu}{\partial t} \right\| dt \\
&= I_{\tilde{\gamma}}(\widetilde{J}, \widetilde{J}) + \left\langle \nabla_{d/ds} \frac{\partial \nu}{\partial s}\Big|_{s=0}, \tilde{\gamma}' \right\rangle \Big|_0^{\rho'} \\
&= f'(\rho')f(\rho') + \left\langle II\big(\widetilde{J}(\rho'), \widetilde{J}(\rho')\big), \tilde{\gamma}'(\rho') \right\rangle,
\end{aligned}$$

where II stands for the second fundamental form of $\partial B_{\rho'}(\tilde{p})$ at $\tilde{x}$. Therefore we find

$$\left\langle II\big(\widetilde{J}(\rho'), \widetilde{J}(\rho')\big), \tilde{\gamma}'(\rho') \right\rangle = -f'(\rho')f(\rho') > 0.$$

Thus we see that $\overline{M} \setminus B_{\rho'}(\tilde{p})$ is strictly convex. $\square$

By a similar discussion, we see that $\widetilde{M} \setminus B_\rho(\tilde{p})$ is convex. Let (M,p) be a compact Riemannian manifold whose radial curvature is bounded from below by K, and suppose $\mathrm{rad}_p > \rho$. Combining the convexity of $\widetilde{M} \setminus B_\rho(\tilde{p})$ with Theorem 2.2, we immediately obtain the following.

Lemma 3.2 *The set $M \setminus B_\rho(p)$ is convex.*

Proposition 3.3 *The function $d(p,\cdot)$ attains its maximum at a unique point $p^* \in M$. In particular, $M \setminus B_\rho(p)$ is a topological disk.*

Proof. We can assume $\mathrm{rad}_p < \ell$. Suppose that there exist two distinct points $x, y \in \partial B_{\mathrm{rad}_p}(p)$. Take a comparison triangle $\tilde{\triangle}(pxy) \subset \widetilde{M}$ corresponding to the triangle $\triangle(pxy)$, and let $\tilde{\eta} : [0,1] \longrightarrow \widetilde{M}$ and $\eta : [0,1] \longrightarrow M$ be minimal geodesics joining $\tilde{x}$ and $\tilde{y}$, x and y, respectively. By Theorem 2.2 and Lemma 3.1, for every $s \in (0,1)$, we have

$$d\big(p, \eta(s)\big) \geq d\big(\tilde{p}, \tilde{\eta}(s)\big) > \mathrm{rad}_p\,.$$

This contradicts to the definition of rad_p, so that $d(p,\cdot)$ attains its maximun at a unique point. The second assertion follows from the first one as $M \setminus B_\rho(p)$ is convex. □

Recall that, for a fixed point $q \in M$, a point $x \in M \setminus \{q\}$ is called a *critical point* for q if, for every nonzero tangent vector $v \in T_xM$, we find a minimal geodesic γ from x to q satisfying $\angle(v, \gamma'(0)) \leq \pi/2$ (see [Gv]). By the isotopy lemma ([Gv], see also [GS]), if there are no critical points for q in $\overline{B_r(q)} \setminus \{q\}$, then $\overline{B_r(q)}$ is a topological disk.

Proposition 3.4 *Assume that p is a critical point for some point $z \in M \setminus \overline{B_\rho(p)}$. Then there are no critical point for p in $\overline{B_\rho(p)} \setminus \{p\}$, in particular, $\overline{B_\rho(p)}$ is a topological disk.*

Proof. Note that no point in $\partial B_\rho(p)$ is critical for p since $M \setminus B_\rho(p)$ is convex (Lemma 3.2). We suppose that there exists a critical point $x \in B_\rho(p) \setminus \{p\}$. Fix a minimal geodesic $\tau : [0,1] \longrightarrow M$ from z to x. As x is a critical point for p, we find a minimal geodesic $\gamma : [0,1] \longrightarrow M$ from p to x for which $\angle(\tau'(1), \gamma'(1)) \leq \pi/2$ holds. Furthermore, since p is a critical point for z, there also exists a minimal geodesic $\sigma : [0,1] \longrightarrow M$ from p to z satisfying $\angle(\sigma'(0), \gamma'(0)) \leq \pi/2$. Consider a comparison triangle $\tilde{\triangle}(pzx) \subset \widetilde{M}$ corresponding to the triangle $\triangle(pzx)$ consisting of γ, τ, and σ, and denote by $\tilde{\gamma}$, $\tilde{\tau}$, and $\tilde{\sigma}$ the edges corresponding to γ, τ, and σ, respectively. Then it follows from Theorem 2.1 that

$$\angle\big(\tilde{\tau}'(1), \tilde{\gamma}'(1)\big) \leq \angle\big(\tau'(1), \gamma'(1)\big) \leq \frac{\pi}{2}, \tag{3.1}$$

$$\angle\big(\tilde{\sigma}'(0), \tilde{\gamma}'(0)\big) \leq \angle\big(\sigma'(0), \gamma'(0)\big) \leq \frac{\pi}{2}. \tag{3.2}$$

By the assumption $d(z,p) > \rho$, we can take $s_- \in (0,1)$ with $\tilde{\tau}(s_-) \in \partial B_\rho(\tilde{p})$. The inequality (3.1) implies that we have $\angle(\tilde{p}\tilde{\tau}(s_0)\tilde{z}) = \pi/2$ for some $s_0 \in (s_-, 1]$. Note that, if we extend $\tilde{\tau}$, then $\tilde{\tau}(s_+) \in \partial B_\rho(\tilde{p})$, where we set $s_+ := 2s_0 - s_-$. It follows from (3.2) that $\angle(\tilde{\tau}(s_-)\tilde{p}\tilde{\tau}(s_+)) < 2\angle(\tilde{z}\tilde{p}\tilde{x}) \leq \pi$, and hence $\tilde{\tau}$ is minimal on $[s_-, s_+]$. However, since $f'(\rho) = 0$, there is an another minimal geodesic between $\tilde{\tau}(s_-)$ and $\tilde{\tau}(s_+)$ contained in $\partial B_\rho(\tilde{p})$, and hence $\tilde{\tau}(s_+) \in \mathrm{Cut}(\tilde{\tau}(s_-))$. This contradicts to the structure of the cut locus $\mathrm{Cut}(\tilde{\tau}(s_-))$ (see Section 2). □

Therefore, by Propositions 3.3 and 3.4, M is homeomorphic to a sphere $\mathbb{S}^n$.

4 Proof of Theorem B

We again start with a lemma on a von Mangoldt surface of revolution $(\widetilde{M}, \tilde{p})$. Let $K : [0, \ell) \longrightarrow \mathbb{R}$ and $f : [0, \ell) \longrightarrow \mathbb{R}$ be as in Section 1 with $\ell < \infty$, and let $\rho \in (0, \ell)$ be the zero of f'.

Lemma 4.1 *It holds that $2\rho \leq \ell$.*

Proof. Fix a meridian $\tilde{\gamma} : [0, \ell] \longrightarrow \widetilde{M}$ emanating from $\tilde{p}$ to $\tilde{q}$. Let $\widetilde{J}(t) = f(t)\widetilde{E}(t)$ be a Jacobi field along $\tilde{\gamma}$, where $\widetilde{E}$ is a unit parallel vector field along $\tilde{\gamma}$ such that $\widetilde{E}(t) \perp \tilde{\gamma}'(t)$ holds for all $t \in [0, \ell)$. As $f'(\rho) = 0$, we know $\widetilde{J}'(\rho) = 0$. Let $\tilde{\gamma}_1 : [0, \ell - \rho] \longrightarrow \widetilde{M}$ be a subarc of $\tilde{\gamma}$ given by $\tilde{\gamma}_1(t) := \tilde{\gamma}(t + \rho)$, and let $\widetilde{J}_1(t)$ be a Jacobi field along $\tilde{\gamma}_1$ such that $\widetilde{J}_1{}''(t) + K(t + \rho)\widetilde{J}_1(t) = 0$ with $\|\widetilde{J}_1(0)\| = f(\rho)$ and $\widetilde{J}_1{}'(0) = 0$. Note that $\|\widetilde{J}_1(\ell - \rho)\| = f(\ell) = 0$. Similarly, let $\tilde{\gamma}_2 : [0, \rho] \longrightarrow \widetilde{M}$ be the converse of a subarc of $\tilde{\gamma}$ given by $\tilde{\gamma}_2(t) := \tilde{\gamma}(\rho - t)$, and let $\widetilde{J}_2(t)$ be a Jacobi field along $\tilde{\gamma}_2$ such that $\widetilde{J}_2{}''(t) + K(\rho - t)\widetilde{J}_2(t) = 0$ with $\|\widetilde{J}_2(0)\| = f(\rho)$ and $\widetilde{J}_2{}'(0) = 0$. Note also that $\|\widetilde{J}_2(\rho)\| = f(0) = 0$. Since $K(\rho - t) \geq K(\rho + t)$, the Berger-Rauch comparison theorem implies that $\|\widetilde{J}_2\| \leq \|\widetilde{J}_1\|$ holds on $[0, \rho] \cap [0, \ell - \rho]$. Therefore we obtain $\rho \leq \ell - \rho$, and hence $2\rho \leq \ell$. □

As in the hypothesis in Theorem B, we assume that

$$\mathrm{vol}(M) > \frac{1}{2}\left\{ \mathrm{vol}\left(B^n_\rho(\tilde{p})\right) + \mathrm{vol}(\widetilde{M}^n) \right\} \tag{4.1}$$

holds.

Proposition 4.2 *The set $M \setminus B_\rho(p)$ is not an empty set, and is a topological disk.*

Proof. It follows from Lemma 4.1 and (4.1) that

$$\mathrm{vol}(M) > \frac{1}{2}\,\mathrm{vol}(\widetilde{M}^n) \geq \frac{1}{2}\,\mathrm{vol}\left(B^n_{2\rho}(\tilde{p})\right) > \mathrm{vol}\left(B^n_\rho(\tilde{p})\right) \geq \mathrm{vol}\left(B_\rho(p)\right),$$

which implies that $M \setminus B_\rho(p)$ is not empty. In particular, we have $\mathrm{rad}_p > \rho$, and hence $M \setminus B_\rho(p)$ is a topological disk by Proposition 3.3. □

For $q \in M$, we denote by $S_qM \subset T_qM$ the unit tangent sphere at q, and set

$$D(p) := \{tv \,|\, v \in S_qM,\ t \geq 0,\ \exp_q([0,t]v) \cap \mathrm{Cut}(q) = \emptyset\}.$$

Define the map $\Pi : T_pM \setminus \{0\} \longrightarrow S_pM$ by $\Pi(v) := v/\|v\|$. Now we see the following and it completes the proof of Theorem B.

Proposition 4.3 *The closed ball $\overline{B_\rho(p)}$ is a topological disk.*

Proof. Suppose there exists a critical point $x \in B_\rho(p)$ for p. For each $r > 0$, we put

$$\Omega_r := \Pi\big(\exp_p^{-1}[M \setminus B_r(p)] \cap D(p)\big) \subset S_pM.$$

By the hypothesis (4.1), we find

$$\begin{aligned}\operatorname{vol}(M) &> \frac{1}{2}\operatorname{vol}(\widetilde{M}) + \frac{1}{2}\operatorname{vol}(B_\rho(\tilde{p})) = \frac{1}{2}\operatorname{vol}\big(\widetilde{M} \setminus B_\rho(\tilde{p})\big) + \operatorname{vol}\big(B_\rho(\tilde{p})\big) \\ &\geq \frac{1}{2}\operatorname{vol}\big(\widetilde{M} \setminus B_\rho(\tilde{p})\big) + \operatorname{vol}\big(B_\rho(p)\big),\end{aligned}$$

and hence

$$\operatorname{vol}\big(M \setminus B_\rho(p)\big) > \frac{1}{2}\operatorname{vol}\big(\widetilde{M} \setminus B_\rho(\tilde{p})\big).$$

This implies that we can choose $\varepsilon > 0$ and $r > \rho$ such that Ω_r is $(\pi/2 - \varepsilon)$-dense in $(S_pM, \angle)$, where we denote by $\angle$ the angle distance on S_pM.

Let γ_1 be a minimizing geodesic emanating from p to the critical point x. By the denseness of $\Omega_r \subset S_pM$, there exist a point $y \in M \setminus B_r(p)$ and a minimizing geodesic σ emanating from p to y such that

$$\angle\big(\sigma'(0), \gamma_1'(0)\big) \leq \frac{\pi}{2} - \varepsilon.$$

Let τ be a minimizing geodesic emanating from y to x. Since x is a critical point for p, there exists a minimizing geodesic γ_2 emanating from p for x such that

$$\angle\big(\gamma_2'(1), \tau'(1)\big) \leq \frac{\pi}{2}.$$

For $i = 1, 2$, consider a comparison triangle $\tilde{\triangle}_i(pyx) \subset \widetilde{M}$ (with the common vertices for $i = 1, 2$) corresponding to the triangle $\triangle_i(pyx) \subset M$ constructed by γ_i, σ, and τ, and denote by $\tilde{\gamma}$, $\tilde{\sigma}_i$, and $\tilde{\tau}_i$ the edges corresponding to them, respectively. By Theorem 2.1, we have

$$\angle\big(\tilde{\gamma}'(1), \tilde{\tau}_2'(1)\big) \leq \angle\big(\gamma_2'(1), \tau'(1)\big) \leq \frac{\pi}{2}, \tag{4.2}$$

$$\angle\big(\tilde{\gamma}'(0), \tilde{\sigma}_1'(0)\big) \leq \angle\big(\gamma_1'(0), \sigma'(0)\big) \leq \frac{\pi}{2} - \varepsilon. \tag{4.3}$$

Using these instead of inequalities (3.1) and (3.2), we see that no point in $B_\rho(p)$ is a critical point for p just as in the proof of Proposition 3.4. Note also that no point in $\partial B_\rho(p)$ is critical for p, for $M \setminus B_\rho(p)$ is convex. Therefore $\overline{B_\rho(p)}$ is a topological disk. □

5 Proof of Theorem C

Let (M, p) be a complete, noncompact Riemannian n-manifold whose radial curvature at $p \in M$ is bounded from below by $K : [0, \infty) \longrightarrow \mathbb{R}$, and denote by $c(\widetilde{M})$ the total curvature of $\widetilde{M}$.

We assume that M has at least two ends. Take two rays $\gamma, \sigma : [0, \infty) \longrightarrow M$ emanating from $p \in M$ to distinct ends of M. For $s, t > 0$, let $\eta_{s,t}$ be a minimizing geodesic from $\gamma(s)$ to $\sigma(t)$. By the definition of the end, there exists a compact set $Z \subset M$ such that $\eta_{s,t}$ passes Z for all $s, t > 0$. Thus, as s and t diverge to the infinity, $\eta_{s,t}$ converges to some geodesic line $\eta : (-\infty, +\infty) \longrightarrow M$. Set $d_0 := d(p, \eta)$. Without loss of generality, we may suppose $d(p, \eta(0)) = d_0$. We define $\eta_r : [-r, r] \longrightarrow M$ as the restriction of η on $[-r, r]$. Let μ_{+r} and μ_{-r} be minimizing geodesics joining p to $\eta(r)$ and to $\eta(-r)$, and let $\mu_{\pm\infty} : [0, \infty) \longrightarrow M$ be their limits as $r \to \infty$, respectively. Consider a comparison triangle $\widetilde{\triangle}(\eta(-r)p\eta(r)) \subset \widetilde{M}$ corresponding to the triangle $\triangle(\eta(-r)p\eta(r))$ consisting of η_r and $\mu_{\pm r}$, and denote by $\tilde{\eta}_r$ and $\tilde{\mu}_{\pm r}$ the corresponding edges. Then Theorem 2.2 implies that

$$d\big(\tilde{p}, \tilde{\eta}_r(0)\big) \leq d\big(p, \eta(0)\big) = d_0$$

holds for all $r > 0$. This means that, for some sequence $\{r_i\}_{i=1}^{\infty}$ diverging to the infinity, the sequence $\{\tilde{\eta}_{r_i}\}_{i=1}^{\infty}$ converges to some line $\bar{\eta} : (-\infty, +\infty) \longrightarrow \widetilde{M}$ as $i \to \infty$ such that $d(\tilde{p}, \bar{\eta}(0)) \leq d_0$. Let $\tilde{\mu}_{\pm\infty}$ be the limits of $\tilde{\mu}_{\pm r_i}$ as $i \to \infty$, respectively.

In this situation, we set

$$\alpha_p := \angle\big(\mu'_{-\infty}(0), \mu'_{+\infty}(0)\big), \qquad \tilde{\alpha}_{\tilde{p}} := \angle\big(\tilde{\mu}'_{-\infty}(0), \tilde{\mu}'_{+\infty}(0)\big).$$

Let $\widetilde{V} \subset \widetilde{M}$ be the domain bounded by $\tilde{\mu}_{\pm\infty}([0, \infty))$ and containing $\bar{\eta}$. Furthermore, let $\widetilde{H} \subset \widetilde{V}$ be the half plane bounded by the geodesic line $\bar{\eta}$, and set $\widetilde{D} := \widetilde{V} \setminus \widetilde{H}$. On one hand, by the Gauss-Bonnet theorem, we observe $c(\widetilde{D}) = \tilde{\alpha}_{\tilde{p}} - \pi$. Recall that $c(\widetilde{D})$ stands for the total curvature of $\widetilde{D}$. On the other hand, it follows from Cohn-Vossen's theorem (cf. [SST, Chapter 2]) that $c(\widetilde{H}) \leq 0$ holds. Therefore we have

$$c(\widetilde{M}) = \frac{2\pi}{\tilde{\alpha}_{\tilde{p}}} c(\widetilde{V}) = \frac{2\pi}{\tilde{\alpha}_{\tilde{p}}} \big\{c(\widetilde{D}) + c(\widetilde{H})\big\} \leq \frac{2\pi}{\tilde{\alpha}_{\tilde{p}}} \big(\tilde{\alpha}_{\tilde{p}} - \pi\big). \tag{5.1}$$

If $c(\widetilde{M}) > 0$, then it follows from (5.1) that $\alpha_p > \pi$, this is a contradiction. So that M has exactly one end, and this completes the proof of (C-i). If $c(\widetilde{M}) \leq 0$, then the inequality (5.1) together with Theorem 2.1 yields that

$$\alpha_p \geq \tilde{\alpha}_{\tilde{p}} \geq \frac{2\pi^2}{2\pi - c(\widetilde{M})}. \tag{5.2}$$

Now let Ω be the set of ends of M and, for each $\mathcal{E} \in \Omega$, choose a unit vector $v_{\mathcal{E}} \in S_pM$ such that the ray $\gamma_{v_{\mathcal{E}}}$ emanating from p with $\gamma'_{v_{\mathcal{E}}}(0) = v_{\mathcal{E}}$ is contained in $\mathcal{E}$. Set $a := (\pi^2)/(2\pi - c(\widetilde{M}))$. Let $B_a(v_{\mathcal{E}})$ be the ball in the unit sphere S_pM with respect to the angle distance. By (5.2), balls in $\{B_a(v_{\mathcal{E}})\}_{\mathcal{E} \in \Omega}$ are mutually disjoint in S_pM. By the well-known Packing Lemma, the number of these balls does not exceed

$$\mathcal{N}\big(n, c(\widetilde{M})\big) := 2\left(\frac{\pi}{2a}\right)^{n-1} = 2\left(1 - \frac{c(\widetilde{M})}{2\pi}\right)^{n-1},$$

which completes the proof of (C-ii).

References

[A] U. Abresch, *Lower curvature bounds, Toponogov's theorem, and bounded topology*, Ann. Sci. École Norm. Sup. (4) **18** (1985), 651–670.

[B] M. Berger, *Les variétés Riemanniennes* (1/4)*-pincées*, Ann. Scuola Norm. Sup. Pisa (3) **14** (1960), 161–170.

[E] D. Elerath, *An improved Toponogov comparison theorem for nonnegatively curved manifolds*, J. Differential Geom. **15** (1980), 187–216.

[GW] R. E. Greene and H. Wu, *Function theory on manifolds which possess a pole*, Lecture Notes in Mathematics **699**, Springer, Berlin, 1979.

[Gv] M. Gromov, *Curvature, diameter and Betti numbers*, Comment. Math. Helv. **56** (1981), 179–195.

[GS] K. Grove and K. Shiohama, *A generalized sphere theorem*, Ann. of Math. (2) **106** (1977), 201–211.

[IMS] Y. Itokawa, Y. Machigashira, and K. Shiohama, *Generalized Toponogov's theorem for manifolds with radial curvature bounded below*, Explorations in complex and Riemannian geometry, 121–130, Contemp. Math. **332**, Amer. Math. Soc., Providence, RI, 2003.

[Ka] A. Kasue, *A compactification of a manifold with asymptotically nonnegative curvature*, Ann. Sci. École Norm. Sup. (4) **21** (1988), 593–622.

[Kl] W. Klingenberg, *Contributions to Riemannian geometry in the large*, Ann. of Math. (2) **69** (1959), 654–666.

[Ko] K. Kondo, *Radial curvature and topology of pointed Riemannian manifolds*, Dissertation, Saga University, 2003.

[Ma1] Y. Machigashira, *Manifolds with pinched radial curvature*, Proc. Amer. Math. Soc. **118** (1993), 979–985.

[Ma2] Y. Machigashira, *Manifolds of roughly non-negative radial curvature*, preprint, 2001.

[MaS] Y. Machigashira and K. Shiohama, *Riemannian manifolds with positive radial curvature*, Japan. J. Math. **19** (1994), 419–430.

[Mh] V. Marenich, *Manifolds with minimal radial curvature bounded from below and big volume*, Trans. Amer. Math. Soc. **352** (2000), 4451–4468.

[MM] V. B. Marenich and S. J. X. Mendonça, *Manifolds with minimal radial curvature bounded from below and big radius*, Indiana Univ. Math. J. **48** (1999), 249–274.

[O] Y. Otsu, *Topology of complete open manifolds with nonnegative Ricci curvature*, Geometry of manifolds (Matsumoto 1988), 295–302, Perspect. Math. **8**, Academic Press, Boston, MA, 1989.

[R] H. E. Rauch, *A contribution to differential geometry in the large*, Ann. of Math. (2) **54** (1951), 38–55.

[SST] K. Shiohama, T. Shioya, and M. Tanaka, The geometry of total curvature on complete open surfaces, Cambridge University Press, Cambridge, 2003.

[T] M. Tanaka, *On the cut loci of a von Mangoldt's surface of revolution*, J. Math. Soc. Japan **44** (1992), 631–641.

[Z] S.-H. Zhu, *A volume comparison theorem for manifolds with asymptotically nonnegative curvature and its applications*, Amer. J. Math. **116** (1994), 669–682.