Topology Optimization of Slightly Compressible Fluids[1]

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Abstract

We consider the problem of optimal design of flow domains for Navier–Stokes flows in order to minimize a given performance functional. We attack the problem using topology optimization techniques, or control in coefficients, which are widely known in structural optimization of solid structures for their flexibility, generality, and yet ease of use and integration with existing FEM software. Topology optimization rapidly finds its way into other areas of optimal design, yet until recently it has not been applied to problems in fluid mechanics. The success of topology optimization methods for the minimal drag design of domains for Stokes fluids (see the study of Borrvall and Petersson [Internat. J. Numer. Methods Fluids, vol. 41, no. 1, pp. 77–107, 2003]) has lead to attempts to use the same optimization model for designing domains for incompressible Navier–Stokes flows. We show that the optimal control problem obtained as a result of such a straightforward generalization is ill-posed, at least if attacked by the direct method of calculus of variations.

We illustrate the two key difficulties with simple numerical examples and propose changes in the optimization model that allow us to overcome these difficulties. Namely, to deal with impenetrable inner walls that may appear in the flow domain we slightly relax the incompressibility constraint as typically done in penalty methods for solving the incompressible Navier–Stokes equations. In addition, to prevent discontinuous changes in the flow due to very small impenetrable parts of the domain that may disappear, we consider so-called filtered designs, that has become a “classic” tool in the topology optimization toolbox. Technically, however, our use of filters differs significantly from their use in the structural optimization problems in solid mechanics, owing to the very unlike design parametrizations in the two models.

We rigorously establish the well-posedness of the proposed model and then discuss related computational issues.

Keywords. Topology optimization, Fluid mechanics, Navier–Stokes flow, Domain identification, Fictitious domain.

AMS subject classification. 76D55, 76N25, 62K05, 49J20, 49J45.

1 Introduction

The optimal control of fluid flows has long been receiving considerable attention by engineers and mathematicians, owing to its importance in many applications involving fluid related technology; see, e.g., the recent monographs [Gun03, MoP01], and articles [Fei03, Ton03b, Ton03a, vBS02, ChG02, GuM02, Kim01, GKM00, OkK00, DzZ99, GuK98, DRSS96, Sue96, BoB95, NRS95, StS94, BeD92, BFCS92], including the pioneering works of Pironneau [Pir73, Pir74] on the optimality conditions for shape optimization in fluid mechanics. According to a well-established classification in structural optimization [BeS03, page 1], the absolute majority of works dealing with optimal design of flow domains fall into the category of shape optimization. (See the bibliographical notes (2) in [BeS03] for classic references in shape optimization.) In the framework of *shape optimization*, the optimization problem formulation can be stated as follows: choose a flow domain out of some family so as to maximize an associated performance functional. The family of domains considered may be as rich as that of all open subsets of a given set satisfying some regularity criterion

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(see, e.g. [Fei03]), or as poor as the ones obtained from a given domain by locally perturbing some part of the boundary in a Lipschitz manner (cf. [Ton03b, GKM00, GuK98]). Unfortunately, it is typically only the problems in the latter group that can be attacked numerically. On the other hand, *topology optimization* (or, control in coefficients) techniques are known for their flexibility in describing the domains of arbitrary complexity (e.g., the number of connected components need not to be bounded), and at the same time require relatively moderate efforts from the computational part. In particular, one may completely avoid remeshing the domain as the optimization algorithm advances, which eases the integration with existing FEM codes, and simplifies and speeds up sensitivity analysis.

While the field of topology optimization is very well established for optimal design of solids and structures, surprisingly little work has been done for optimal design of fluid domains. Borrvall and Petersson [BoP03] considered the optimal design of flow domains for minimizing the total power of the incompressible Stokes flows, using inhomogeneous porous materials with a spatially varying Darcy permeability tensor, under a constraint on the total volume of fluid in the control region. Later, this approach has been generalized to include both limiting cases of the porous materials, i.e., pure solid and pure flow regions have been allowed to appear in the design domain as a result of the optimization procedure [Evg03]. (We also cite the work of Klarbring et al. [KPTK03], which however studies the problem of optimal design of flow networks, where design and state variables reside in finite-dimensional spaces; in some sense this is an analogue of truss design problems if one can carry over the terminology and ideas from the area of optimal design of structures and solids.)

However, applications of the Stokes flows are rather limited, while the Navier–Stokes equations are now regarded as the universal basis of fluid mechanics [Dar02]. Therefore, it has been suggested that the optimization model proposed by Borrvall and Petersson [BoP03] (with straightforward modifications), in particular the same design parametrization should be used for the topology optimization of the incompressible Navier–Stokes equations [GH03]. Essentially, in this model we control the Brinkman-type equations including the nonlinear convection term [All90a] (which will be referred to as nonlinear Brinkman equations in the sequel) by varying a coefficient before the zeroth order velocity term. Setting the control coefficient to zero is supposed to recover the Navier–Stokes equations; at the same time, infinite values of the coefficient are supposed to model the impenetrable inner walls in the domain. In Section 3 we illustrate the difficulties inherent in this approach, namely that the design-to-flow mapping is not closed, leading to ill-posed control problems.

It turns out that if we employ the idea of *filter* [Sig97, SiP98] (which has become quite a standard technique in topology optimization, see [Bou01, BrT01] for the rigorous mathematical treatment) *in addition* to relaxing the incompressibility constraint (which is unique to the topology optimization of fluids) we can establish the continuity of the resulting design-to-flow mapping, and therefore the existence of optimal designs for a great variety of design functionals; this is discussed in Section 4. Not going into details yet, we comment that our use of filters significantly differs from the traditional one in the topology optimization. Namely, not only do we use filters to forbid small features from appearing in our designs and thus to transform weak(-er) design convergence into a strong(-er) one (cf. Proposition 4.1), but also to verify certain growth conditions near impenetrable walls [see inequality (4) and Proposition A.1], which later guarantees the embedding of certain weighted Sobolev spaces into classic ones (see inequality (14) in the proof of Proposition 5.3), and finally allows us to prove the continuity of design-to-flow mappings in Section 5. The existence of optimal designs, formally established in Section 6, is an easy corollary of the continuity of the design-to-flow mappings.

Some computational techniques are introduced in Section 7. Namely, in Subsection 7.1 we discuss a standard topic of approximating the topology optimization problems with so-called sizing optimization problems (also known as "ε-perturbation"), which in our case reduces to approximation of the impenetrable walls with materials of very low permeability. In Subsection 7.2 we touch upon techniques aimed at reducing the amount of porous material in the optimal design. We conclude the paper by discussing possible extensions of the presented results, open questions, and further research topics in Section 8. Proofs of some results are found in Appendix A.

2 Prerequisites

2.1 Notation

We follow standard engineering practice and will denote vector quantities, such as vectors and vector-valued functions, using the **bold** font. However, for functional spaces of both scalar- and vector-valued functions we will use regular font.

Let Ω be a connected bounded domain of $\mathbb{R}^d$, $d \in \{2, 3\}$ with a Lipschitz continuous boundary Γ. In this domain we would like to control the nonlinear Brinkman equations [All90a] with the prescribed flow velocities $\boldsymbol{g}$ on the boundary, and forces $\boldsymbol{f}$ acting in the domain by adjusting the inverse permeability α of the medium occupying Ω, which depends on the control function ρ:

$$\begin{cases} \left.\begin{aligned} -\nu\boldsymbol{\Delta u} + \boldsymbol{u}\cdot\boldsymbol{\nabla u} + \alpha(\rho)\boldsymbol{u} + \boldsymbol{\nabla}p = \boldsymbol{f}, \\ \operatorname{div}\boldsymbol{u} = 0 \end{aligned}\right\}, & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g}, & \text{on } \Gamma. \end{cases} \tag{1}$$

In the system (1), $\boldsymbol{u}$ is the flow velocity, p is the pressure, and ν is the kinematic viscosity. Of course, owing to the incompressibility of $\boldsymbol{u}$, the function $\boldsymbol{g}$ must satisfy the compatibility condition

$$\int_\Gamma \boldsymbol{g}\cdot\boldsymbol{n} = 0, \tag{2}$$

where $\boldsymbol{n}$ denotes the outward unit normal. If $\alpha(\rho(\boldsymbol{x})) = +\infty$ for some $\boldsymbol{x} \in \Omega$, we simply require $\boldsymbol{u}(\boldsymbol{x}) = 0$ in the first equation of (1).

Our control set $\mathcal{H}$ is defined as follows:

$$\mathcal{H} = \{\, \rho \in L^\infty(\Omega) \mid 0 \le \rho \le 1, \text{a.e. in } \Omega, \int_\Omega \rho \le \gamma|\Omega| \,\},$$

where $0 < \gamma < 1$ is the maximal volume fraction that can be occupied by the fluid. Every element $\rho \in \mathcal{H}$ describes the scaled Darcy permeability tensor of the medium at a given point $\boldsymbol{x} \in \Omega$ in the following (informal) way: $\rho(\boldsymbol{x}) = 0$ corresponds to zero permeability at $\boldsymbol{x}$ (i.e., solid, which does not permit any flow at a given point), while $\rho(\boldsymbol{x}) = 1$ corresponds to infinite permeability (i.e., 100% flow region; no structural material is present). Formally, we relate the permeability α^{-1} to ρ using a convex, decreasing, and nonnegative function (cf. [BoP03, Evg03]) $\alpha : [0, 1] \to \mathbb{R}_+ \cup \{+\infty\}$, defined as

$$\alpha(\rho) = \rho^{-1} - 1.$$

In the rest of the paper we will use the symbol χ_A for $A \subset \Omega$ to denote the characteristic function of A: $\chi_A(\boldsymbol{x}) = 1$ for $\boldsymbol{x} \in A$; $\chi_A(\boldsymbol{x}) = 0$ otherwise.

2.2 Variational formulation

To state the problem in a more analytically suitable way and to incorporate the special case $\alpha = +\infty$ into the first equation of the system (1), we introduce a weak formulation of the equations. Let us consider the sets of admissible flow velocities:

$$\begin{aligned} \mathcal{U} &= \{\, \boldsymbol{v} \in H^1(\Omega) \mid \boldsymbol{v} = \boldsymbol{g} \text{ on } \Gamma \,\}, \\ \mathcal{U}_{\text{div}} &= \{\, \boldsymbol{v} \in \mathcal{U} \mid \operatorname{div}\boldsymbol{v} = 0, \text{weakly in } \Omega \,\}. \end{aligned}$$

Let $\mathcal{J}^S : \mathcal{U} \to \mathbb{R}$ denote the potential power of the viscous flow:

$$\mathcal{J}^S(\boldsymbol{u}) = \frac{\nu}{2}\int_\Omega \boldsymbol{\nabla u}\cdot\boldsymbol{\nabla u} - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}.$$

Let us further define the additional power dissipation $\mathcal{J}^{\mathcal{D}} : \mathcal{H} \times \mathcal{U} \to \mathbb{R} \cup \{+\infty\}$, due to the presence of the porous medium (we use the standard convention $0 \cdot +\infty = 0$):

$$\mathcal{J}^{\mathcal{D}}(\rho, \boldsymbol{u}) = \frac{1}{2} \int_{\Omega} \alpha(\rho) \boldsymbol{u} \cdot \boldsymbol{u}.$$

Finally, let $\mathcal{J}(\rho, \boldsymbol{u}) = \mathcal{J}^{\mathcal{S}}(\boldsymbol{u}) + \mathcal{J}^{\mathcal{D}}(\rho, \boldsymbol{u})$ denote the total power of the Brinkman flow. Then, the requirement "$\alpha(\rho) = +\infty \implies \boldsymbol{u} = 0$" is automatically satisfied if $\mathcal{J}^{\mathcal{D}}(\rho, \boldsymbol{u}) < +\infty$.

We will use epi-convergence of optimization problems as a main theoretical tool in the subsequent analysis, thus it is natural to study the following variational formulation (cf., e.g., [Evg03]) for Darcy-Stokes flows [i.e., obtained by neglecting the convection term $\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}$ in (1)]: for $\boldsymbol{f} \in L^2(\Omega)$, compatible $\boldsymbol{g} \in H^{1/2}(\Gamma)$, and $\rho \in \mathcal{H}$, find $\boldsymbol{u} \in \mathcal{U}_{\text{div}}$ such that

$$\boldsymbol{u} \in \operatorname*{argmin}_{\boldsymbol{v} \in \mathcal{U}_{\text{div}}} \mathcal{J}(\rho, \boldsymbol{v}).$$

Naturally, taking convection into account, this can be generalized to the following fixed point-type formulation of (1) (see Subsection 5.2 for the rigorous discussion of its well-posedness): for $\boldsymbol{f} \in L^2(\Omega)$, compatible $\boldsymbol{g} \in H^{1/2}(\Gamma)$, and $\rho \in \mathcal{H}$ find $\boldsymbol{u} \in \mathcal{U}_{\text{div}}$ such that

$$\boldsymbol{u} \in \operatorname*{argmin}_{\boldsymbol{v} \in \mathcal{U}_{\text{div}}} \left\{ \mathcal{J}(\rho, \boldsymbol{v}) + \int_{\Omega} (\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}) \cdot \boldsymbol{v} \right\}. \tag{3}$$

3 Problems with the existing approach

When we allow impenetrable walls to appear and to disappear in the design domain, we create two particular types of difficulties, each related to a corresponding change in topology (see Subsection 3.1 and 3.2). We note that in the "sizing" case, which can be modeled by introducing an additional design constraint $\rho \geq \varepsilon$, a.e. in Ω (for some small $\varepsilon > 0$) these difficulties do not appear. (In fact, it is an easy exercise to verify that under such circumstances the design-to-flow mapping is closed w.r.t. strong convergence of designs, e.g., in $L^1(\Omega)$, and $H^1(\Omega)$-weak convergence of flows.) Such a distinct behavior of the sizing and topology optimization problems may indicate that the former is not a useful approximation of the latter in this case.

3.1 Disappearing walls

For the sake of simplicity, in this subsection we assume that the objective functional in our control problem (which is not formally stated yet) is the power $\mathcal{J}$ of the incompressible Navier–Stokes flow. This functional is interesting from at least two points of view. Firstly, in many cases the resulting control problem is equivalent to the minimization of the drag force or pressure drop, which is very important in engineering applications [BoP03]. Secondly, while it is intuitively clear that impenetrable inner walls of vanishing thickness change the flow in a discontinuous way, for the Stokes flows the total potential power is lower semi-continuous w.r.t. such changes, which allows us to apply the Weierstrass theorem and ensure the existence of optimal designs (cf. [Evg03, Theorem 3.3]). In this subsection we consider two examples illustrating the discontinuity of the flow as well as non-lower semicontinuity of the power functional in the case of the incompressible Navier–Stokes equations; this means that the corresponding control problem of minimizing the potential power is ill-posed, at least from the point of view of the direct method of calculus of variations.

Example 3.1 (Infinitely thin wall). We consider a variant of the backstep flow with $\nu = 1.0 \cdot 10^{-3}$ (which corresponds to the Reynolds number Re = 1000), as shown in Figure 1. We specify $\boldsymbol{u}$ on the inflow boundary to be $(0.25 - (y - 0.5)^2, 0.0)^t$, on the outflow boundary we require $u_y = 0$ as well as $p = 0$; on the rest of the boundary the no-slip condition $\boldsymbol{u} = \boldsymbol{0}$ is assumed. We consider a sequence of the domains containing a thin but impenetrable wall of vanishing thickness (as shown in Figure 1 by dashed line). The limiting domain is the usual backstep shown with the solid line. Direct numerical computation in Femlab (see Figure 2 showing the flows) shows that for the domains with thin wall we have $\mathcal{J} \approx 0.8018$, while for the limiting domain $\mathcal{J} \approx 0.8263$. This demonstrates the non-lower semicontinuity of the total power functional in the case of incompressible Navier–Stokes equations.

We note that while the "jump" of the power functional may seem negligible in this example, other examples may be constructed where this jump is much bigger.

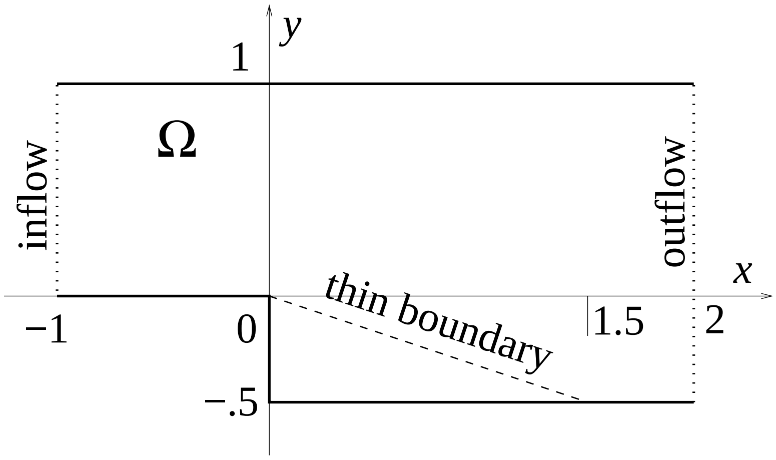


Figure 1: Flow domain for the backstep flow.

It may be argued that in the example above the thin wall may be substituted by the complete filling of the resulting isolated subdomain with impenetrable material, and the following example is more peculiar and demonstrates that we can control the behavior of the Navier–Stokes flow with an infinitesimal amount of material. It is interesting to note that the example is based on the construction of Allaire [All90a], which in some sense is "opposite" to our design parametrization. Namely, we try to control the Navier–Stokes equations by adjusting the coefficients in the nonlinear Brinkman equations, while the sequence of perforated domains considered in Example 3.2 has been used to obtain the nonlinear Brinkman equations starting from the Navier–Stokes equations in a periodically perforated domain as a result of the homogenization process.

Example 3.2 (Perforated domains with tiny holes). We assume that the boundary Γ is *smooth* and impenetrable (i.e., the homogeneous boundary conditions $\boldsymbol{g} = \mathbf{0}$ hold), and that the viscosity ν is large enough relatively to the force $\boldsymbol{f}$ to guarantee the existence of a unique solution to the Navier–Stokes system in Ω. Let Ω^ε denote a perforated domain, obtained from Ω by taking out spheres of radius $r_d(\varepsilon)$ with centers $\varepsilon\mathbb{Z}^d$, where $\lim_{\varepsilon\to+0} r_d(\varepsilon)/\varepsilon = 0$; see Figure 3. Let $(\widetilde{\boldsymbol{u}}^\varepsilon, \widetilde{p}^\varepsilon)$ denote a solution to the Navier–Stokes problem inside Ω^ε with homogeneous boundary conditions $\widetilde{\boldsymbol{u}}^\varepsilon = \mathbf{0}$ on $\partial\Omega^\varepsilon$. We extend $\widetilde{\boldsymbol{u}}^\varepsilon$ onto the whole Ω by setting it to zero inside each sphere; we further denote by $\boldsymbol{u}^\varepsilon$ this extended solution. For every small $\varepsilon > 0$ it holds that $\boldsymbol{u}^\varepsilon$ solves the problem (3) for $\rho^\varepsilon = \chi_{\Omega^\varepsilon}$. Allaire [All90a] has shown that depending on the limit $C = \lim_{\varepsilon\to+0} r_d(\varepsilon)/\varepsilon^3$ for $d = 3$, or $C = \lim_{\varepsilon\to+0} -\varepsilon^2 \log(r_d(\varepsilon))$ for $d = 2$, there are three limiting cases:

- $C = 0$: $\{\boldsymbol{u}^\varepsilon\}$ converges strongly in $H^1(\Omega)$ towards the solution to the Navier–Stokes problem in the unperforated domain Ω, i.e., the solution to the problem (3) corresponding to $\rho = 1$ (see [All90b, Theorem 3.4.4]);
- $C = +\infty$: $\{\boldsymbol{u}^\varepsilon\}$ converges towards 0 strongly in $H^1(\Omega)$ (in fact, there is more information about $\{\boldsymbol{u}^\varepsilon\}$ available, see [All90b, Theorem 3.4.4]);
- $0 < C < +\infty$: $\{\boldsymbol{u}^\varepsilon\}$ converges weakly in $H^1(\Omega)$ towards the solution to the nonlinear Brinkman problem in the unperforated domain Ω, i.e., the solution of the problem (3) corresponding to $\rho = \sigma$, for a computable constant $\sigma(d, \nu, C) > 0$ (see [All90a, Main Theorem]).

We note that in all three cases the sequence of designs $\{\rho^\varepsilon\}$ strongly converges to zero in $L^1(\Omega)$, while only in the case $C = 0$ the corresponding sequence of flows converges to the "correct" flow. As for the other two cases, we can either completely stop ($C = +\infty$) or just slow ($0 < C < +\infty$) the flow using only infinitesimal amounts of structural material (recall that $r_d(\varepsilon)/\varepsilon \to +0$). Moreover, the sequence of perimeters of ρ^ε converges to zero, and therefore the perimeter constraint cannot enforce the convergence of flows in this case (contrary to the situation in linear elasticity, cf. [BeS03, p. 31]). In the same spirit, the regularized intermediate density control method considered by Borrvall and Petersson [BoP01] classifies the designs ρ^ε as regular for all enough small $\varepsilon > 0$ (since they are indeed close to a regular design $\rho \equiv 0$ in the strong topology of $L^p(\Omega)$, $1 \le p < \infty$); thus the latter method also fails to recognize the pathological cases illustrated in the present example.

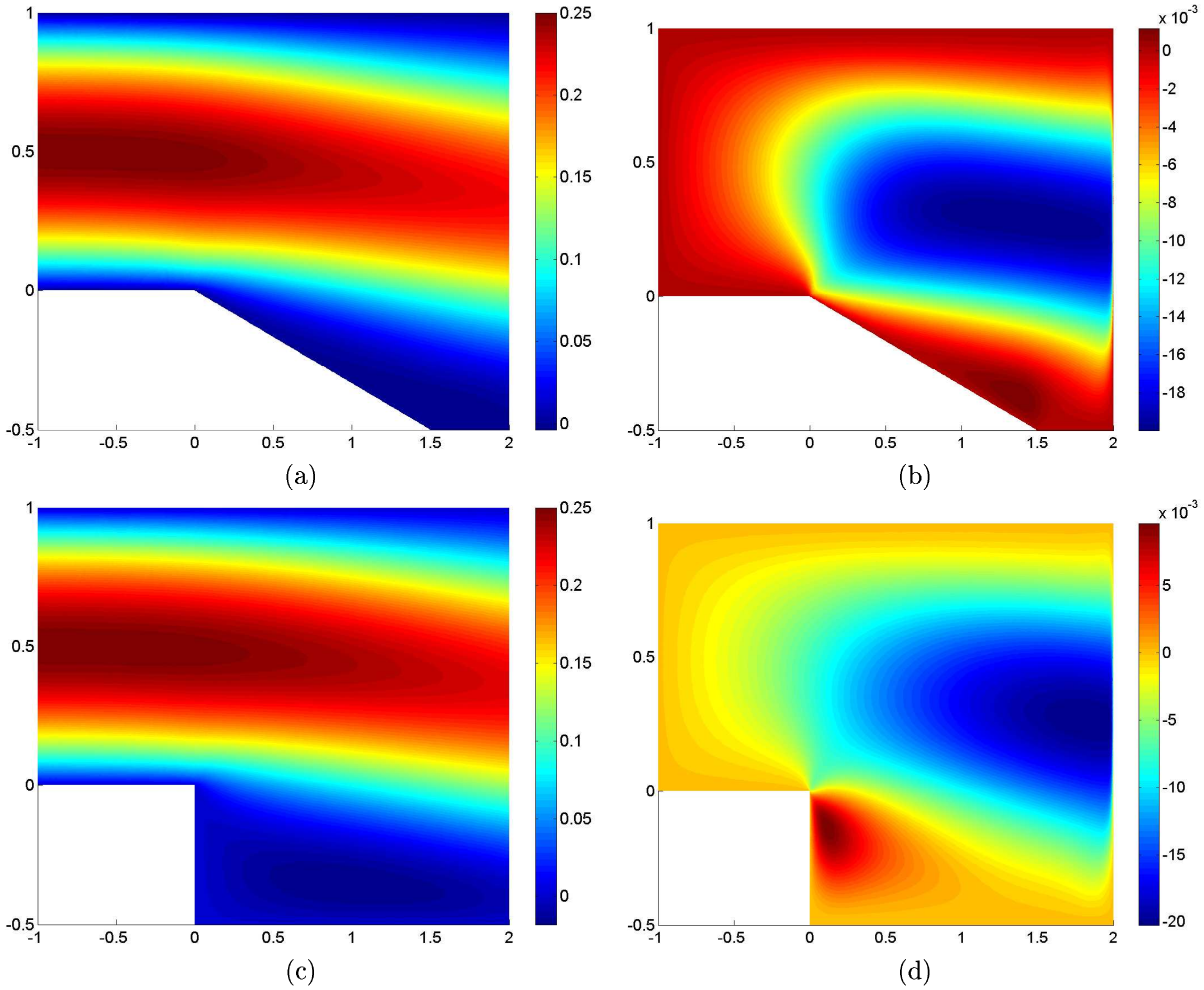


Figure 2: Backstep flow: Example 3.1. (a), (b): x- and y-components, respectively, of the flow velocity when the impenetrable wall has arbitrary but positive thickness (only the part of the domain with nontrivial flow is shown); (c), (d): x- and y-components, respectively, of the flow velocity as the impenetrable wall disappears. *Note the different color scales.*

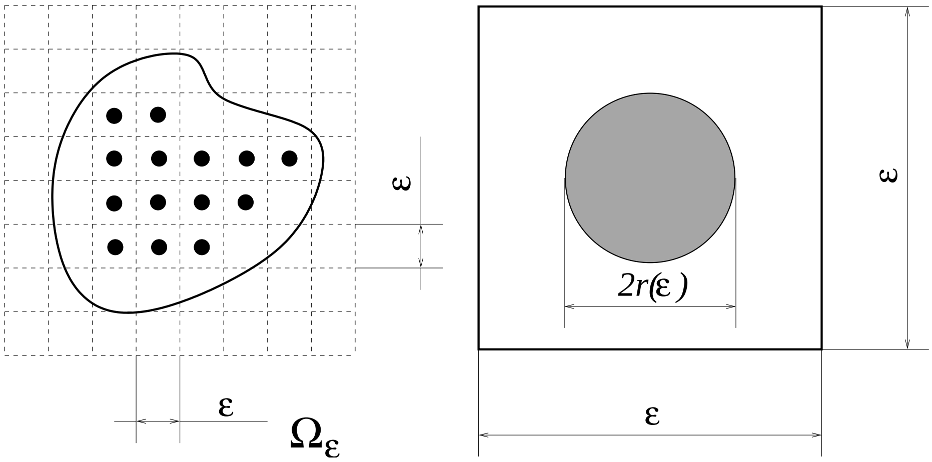
3.2 Appearing walls

Walls that appear in the domain as a result of the optimization process may break the connectivity of the flow domain (or some parts of it), so that the incompressible Navier–Stokes system may not admit any solutions in the limiting domain (resp., some parts of it). While obtaining such results may seem to be a failure of the optimization procedure, completely stopping the flow might be interesting (or even optimal) with respect to some engineering design functionals.

The following example is purely artificial and its only purpose is to demonstrate the possible non-closedness of the design-to-flow mapping when new walls appear in the domain. It essentially repeats [Evg03, Example 2.1], but we include it here for convenience of the reader.

Example 3.3 (Domain with diminishing permeability). Let $\Omega = (0,1)^2$, $\boldsymbol{g} \equiv (1,0)^t$, and $\boldsymbol{f} \equiv \mathbf{0}$. Let further $\rho_k \equiv 1/k$ in Ω, $k = 1,2,\ldots$, $\rho \equiv 0$ in Ω, so that $\rho_k \to \rho$, strongly in $L^\infty(\Omega)$ as $k \to \infty$. Then, $\boldsymbol{u} \equiv (1,0)^t$ is a solution of the problem (3) for all $k = 1,2,\ldots$; clearly, $(\rho_k, \boldsymbol{u}) \to (\rho, \boldsymbol{u})$, strongly in $L^\infty(\Omega) \times H^1(\Omega)$. At the same time, it is not difficult to verify that the problem (3) has no solutions for the limiting design ρ, which means that the design-to-flow mapping is not closed even in the strong topology of $L^\infty(\Omega) \times H^1(\Omega)$!

The problem related to the appearence of walls completely stopping the flow in some domains has been solved for Stokes flows by (implicitly) introducing an additional constraint $\mathcal{J}(\rho, \boldsymbol{u}) \leq C$, for a suitable constant C. Owing to the coercivity of $\mathcal{J}$ on $H^1_0(\Omega)$, this keeps the flows in some bounded set. However, in



(a) (b)

Figure 3: The perforated domain (a) and a periodic cell (b).

view of the non-lower semicontinuity of the power functional for the Navier–Stokes flows (see Example 3.1), this set is not necessarily closed, making the problems with appearing walls much more severe in the present case.

We consider the next example in some detail, even though it is quite similar to the previous one, because we will return to it later in Subsection 4.2.

Example 3.4 (Channel with a porous wall). We consider a channel flow at Reynolds number Re $=$ 1000 ($\nu = 1.0 \cdot 10^{-3}$) through a wall made of porous material with vanishing permeability appearing in the middle of the channel (see Figure 4). We specify $\boldsymbol{u}$ on the inflow boundary to be $(1-y^2, 0.0)^t$, on the outflow

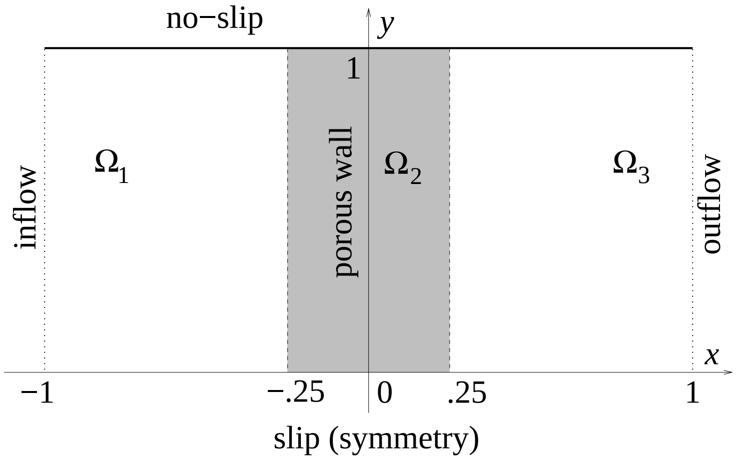


Figure 4: Flow domain of Example 3.4.

boundary we require $u_y = 0$ as well as $p = 0$; on the rest of the boundary the no-slip condition $\boldsymbol{u} = \boldsymbol{0}$ is assumed except that on the "lower" edge we have slip (i.e., only $u_y = 0$) due to the symmetry.

We choose ρ so that $\alpha(\rho) = 0$ on $\Omega_1 \cup \Omega_3$ and $\alpha(\rho) = \alpha$ on Ω_2, where α assumes values 1.0, $1.0 \cdot 10^2$, $1.0 \cdot 10^4$, $+\infty$. The corresponding flows (calculated in Femlab) are shown in Figure 5; the incompressible Navier–Stokes problem in the last (limiting as $\alpha \to +\infty$) domain admits no solutions.

To summarize, even though the sequence of designs $\rho_\alpha \to \chi_{\Omega_1 \cup \Omega_3}$, strongly in $L^\infty(\Omega)$, the corresponding sequence of flows does not converge to the flow corresponding to the limiting design, simply because the latter does not exist.

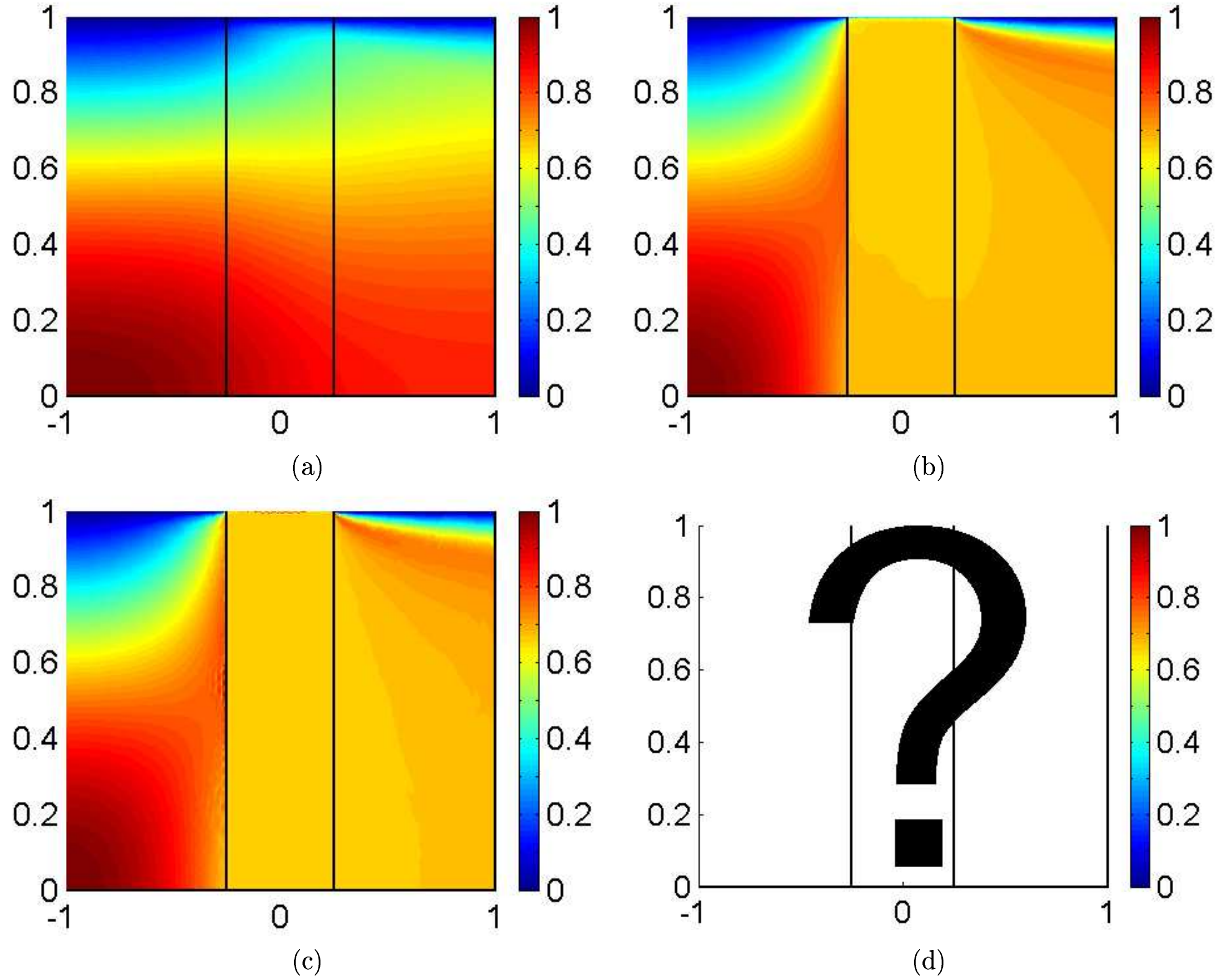


Figure 5: Incompressible flow through the porous wall: (a) $\alpha = 1.0$, (b) $\alpha = 1.0 \cdot 10^2$, (c) $\alpha = 1.0 \cdot 10^4$, (d) $\alpha = +\infty$.

4 Proposed solutions to the difficulties outlined

Difficulties inherent in the straightforward generalization of the methodology proposed by Borrvall and Petersson [BoP03] for Stokes flows to incompressible Navier–Stokes flows have been outlined in Section 3. One possible solution, which allows us to avoid these difficulties, is simply to forbid topological changes and to perform sizing optimization, interpreting optimal designs as distributions of porous materials with spatially varying permeability (cf. [All90a, All90b]; see also [Hor97]). As it has already been mentioned the resulting designs may or may not accurately describe the domains obtained by substituting the materials with high permeability by void, and those with low permeability by impenetrable walls. Furthermore, if we decide to keep the porous material, it is questionable whether such designs can be easily manufactured and thus it is unclear whether they are "better" from the engineering point of view. Thus we do not employ this approach but instead try to slightly modify the design parametrization as well as the underlying state equations with the ultimate goal to rigorously obtain a closed design-to-flow mapping while maintaining a clear engineering/physical meaning of our optimization model.

4.1 Filters in the topology optimization

In both examples in Subsection 3.1 we constructed the sequences of designs having very small details, which disappear in the limit. Using the notion of a filter [Sig97, SiP98] we can control the minimal scale of our designs; we will employ this technique, which has become quite standard in topology optimization of linearly elastic materials [BeS03].

Following Bourdin [Bou01], and Bruns and Tortorelli [BrT01], we define a *filter* $F : \mathbb{R}^d \to \mathbb{R}$ *of charac-*

teristic radius $R > 0$ to be a function verifying the following properties:

$$F \in C^{0,1}(\mathbb{R}^d), \qquad \operatorname{supp} F \Subset B_R,$$
$$F \geq 0 \text{ in } B_R, \qquad \int_{B_R} F = 1,$$

where B_R denotes the open ball of radius R centered in origo. We denote the convolution product by a $*$ sign, i.e.

$$(F * \rho)(\boldsymbol{x}) = \int_{\mathbb{R}^d} F(\boldsymbol{x} - \boldsymbol{y})\rho(\boldsymbol{y})d\boldsymbol{y}.$$

Owing to the Lipschitz continuity of F, $F * \rho$ is a continuous function (cf. [Bre83, Proposition IV.19]).

In order to compute the convolution between the filter and a given design ρ the latter must be defined not only on Ω, but also on the whole space $\mathbb{R}^d$. Therefore, in the sequel we consider the following redefined design domain:

$$\mathcal{H} = \{\, \rho \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \mid 0 \leq \rho \leq 1, \text{a.e. in } \mathbb{R}^d, \int_{\mathbb{R}^d} \rho \leq V \,\},$$

for a given $V > 0$.

One of the consequences of the fact that F is Lipschitz continuous in $\mathbb{R}^d$ and not just in B_R is that the following important *growth condition* is verified (see Proposition A.1):

$$(F * \chi_{\mathbb{R}^d \setminus \operatorname{supp} F})(\boldsymbol{x}) \leq C|\boldsymbol{x}|^2, \tag{4}$$

as $|\boldsymbol{x}| \to 0$, for some appropriate constant $C > 0$, which implies that $\alpha((F * \rho)(\cdot))$ grows at least as fast as $\operatorname{dist}^{-2}(\cdot, \{\, F * \rho = 0 \,\})$ arbitrarily near to impenetrable walls. It is this condition that allows us to prove an approximation result, Proposition 5.3, which is in turn the key ingredient in the proof of our closedness theorems.

For notational convenience we set $\mathcal{J}^F(\rho, \boldsymbol{u}) = \mathcal{J}(F * \rho, \boldsymbol{u})$. As a consequence of the introduction of the filter, we can demonstrate the following simple claim, which translated to normal language says that impenetrable walls cannot disappear in the limit. In the following Proposition, Lim sup is understood in the sense of Painlevé-Kuratowski, see [AuF90, Definition 1.4.6], or [BoS00, Definition 2.52].

Proposition 4.1. *Consider an arbitrary sequence of designs* $\{\rho_k\} \subset \mathcal{H}$*, such that* $\rho_k \rightharpoonup \rho$*, weakly in* $L^1_{\text{loc}}(\mathbb{R}^d)$*, for some* $\rho \in \mathcal{H}$*. Define a sequence* $\{\Omega_0^k\}$ *of subsets of* Ω *as*

$$\Omega_0^k = \{\, \boldsymbol{x} \in \Omega \mid (F * \rho_k)(\boldsymbol{x}) = 0 \,\},$$
$$\Omega_0^\infty = \{\, \boldsymbol{x} \in \Omega \mid (F * \rho)(\boldsymbol{x}) = 0 \,\}.$$

Then, $\operatorname{Lim\,sup}_{k\to\infty} \Omega_0^k \subset \Omega_0^\infty \cup \Gamma$.

Proof. Let $I \subset \mathbb{N}$ be an infinite subsequence of indices, such that for some $\boldsymbol{x}_k \in \Omega_0^k$, $k \in I$, there exists $\boldsymbol{x} \in \mathbb{R}^d$ such that $\boldsymbol{x} = \lim_{k \in I} \boldsymbol{x}_k$. We know that $\rho_k \equiv 0$ a.e. on $\boldsymbol{x}_k + \operatorname{supp} F$, $k \in I$. Then, $\rho \equiv 0$ a.e. on $\boldsymbol{x} + \operatorname{supp} F$, i.e., $(F * \rho)(\boldsymbol{x}) = 0$. Clearly, $\boldsymbol{x} \in \operatorname{cl} \Omega$, which finishes the proof. □

Remark 4.2. The convergence of flow domains $\Omega \setminus \Omega_0^k$ induced by the weak convergence of designs (which implies strong convergence of filtered designs) can be compared to the convergence of domains in some topology defined for set convergence, e.g., the complementary Hausdorff topology. It is known, in general, that the latter topology is weaker (see, e.g., [SoZ92, Section 2.6.2]). However, such a comparison is not quite fair in the present situation, where the domains we deal with can be rather irregular (e.g., lie on two sides of their boundaries), and, more importantly, the domains in the sequence may have different connectivity compared to the "limiting" domain.

Later we will see that we need even stronger convergence of $\Omega_0^k \to \Omega_0^\infty$ to obtain closedness of the design-to-flow mappings.

The use of filtered designs $F * \rho$ in place of ρ in problem (3) allows us to overcome the difficulties caused by disappearing walls. While we delay the formal statement of this fact until Section 5, at this point we can consider an example that illustrates the effect of using filters.

Example 4.3 (Example 3.2 revisited). Consider an arbitrary filter F and a sequence of designs $\{\rho_\varepsilon\}$ defined in Example 3.2. Let for every $\varepsilon > 0$ extend the definition of ρ_ε (that has been defined only on Ω) by setting $\rho_\varepsilon(\boldsymbol{x}) = 1$ for all $\boldsymbol{x} \in (\Omega + \operatorname{supp} F) \setminus \Omega$, and $\rho_\varepsilon(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \mathbb{R}^d \setminus (\Omega + \operatorname{supp} F)$. Then, $F * \rho_\varepsilon \to 1$ as $\varepsilon \to +0$, uniformly in $\operatorname{cl}\Omega$, and the corresponding sequence of flows converges to a pure Navier–Stokes flow in the domain Ω (case $C = 0$ in Example 3.2).

4.2 Slightly compressible fluids

While it seems difficult to imagine a reasonable cure for Example 3.3, because the limiting flow must be zero on Ω with nonzero trace on Γ, we can at least try to get a closed design-to-flow mapping if impenetrable walls do not appear too close to the boundary with non-homogeneous Dirichlet conditions on velocity, as in Example 3.4. The difficulty in the latter example is that in our model the porous wall does not stop, or slow, the incompressible fluid while we use material with positive permeability. At the same time, the limiting domain does not permit any incompressible flow through it, because it is not connected.

We can solve this problem by relaxing the incompressibility requirement $\operatorname{div} \boldsymbol{u} = 0$ in the system (1) [of course, we do not need to require the compatibility condition (2) in this case]. For example, we may assume that the fluid is *slightly compressible*, i.e., choose a small $\delta > 0$ and let $\operatorname{div} \boldsymbol{u} + \delta p = 0$. In fact, it is known that for a fixed domain admitting an incompressible flow, the difference between the regular incompressible and slightly compressible flows is of order δ, i.e., we change model only slightly if δ is small enough. The slightly compressible Navier–Stokes equations are often used as approximations of incompressible ones in so-called *penalty algorithms* [Gun89, Chapter 5]. On the other hand, with the gained maturity of mixed finite element methods, the incompressible system can be equally well solved to approximate the behavior of slightly compressible fluids [Tem01].

Whether one considers slightly compressible Navier–Stokes fluids to be the most suitable mathematical model of the underlying physical flow (see Remark 4.5) or just an accurate approximation of the incompressible Navier–Stokes equations, we make an assumption of slight compressibility because it allows us to achieve the ultimate goal of this paper: to obtain a closed design-to-flow mapping. Again, delaying the precise formulations until Section 5, we revisit Example 3.4 to illustrate our point.

Example 4.4 (Example 3.4 revisited). We choose $\delta = 1.0 \cdot 10^{-3}$ and resolve the flow problem of Example 3.4 for $\alpha \in \{\, 1.0, 1.0 \cdot 10^2, 1.0 \cdot 10^4, +\infty \,\}$. The corresponding flows (calculated in Femlab) are shown in Figure 6; in contrast with the incompressible Navier–Stokes case we can see the convergence of flows as domains converge (i.e., as α increases) to a limiting flow, which exists in the compressible case. Note that for small values of α and δ the incompressible and the slightly compressible flows look similar.

Remark 4.5. It is known that the pseudo-constitutive relation $\operatorname{div} \boldsymbol{u} + \delta p = 0$ lacks an adequate physical interpretation for many important physical flows (e.g., see [HeV95]). In particular, there is no physical pressure field compatible with the flow shown in Figure 6 (d). On the other hand, the pseudo-constitutive relation resulting from the penalty method can still be used as a mathematical method of generating flows approximating those of incompressible viscous fluids. Moreover, the idea of relaxing the incompressibility contraint may also be useful for topology optimization in fluid *dynamics*, where the corresponding relation $\operatorname{div} \boldsymbol{u} + \delta dp/dt = 0$ is known to be physical.

5 Continuity of the design-to-flow mapping

5.1 Stokes flows

We start by showing the closedness of the design-to-flow mapping for slightly compressible Stokes flows with homogeneous boundary conditions, and then show the necessary modifications for the inhomogeneous boundary conditions. For the compressible Stokes system the variational formulation is as follows. Given $\rho \in \mathcal{H}$, find the solution to the following minimization problem:

$$\min_{\boldsymbol{v} \in \mathcal{U}} \left\{ \mathcal{J}^F(\rho, \boldsymbol{v}) + (2\delta)^{-1} \int_\Omega (\operatorname{div} \boldsymbol{v})^2 \right\}. \tag{5}$$

We note that in the case of homogeneous boundary conditions we have $\mathcal{U} = H_0^1(\Omega)$.

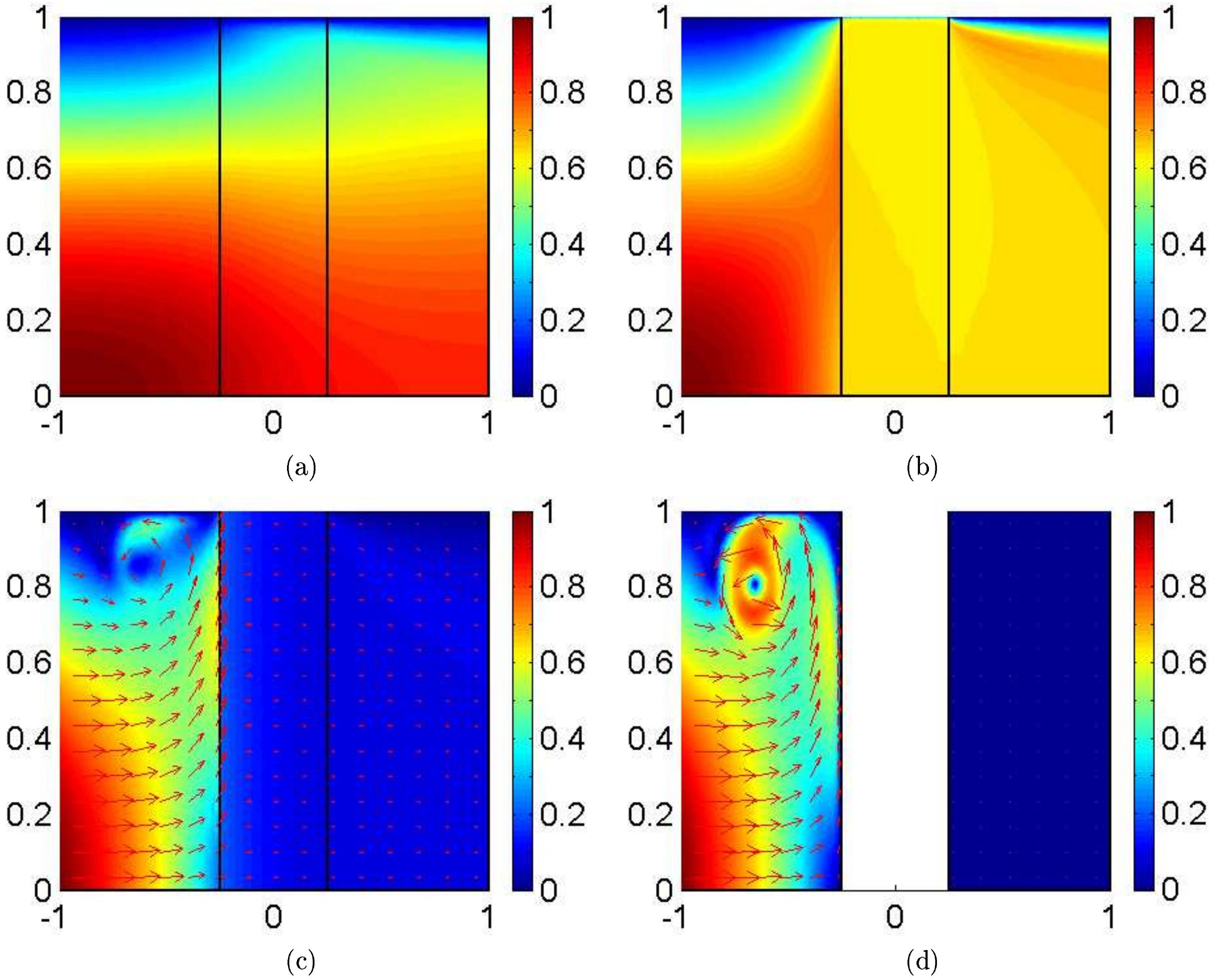


Figure 6: Compressible flow through the porous wall: (a) $\alpha = 1.0$, (b) $\alpha = 1.0 \cdot 10^2$, (c) $\alpha = 1.0 \cdot 10^4$, (d) $\alpha = +\infty$. Compare with Figure 5.

Remark 5.1. Since the condition $\operatorname{div} \boldsymbol{u} = 0$ is violated, we should replace the term $\int_\Omega |\nabla \boldsymbol{u}|^2$ in the definition of $\mathcal{J}^S$ with $\int_\Omega |E(\boldsymbol{u})|^2$, where $E(\boldsymbol{u}) = (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^t)/2$ is the linearized rate of strain tensor (cf. [Gun89, Section 4.3]). However, both quadratic forms give rise to equivalent norms on $H_0^1(\Omega)$ (cf. [CaK84, Bre83]) and thus do not affect our theoretical developments in any way. Therefore, we choose to keep the definition of $\mathcal{J}^S$ for notational simplicity.

In fact, one can go one step further and replace the term $\int_\Omega |\nabla \boldsymbol{u}|^2$ with $\int_\Omega \mathcal{P}(|E(\boldsymbol{u})|)$, where $\mathcal{P}$ is a positive convex function verifying certain growth assumptions, thus including non-Newtonian flows into the discussion [FuS00, Chapters 3 and 4]. For some functionals this will not affect the discussion, while for others (e.g., Prandtl-Eyring fluids) we must reconsider the very basic problem statements [such as (5)]. Therefore, in this paper we consider Newtonian fluids only (that is, the case $\mathcal{P}(x) = x^2$) and discuss possible extensions in Section 8.

Proposition 5.2. *For every design $\rho \in \mathcal{H}$ the optimization problem (5) has a unique solution $\boldsymbol{v} \in H^1(\Omega)$ whenever its objective functional is proper w.r.t. $\mathcal{U}$, in particular if $\mathcal{U} = H_0^1(\Omega)$.*

Proof. See Appendix A. □

The proof of the main theorem of this section, Theorem 5.4, which establishes the continuity of the design-to-flow mapping in the case of Stokes flow with homogeneous boundary conditions, heavily depends on the following approximation result. Its proof can be found in the Appendix A.

Proposition 5.3. *Let $\boldsymbol{u} \in H_0^1(\Omega)$, $\rho \in \mathcal{H}$, and $\mathcal{J}^F(\rho, \boldsymbol{u}) \le M < +\infty$. Define also $\Omega_0 = \{\, \boldsymbol{x} \in \Omega \mid (F * \rho)(\boldsymbol{x}) = 0 \,\}$. Then, there exists a sequence $\{\boldsymbol{u}_k\} \subset H_0^1(\Omega)$ such that:*

(*i*) $\operatorname{supp} \boldsymbol{u}_k \Subset (\Omega \setminus \Omega_0)$;
(*ii*) $\lim_{k\to+\infty} \boldsymbol{u}_k = \boldsymbol{u}$, *strongly in* $H^1_0(\Omega)$;
(*iii*) $\limsup_{k\to+\infty} \mathcal{J}^F(\rho, \boldsymbol{u}_k) \le M$.

Theorem 5.4. *Consider a sequence of designs* $\{\rho_k\} \subset \mathcal{H}$ *and the corresponding sequence of flows* $\{\boldsymbol{u}_k\} \subset H^1_0(\Omega)$, $k = 1, 2, \dots$ *(i.e.,* $\boldsymbol{u}_k$ *solves the problem* (5) *for* ρ_k*). Assume that* $\rho_k \to \rho_0$*, strongly in* $L^1(\Omega + B_R)$*, and* $\boldsymbol{u}_k \rightharpoonup \boldsymbol{u}_0$*, weakly in* $H^1_0(\Omega)$*. Then,* $\boldsymbol{u}_0$ *is the flow corresponding to the limiting design* ρ_0*.*

Proof. Throughout the proof we denote the optimal value of the optimization problem (5) for a given design ρ as $\operatorname{val}(\rho)$. Owing to the weak lower-semicontinuity of $\mathcal{J}$ (cf. [Evg03, Lemma 3.2]) and the weak lower-semicontinuity of $\int_\Omega (\operatorname{div} \boldsymbol{u})^2$ (cf. [EkT99, Corollary 2.2]) we have that

$$\operatorname{val}(\rho_0) \le \mathcal{J}^F(\rho_0, \boldsymbol{u}_0) + (2\delta)^{-1} \int_\Omega (\operatorname{div} \boldsymbol{u}_0)^2 \le \liminf_{k\to\infty} \operatorname{val}(\rho_k). \tag{6}$$

If we can also show that $\operatorname{val}(\rho_0) \ge \limsup_{k\to\infty} \operatorname{val}(\rho_k)$, then since $\operatorname{val}(\rho_0) < +\infty$ (owing to Proposition 5.2) we must have equality throughout in (6), which means that $\boldsymbol{u}_0$ solves (5) for ρ_0.

Without any loss of generality, we assume that $\operatorname{val}(\rho_0) = \lim_{k\to\infty} \operatorname{val}(\rho_k)$. Let $\widetilde{\boldsymbol{u}}_0$ be the optimal solution of (5), and consider a sequence $\{\boldsymbol{u}^n_0\} \subset H^1_0(\Omega)$ constructed in Proposition 5.3 for ρ_0 and $\widetilde{\boldsymbol{u}}_0$. Due to the properties of $\{\boldsymbol{u}^n_0\}$, for every $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that for all $n > N(\varepsilon)$ it holds that

$$\operatorname{val}(\rho_0) + \varepsilon > \mathcal{J}^F(\rho_0, \boldsymbol{u}^n_0) + (2\delta)^{-1} \int_\Omega (\operatorname{div} \boldsymbol{u}^n_0)^2. \tag{7}$$

Moreover, strong convergence of ρ_k together with Lipschitz continuity of F imply uniform convergence of $F * \rho_k$ towards $F * \rho_0$ on $\operatorname{cl}\Omega$ (cf. [Bre83, Théorème IV.15]). Since $\boldsymbol{u}^n_0 \Subset \Omega \setminus \{\, \boldsymbol{x} \in \Omega \mid (F * \rho_0)(\boldsymbol{x}) = 0 \,\}$, it holds that $\alpha(F * \rho_k)$ uniformly converges towards $\alpha(F * \rho_0)$ on $\operatorname{supp} \boldsymbol{u}^n_0$, and thus there is $K(n, \varepsilon) \in \mathbb{N}$ such that for all $k > K(n, \varepsilon)$ we have

$$\mathcal{J}^F(\rho_0, \boldsymbol{u}^n_0) + (2\delta)^{-1} \int_\Omega (\operatorname{div} \boldsymbol{u}^n_0)^2 + \varepsilon > \mathcal{J}^F(\rho_k, \boldsymbol{u}^n_0) + (2\delta)^{-1} \int_\Omega (\operatorname{div} \boldsymbol{u}^n_0)^2 \ge \operatorname{val}(\rho_k), \tag{8}$$

where the last inequality is due to the feasibility of $\boldsymbol{u}^n_0$ in (5) for the design ρ_k. Combining (7) and (8), and letting k grow to infinity in the latter we get

$$\operatorname{val}(\rho_0) + 2\varepsilon > \lim_{k\to\infty} \operatorname{val}(\rho_k).$$

Finally, letting ε go to zero, we finish the proof. □

Remark 5.5. Theorem 5.4 shows the epi-convergence of the objective functionals of the ρ-parametric optimization problem (5) as the parameters strongly converge in $L^1(\Omega + B_R)$ (cf. [BoS00, p. 41]).

Remark 5.6. We use strong convergence on the space of designs in order to guarantee the Lipschitz continuity (cf. [AuF90, Definition 1.4.5]) of the family of walls $\{\, \boldsymbol{x} \in \Omega \mid (F * \rho_k)(\boldsymbol{x}) = 0 \,\}$, parametrized by $k \in \mathbb{N}$, which is a stronger property than upper-semicontinuity (cf. Proposition 4.1). We need Lipschitz continuity to justify (8).

In the case of non-homogeneous boundary conditions, the proof is essentially the same provided we can keep the walls away from the regions of the boundary where injection/suction of the fluid is performed; see Subsection 4.2 and Example 3.3 for motivations.

Theorem 5.7. *Consider a sequence of designs* $\{\rho_k\} \subset \mathcal{H}$ *and the corresponding sequence of flows* $\{\boldsymbol{u}_k\} \subset \mathcal{U}$, $k = 1, 2, \dots$ *(i.e.,* $\boldsymbol{u}_k$ *solves the problem* (5) *for* ρ_k*). Assume that* $\rho_k \to \rho_0$*, strongly in* $L^1(\Omega + B_R)$*, and* $\boldsymbol{u}_k \rightharpoonup \boldsymbol{u}_0$*, weakly in* $H^1(\Omega)$*. Further assume that for some positive constants* ε, τ *it holds that*

$$\inf\{\, (F * \rho_k)(\boldsymbol{x}) \mid k \in \mathbb{N}, \boldsymbol{x} \in \Omega \cap (\operatorname{supp} \boldsymbol{g} + B_\varepsilon) \,\} \ge \tau. \tag{9}$$

Then, $\boldsymbol{u}_0$ *is the flow, corresponding to the limiting design* ρ_0 *(i.e.,* $\boldsymbol{u}_0$ *solves the problem* (5) *for* ρ_0*).*

Proof. Let $\boldsymbol{w} \in \mathcal{U}$ be a function with $\operatorname{supp} \boldsymbol{w} \Subset \Omega \cap (\operatorname{supp} \boldsymbol{g} + B_\varepsilon)$. Then, owing to the additional condition (9), the objective functional of (5) is finite when evaluated at $\boldsymbol{w}$, for every ρ_k, $k \in \mathbb{N}$, as well as for ρ_0. Therefore, for every ρ_k, $k \in \mathbb{N}$, (resp., for the limiting design ρ_0) the optimization problem (5) admits a unique optimal solution, which can be written as $\boldsymbol{u}_k = \boldsymbol{w} + \boldsymbol{v}_k$, $\boldsymbol{v}_k \in H_0^1(\Omega)$ (resp., $\widetilde{\boldsymbol{u}}_0 = \boldsymbol{w} + \widetilde{\boldsymbol{v}}_0$, $\widetilde{\boldsymbol{v}}_0 \in H_0^1(\Omega)$).

The epi-convergence of the mappings $H_0^1(\Omega) \ni \boldsymbol{v} \to \mathcal{J}^F(\rho, \boldsymbol{w} + \boldsymbol{v}) + (2\delta)^{-1} \int_\Omega (\operatorname{div}(\boldsymbol{w} + \boldsymbol{v}))^2$ as the parameters ρ strongly converge in $L^1(\Omega + B_R)$ keeping (9) true, can be shown exactly as in the proof of Theorem 5.4. The latter implies the claim. □

Remark 5.8. We note that the condition (9) is automatically verified for Stokes problems with homogeneous boundary conditions, because the infimum is taken over the empty set in this case ($\operatorname{supp} \boldsymbol{g} = \emptyset$).

5.2 Navier–Stokes flows

In the case of the Navier–Stokes equations things get much more complicated, because we do not seek a minimizer of some functional anymore, and we cannot apply epi-convergence results directly. Nevertheless, we can utilize them to show the closedness of the design-to-flow mappings even in the Navier–Stokes case.

We introduce a general fixed-point framework related to the optimization problem (5), and then show (at least for the case of homogeneous boundary conditions) that the slightly compressible Navier–Stokes equations can be considered in this framework.

Let $A(\boldsymbol{u}, \boldsymbol{v}) : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ be a weakly continuous functional, and consider the problem of finding a fixed point of the point-to-set mapping $T_\rho : \mathcal{U} \rightrightarrows \mathcal{U}$ defined for $\rho \in \mathcal{H}$ as

$$T_\rho(\boldsymbol{u}) = \operatorname*{argmin}_{\boldsymbol{v} \in \mathcal{U}} \left\{ \mathcal{J}^F(\rho, \boldsymbol{v}) + (2\delta)^{-1} \int_\Omega (\operatorname{div} \boldsymbol{v})^2 + A(\boldsymbol{u}, \boldsymbol{v}) \right\}. \tag{10}$$

Theorem 5.9. *Consider a sequence of designs $\{\rho_k\} \subset \mathcal{H}$ and the corresponding sequence of fixed points $\{\boldsymbol{u}_k\} \subset \mathcal{U}$, $k = 1, 2, \ldots$ (i.e., $\boldsymbol{u}_k \in T_{\rho_k}(\boldsymbol{u}_k)$ for T_{ρ_k} defined by (10)). Assume that $\rho_k \to \rho_0$, strongly in $L^1(\Omega + B_R)$, $\boldsymbol{u}_k \rightharpoonup \boldsymbol{u}_0$, weakly in $H^1(\Omega)$, and $T(\boldsymbol{u}_0) \neq \emptyset$. Further assume that for some positive constants ε, τ the condition (9) is satisfied. Then, $\boldsymbol{u}_0 \in T_{\rho_0}(\boldsymbol{u}_0)$.*

Proof. It is enough to show that the objective functionals of the parametric optimization problems (10) epi-converge as $(\rho_k, \boldsymbol{u}_k)$ converge towards $(\rho_0, \boldsymbol{u}_0)$. This follows from Theorem 5.7, the continuity of A, and [RoW98, Exercise 7.8.(a)]. □

Remark 5.10. In fact, weak continuity of $A(\boldsymbol{u}, \boldsymbol{v})$ is an unnecessarily strong requirement. We can repeat the arguments of Theorem 5.4 with straightforward modifications and prove Theorem 5.9 under the following weaker assumptions on A:

(*i*) $A(\boldsymbol{u}, \boldsymbol{u}) \le \liminf_{k\to\infty} A(\boldsymbol{u}_k, \boldsymbol{u}_k)$ whenever $\boldsymbol{u}_k \rightharpoonup \boldsymbol{u}$, weakly in $\mathcal{U}$; and
(*ii*) $A(\boldsymbol{u}, \boldsymbol{v}) \ge \limsup_{k\to\infty} A(\boldsymbol{u}_k, \boldsymbol{v}_k)$ whenever $\boldsymbol{u}_k \rightharpoonup \boldsymbol{u}$, weakly in $\mathcal{U}$, and $\boldsymbol{v}_k \to \boldsymbol{v}$, strongly in $\mathcal{U}$.

As an example application of Theorem 5.9, we consider a particular penalty formulation of the incompressible Navier–Stokes equations with homogeneous boundary conditions studied in [CaK84]. A more general treatment is of course possible, including inhomogeneous boundary conditions and variants of slightly compressible Navier–Stokes equations; the main difference is in the number of technical details to be covered.

To put the penalty formulation considered in [CaK84] (of course, without the control term α) into the framework of (10) we define

$$A(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega (\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}) \cdot \boldsymbol{v} + 2^{-1} \int_\Omega (\boldsymbol{u} \cdot \boldsymbol{v}) \operatorname{div} \boldsymbol{u}. \tag{11}$$

We note that the last integral adds an additional stability to the penalty algorithm [CaK84] and identically equals zero in the incompressible case; we can thus expect that the effects of its presence can be almost neglected in the slightly compressible case. Owing to [CaK84, Lemma 2.7], the functional A defined in (11) is weakly continuous on $H_0^1(\Omega) \times H_0^1(\Omega)$, and in order to apply Theorem 5.9 it remains to establish an analogue of Proposition 5.2.

Proposition 5.11. *With $\mathcal{U} = H_0^1(\Omega)$ and A defined by (11), the fixed-point problem (10) admits solutions for every $\rho \in \mathcal{H}$.*

Proof. The functional $A(\boldsymbol{u},\cdot)$ is linear and continuous on $H_0^1(\Omega)$. Applying Lemma A.3 to the "force" $\langle \boldsymbol{f},\cdot\rangle + A(\boldsymbol{u},\cdot) \in H^{-1}(\Omega)$, we conclude that for every $\rho \in \mathcal{H}$ the operator $T_\rho(\boldsymbol{u})$ is single-valued and completely continuous.

Now, assume that $\boldsymbol{w} = \sigma T_\rho(\boldsymbol{w})$ for some $\boldsymbol{w} \in H_0^1(\Omega)$ and $0 < \sigma \le 1$. Then, using the fact that $A(a\boldsymbol{w}, b\boldsymbol{w}) = a^2 b A(\boldsymbol{w},\boldsymbol{w}) = 0$ for all $a, b \in \mathbb{R}$, where the last equality is by [CaK84, Lemma 2.4], and evaluating the objective function of (10) at $\sigma^{-1}\boldsymbol{w}$ (the optimal solution) and $\mathbf{0} \in H_0^1(\Omega)$ we get the inequality

$$\frac{\nu}{2\sigma^2}\int_\Omega |\nabla \boldsymbol{w}|^2 + \frac{1}{2\sigma^2\delta}\int_\Omega (\operatorname{div}\boldsymbol{w})^2 + \frac{1}{2\sigma^2}\int_\Omega \alpha(F*\rho)|\boldsymbol{w}|^2 - \frac{1}{\sigma}\int_\Omega \boldsymbol{f}\cdot\boldsymbol{w} \le 0,$$

which implies that $\|\boldsymbol{w}\|_{H_0^1(\Omega)} \le C\|\boldsymbol{f}\|_{L^2(\Omega)}$ holds, for some constant C independent of σ. An application of the Leray-Schauder Theorem (cf. [GrD03, § 6, Theorem 5.4]) concludes the proof. □

Remark 5.12. While the mapping $(\rho,\boldsymbol{u}) \to T_\rho(\boldsymbol{u})$ is in many cases single-valued for every pair $(\rho,\boldsymbol{u})$, there might be more than one solution to the fixed point problem (10) with this operator. In other words, we do not assume that the compressible Navier-Stokes system admits a unique solution.

Remark 5.13. We can use another popular weak formulation of slightly compressible Navier–Stokes equations (e.g., see [LCW95]), identifying

$$A(\boldsymbol{u},\boldsymbol{v}) = \frac{1}{2}\left(\int_\Omega (\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{v} - \int_\Omega (\boldsymbol{u}\cdot\nabla\boldsymbol{v})\cdot\boldsymbol{u}\right).$$

Our results hold even in this case without any changes.

Remark 5.14. Of course, the fixed-point framework (10) is not bounded to Navier–Stokes equations. For example, putting, for some $\boldsymbol{u}_0 \in \mathbb{R}^d$,

$$A(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega (\boldsymbol{u}_0\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{v},$$

we can show continuity results for Oseen flows [Gal94]. This type of flow is probably not very interesting in bounded domains Ω, but illustrates the possible uses of the fixed-point formulation (10). Finally, we note that setting $A \equiv 0$ we recover the original Stokes problem.

6 Existence of optimal solutions

6.1 Ensuring strong convergence of designs and condition (9)

The results established in Section 5 all require strong convergence of designs in $L^1(\Omega + B_R)$. In order to guarantee convergence we need to embed our controls $\mathcal{H}$ into some space that is more regular than $L^\infty(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$. The most appropriate choice, in our opinion, is the space $SBV(\mathbb{R}^d)$ (cf. [AFP00]), which is typically used for perimeter constrained topology optimization (see [BeS03, p. 31] and references therein; see also [FGR99, Pet99, HBJ96]). Other choices are possible, including $W^{1,1}(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ (that is, imposing "slope constraints" on the design space; see [PeS98], but also [Bor01, BoC03]). Bounds on the perimeter, or slope, may be introduced into the problem directly as constraints, or added as penalties to the objective function.

Regardless of the particular method used, we get the required property: $\rho_k \rightharpoonup \rho$, weakly in $\mathcal{H}$, implies $\rho_k \to \rho$, strongly in $L^1(\Omega + B_R)$, allowing us to establish the closedness of the design-to-flow mappings.

As for the condition (9), it can be easily verified if we require in addition that every design $\rho \in \mathcal{H}$, satisfying the bounds $0 \le \rho \le 1$, a.e. on $\mathbb{R}^d$, also satisfies $\rho \ge \tau$, a.e. on $\operatorname{supp}\boldsymbol{g} + B_{R+\varepsilon}$, for some positive constants ε, τ.

6.2 An abstract flow topology optimization problem

Now we are ready to formally discuss the well-posedness of an abstract flow topology optimization problem:

$$\begin{aligned} &\min_{(\rho,\boldsymbol{u})} \mathcal{F}(\rho,\boldsymbol{u}),\\ &\text{s.t.} \begin{cases} (\rho,\boldsymbol{u}) \in \mathcal{Z}, \\ \quad \boldsymbol{u} \in T_\rho(\boldsymbol{u}). \end{cases} \end{aligned} \tag{12}$$

The previous results imply the following theorem.

Theorem 6.1. *Let $\mathcal{Z}$ be a nonempty weakly compact subset of $\mathcal{H} \times \mathcal{U} \subset SBV(\mathbb{R}^d) \times H^1(\Omega)$, and let for all $\rho \in \mathcal{H}$ the assumption* (9) *be verified* (*see the discussion in Subsection 6.1*). *We also assume that A* [*which defines the mapping T_ρ via* (10)] *enjoys the conditions of Remark 5.10, and that for every $\rho \in \mathcal{H}$ the fixed-point problem* (10) *admits solutions. Finally, let $\mathcal{F} : SBV(\mathbb{R}^d) \times H^1(\Omega) \to \mathbb{R}$ be weakly l.s.c. Then, there exists at least one optimal solution to the abstract flow topology optimization problem* (12).

Proof. Essentially this is a Weierstrass Theorem adapted to our specific notation, because the hypotheses and Theorem 5.9 imply that the feasible set of (12) is nonempty and weakly compact. □

Remark 6.2. If the assumptions of Theorem 6.1 about the flow model are satisfied, we may set

$$\mathcal{Z} = \{\, (\rho,\boldsymbol{u}) \in \mathcal{Z}_0 \times \mathcal{U} \mid \mathcal{G}(\rho,\boldsymbol{u}) \le C \,\},$$

where $\mathcal{Z}_0$ is a nonempty weakly compact subset of $\mathcal{H} \subset SBV(\mathbb{R}^d)$ verifying condition (9), $\mathcal{G}(\rho,\boldsymbol{u})$ is an arbitrary weakly l.s.c. functional, which is in addition coercive in $\boldsymbol{u}$, uniformly w.r.t. ρ, and $C \in \mathbb{R}$ is an arbitrary constant but such that $\mathcal{Z} \neq \emptyset$.

In particular, we may set $\mathcal{G} = \mathcal{J}$, or $\mathcal{G} = \mathcal{J}^F$ (cf. [Evg03, Lemma 3.2]).

At last, we note that assumptions of Theorem 6.1 about the solvability of the fixed-point problem for every feasible design ρ are verified in many practical situations. For example, we have shown that they are satisfied for Stokes equations (see Proposition 5.2 and Remark 5.14) and for Navier–Stokes equations with homogeneous boundary conditions (see Proposition 5.11).

7 Computational issues

In this section we briefly discuss two topics that are standard in topology optimization with specialization to flow topology optimization problems. Throughout the section we will use problem (12) as a model example, and we assume that the assumptions of Theorem 6.1 are verified without further notice.

7.1 Approximation with sizing optimization problems

Clearly, no finite element software can be applied to the problem (10) if $\alpha(F * \rho)$ is allowed to become arbitrarily large; from the practical point of view the theory of Section 5 implying the existence of optimal solutions to (12) is pointless, unless we can describe a computational procedure capable of finding approximations of these optimal solutions. In fact, once we have proved Theorem 5.9 the latter goal can be easily accomplished. For arbitrary $\varepsilon > 0$, consider the set $\mathcal{Z}_\varepsilon = \{\, (\rho,\boldsymbol{u}) \in \mathcal{Z} \mid \rho \ge \varepsilon, \text{a.e.} \,\}$, i.e., only designs with porosity uniformly bounded away from zero are allowed, implying in particular the uniform bound $\alpha(F * \rho) \le \varepsilon^{-1} - 1$, for every $(\rho,\boldsymbol{u}) \in \mathcal{Z}_\varepsilon$.

Then, the following easy statement holds.

Proposition 7.1. *Assume that the sequence $\{\mathcal{Z}_\varepsilon\}$ is lower-semicontinuous in Painlevé-Kuratowski sense* (*topology in $\mathcal{H} \times \mathcal{U}$ being the strong one*), *namely*

$$\operatorname*{Lim\,inf}_{\varepsilon \to +0} \mathcal{Z}_\varepsilon = \mathcal{Z}, \tag{13}$$

(*in particular, $\mathcal{Z}_\varepsilon \neq \emptyset$ for all small $\varepsilon > 0$*). *Let further, for every small $\varepsilon > 0$, $(\rho_\varepsilon, \boldsymbol{u}_\varepsilon)$ denote a globally optimal solution of an approximating problem, obtained from* (12) *substituting $\mathcal{Z}_\varepsilon$ in place of $\mathcal{Z}$. Then, an* *arbitrary limit point of* $\{(\rho_\varepsilon, \boldsymbol{u}_\varepsilon)\}$ *(and there is at least one) is a globally optimal solution of the limiting problem* (12).

Proof. All claims easily follow from the uniform inclusion $\mathcal{Z}_\varepsilon \subset \mathcal{Z}$, Theorems 6.1 and 5.9, and finally [BoS00, Proposition 4.4]. □

The assumption (13) is probably easier to check in every particular case rather than to develop a general sufficient condition implying it; we only mention that for typical constraints in topology optimization, such as constraints on volume and on the perimeter, it is easily verified.

In general, there is a substantial amount of literature on the topic of approximation of topology optimization problems using sizing ones. (See the bibliographical notes (16) in [BeS03] for a survey of the situation in the topology optimization of linearly elastic materials; also see [Evg03, Section 6] results on incompressible stokesian flows, and [KPTK03, Appendix A.2] for a similar problem arising in the design of flow networks.) Cases of interest in such literature are when some of the underlying assumptions of Proposition 7.1 are violated, such as the compactness of $\mathcal{Z}$ or $\mathcal{Z}_\varepsilon$, or the assumption (13); in some particular situations it is nevertheless possible to prove statements similar to Proposition 7.1. We do not try to generalize our result in this direction, because computationally the problem (12) is already extremely demanding for realistic flows, and complicated constraints violating (13) are hardly necessary in practical situations.

7.2 Control of intermediate densities

Starting with the problem of distributing the solid material inside a control volume Ω so as to minimize some objective functional dependent on the flow, we expect an optimal design of the type $\rho = \chi_A$, where $A \subset \Omega$ is a flow region ("black–white" designs). Usually, this is a very naïve expectation [BeS03, Section 1.3.1]; however, there are some exceptions, such as the minimum-power design of domains for Stokes flows [BoP03, Evg03], or the design of one-dimensional wave-guides for stopping wave propagation [Bel03].

However, if we use a filter, it is simply impossible to obtain optimal distributions of material assuming *only* values zero or one (not counting the trivial designs $F * \rho \equiv 0$ and $F * \rho \equiv 1$), because $F * \chi_A$ is a continuous function, and the "edges" ∂A will be "smoothed out" by the filter. One possible way to reduce the amount of porous material in the final optimal design $F * \rho$ is to use a filter of a smaller radius. This may or may not work as expected — since the control problem (12) is non-convex, the optimal designs may change significantly as we vary the radius only slightly.

Another possibility is to add a penalty term $\mu\mathcal{J}^{\mathcal{D}}(F * \rho, \boldsymbol{u})$, for some positive μ, requiring that the power dissipation due to the flow through the porous part of the domain should be relatively small [Evg03, Section 5]. We must warn that increasing penalty μ might lead to unexpected results, because as we have already mentioned, the presence of the filter *requires* the presence of porous regions in the domain (except for trivial cases), thus the sequence of designs may converge to either one of those trivial designs, or $\mu\mathcal{J}^{\mathcal{D}}(F * \rho, \boldsymbol{u})$ may grow indefinitely. Therefore, suitable values of μ should be obtained in each case experimentally.

At last, various restriction or regularization techniques that are designed to control the amount of "microstructural material" in topology optimization of linearly elastic structures may be used for similar purposes in our case. We already mentioned the regularized intermediate density control method [BoP01]; other possible choices may be found in [BeS03, bibliographical notes (8)].

8 Conclusions and further research

We have considered the topology optimization of fluid domains in a rather abstract setting, and established the closedness of design-to-flow mappings for a general family of slightly compressible fluids, whose behavior is characterized by the fixed-point formulation (10). We used the notion of epi-convergence of optimization problems as a main analytical tool (cf. [BoS00, Proposition 4.6]) that allows us to treat very ill-behaving functionals, which arise due to the fact that we allow completely impenetrable walls to appear in the design domain.

It is of course of great engineering interest to perform numerical experiments with topology optimization of slightly compressible fluids for various objective functionals, theoretical foundations for which are established in this paper. Provided a stable solver of the underlying flow problem is available, it should not be a difficult task to combine it with the optimization code; in the end, the ease of integration with FEM software is one of the main reasons why topology optimization techniques are widely accepted and still gain popularity in many fields of physics and engineering [BeS03]. In fact, one such successful attempt of integrating topology optimization with Femlab is done for incompressible Navier–Stokes fluids [GH03]. Unfortunately, at the time of writing this code was not available to the author. We hope to be able to perform numerical computations in the near future.

The motivation for relaxing the incompressibility requirement is found in Section 3.2; however, if one is not convinced, and for the sake of completeness it would be interesting to prove the main approximation result, Proposition 5.3, for divergence-free functions, from which the rest of the theory should follow for incompressible fluids as well.

The method we used is of course not bound to Newtonian fluids. It seems that our results should hold for many common non-Newtonian fluids, including power-law, Bingham, and Powell-Eyring models (cf. [FuS00, Chapter 3]), without any major modifications (cf. Remark 5.1). Additional work is obviously needed for fluids of Prandtl-Eyring type [FuS00, Chapter 4]; we however feel that the special treatment this (mathematically) exotic type of fluids deserves lies well outside the scope of this paper.

At last, but not the least, we feel it is important to establish the existence of solutions, or construct a disproving counter-example, for the "original" problem of power minimization for incompressible Navier–Stokes fluids without the use of filtered designs. While we have shown that this problem looks ill-posed and is probably unsuitable for practical numerical computations, knowing whether optimal solutions exist would greatly contribute to the deeper understanding of Navier–Stokes flows and affect the further development in the area of topology optimization of fluids.

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A Appendix

Proposition A.1. *Under the assumptions on the filter F made in this paper the condition* (4) *is satisfied.*

Proof. Let $|\boldsymbol{x}| = h$ and define $S_h = \{\, \boldsymbol{y} \in \operatorname{supp} F \mid \operatorname{dist}(\boldsymbol{y}, \partial \operatorname{supp} F) \leq h \,\}$. $|S_h| \leq h\mathcal{H}_{d-1}(\partial \operatorname{supp} F)$, where $\mathcal{H}_{d-1}(\partial \operatorname{supp} F)$ is the Hausdorff measure of $\partial \operatorname{supp} F$ (i.e., perimeter of $\partial \operatorname{supp} F$). Moreover, $\sup_{\boldsymbol{y} \in S_h} F(\boldsymbol{y}) \leq Lh$, where L is the Lipschitz constant for F. Thus,

$$(F * \chi_{\mathbb{R}^d \setminus \operatorname{supp} F})(\boldsymbol{x}) \leq \int_{S_h} F \leq h^2 L \mathcal{H}_{d-1}(\partial \operatorname{supp} F). \qquad \square$$

Proof (of Proposition 5.2). The function $\mathcal{U} \ni \boldsymbol{v} \to \mathcal{J}^F(\rho, \boldsymbol{v}) + (2\delta)^{-1} \int_\Omega (\operatorname{div} \boldsymbol{v})^2$ is strongly convex and l.s.c. (in particular, owing to Poincaré inequality [Bre83, Corollaire IX.19]). Of course, if it is also proper w.r.t. $\mathcal{U}$ we get both existence and uniqueness of solutions. In particular, the last property holds in the case $\mathbf{0} \in \mathcal{U}$.$\square$

We will make use of the following statement, which can be found in the proof of [Kuf80, Theorem 9.7]. We remark that Ω needs not to be regular for this to hold (cf. [Tri78, Section 3.2.3]).

Lemma A.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Then, for every $h > 0$ there is a cut-off function $F_h \in C_0^\infty(\Omega)$ such that:*
(i) $0 \le F_h \le 1$,
(ii) $\forall \boldsymbol{x} \in \Omega : |\boldsymbol{\nabla} F_h(x)| \le C_1 h^{-1}$ *for a suitable constant* $C_1 > 0$, *and*
(iii) $F_h \equiv 1$ *on* $\Omega \setminus \Omega_h$, *where* $\Omega_h = \{ \boldsymbol{x} \in \Omega \mid \operatorname{dist}(\boldsymbol{x}, \partial\Omega) \le h \}$.
We can always (and in fact will) assume that $F_h \in C_0^\infty(\mathbb{R}^d)$, extending F_h by zero on $\mathbb{R}^d \setminus \Omega$.

The proof of the Proposition 5.3 essentially mimics the proof of [Kuf80, Theorem 9.7]; however, we adapt it to our notation. The most important difference is the fact that the specific growth condition holds not on the whole boundary of our domain but rather only on part of it, therefore we cannot apply the cited theorem directly.

Proof (of Proposition 5.3). We apply Lemma A.2 to a set $(\Omega + B_R) \setminus \Omega_0$ to obtain a family of "cut-off" functions $\{F_h\} \subset C_0^\infty((\Omega + B_R) \setminus \Omega_0)$, $h > 0$, and set $\boldsymbol{u}_h = F_h \boldsymbol{u}$ on Ω.

Defining u_h in such a way implies that $\{\boldsymbol{u}_h\} \subset H_0^1(\Omega)$ and clearly gives us (i) and the uniform estimation $\int_\Omega \alpha(F * \rho)|\boldsymbol{u}_h|^2 \le \int_\Omega \alpha(F * \rho)|\boldsymbol{u}|^2$. Thus it suffices to verify (ii) to obtain (iii) as well.

Define $\Omega_h = \{ \boldsymbol{x} \in \Omega \setminus \Omega_0 \mid \operatorname{dist}(\boldsymbol{x}, \Omega_0) \le h \}$. Since $\boldsymbol{u} - \boldsymbol{u}_h = (1 - F_h)\boldsymbol{u}$, $\boldsymbol{\nabla u} - \boldsymbol{\nabla u}_h = (1 - F_h)\boldsymbol{\nabla u} - \boldsymbol{\nabla}(1 - F_h) \cdot \boldsymbol{u}$, and $\operatorname{supp}(1 - F_h) \subset \Omega_h$, it is necessary to estimate the differences only on Ω_h.

$$\lim_{h \to +0} \|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{L^2(\Omega)} \le \lim_{h \to +0} \int_{\Omega_h} (1 - F_h)^2 |\boldsymbol{u}|^2 = 0,$$

because $0 \le F_h \le 1$, $\boldsymbol{u} \in H^1(\Omega)$, and $|\Omega_h| \to 0$ as $h \to +0$.

Similarly,

$$\lim_{h \to +0} \|\boldsymbol{\nabla u} - \boldsymbol{\nabla u}_h\|^2_{L^2(\Omega)} \le \lim_{h \to +0} \int_{\Omega_h} (1 - F_h)^2 |\boldsymbol{\nabla u}|^2 + \lim_{h \to +0} \int_{\Omega_h} |\boldsymbol{\nabla} F_h|^2 |\boldsymbol{u}|^2,$$

and the first limit is zero, as before, since $0 \le F_h \le 1$, $\boldsymbol{u} \in H^1(\Omega)$, and $|\Omega_h| \to 0$ as $h \to +0$. We estimate the last integral as

$$\int_{\Omega_h} |\boldsymbol{\nabla} F_h|^2 |\boldsymbol{u}|^2 \le C_1^2 \int_{\Omega_h} h^{-2} |\boldsymbol{u}|^2 \le C_1^2 C^{-1} \int_{\Omega_h} \alpha(F * \rho) |\boldsymbol{u}|^2, \tag{14}$$

where the constant C_1 is given by Lemma A.2, and the last inequality holds owing to the filter growth condition (4). Owing to the bound $\mathcal{J}^F(\rho, \boldsymbol{u}) < +\infty$, the last integral converges to zero as h does, and thus the proof is complete. □

The following fact is very well known for elliptic forms; we only show that possible infinite values if α do not change it.

Lemma A.3. *For every $\rho \in \mathcal{H}$,*

$$H^{-1}(\Omega) \ni \boldsymbol{f} \rightrightarrows \operatorname*{argmin}_{\boldsymbol{v} \in H_0^1(\Omega)} \left\{ \frac{\nu}{2} \int_\Omega \boldsymbol{\nabla v} \cdot \boldsymbol{\nabla v} + \frac{1}{2\delta} \int_\Omega (\operatorname{div} \boldsymbol{v})^2 + \frac{1}{2} \int_\Omega \alpha(F * \rho) \boldsymbol{v} \cdot \boldsymbol{v} - \langle \boldsymbol{f}, \boldsymbol{v} \rangle \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between the $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ is single-valued, linear, and completely continuous.

Proof. Both existence and uniqueness follow from Proposition 5.2. It is an easy exercise to verify the linearity of $\boldsymbol{f} \to \boldsymbol{u}(\boldsymbol{f})$. Furthermore, comparing objective functionals at $\boldsymbol{u}(\boldsymbol{f})$ and at $\boldsymbol{0}$, and using Poincaré inequality [Bre83, Corollaire IX.19] we get the inequality $\|\boldsymbol{u}(\boldsymbol{f})\|^2_{H_0^1(\Omega)} \le C\langle \boldsymbol{f}, \boldsymbol{u}(\boldsymbol{f}) \rangle$, for some C independent of $\boldsymbol{f}$. Of course, it implies complete continuity at zero, which owing to the linearity is equivalent to complete continuity at every point. □