

Topology Optimization of structures with stress constraints: Aeronautical Applications

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Abstract. Topology optimization of structures is nowadays the most active and widely studied branch in structural optimization. This paper develops a minimum weight formulation for the topology optimization of continuum structures. This approach also includes stress constraints and addresses important topics like the efficient treatment of a large number of stress constraints, the approach of discrete solutions by using continuum design variables and the computational cost. The proposed formulation means an alternative to maximum stiffness formulations and offers additional advantages. The minimum weight formulation proposed is based on the minimization of the weight of the structure. In addition, stress constraints are included in order to guarantee the feasibility of the final solution obtained. The objective function proposed has been designed to force the convergence to a discrete solution in the final stages of the optimization process. Thus, near discrete solutions are obtained by using continuum design variables. The robustness and reliability of the proposed formulation are verified by solving application examples related to aeronautical industry.

1. Introduction

Topology optimization of continuum structures is a recent field in structural optimization. However, an increasing research activity in this area has been developed since the statement of the first formulations [1, 2]. The traditional statements of this problem try to obtain the optimal distribution of material that maximizes the stiffness of the solution (minimize the compliance) by adequately distributing a predefined amount of material in a predefined domain [1, 2, 3]. Thus, the main objective is to determine the parts of the domain where material should exist or not in order to produce the most stiff structure possible. The existence or absence of material is usually defined by using a continuum design variable (the relative density) in order to avoid dealing with a discrete optimization problem since the optimization of discrete problems is not affordable in practical topology optimization applications due to the large number of design variables. This continuum approach of the material properties present important advantages since conventional optimization algorithms can be used to obtain the optimal material distribution. However, numerical models must be defined in order to compute the structural analysis for intermediate values of the relative densities. This difficulty is usually solved by using predefined material microstructure models like SIMP (Solid Isotropic Material with Penalty) [3], for example, and by using homogenization techniques.

In this paper, we propose a different approach of the topology optimization problem that

minimizes the weight of the structure with stress constraints [4, 5, 6, 7]. This formulation offers important advantages versus the maximum stiffness approach since the most important instabilities related to this kind of formulations are avoided. On the other hand, the minimum weight approach with stress constraints requires much larger computing resources than the maximum stiffness approaches. Thus, computational aspects must be addressed in order to reduce the computing effort and to increase the profits of the minimum weight with stress constraints formulation.

The main objective of this paper is to present an improved formulation that tries to reach binary 0-1 material distributions by using a continuum design variable (the relative density). Binary 0-1 material distributions are obtained by using a modified objective function based on the weight of the structure. This approach presents two important advantages since the continuum approximation allows to deal with continuum design variables and essentially binary solutions are finally obtained.

2. Topology optimization problem

The minimum weight topology optimization problem with stress constraints can be formulated from a generic point of view as

$$\begin{aligned}
& \text{Find} && \boldsymbol{\rho} = \{\rho_e\} \\
& \text{Minimize} && F(\boldsymbol{\rho}) = \text{Cost}(\boldsymbol{\rho}) \\
& \text{subject to:} && g_j(\boldsymbol{\rho}) \leq 0 \quad j = 1, \dots, m \\
& && 0 < \rho_{min} \leq \rho_e \leq 1, \quad e = 1, \dots, N_e
\end{aligned} \tag{1}$$

where $\boldsymbol{\rho}$ is the vector of design variables (relative densities), g_j are the stress constraints, m is the number of constraints considered, N_e is the number of elements of the finite element mesh [7, 8] and ρ_{min} is usually stated as $\rho_{min} = 0.001$.

The main issues of this kind of formulations consist in the definition of the stress constraints (in order to guarantee that the final design satisfies the limitations imposed) and the definition of the most adequate objective function (in order to obtain the benefits expected).

3. Stress Constraints

The authors have studied three different formulations in order to deal with stress constraints in the topology optimization problem: the local approach, the global approach and the block aggregation of stress constraints [4, 5, 7, 8].

The local approach of the stress constraints states one stress constraint in the central point of each element of the mesh by comparing a reference stress with the maximum elastic limit of the material being used [7, 8, 9, 10, 11, 12, 13]. Thus, if we introduce the required modifications to deal with the singularity phenomena [9, 14, 15], the local stress constraints can be stated as:

$$g_e(\boldsymbol{\rho}) = \left[\hat{\sigma}(\boldsymbol{\sigma}_e^h(\boldsymbol{\rho})) - \hat{\sigma}_{max} \varphi_e \right] (\rho_e)^q \leq 0, \quad \varphi_e = 1 - \varepsilon + \frac{\varepsilon}{\rho_e}. \tag{2}$$

The exponent q can be predefined to deal with real stresses ($q = 0$) or to deal with effective stresses ($q = 1$) [6]. The reference stress $\hat{\sigma}$ is the Von Mises criterion and $\boldsymbol{\sigma}_e^h$ is the stress tensor of the material in the central point of element e . The relaxation factor ε usually takes the values $\varepsilon \in (0.001, 0.1)$ [4, 7, 8].

The global approach of the stress constraints is defined to avoid the high computing resources required by the local approach of the stress constraints. Thus, the local stress constraints are aggregated in only one global function by using the Kreisselmeier-Steinhauser approach [4, 7, 16]. Thus, the global constraint can be stated as

$$G_{KS}(\boldsymbol{\rho}) = \frac{1}{\mu} \left[\ln \left(\sum_{e=1}^{N_e} \exp \mu(\hat{\sigma}_e^* - 1) \right) - \ln(N_e) \right] \leq 0, \quad (3)$$

being

$$\hat{\sigma}_e^* = \frac{\hat{\sigma}(\boldsymbol{\sigma}_e^h(\boldsymbol{\rho}))}{\hat{\sigma}_{max} \varphi_e}. \quad (4)$$

The aggregation parameter μ allows to specify the degree of approximation to the maximum local stress constraint [4]. In the limit, when $\mu \rightarrow \infty$ the global approach of the stress constraints is equivalent to the highest local stress constraint. In practical applications, this parameter must take the largest value that does not introduce numerical instabilities. These numerical instabilities are usually related to the high non-linearity of the aggregation function [4]. The most appropriate range of values of the aggregation parameter μ is (20,40). These values have demonstrated to work properly in the application examples solved [4, 5, 7, 8].

The block aggregation of the stress constraints is a more general method that includes both previous formulations. In this approach the domain of the structure is divided in a predefined number of groups of elements such that all the groups contain approximately an equal number of elements. Thus, a global-type constraint can be imposed over the elements of each block of the mesh [5, 8]. Thus,

$$G_{KS}^b(\boldsymbol{\rho}) = \frac{1}{\mu} \left[\ln \left(\sum_{e \in B_b} \exp \mu(\hat{\sigma}_e^* - 1) \right) - \ln(N_e^b) \right] \leq 0 \quad (5)$$

where B_b is the set of elements contained in block b and N_e^b is the number of elements contained in block b .

4. Objective function

The goal of the Topology Optimization problem with stress constraints is to determine the minimum weight structure that supports the applied forces. Thus, the objective function of the optimization problem is the weight of the structure:

$$F(\boldsymbol{\rho}) = \sum_{e=1}^{N_e} \int_{\Omega_e} \rho_e \gamma_{mat} d\Omega \quad (6)$$

where γ_{mat} is the material density, ρ_e is the relative density of element e and Ω_e is the domain occupied by element e .

This objective function introduces an unwanted phenomenon since the material distributions obtained present a large number of elements with intermediate values of relative density. Thus, a different formulation of this objective function has been proposed in order to penalize the material distributions with intermediate values of relative density. This formulation has been obtained by following the same idea introduced in the SIMP model of microstructure. Thus, this modified objective function can be stated as:

$$F_p(\boldsymbol{\rho}) = \sum_{e=1}^{N_e} \int_{\Omega_e} \Psi_p(\rho_e) \gamma_{mat} d\Omega, \quad \Psi_p(\rho_e) = \rho_e^{\frac{1}{p}}, \quad p \geq 1 \quad (7)$$

where the exponent p is the penalization factor and takes the values $p \geq 1$ (figure 1 left).

This formulation of the objective function has been widely analyzed and tested in previous publications of the authors by solving some application examples [4, 5, 6, 7, 8]. However, this

formulation does not guarantee a final solution with 0-1 material distribution. This fact can be easily understood if we consider that the derivatives of the proposed objective function are always positive. Thus, a reduction of the design variable always introduces a reduction in the objective function. This fact is the expected one from a physical point of view. However, the solutions obtained with this approach are not 0-1 when we use small values of the relaxation parameter (*e.g.* $\varepsilon = 0.01$) since stress constraints impose lower limits of the relative densities. These lower limits are usually higher than the minimum value allowed. Thus, the lower limits of relative densities imposed by stress constraints and the positive value of the derivatives of the objective function define an optimal solution that usually presents intermediate values of relative density. This issue can be explained by analyzing a simplified problem with one design variable and one stress constraint. The optimization algorithm tries to reduce the value of the design variable but the stress constraint avoids the reduction. The optimal solution obtained presents an intermediate value of the relative density although the penalization parameter is used. This fact can be easily observed in figure 1 (right). If large values of the relaxation parameter ε are used the lower limits imposed by the stress constraints usually disappear. However, some important parts of the domain may be removed inadequately when the relaxation parameter is increased.

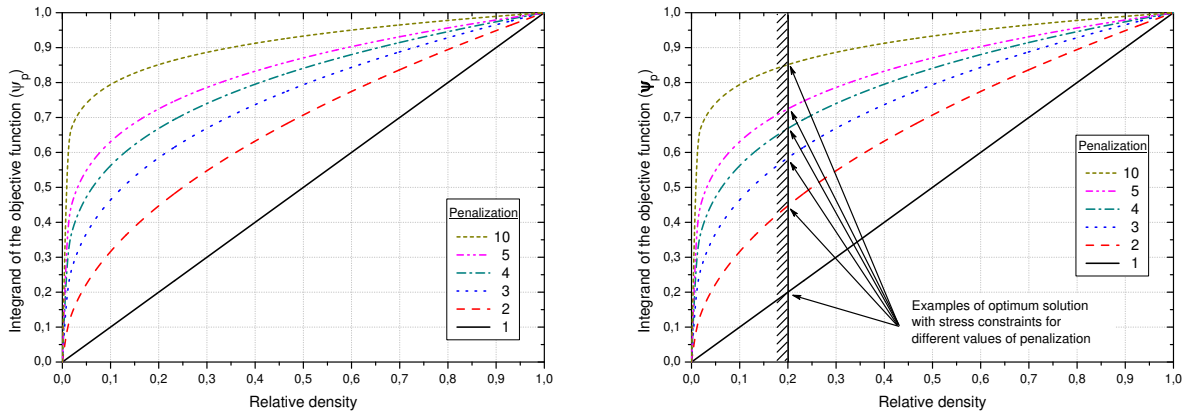


Figure 1. Integrand of the objective function ($\Psi_p(\rho_e)$) for different values of the penalization parameter (p) (left) and examples of optimum design variables when stress constraints are considered (right).

Thus, the proposed penalization of the intermediate relative densities does not guarantee 0-1 final solutions when small values of the relaxation parameter are used. The values of the relaxation parameter must be as low as possible in order to avoid removing useful material in the final designs but, on the other hand, it must be large enough to avoid singularity phenomena.

The optimal solutions can be forced to reach 0-1 binary values by adequately modifying the objective function proposed in (7). In this paper, we propose a modified objective function to obtain 0-1 optimal solutions as:

$$F_b(\boldsymbol{\rho}) = \sum_{e=1}^{N_e} \int_{\Omega} \Psi_b(\rho_e) \gamma_{mat} d\Omega, \quad (8)$$

where

$$\Psi_b(\rho_e) = \rho_e^{1/(1+\beta)^2} + \beta \exp^{-\beta\rho_e} \sin(\pi\rho_e), \quad (9)$$

This modification introduces severe changes in the mathematical function since the sign of the derivatives becomes negative in part of the range of relative density (see figure 2). This modification produces the expected benefits but it also introduces local minima and other unwanted phenomena. Thus, a whole procedure must be analyzed in order to include this modification in the optimization procedure.

Most of the initial solutions for the topology optimization problems with stress constraints use the maximum value of the relative density. Thus, if we use the modified objective function proposed in (8), the optimization algorithm will not modify the initial solution since the value of the objective function will rise for a reduction of the design variables. In this case, the optimum solution is the initial distribution and, obviously, this may not correspond to the real optimum solution. Thus, the modified objective function must be specifically used to obtain a final material distribution. The optimization process must be developed in two stages. First stage obtains an optimal material distribution with the objective function proposed in (7) by penalizing the intermediate densities. The second stage starts with the solution obtained in the first stage and incorporates the modified objective function in order to force the elements with intermediate densities to reach 0-1 values. This second stage must be developed by increasing progressively the value of the parameter β proposed in (8).

The values of the parameter β must be in the range $[0.260, 7.404]$ in order to obtain the expected solutions. Values smaller than, approximately, 0.260 introduce positive derivatives when $\rho \rightarrow 1$. Values higher than, approximately, 7.404 also introduces positive derivatives when $\rho \rightarrow 1$.

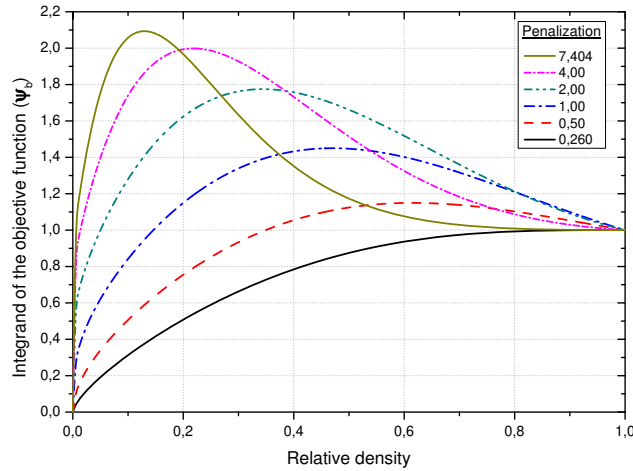


Figure 2. Integrand of the objective function ($\Psi_b(\rho_e)$) (8) for different values of penalization.

This formulation of the objective function does not produce minimum weight designs (from a continuum point of view of the design variables) since the modifications introduced forces 0-1 material distributions. These solutions are not the optimal ones in minimum weight terms but they produce important benefits since elements with intermediate relative densities are forced to disappear. Thus, the optimal solutions obtained present the lowest cost in practice.

5. Parallelization

The resulting optimization problem requires the use of efficient algorithms that allow to deal with a large number of highly non-linear stress constraints and a non-linear objective function. In this paper, we have used a Sequential Linear Programming algorithm [17, 18] with Quadratic Line Search proposed in [6, 7, 19].

This algorithm has demonstrated to produce optimal solutions for the optimization problems proposed in this paper. However, this optimization algorithm and the sensitivity analysis involved [5, 8, 20] require high computing resources (computing time) when a large number of design variables and stress constraints are imposed. Consequently, it is necessary to propose computational techniques that allow to reduce the computational effort. According to this idea, parallelization techniques have been implemented in the sensitivity analysis and the optimization algorithm. First order derivatives can be computed independently for each stress constraint. Thus, the parallelization of this procedure is feasible and it produces suitable performance [8]. The Optimization algorithm can be also computed in parallel since the Simplex Algorithm involved develops a large number of matrix operations that can be computed in parallel. The performance of this parallelization is not as effective as in the sensitivity analysis computation since a number of sequential operations must be developed between each matrix modification. This fact can be observed in figure 3. However, the total speed-up of the optimization process allows to reduce the computing time about 6 times when 8 processors are used in a problem with 7200 design variables and constraints.

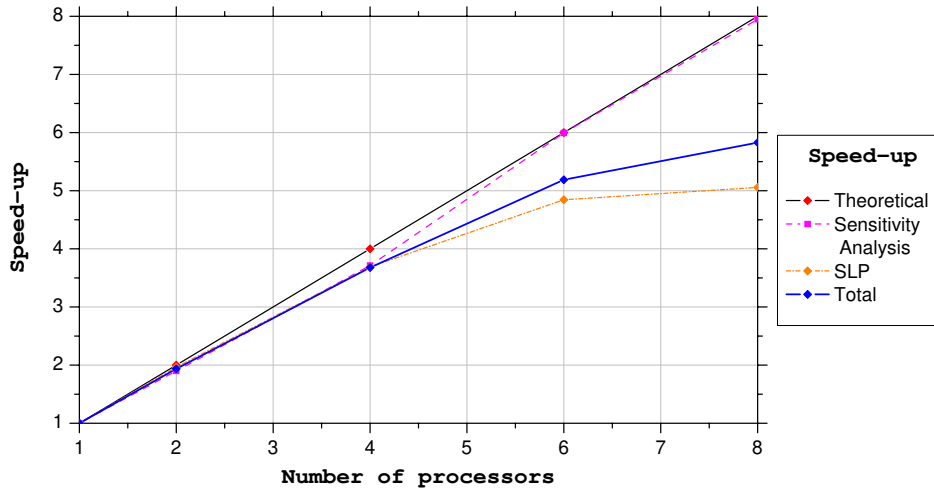


Figure 3. Speed-up obtained for a problem with 7200 design variables by using the local approach of stress constraints in a computer with 4 Intel Xeon 7120M Dual Core processors.

6. Application examples

We present two structural problems related to aeronautical industry. These examples are 2D structures in plane stress. In addition, the effect of the modified objective function proposed in (8) is analyzed.

6.1. Optimization of the structural section of a tail plane

The first example corresponds to the structural section of an aerodynamic airfoil like the ones used in the vertical/horizontal tail plane (VTP/HTP) of an airplane. Only a half of the structure is analyzed due to symmetry of the geometry and the applied loads. The geometry is not exact since it does not correspond to a real aerodynamic airfoil and the applied loads have not been obtained by considering the real pressure of the fluid around the airfoil (by using Computational Fluid Dynamics for example). However, this example is devoted to show the wide range of application and the robustness of the formulation proposed.

Figure 4 (left) shows the dimensions of the domain and the position of the external forces. In this case, 12 kN have been distributed along 76.6 cm on the lower surface of the airfoil.

The domain of the structure is discretized by using $N_e = 5200$ eight-node quadrilateral elements.

The material being used is steel with density $\gamma_{mat} = 76500$ kN/m³, Young's modulus $E = 2.1 \cdot 10^5$ MPa, Poisson's ratio $\nu = 0.3$ and elastic limit $\hat{\sigma}_{max} = 230$ MPa. The thickness of the structure is 1 cm

Figure 5 (left) shows the solution obtained by using the local approach of the stress constraints and the objective function proposed in (7). Figure 5 (right) shows the optimal solution obtained by using the local approach and the modified objective function proposed in (8) in order to obtain binary 0-1 solutions.

Table 1 shows the most important parameters used in the optimum design formulation and the value of the final objective function as a percentage of the original weight of the structure. Note that the modified objective function does not produce minimum weight designs since the final weight obtained is higher than the one obtained with the penalization of the intermediate relative densities. However, the optimal solution approaches a binary distribution of material and this fact introduces important benefits from a practical point of view. A more accurate solution to minimum weight design with binary distribution of material can be obtained by using more refined finite element meshes.

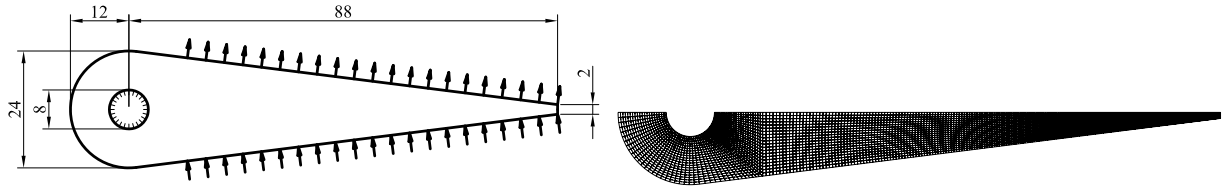


Figure 4. Geometric scheme of the tail plane (units in mm) (left) and FEM mesh used (right).

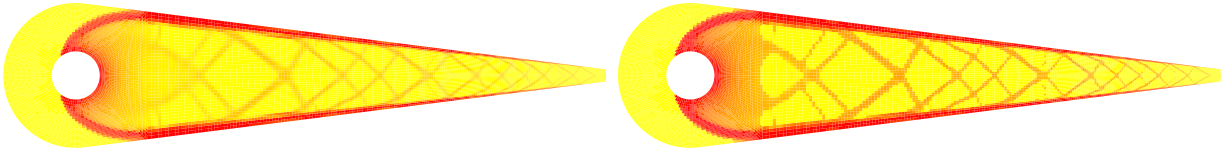


Figure 5. Optimal solution for the tail plane problem by using the local approach of the stress constraints ($\varepsilon = 0.01$, $q = 0$) (left) and by using the modified objective function (8) ($\varepsilon = 0.02$, $q = 0$) (right).

Table 1. Summary of the most important parameters of the tail plane problem.

Tail plane	Local Appr. (Fig. 5 left)	Modif. Obj. F. (Fig. 5 right)
Number of elements	5200	5200
Penalization (p, β)	4.00	7.00
Final weight/Initial weight	17.70 %	22.47 %

6.2. Optimization of the structure of an undercarriage

The second example corresponds to the optimization of the structure of an undercarriage.

Figure 6 (left) shows the dimensions of the domain and the position of the external load (74 kN/m) applied on the lower vertical edge on the right.

The domain of the structure is discretized in $N_e = 4014$ eight-node quadrilateral elements.

The material being used is steel with density $\gamma_{mat} = 76500 \text{ kN/m}^3$, Young's modulus $E = 2.1 \cdot 10^5 \text{ MPa}$, Poisson's ratio $\nu = 0.3$ and elastic limit $\hat{\sigma}_{max} = 230 \text{ MPa}$. The thickness of the structure is 0.1 m.

Figure 7 (left) shows the optimal solution obtained by using the local approach of the stress constraints. Figure 7 (right) shows the solution obtained by using the local approach of the stress constraints and a final stage with the modified objective function proposed in (8).

Table 2 presents the most important parameters involved in the optimum design formulation and the value of the final objective function as a percentage of the original weight of the structure.

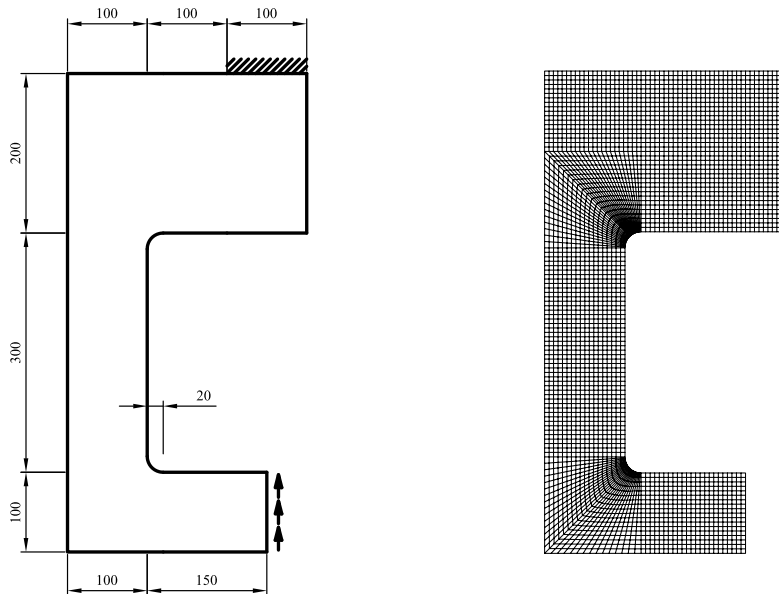


Figure 6. Geometric scheme of the undercarriage (units in mm) (left) and FEM mesh used (right).

Table 2. Summary of the most important parameters of the undercarriage problem.

Undercarriage	Local Appr. (Fig. 7 left)	Modif. Obj. F. (Fig. 7 right)
Number of elements	4014	4014
Penalization (p, β)	4.00	7.30
Final weight/Initial weight	8.76 %	11.95 %

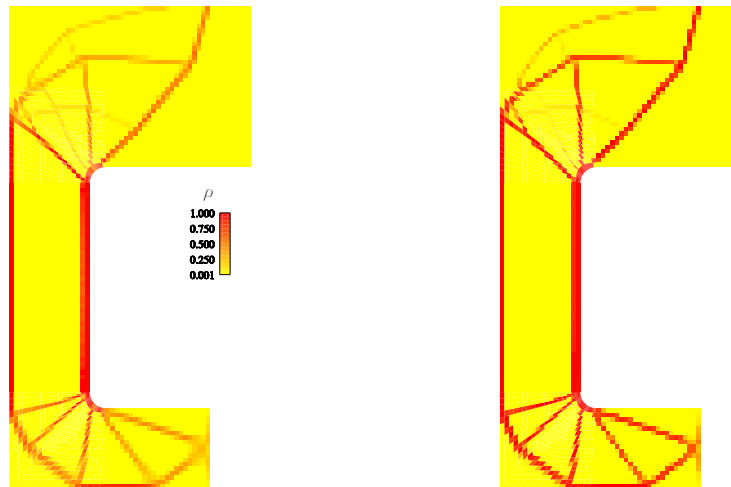


Figure 7. Optimal solution of the undercarriage problem by using the local approach of the stress constraints ($\varepsilon = 0.01$, $q = 1$) (left) and by using the modified objective function (8) ($\varepsilon = 0.01$, $q = 1$) (right).

7. Conclusions

In this paper we present a topology optimization of structures formulation with a continuum approach of the design variables that produces essentially binary solutions by using a continuum approach of the relative density and a modified minimum weight objective function. This modified objective function allows to deal with continuum design variables during the optimization process but it produces final binary distributions of material.

This formulation can be analyzed by considering three different formulations in order to deal with stress constraints: the local approach of the stress constraints, the global approach of the stress constraints and a more general formulation that defines groups of elements and imposes one global constraint per group (the block aggregation approach).

Finally, some application examples related to the aeronautical industry are solved in order to verify the validity of the algorithms and formulations proposed in this paper.

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