Topology optimization of trusses with stress and local constraints on nodal stability and member intersection[1]

M. Ohsaki

Department of Architecture and Architectural Engineering,
Kyoto University Kyotodaigaku-Katsura, Nishikyo, Kyoto 615-8540, Japan
E-mail: ohsaki@archi.kyoto-u.ac.jp

N. Katoh

Department of Architecture and Architectural Engineering,
Kyoto University Kyotodaigaku-Katsura, Nishikyo, Kyoto 615-8540, Japan

Abstract

A truss topology optimization problem under stress constraints is formulated as a Mixed Integer Programming (MIP) problem with variables indicating existence of nodes and members. The local constraints on nodal stability and intersection of members are considered, and a moderately large lower bound is given for the cross-sectional area of an existing member. A lower-bound objective value is found by neglecting the compatibility conditions, where linear programming problems are successively solved based on a branch-and-bound method. An upper-bound solution is obtained as a solution of NonLinear Programming (NLP) problem for the topology satisfying the local constraints. It is shown in the examples that upper- and lower-bound solutions with small gap in the objective value can be found by the branch-and-bound method, and the computational cost can be reduced by using the local constraints.

Keywords Topology optimization; Mixed integer programming; Truss; Local constraints; Stress constraints; Branch-and-bound method

1 Introduction.

In the widely used *ground structure approach* for topology optimization of trusses, the necessary members and nodes are selected from the highly connected ground structure with many nodes and members (Dobbs and Felton, 1969; Dorn *et al.*, 1964). In this approach, the cross-sectional areas are considered as continuous design variables and a member with null cross-sectional area is to be removed after optimization with fixed nodal locations.

One of the main difficulties in topology optimization under stress constraints is that the constraints need not be satisfied by the member with vanishing cross-sectional

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area; i.e. the constraint does not exist for a member that does not exist (Kirsch, 1989, 1990; Sved and Ginos, 1968). If we consider a process of continuously decreasing the cross-sectional area of a member from a finite positive value to 0, the constraint suddenly disappears at the final state with null cross-sectional area. Therefore, the optimization problem has discontinuity in the formulation of constraints. As a result, the feasible region is not convex in general, and the optimal solution is often located at a cusp or a ridge of the feasible region.

To overcome the difficulty due to discontinuity of the problem, several relaxation methods as well as branch-and-bound-type iterative methods have been presented. Sheu and Schmit (1972) developed a method for obtaining the lower-bound objective value by ignoring the compatibility conditions and by solving a Linear Programming (LP) problem with the member forces as design variables. An upper bound is computed for the obtained topology by solving a NonLinear Programming (NLP) problem considering the compatibility conditions.

Ringertz (1985) presented a method for problems with stress and displacement constraints, where a compatible set of strains and displacements is first calculated for specified cross-sectional areas. An NLP problem is solved under stress constraints while fixing the displacements. Ringertz (1986) proposed, in another paper, an approach for obtaining the lower-bound solution by solving an NLP problem under displacement constraints only. The stress constraints are successively given for members that violate the constraints.

A relaxation method has been presented by Cheng (1995) and Cheng and Guo (1997) for obtaining a good approximate solution. In their approach called *ε-relaxation method*, the stress constraint is relaxed for a member with small cross-sectional area. It is very difficult, however, to determine appropriate value of the parameter for relaxing the constraints, and to assign the initial solution to reach the globally optimal solution as noted by Stolpe and Svanberg (2001). Therefore, they extended the method to use an extrapolation approach (Guo and Cheng, 2000) which is similar to those in Nakamura and Ohsaki (1992) and Ohsaki and Nakamura (1996). Stolpe and Svanberg (2003) showed that stress constrained problem can be solved by a traditional nonlinear programming approach. The solution obtained by them, however, is not a singular optimum, and the performance of the method cannot be verified by their example.

Another difficulty in topology optimization is that the solution often turns out to be an unrealistic design due to existence of unstable nodes, intersection of members, and existence of extremely slender members. Those unrealistic designs cannot be prevented if only the cross-sectional areas are considered as design variables. Topology cannot be optimized if a moderately large lower bound is given for the cross-sectional area of each member. An unstable node may be stabilized by simply fixing the pin-joint or by adding a member that connects to the node. There is no proof, however, that the design after modification is also a good approximate solution. Nakamura and Ohsaki (1992) investigated the characteristics of optimal topologies under eigenvalue constraints and showed that local instability and multiplicity of eigenvalues lead to serious difficulties in finding optimal topologies. Since a solution that satisfies the necessary conditions for optimality is also a globally optimal solution for eigenvalue constraints, there is no difficulty due to singularity or nonconvexity contrary to the problem with stress constraints.

In this paper, the topology optimization problem under stress constraints is first formulated as a Mixed Integer Programming (MIP) problem (Kravanja *et al.*, 1998). The local constraints such as constraints on nodal instability and intersection of members are considered, and a moderately large lower bound is given for the cross-sectional area of an existing member. The integer variables for indicating existence of nodes and members are used. An LP problem is formulated by relaxing the integer variables and by ignoring the compatibility conditions to obtain a lower-bound solution. An NLP with fixed topology satisfying the local constraints is solved to find an upper-bound solution. It is shown in the examples that upper- and lower-bound solutions with small gap in objective value can be found by using the branch-and-bound method.

2 Topology optimization problem.

2.1 Governing equations

Consider an elastic truss subjected to multiple static loading conditions. The problem is to obtain an optimal topology as well as member cross-sectional areas that minimizes the total structural volume under constraints on stresses of members. The conventional ground structure approach is used.

Let $\mathbf{P}^k$ denote the vector of kth set of nodal loads. The vector of axial forces for $\mathbf{P}^k$ is written as $\mathbf{N}^k = \{N_i^k\}$. In the following, a subscript is used for indicating a component of a vector. The equilibrium equation is given as

$$\mathbf{B}\mathbf{N}^k = \mathbf{P}^k, \quad (k = 1, 2, \dots, f) \tag{1}$$

where f is the number of loading conditions, and $\mathbf{B}$ is called equilibrium matrix.

Let $\mathbf{U}^k$ denote the vector of nodal displacements against $\mathbf{P}^k$. The elongation of the ith member corresponding to $\mathbf{U}^k$ is denoted by d_i^k. The compatibility condition between $\mathbf{U}^k$ and d_i^k is written as

$$\mathbf{B}_i^\top \mathbf{U}^k = d_i^k, \quad (i = 1, 2, \dots, m; k = 1, 2, \dots, f) \tag{2}$$

where $\mathbf{B}_i$ is the ith column of $\mathbf{B}$, m is the number of members, and $(\ )^\top$ indicates the transpose of a vector. Hence the stress σ_i^k and axial force N_i^k of the ith member are obtained from $\mathbf{U}^k$ as

$$\sigma_i^k = \frac{E}{L_i}\mathbf{B}_i^\top \mathbf{U}^k, \quad N_i^k = A_i \sigma_i^k, \\ (i = 1, 2, \dots, m; k = 1, 2, \dots, f) \tag{3}$$

where L_i is the length of the ith member, and E is the elastic modulus. Eqs. (1) and (3) are combined into the following stiffness equation:

$$\mathbf{K}(\mathbf{A})\mathbf{U}^k = \mathbf{P}^k, \quad (k = 1, 2, \dots, f) \tag{4}$$

where $\mathbf{K}(\mathbf{A})$ is the stiffness matrix that is a function of the vector $\mathbf{A}$ of the cross-sectional areas.

Note that σ_i^k may easily be obtained by using (3) even for a member with $A_i = 0$ if the nodes at the two ends of the member exist and d_i^k can be calculated from the displacements of those nodes. If σ_i^k is defined as the axial force divided by A_i, however, it cannot be computed for a member with $A_i = 0$.

2.2 Problem formulation

Consider a problem of minimizing the total structural volume $V(\mathbf{A})$ defined by

$$V(\mathbf{A}) = \sum_{i=1}^{m} A_i L_i \tag{5}$$

The upper and lower bounds for σ_i^k are denoted by σ_i^{U} and σ_i^{L}, respectively. As it is well discussed in the literature (Kirsch, 1989; Sved and Ginos, 1968), the main difficulty of topology optimization under stress constraints exists in the discontinuity of the problem formulation due to the fact that the constraint need not be satisfied by a non-existent member.

Let $y_i \in \{0, 1\}$ denote a variable indicating by $y_i = 1$ and $y_i = 0$, respectively, the existence and non-existence of the ith member. Stress constraints should be assigned only for members with $y_i = 1$.

One of the drawbacks in topology optimization based on the ground structure approach is that an unstable optimal truss is often obtained. A node connecting only two colinear members is unstable to the transverse direction of the members. An unstable solution can be avoided by introducing the lower bound for the number of members connected to an existing node, and by assigning a moderately large lower-bound A_i^{L} for the cross-sectional area of an existing member. Then the constraints for A_i are given as

$$A_i^{\mathrm{L}} y_i \leq A_i \leq A_i^{\mathrm{U}} y_i, \quad (i = 1, 2, \cdots, m) \tag{6}$$

where A_i^{U} is the upper bound for A_i. Note from (6) that $A_i = 0$ should be satisfied if $y_i = 0$.

Let $x_r \in \{0, 1\}$ be the variable indicating non-existence and possible existence of the rth node, respectively, by $x_r = 0$ and $x_r = 1$. The upper and lower bounds for the number of members connected to the rth node, if exists, are denoted by C_r^{U} and C_r^{L}, respectively. Note that C_i^{U} is given to prevent existence of a highly connected node. The set of indices of members connected to the rth node in the initial ground structure is denoted by J_r, and the following constraints are given:

$$x_r C_r^{\mathrm{L}} \leq \sum_{i \in J_r} y_i \leq x_r C_r^{\mathrm{U}}, \quad (r = 1, 2, \ldots, s) \tag{7}$$

where s is the number of nodes including the supports. Note from (7) that $y_i = 0$ should be satisfied by all the members connected to a removed node with $x_r = 0$. $x_r = 1$ indicates existence of the rth node if one of A_i^{L} $(i \in J_r)$ has moderately large value.

In practical situations, intersection of members should also be avoided although those members are needed in the initial ground structure so as not to restrict the design space. The ith pair of mutually intersecting members in the ground structure is denoted by S_i $(i = 1, \dots, q)$. The following constraints are to be satisfied:

$$\sum_{j \in S_i} y_j \leq 1, \quad (i = 1, 2, \dots, q) \tag{8}$$

The topology optimization problem is then formulated as a mixed integer programming problem as

MIP :

$$\begin{aligned}
&\underset{\mathbf{A},\mathbf{y},\mathbf{x},\mathbf{U}^k,\boldsymbol{\sigma}^k,\mathbf{N}^k}{\text{minimize}} && V(\mathbf{A}) = \sum_{i=1}^{m} A_i L_i \\
&\text{subject to} && \sigma_i^{\mathrm{L}} y_i \leq \sigma_i^k y_i \leq \sigma_i^{\mathrm{U}} y_i, \\
& && \quad (i = 1, 2, \cdots, m; k = 1, 2, \cdots, f) && (9) \\
& && \sigma_i^k = \frac{E}{L_i} \mathbf{B}_i^{\top} \mathbf{U}^k, \\
& && \quad (i = 1, 2, \dots, m; k = 1, 2, \dots, f) && (10) \\
& && N_i^k = A_i \sigma_i^k \\
& && \quad (i = 1, 2, \dots, m; k = 1, 2, \dots, f) && (11) \\
& && \mathbf{B}\mathbf{N}^k = \mathbf{P}^k, \quad (k = 1, 2, \dots, f) && (12) \\
& && A_i^{\mathrm{L}} y_i \leq A_i \leq A_i^{\mathrm{U}} y_i, \quad (i = 1, 2, \cdots, m) && (13) \\
& && x_r C_r^{\mathrm{L}} \leq \sum_{i \in J_r} y_i \leq x_r C_r^{\mathrm{U}}, \quad (r = 1, 2, \dots, s) && (14) \\
& && \sum_{j \in S_i} y_j \leq 1, \quad (i = 1, 2, \dots, q) && (15) \\
& && y_i \in \{0, 1\}, \quad (i = 1, 2, \cdots, m) && (16) \\
& && x_r \in \{0, 1\}, \quad (r = 1, 2, \cdots, s) && (17)
\end{aligned}$$

The objective value of MIP is denoted by V^{MIP}. This way, by using the integer variables x_r and y_i, various local and practical constraints can be incorporated. In the following, (15) is called constraint on *member intersection*. For the case where at least one of the A_i^{L} has a moderately large value, (14) with (13) is called constraint on *nodal instability*, and (13)-(15) are referred to as *local constraints*.

2.3 Lower-bound solution

A relaxed problem of MIP is to be formulated as an LP to find the lower bound of V^{MIP}. It is obtained by relaxing integer variables x_r and y_i to continuous ones and by neglecting the compatibility constraint (10). The other constraints (12),(13), (14), and (15) remain imposed in the relaxed problem. The relaxed problem so defined is reformulated so as not to use $\boldsymbol{\sigma}^k$. The constraint (9) is rewritten by using variables N_i^k. In fact, if $y_i > 0$, (9) leads to

$$\sigma_i^{\mathrm{L}} \leq \sigma_i^k \leq \sigma_i^{\mathrm{U}}, \quad (i = 1, 2, \cdots, m; k = 1, 2, \cdots, f) \tag{18}$$

and by multiplying A_i,

$$A_i \sigma_i^{\mathrm{L}} \leq N_i^k \leq A_i \sigma_i^k \quad (i = 1, 2, \cdots, m; k = 1, 2, \cdots, f) \tag{19}$$

is derived. Note that (19) is satisfied for $y_i = 0$ because the constraint (13) is imposed in the relaxed problem. Therefore, (19) is satisfied for $0 \leq y_i \leq 1$ if (9) is satisfied.

Hence, the relaxed LP of MIP is formulated as

$$\begin{aligned}
&\mathrm{LP}: \\
&\underset{\mathbf{A},\mathbf{y},\mathbf{x},\mathbf{N}^k}{\text{minimize}} \quad V(\mathbf{A}) = \sum_{i=1}^{m} A_i L_i \\
&\text{subject to} \quad \mathbf{B}\mathbf{N}^k = \mathbf{P}^k, \quad (k = 1, 2, \ldots, f) \\
&\qquad A_i \sigma_i^{\mathrm{L}} \leq N_i^k \leq A_i \sigma_i^{\mathrm{U}} \\
&\qquad\quad (i = 1, 2, \cdots, m; k = 1, 2, \cdots, f) \\
&\qquad A_i^{\mathrm{L}} y_i \leq A_i \leq A_i^{\mathrm{U}} y_i, \quad (i = 1, 2, \cdots, m) \\
&\qquad x_r C_r^{\mathrm{L}} \leq \sum_{i \in J_r} y_i \leq x_r C_r^{\mathrm{U}}, \quad (r = 1, 2, \ldots, s) \\
&\qquad \sum_{j \in S_i} y_j \leq 1, \quad (i = 1, 2, \ldots, q) \\
&\qquad 0 \leq y_i \leq 1, \quad (i = 1, 2, \cdots, m) \\
&\qquad 0 \leq x_r \leq 1, \quad (r = 1, 2, \cdots, s)
\end{aligned}$$

which is a linear programming problem, where the global optimality of the solution is guaranteed. The variables of LP are $\mathbf{A}, \mathbf{y}, \mathbf{x}$ and $\mathbf{N}^k$.

Remark 1 *In the case where a statically determinate truss satisfying the local constraints is obtained by solving LP, the solution gives the globally optimal topology. If a statically indeterminate truss is obtained, V^{LP} is a lower bound of the true optimal objective value of MIP, because the solution of MIP satisfies all the constraints of LP.*

2.4 Upper-bound solution

Given a set $\mathcal{I}$ of existing members, the following NLP0$_\mathcal{I}$ is defined. If NLP0$_\mathcal{I}$ is feasible, its objective value gives an upper bound of V^{MIP} since the solution of NLP0$_\mathcal{I}$ satisfies all the constraints of MIP.

$$
\begin{aligned}
&\text{NLP0}_\mathcal{I}: \\
&\underset{\mathbf{A},\mathbf{U}^k,\boldsymbol{\sigma}^k,\mathbf{N}^k}{\text{minimize}} \quad V(\mathbf{A}) = \sum_{i\in\mathcal{I}} A_i L_i \\
&\text{subject to} \quad \sigma_i^{\text{L}} \leq \sigma_i^k \leq \sigma_i^{\text{U}}, \\
&\qquad (i \in \mathcal{I}; k = 1, 2, \dots, f) && (20) \\
&\qquad \sigma_i^k = \frac{E}{L_i}\mathbf{B}_i^\top \mathbf{U}^k, \\
&\qquad (i \in \mathcal{I}; k = 1, 2, \dots, f) && (21) \\
&\qquad N_i^k = A_i \sigma_i^k, \quad (i \in \mathcal{I}; k = 1, 2, \dots, f) && (22) \\
&\qquad \mathbf{B}\mathbf{N}^k = \mathbf{P}^k, \quad (k = 1, 2, \dots, f) && (23) \\
&\qquad A_i^{\text{L}} \leq A_i \leq A_i^{\text{U}}, \quad (i \in \mathcal{I}) && (24)
\end{aligned}
$$

Alternatively, as stated in Section 2.1, $\mathbf{U}^k$ can be found from (4) by combining (1) and (3), and σ_i^k can be found from (3). Hence σ_i^k can be regarded as implicit functions of $\mathbf{A}$. If the stiffness matrix $\mathbf{K}$ is singular, i.e. the truss is unstable, $\mathbf{U}^k$ need not be found because the solution is infeasible. Therefore, NLP0$_\mathcal{I}$ is alternatively written as

$$
\begin{aligned}
&\text{NLP}_\mathcal{I}: \\
&\underset{\mathbf{A}}{\text{minimize}} \quad V(\mathbf{A}) = \sum_{i\in\mathcal{I}} A_i L_i \\
&\text{subject to} \quad \sigma_i^{\text{L}} \leq \sigma_i^k(\mathbf{A}) \leq \sigma_i^{\text{U}} \\
&\qquad (i \in \mathcal{I}; k = 1, 2, \cdots, f) \\
&\qquad A_i^{\text{L}} \leq A_i \leq A_i^{\text{U}}, \quad (i \in \mathcal{I})
\end{aligned}
$$

The optimal objective value of NLP$_\mathcal{I}$ is denoted by $V^{\text{NLP}_\mathcal{I}}$. In the following, the subscript $\mathcal{I}$ of NLP$_\mathcal{I}$ is often omitted when $\mathcal{I}$ can be understood from the context.

Stolpe and Svanberg (2003) stated that the discontinuity in stress constraints can be avoided, for a problem with single loading condition, by simply assigning constraints as (19) on member forces instead of stresses. If $A_i = 0$, (19) is obviously satisfied by $N_i^1 = 0$. However, stresses should be between σ_i^{L} and σ_i^{U} if A_i has a small positive value. In fact, the solution found by Stolpe and Svanberg (2003) is not singular, and satisfies all the stress constraints, and a singular optimal topology with violating stress constraints by vanishing members cannot be found by a conventional NLP algorithm for a multiple loading case.

3 A branch-and-bound algorithm.

A branch-and-bound algorithm for solving MIP is proposed. The idea is based on the standard principle of branch-and-bound algorithms.

Throughout the entire process of the algorithm, the best feasible solution and its objective value (denoted by V^{U}) is maintained which gives an upper bound of the original problem. The algorithm successively decomposes the problem into subproblems by fixing integer variable y_i to 0 and 1, respectively, and maintains a set $\mathcal{A}$ of active subproblems.

In a general step, the algorithm first picks up a problem P from $\mathcal{A}$, and solves the relaxed LP of P which gives a lower-bound solution. Let $\mathcal{R}$ and $\mathcal{I}$ denote the set of nodes and members, respectively, where $x_r > 0$ and $y_i > 0$ hold in the lower bound solution. *The topology conforming to the LP solution* is defined by regarding the set of members $i \in \mathcal{I}$ as the existing members and the set of nodes $r \in \mathcal{R}$ as the existing nodes. If such topology satisfies (14) and (15), then an upper bound V^{U} of the optimal objective value of MIP is obtained by solving $\mathrm{NLP}_{\mathcal{I}}$ after specifying the members $\mathcal{I}$ corresponding to the topology. In particular, if this upper bound is smaller than V^{U}, V^{U} is updated. If the lower bound solution does not satisfy (14) or (15), the algorithm does not solve $\mathrm{NLP}_{\mathcal{I}}$.

In any case, the problem is then decomposed into two subproblems P_0 and P_1, respectively, by specifying $y_j = 0$ and 1 for some $j \in \mathcal{I}$ with $0 < y_j < 1$, and add these two subproblems to $\mathcal{A}$. The entire procedure terminates when all subproblems are terminated. The termination condition for the subproblem is (i) the optimal value of the relaxed LP problem is larger than the current upper bound V^{U} or (ii) the relaxed LP is infeasible.

Let V^{L} denote the lower bound given by the LP solution, and V_0 and V_1 the lower bounds obtained for P_0 and P_1, respectively. Since P_0 and P_1 are obtained by restricting y_j to 0 and 1, respectively, the feasible region of P strictly includes the union of those for P_0 and P_1. Thus, $V^{\mathrm{L}} \leq \min\{V_0, V_1\}$ follows, and the value $\min\{V_0, V_1\}$ may be used as the more exact lower bound. Therefore, in the algorithm, when both of subproblems P_0 and P_1 terminate, V^{L} is updated to $\min\{V_0, V_1\}$. Notice that such update scheme of the lower bound does not improve the output quality, but the value V^{L} obtained after the termination of all subproblems will be used to evaluate how close the upper bound solution obtained by the proposed algorithm is to the exact optimal solution.

The algorithm is summarized as follows:

Step 0 Initialize the upper bound V^{U} as $V^{\mathrm{U}} = \infty$. Let the set $\mathcal{A}$ of the active problems consist of the original MIP.

Step 1 Select a problem P from $\mathcal{A}$ and remove it from $\mathcal{A}$.

Step 2 Select a member j from $\mathcal{I}$ with $0 < y_j < 1$ of the solution of the relaxed LP $\bar{P}$ of P, and solve LP denoted by $\bar{P}_0$ and $\bar{P}_1$ of the subproblems P_0 and P_1, respectively, of P by specifying $y_j = 0$ and 1.

Step 3 Let V_0 and V_1 denote the optimal objective values of $\bar{P}_0$ and $\bar{P}_1$. If $V_0 > V^{\mathrm{U}}$, $y_i = 1$ and terminate P_0. If $V_1 > V^{\mathrm{U}}$, $y_i = 0$ and terminate P_1.

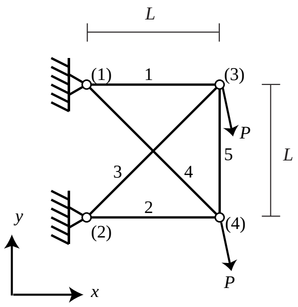

Fig. 1: A 5-bar truss.

Step 4 If the topology conforming to the LP solution satisfies all the local constraints, compute the objective value of $\mathrm{NLP}_{\mathcal{I}}$, denoted by $V^{\mathrm{NLP}_{\mathcal{I}}}$, after specifying the set $\mathcal{I}$ of the existing members as $\mathcal{I} = \{j \mid j = 1, 2, \ldots, m, y_j > 0\}$.
Step 5 If $V^{\mathrm{NLP}_{\mathcal{I}}} < V^{\mathrm{U}}$, update V^{U} to $V^{\mathrm{NLP}_{\mathcal{I}}}$.
Step 6 If V_0 and/or V_1 is less than V^{U}, add P_0 and/or P_1, respectively, to $\mathcal{A}$ and go to Step 1.
Step 7 If $\mathcal{A} \neq \emptyset$ go to Step 1.
Step 8 Output the best value of V^{U}. Compute the lower bound V^{L} backward from the bottom of the branching tree so that V^{L} of the parent problem P is updated by $\min\{V_0, V_1\}$ if $V^{\mathrm{L}} < \min\{V_0, V_1\}$.
Remark 2 *At the first stage of the branching process, we can obtain an upper bound* V^{U} *by fixing* y_i *appropriately for all the members existing in the lower bound solution so that the local constraints are satisfied. Note that it is not recommended to try all the possible combinations of* y_i *satisfying local constraints. Letting* $y_i = 1$ *for the member with the larger cross-sectional area between the pair of intersecting members in an LP solution will lead to a good upper-bound solution. If this process is skipped, simply set* $V^{\mathrm{U}} = \infty$ *at the beginning of the branching step.*
Remark 3 *We can use the local constraints before solving LP: i.e.* $x_r = 1$ *if a member exists such that* $y_i = 1$ $(i \in J_r)$*;* $x_r = 0$ *if* $y_i = 0$ *for all the members in* J_r*;* $y_i = 0$ $(i \in S_r)$ *if* $y_j = 1$ $(j \in S_r, j \neq i)$.
Remark 4 *Under the assumption that exact optimal solutions for all* $NLP_{\mathcal{I}}$*s can be obtained, global optimality of the solution found by the proposed branch-and-bound algorithm is guaranteed. However, since the* $NLP_{\mathcal{I}}$ *solved is a nonconvex problem, it is hard to theoretically guarantee that the* $NLP_{\mathcal{I}}$ *can be solved exactly. If it is known that the obtained solution of* $NLP_{\mathcal{I}}$ *is within error* ϵ *from the optimal, the solution output by the proposed branch-and-bound algorithm is also within error* ϵ *from the optimal.*
Remark 5 *Since different feasible solutions of LP might lead to the same topology for solving the* $NLP_{\mathcal{I}}$*, the topologies solved by NLP are stored in a list, and the* $NLP_{\mathcal{I}}$ *is not solved if the topology matches one in the list.*

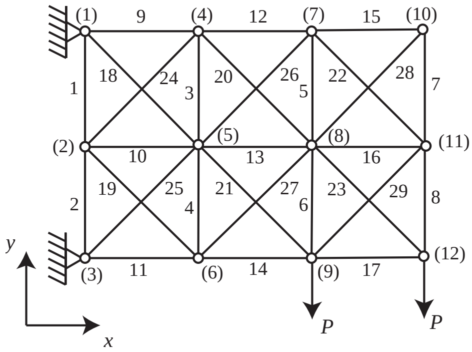

Fig. 2: A 3 × 2 plane grid.

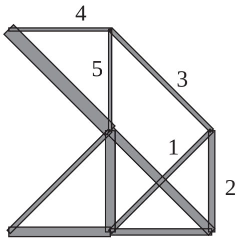

Fig. 3: Initial LP solution for the 2 × 2 plane grid.

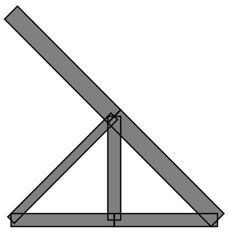

Fig. 4: Initial upper-bound solution for the 2 × 2 plane grid.

4 Examples.

In the following examples, the units of force, length, area, volume and stress are kN, cm, cm^2, cm^3 and MPa, respectively.

LP is solved by HOPDM Ver. 2.13 (Gondzio, 1995) that uses higher-order primal-dual method. NLP is carried out by NLPQL implemented as DNCONG in IMSL library (Visual Numerics, 1997), where the sequential quadratic programming is used. It has been confirmed that the same optimal solutions can be found in the following examples by using IDESIGN Ver. 3.5 (Arora and Tseng, 1987) starting from different initial solutions. Therefore, the nonconvexity of the $\mathrm{NLP}_{\mathcal{I}}$ is small, and the globally optimal solution of $\mathrm{NLP}_{\mathcal{I}}$ can be found for almost all the cases. Sensitivity coefficients of stresses and displacements with respect to the cross-sectional areas are computed analytically using well-established method of design sensitivity analysis (Haug *et al.*, 1986). Optimization has been carried out on Xeon 2.8GHz with 1GB memory.

Consider for verification purpose the 5-bar truss as shown in Fig. 1 which was solved in Cheng and Guo (1997). The number with and without parentheses indicate the member number and node number, respectively. Two loading conditions are considered, where the loads $(P_x, P_y) = (5, -50)$ are applied at nodes 3 and 4, respectively. The upper and lower bounds for stresses are ± 20 for member 2 and ± 5 for the remaining members. The bounds of the cross-sectional areas are $A_i^{\mathrm{L}} = 0.0$, $A_i^{\mathrm{U}} = 20.0$. The local constraints are not given for comparison purpose. However, from stability requirements, C_r^L should be 1 for the supports and 2 for the loaded nodes. x_r should be 1 for all the nodes and supports.

The initial LP solution has intersecting members 3 and 4. The member 4 has been selected as the branching member. The optimal objective value of $\bar{P}_0$ is 33.500. Since the optimal truss of $\bar{P}_0$ is statically determinate, V^{U} is also 33.500. The optimal objective value of $\bar{P}_1$ is 32.500. All the members exist in the solution of $\bar{P}_1$, and the value of $V^{\mathrm{NLP}_{\mathcal{I}}}$ is 39.986. After solving LP and $\mathrm{NLP}_{\mathcal{I}}$ 15 times and 4 times, respectively, the best upper bound solution is $(A_1, A_2, A_3, A_4, A_5) = (1.0000,\ 2.5000,\ 10.0000,\ 0.0,\ 14.1421)$ with $V^{\mathrm{U}} = 33.500$, which agrees with the result in Cheng and Guo (1997). The lower bound V^{L} is 32.500 which is slightly smaller than V^{U}. Note that the numbers of LP and NLP problems to be solved are large for this small example, because the local constraints are not considered for comparison purpose. It is shown in the following example that the numbers of LP and NLP solutions are drastically reduced by incorporating the local constraints.

Next we consider a 3×2 grid as shown in Fig. 2. The lengths of the members in x- and y-directions are 200. Irrespective of the numbers of divisions, two loading conditions are considered, where the loads in the negative y-directions are applied at the node at the lowest end (node 12 in Fig. 2) and the node left to the lowest end (node 9 in Fig. 2), respectively. The magnitude of each load is 1000.

In the following examples, the bounds for the stress are ± 200.0, and $C_r^{\mathrm{U}} = 6$. The value of C_r^{L} is 1 for the supports, 2 for the node at the lowest end, and 3 for the remaining nodes. Note that these lower bound are defined naturally from the requirements of stability and equilibrium, and they do not unnecessarily restrict the design space.

Optimal topology has been first found for the 2×2 grid. The LP solution at the first step is as shown in Fig. 3, where the width of a member is proportional to its cross-sectional area. The objective value V^{LP} is 7.0000×10^3. To obtain an initial upper-bound solution, member 1 indicated in Fig. 3 is removed because it has smaller cross-sectional area in the pair of intersecting members. Note that this selection is heuristic. However, only a good upper bound is to be found at this stage as commented in Remark 2. After removing member 1, the node connected by members 2 and 3 is removed because it violates the local constraint (14) with $C_r^{\mathrm{L}} = 3$. Hence, members 2 and 3 are removed, and members 4 and 5 are to be removed in a similar manner as described in Remark 3. The $\mathrm{NLP}_{\mathcal{I}}$ is solved by fixing the values of y_i and x_r to find an upper bound solution as shown in Fig. 4, where $V^{\mathrm{NLP}_{\mathcal{I}}}$ is 8.0000×10^3.

The branch-and-bound process is carried out to find the final upper-bound solution as shown in Fig. 5, where $V^{\mathrm{U}} = 7.9000 \times 10^3$. The optimization results are listed

Table 1: Optimization results.

No. of division	m	DOF	$A_i^{\rm L}$	$A_i^{\rm U}$	No. of steps	No. of LP	No. of NLP	$V^{\rm U}$	$V^{\rm L}$	CPU time (s)
2×2	20	14	200	800	121	64	5	7.9000×10^3	7.8000×10^3	2.02
3×2	29	20	200	800	942	571	6	1.2900×10^4	1.2800×10^4	18.95
3×3	42	28	200	800	5874	3483	23	1.2467×10^4	1.2467×10^4	147.84
4×4	72	46	200	800	64890	42831	7	1.7067×10^4	1.7067×10^4	3072.84
4×4	72	46	200	600	68656	42707	73	1.8373×10^4	1.7916×10^4	2513.06
4×4	72	46	400	800	41001	26580	3	2.1507×10^4	2.1507×10^4	1794.08

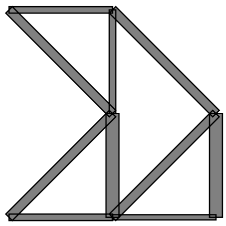

Fig. 5: Final upper-bound solution for the 2×2 plane grid.

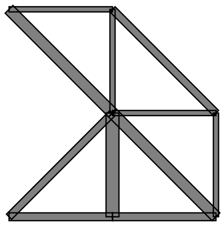

Fig. 6: Final lower-bound solution for the 2×2 plane grid.

in the first row of Table 1, where *No. of steps* means the number of nodes of the branching tree. In this example, only 121 topologies have been searched, whereas the total number of the possible topologies are $2^{20} \simeq 1.0 \times 10^6$. The final lower-bound solution is as shown in Fig. 6, where $V^{\rm L} = 7.8000 \times 10^3$. Since this truss is statically indeterminate, the axial forces obtained by solving LP are not correct. The maximum absolute value of the ratio of stress to the upper or lower bounds is 1.1111 if the compatibility conditions are considered; i.e. the solution does not satisfy stress constraints. Hence $V^{\rm L}$ has smaller value than $V^{\rm U}$. However, the difference between $V^{\rm L}$ and $V^{\rm U}$ is very small and the good upper-bound solution has been found after solving NLP$_{\mathcal{I}}$ only 5 times.

The problem without local constraints have been also solved to compare the computational costs. The numbers of steps, LP, and NLP are 529, 350, and 58, respectively, and CPU time is 5.05.

The optimization results for 3×2, 3×3 and 4×4 grids are shown in the second, third and fourth rows of Table 1, respectively. The final upper-bound solutions are as

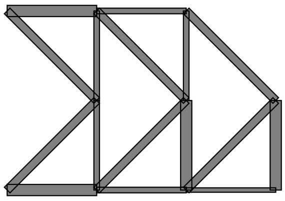

Fig. 7: Final upper-bound solution for the 3×2 plane grid.

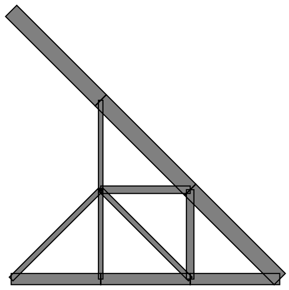

Fig. 8: Final upper-bound solution for the 3×3 plane grid.

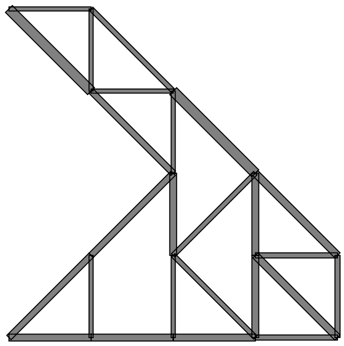

Fig. 9: Final upper-bound solution for the 4×4 plane grid.

shown in Figs. 7-9. Note that $V^{\mathrm{L}} = V^{\mathrm{U}}$ is satisfied for 3×3 and 4×4, because the lower-bound solutions are statically determinate.

The number of LP steps and CPU time increase drastically as the size of the problem such as the number of members is increased. The number of NLP steps, however, is independent of the problem size, because it depends on the quality of the initial upper-bound solution.

If A_i^{U} is decreased to 600 for the 4×4 grid, the optimization results are as shown in the fifth row of Table 1. The number of NLP steps is increased to 73. However, it is difficult to suggest a relation between the computational cost and the constraints or the size of the feasible region, because the CPU time for $A_i^{\mathrm{L}} = 400$ is almost half of that for $A_i^{\mathrm{L}} = 200$ as shown in the last row of Table 1.

If we do not consider the local constraints for the 3×2 truss, the numbers of steps, LP, and NLP are 2364, 1468, and 391, respectively, and CPU time is 72.28. We can observe from these results that the computational cost can be drastically reduced by using the local constraints.

5 Conclusions.

A branch-and-bound method has been presented for obtaining upper- and lower-bound solutions of optimal topology of trusses under stress constraints. A rigorous problem is first defined as a MIP problem with 0-1 variables indicating existence of nodes and members. The constraints on member intersection and nodal instability, which are referred to as local constraints, are also considered. A moderately large lower bound is given for the cross-sectional area of an existing member to prevent an unrealistic optimal solution.

The MIP problem is converted to an LP problem by relaxing the integer variables and by ignoring the compatibility conditions between stresses and displacements. It has been shown in the examples that good upper and lower bounds can be found by using the proposed method. Computational cost can be drastically reduced by introducing local constraints to obtain practically acceptable optimal topologies.

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