Topp–Leone Family of Distributions: Some Properties and Application

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Abstract

In this paper we have proposed a new family of distributions; the Topp–Leone family of distributions. We have given general expression for density and distribution function of the new family. Expression for moments and hazard rate has also been given. We have also given an example of the proposed family.

Keywords: Topp–Leone distribution, Hazard rate, Moments.

1. Introduction

Topp–Leone distribution is a simple bounded J-shaped distribution that has attracted various statisticians as an alternative to Beta distribution. The density and distribution function of Topp-Leone distribution is given by Nadarajah and Kotz (2003) as

$$f_{TL}(x) = 2\alpha x^{\alpha-1}(1-x)(2-x)^{\alpha-1}; 0 \le x \le 1, \alpha > 0 ; \qquad (1.1)$$

and $\quad F_{TL}(x) = x^{\alpha}(2-x)^{\alpha}; 0 \le x \le 1, \alpha > 0 . \qquad (1.2)$

Since its emergence the distribution has been studied by number of authors, see for example Ghitany et al. (2005), van Dorp and Kotz (2006), Zhou et al. (2006), Kotz and Seier (2007), Nadarajah (2009), and Genç (2012).

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Generalizing probability distributions has been area of study by number of authors. Various families of distributions have also been proposed by using different bounded and unbounded distributions. Eugene et al. (2002) proposed Beta family of distributions by using distribution function of a Beta random variable. The distribution function of the proposed family is

$$F_{B-G}(x) = \frac{1}{B(a,b)} \int_0^{G(x)} w^{a-1} (1-w)^{b-1} \, dw;$$

(1.3)

where $B(a,b)$ is the Beta function and $G(x)$ is distribution function of any baseline distribution. The Beta family of distribution has been widely studied for different choices of $G(x)$, see for example Famoye et al. (2005), Hanook et al. (2013), Nadarajah and Kotz (2004) among many others.

Kumaraswamy (1980) has proposed another bounded distribution which is simpler than the Beta distribution. The Kumaraswamy distribution has been used by Cordiero and Castro (2011) to propose the Kumaraswamy family of distributions. The distribution function of Kumaraswamy family of distribution is

$$F_{K-G}(x) = 1 - \left[1 - \left\{ G(x) \right\}^a \right]^b.$$

(1.4)

Shahbaz et al. (2012) have used distribution function of Inverse Weibull distribution in (1.4) to propose the Kumaraswamy Inverse Weibull distribution.

Alzaatreh et al. (2014) have used distribution function of Gamma random variable to propose the Gamma family of distribution. The distribution function of proposed family has the form

$$F_{G-G}(x) = \frac{1}{\Gamma(\alpha)} \int_0^{-\ln[1-G(x)]} t^{\alpha-1} e^{-t} dt \ ;$$

(1.5)

where $\Gamma(a)$ is the Gamma function. The Gamma family of distribution has also been studied by number of authors.

We propose a new family of distributions by using distribution function of Topp-Leone distribution in the following section.

2. Topp–Leone Family of Distributions

The distribution function of Topp–Leone distribution is given in (1.2) as

$$F_{TL}(x) = x^\alpha (2-x)^\alpha ; 0 \le x \le 1, \alpha > 0.$$

Using above distribution function we propose the Topp–Leone family of distributions as one having the distribution function given as

$$F_{TL-G}(x) = \left[G(x) \right]^\alpha \left[2 - G(x) \right]^\alpha ; x \in \Re, \alpha > 0.$$

(2.1)

The Topp–Leone family of distributions reduces to the baseline distribution $G(x)$ if

$$\alpha = \frac{\ln\left[G(x)\right]}{\ln\left[\{G(x)\}\{2-G(x)\}\right]}.$$

The density function of Topp–Leone family of distributions is readily written as

$$f_{TL-G}(x) = 2\alpha g(x)\bar{G}(x)\left[G(x)\right]^{\alpha-1}\left[2-G(x)\right]^{\alpha-1}, \alpha > 0; \tag{2.2}$$

where $g(x) = G'(x)$ and $\bar{G}(x) = 1 - G(x)$. Various choices of $G(x)$ lead to various members of Topp–Leone family. We will denote Topp–Leone family by TL–G distributions.

3   Expansion for Density and Distribution Function

The distribution function of Topp–Leone family of distributions is given in (2.1) as

$$F_{TL-G}(x) = \left[G(x)\right]^{\alpha}\left[2-G(x)\right]^{\alpha} = \left[1-\{\bar{G}(x)\}^2\right]^{\alpha}. \tag{3.1}$$

Now using the following identity given in Prudnikov *et al.* (1986)

$$(1+x)^{\alpha} = \sum_{j=0}^{\infty} \frac{\Gamma(\alpha+1)}{j!\,\Gamma(\alpha+1-j)}x^j;$$

the distribution function of TL–G family is written as

$$F_{TL-G}(x) = \left[1-\{\bar{G}(x)\}^2\right]^{\alpha} = \sum_{j=0}^{\infty} \frac{(-1)^j\,\Gamma(\alpha+1)}{j!\,\Gamma(\alpha+1-j)}\{\bar{G}(x)\}^{2j}. \tag{3.2}$$

or $\quad F_{TL-G}(x) = \sum_{j=0}^{\infty}\sum_{m=0}^{2j} \frac{(-1)^{j+m}\,\Gamma(\alpha+1)}{j!\,\Gamma(\alpha+1-j)}\binom{2j}{m}\{G(x)\}^m = \sum_{j=0}^{\infty}\sum_{m=0}^{2j} a(j,m)\{G(x)\}^m \tag{3.3}$

where $a(j,m) = \dfrac{(-1)^{j+m}\,\Gamma(\alpha+1)}{j!\,\Gamma(\alpha+1-j)}\binom{2j}{m}.$

Again from (3.1), the density function of TL–G family is

$$f_{TL-G}(x) = 2\alpha g(x)\bar{G}(x)\left[1-\{\bar{G}(x)\}^2\right]^{\alpha-1}, \alpha > 0. \tag{3.4}$$

Using series expansion the density can be written as

$$f_{TL-G}(x) = \sum_{j=0}^{\infty} \frac{(-1)^j\,2\Gamma(\alpha+1)}{j!\,\Gamma(\alpha-j)}g(x)\{\bar{G}(x)\}^{2j+1} \tag{3.5}$$

or $\quad f_{TL-G}(x) = \sum_{j=0}^{\infty} \sum_{m=0}^{2j+1} \frac{(-1)^{j+m} 2\Gamma(\alpha+1)}{j!\Gamma(\alpha-j)} \binom{2j+1}{m} g(x)\{G(x)\}^m$

or $\quad f_{TL-G}(x) = \sum_{j=0}^{\infty} \sum_{m=0}^{2j+1} b(j,m)(m+1) g(x)\{G(x)\}^m = \sum_{j=0}^{\infty} \sum_{m=0}^{2j+1} b(j,m) h_{m+1}(x); \quad$ (3.6)

where $\quad b(j,m) = \frac{(-1)^{j+m} 2\Gamma(\alpha+1)}{j!\Gamma(\alpha-j)(m+1)} \binom{2j+1}{m} \quad$ and $\quad h_{m+1}(x) = (m+1) g(x)\{G(x)\}^m \quad$ is

exponentiated $-$ G distribution with power parameter m.

The distribution function and density function of TL–G family given in (3.3) and (3.6) respectively immediately show that the TL–G family can be viewed as weighted sum of exponentiated class of distributions.

Using (3.1) and (3.4), the hazard rate function of TL–G family is readily written as

$$h_{TL-G}(x) = \frac{f_{TL-G}(x)}{1 - F_{TL-G}(x)} = \frac{2\alpha g(x)\bar{G}(x)\left[1 - \{\bar{G}(x)\}^2\right]^{\alpha-1}}{1 - \left[1 - \{\bar{G}(x)\}^2\right]^{\alpha}}.$$ (3.7)

Further, from (3.3) we see that the TL–G family is actually weighted sum of exponentiated–G distribution hence moments of TL–G family can be written from exponentiated base line distribution G. Also from (3.3), the moments of TL–G family can be written as weighted sum of probability weighted moments (PWMs) of the baseline distribution G as

$$\mu_{TL-G}^p = E_{TL-G}(X^p) = \sum_{j=0}^{\infty} \sum_{m=0}^{2j+1} b^*(j,m) \tau_{p,m};$$ (3.8)

where $b^*(j,m) = (m+1) b(j,m)$ and $\tau_{p,m}$ is probability weighted moment of baseline distribution given as

$$\tau_{p,m} = E_G\left[X^p\{G(x)\}^m\right] = \int_{-\infty}^{\infty} x^p g(x)\{G(x)\}^m dx.$$

The moments for TL–G class can be computed from (3.8) for certain special cases.

The quantile function of TL–G family can be readily written from quantile function of Topp-Leone distribution as

$$Q_{TL-G}(u) = G^{-1}\left[1 - \sqrt{1 - \sqrt[\alpha]{u}}\right]; 0 < u < 1.$$ (3.9)

The quantile function can be used to generate the random data for TL–G family of distributions for certain special cases.

4. Parameter Estimation

The density function of TL–G family of distribution is given in (3.4) as

$$f_{TL-G}(x) = 2\alpha g(x)\bar{G}(x)\left[1 - \{\bar{G}(x)\}^2\right]^{\alpha-1}, \alpha > 0.$$

The log of likelihood function for TL – G family is hence

$$L = \ln(LF) = n\ln(2) + n\ln(\alpha) + \sum_{i=1}^{n}\ln\{g(x_i)\} + \sum_{i=1}^{n}\ln\{\bar{G}(x_i)\}$$

$$+ (\alpha - 1)\sum_{i=1}^{n}\ln\left[\left\{1 - \{\bar{G}(x_i)\}^2\right\}\right]$$

The likelihood equation for estimation of α is $\partial L/\partial\alpha = 0$ and is given as

$$\frac{n}{\hat{\alpha}} + \sum_{i=1}^{n}\ln\left[\left\{1 - \{\bar{G}(x_i)\}^2\right\}\right] = 0;$$

which provide the maximum likelihood estimator of α as

$$\hat{\alpha} = -n\left(\sum_{i=1}^{n}\ln\left[\left\{1 - \{\bar{G}(x_i)\}^2\right\}\right]\right)^{-1}. \tag{4.1}$$

The MLE of α for special cases can be readily obtained from (4.1).

5. The Topp–Leone Exponential Distribution

In this section we have given an illustrated example of TL–G family of distributions by using $G(x) = 1 - e^{-\lambda x}$, that is the Exponential distribution and we will call the overall distribution the Topp–Leone–Exponential distribution or TL–E distribution for short. The distribution function of TL–E distribution is readily written from (3.1) as
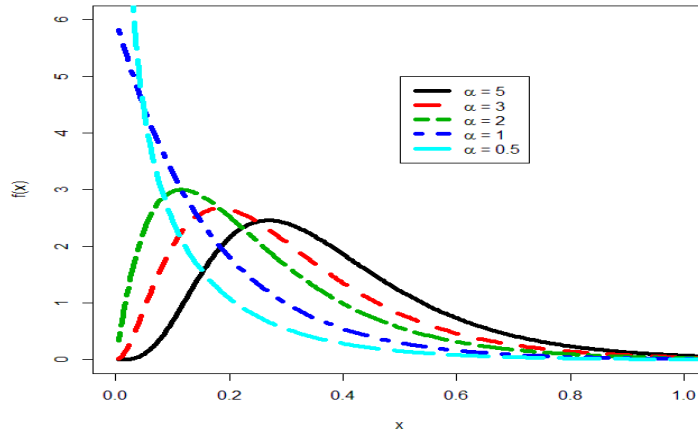
$$F_{TL-E}(x) = \left[1 - e^{-2\lambda x}\right]^{\alpha} \tag{5.1}$$

The density function of TL–E distribution is

$$f_{TL-E}(x) = 2\alpha\lambda e^{-2\lambda x}\left(1 - e^{-2\lambda x}\right)^{\alpha-1}; x, \alpha, \lambda > 0. \tag{5.2}$$

The plot of density function for various values α and $\lambda = 3$ is given below

Figure 1: Density function of TL–E for $\lambda = 3$ and various values of α
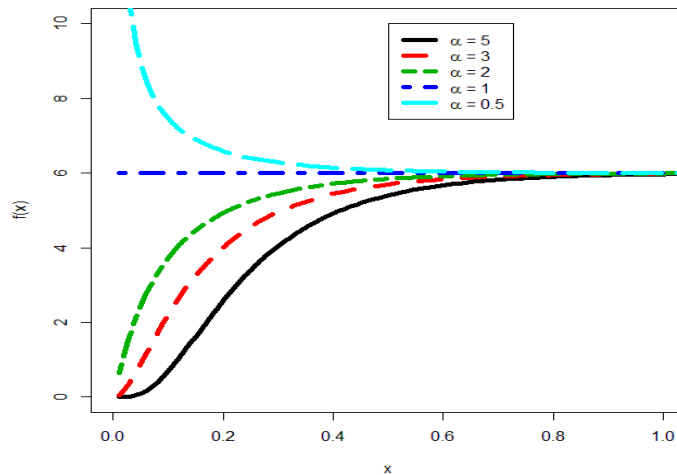

The hazard rate function for TL–E distribution is given as

$$h_{TL-E}(x) = \frac{2\alpha\lambda e^{-2\lambda x}\left(1-e^{-2\lambda x}\right)^{\alpha-1}}{1-\left(1-e^{-2\lambda x}\right)^{\alpha}}.$$

$$(5.3)$$

The plot of hazard rate function for various values α and $\lambda = 3$ is given below

Figure 2: Hazard rate function of TL–E for $\lambda = 3$ and various values of α

From above figure we can see that the TL–E is increasing failure rate distribution for $\alpha > 1$ and is decreasing failure rate distribution for $\alpha < 1$.

The log of likelihood function for TL–E distribution is

$$L = n\ln(2) + n\ln(\alpha) + n\ln(\lambda) - 2\lambda\sum_{i=1}^{n}x_i + (\alpha-1)\sum_{i=1}^{n}\ln\left\{\left(1-e^{-2\lambda x_i}\right)\right\}$$

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The likelihood equations to obtain MLE's of unknown parameters are

$$\frac{n}{\hat{\alpha}} + \sum_{i=1}^{n} \ln\left(1 - e^{-2\hat{\lambda}x_i}\right) = 0$$

$$\frac{n}{\hat{\lambda}} - 2\sum_{i=1}^{n} x_i + \left(\hat{\alpha} - 1\right)\sum_{i=1}^{n} \frac{2x_i e^{-2\hat{\lambda}x_i}}{1 - e^{-2\hat{\lambda}x_i}} = 0$$

which can be solved numerically to obtain MLE's of α and λ.

The entries of information matrix are obtained from following

$$\frac{\partial^2 L}{\partial \alpha^2} = -\frac{n}{\alpha^2};$$

$$\frac{\partial^2 L}{\partial \alpha \partial \lambda} = \sum_{i=1}^{n} \frac{2x_i e^{-2\lambda x_i}}{1 - e^{-2\lambda x_i}};$$

$$\frac{\partial^2 L}{\partial \lambda^2} = -\frac{n}{\lambda^2} - \left(\alpha - 1\right)\sum_{i=1}^{n} \frac{4x_i^2 e^{-2\lambda x_i}}{\left(1 - e^{-2\lambda x_i}\right)^2}.$$

These entries can be obtained numerically.

6. Application

In this section we have given a numerical application of the TL − E distribution. We have used data from Nigm et al. (2003) and is about ordered failure of components. The data is given below

> 0.0009, 0.004, 0.0142, 0.0221, 0.0261, 0.0418, 0.0473, 0.0834, 0.1091, 0.1252, 0.1404, 0.1498, 0.175, 0.2031, 0.2099, 0.2168, 0.2918, 0.3465, 0.4035, 0.6143

The data is highly positively skewed with coefficient of skewness of 1.44. The maximum likelihood estimates of TL–E distribution for the data; standard errors are in parenthesis; are

$$\hat{\alpha} = 0.793\left(0.221\right) \text{ and } \hat{\lambda} = 2.664\left(0.824\right).$$

The Kolmogorov–Smirnov goodness of fit statistic has a value of 0.072 with a p-value of 0.999. This indicates that the data is adequately fitted by TL–E distribution. The plot of data with TL–E distribution is given below

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Figure 3:   Histogram of Data and Fitted Topp–Leone Exponential Distribution
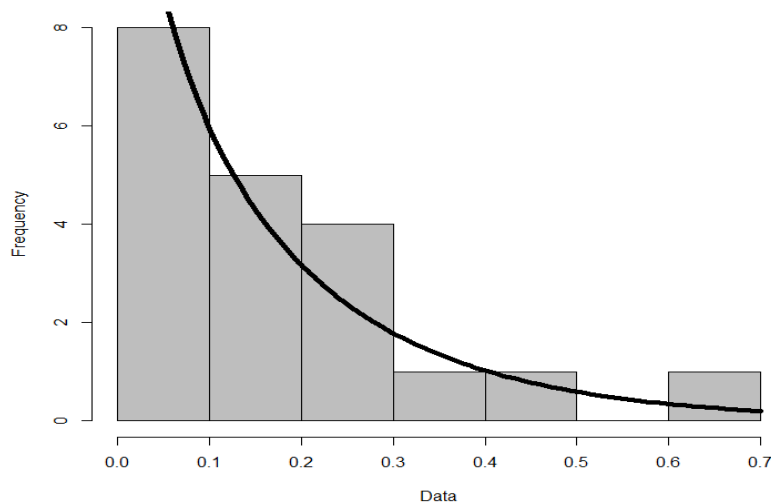
The graph also shows that the data is adequately fitted by TL–E distribution.

7.   Conclusion and Further Work

New family of distributions are proposed; the Topp-Leone family of distributions. The density function and distribution function for new family are given as weighted sum of exponentiated base distribution. In addition, expression for moments and hazard rate are given. Exponential distribution is applied as special case of TL-G family of distributions. Numerical application of the TL–E distribution is discussed. This example indicate that the data is adequately fitted by TL–E distribution.

The present work can be expanded by using other base distributions like Rayleigh, Weibull etc.

References

1.   Alzaatreh, A., Lee, C., Famoye, F. (2014). The Gamma-Normal distribution: Properties and application, *Comp. Stat. and Data Analysis*, 69, 67-80.

2.   Cordeiro, G. M. and Castro, M. de (2011). A new family of generalized distributions. *J. of Stat. Comp. and Sim*. Vol. 81(7), 883–898.

3.   Eugene, N., Lee, C. and Famoye, F. (2002). Beta-Normal distribution and its applications, *Comm. in Statist: Theory and Methods*, Vol. 31, 497–512.

4.   Famoye, F., Lee, C. and Olumolade, O. (2005). The Beta-Weibull Distribution, *Journal of Statistical Theory and Applications*, 4(2), 121-136.

5.   Genç, A. I. (2012). Moments of order statistics of Topp-Leone distribution. *Statistical Papers*, 53, 117-13.

6.   Ghitany, M. E., Kotz, S. and Xie, M. (2005). On some reliability measures and their stochastic orderings for the Topp-Leone distribution. *Journal of Applied Statistics*, 32, 715-722.

7.   Hanook, S., Shahbaz, M. Q., Mohsin, M. and Kibria, G. (2013). A Note on Beta Inverse Weibull Distribution, *Comm. In Statist: Theory and Methods*, 42(2), 320-335.

8.   Kotz, S. and Seier, E. (2007). Kurtosis of the Topp-Leone distributions. *InterStat*, July, #001.

9.   Kumaraswamy, P. (1980). A generalized probability density function for double-bounded random processes, *J. of Hydrology*, Vol. 46, 79–88.

10.   Nadarajah, S. (2009). Bathtub-shaped failure rate functions. *Quality and Quantity*, 43, 855-863.

11.   Nadarajah, S. and Kotz, S. (2003). Moments of some J-shaped distributions. *Journal of Applied Statistics*, 30, 311-317.

12.   Nadarajah, S., Kotz, S. (2004). The Beta Gumbel Distribution, *Mathematical Problems in Engineering*, 4, 323-332.

13.   Nigm, A. M. (2003), AL-Hussaini, E. K. and Jaheen, Z. F. (2003), Bayesian one-sample prediction of future observations under Pareto distribution. *Statistics*, 37, (6), 527–536.

14.   Prudnikov, A. P., Brychkov, Y. A. and Marichev, O. I. (1986) *Integrals and Series. Vol. 3* Gordon and Breach Science Publishers, New York.

15.   Shahbaz, M. Q., Shahbaz, S. and Butt, N. S. (2012). The Kumaraswamy Inverse Weibull distribution, *Pak. J. Stat. & O.R.*, 8(3), 479-489.

16.   van Dorp, J. R. and Kotz, S. (2006). *Modeling income distributions using elevated distributions on a bounded domain*. In R. H. Pleguezuelo, J. C. Céspedes and J. M. Velasco (eds.), Distribution Models Theory, World Scientific, Singapore, 1-25.

17.   Zhou, M., Yang, D. W., Wang, Y. and Nadarajah, S. (2006). Some J-shaped distributions: Sums, products and ratios. RAMS '06. *Annual Reliability and Maintainability Symposium*, 175-18.