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Toric cohomological rigidity of simple convex polytopes

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Abstract

A simple convex polytope P is *cohomologically rigid* if its combinatorial structure is determined by the cohomology ring of a quasitoric manifold over P. Not every P has this property, but some important polytopes such as simplices or cubes are known to be cohomologically rigid. In this paper we investigate the cohomological rigidity of polytopes and establish it for several new classes of polytopes, including products of simplices. The cohomological rigidity of P is related to the *bigraded Betti numbers* of its *Stanley–Reisner ring*, another important invariant coming from combinatorial commutative algebra.

1. *Introduction*

Quasitoric manifolds were defined by Davis and Januszkiewicz [**7**] as a topological analogue of non-singular toric varieties; namely, a *quasitoric manifold* over a simple convex polytope P is a closed $2n$-dimensional manifold M with a locally standard action of an n-torus $G = (S^1)^n$ (that is, the action locally looks like a faithful real $2n$-dimensional representation of G) and a surjective map $\pi\colon M \to P$ whose fibres are the G-orbits. The combinatorial structure of P is completely determined by the equivariant cohomology ring $H^*_G(M)$ because $H^*_G(M)$ is isomorphic to the Stanley–Reisner ring (or the face ring) of P. On the other hand, the $2i$th Betti number of M is equal to the ith component of the h-vector of P. Therefore the usual cohomolgy $H^*(M)$ contains some combinatorial information of P.

In general, the cohomology ring of a quasitoric manifold does not contain sufficient information to determine the combinatorial structure of the base polytope P, as in [**10**, Example 4.3], which we shall discuss briefly for the reader's convenience. To do this, let us fix some notation. For an n-dimensional simple convex polytope P and a vertex v of it, let $\mathrm{vc}(P, v)$ denote the connected sum of P with the n-simplex Δ^n at the vertex v. Hence $\mathrm{vc}(P, v)$ is the simple convex polytope obtained from P by cutting a small n-simplex neighbourhood of the vertex v. We call $\mathrm{vc}(P, v)$ the *vertex cut* of P at v. When the combinatorial structure of $\mathrm{vc}(P, v)$ does not depend on the vertex v, we simply denote it by $\mathrm{vc}(P)$. For example, when P is a product of simplices, the vertex cut $\mathrm{vc}(P, v)$ does not depend on the choice of a vertex v.

The following example explains a phenomenon leading to our main definition.

Example 1.1. We consider $M = \mathbb{C}P^2 \times \mathbb{C}P^1$ with the standard $(S^1)^3$-action. It is a quasitoric manifold over the triangular prism $P = \Delta^2 \times \Delta^1$. The equivariant blow-up M' of M at a fixed point x is a quasitoric manifold over $P' = \mathrm{vc}(P)$, which does not depend on the choice of a fixed point x. Now, if we blow up M' equivariantly at a fixed point y in M', then

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the resulting manifold M'' is a quasitoric manifold over $P'' = \mathrm{vc}(P', v)$. The manifold M'' is no longer independent of a fixed point y; in fact, there are three equivariantly different manifolds corresponding to three combinatorially different vertex cuts $\mathrm{vc}(P', v)$ (these correspond to the first three simplicial complexes in the second line on p. 192 of [**11**]).

On the other hand, the cohomology ring of M'' does not depend on the choice of a fixed point y because M'' is homeomorphic to the connected sum of $\mathbb{C}P^2 \times \mathbb{C}P^1$ with two copies of $\mathbb{C}P^3$. We are therefore in the situation when the cohomology ring of a quasitoric manifold does not determine the combinatorial structure of the base polytope.

Nevertheless, in many cases the combinatorial type of P is determined by $H^*(M)$. We therefore naturally come to the following definition, first introduced in [**10**].

Definition 1.2. A simple polytope P is *cohomologically rigid* if there exists a quasitoric manifold M over P, and, whenever there exists a quasitoric manifold N over another polytope Q with a graded ring isomorphism $H^*(M) \cong H^*(N)$, there is a combinatorial equivalence $P \approx Q$. We shall refer to such P simply as *rigid* throughout the paper.

We shall extend this definition to arbitrary Cohen–Macaulay complexes in Definition 3.10. In [**10**] the rigidity property is expressed in terms of toric manifolds, but here we modify the original definition to make use of a wider class of quasitoric manifolds. The interval I is trivially rigid. More generally, it was shown in [**9**] that any cube I^n is rigid. In Section 2 we give more classes of rigid polytopes, as described in the following results.

Theorem A (Theorem 2.2). *Let P be a simple polytope supporting a quasitoric manifold. If there is no other simple polytope with the same numbers of i-faces as those of P for all i, then P is rigid.*

Corollary B (Corollary 2.3). *Every polygon, that is, 2-dimensional convex polytope, is rigid.*

A simple convex polytope is called *triangle-free* if it has no triangular 2-face. The following result, proved in Section 4, establishes the rigidity for a triangle-free polytope with few facets.

Theorem C (Theorem 4.3). *Every triangle-free n-dimensional simple convex polytope with less than $2n + 3$ facets is rigid.*

Since a cube I^n has $2n$ facets, Theorem 4.3 gives a different proof of the rigidity for cubes from that in [**9**]. In Section 5 this result is generalized as follows.

Theorem D (Theorem 5.3). *A finite product of simplices is rigid.*

From the argument in Example 1.1, one can see immediately that, if the vertex cut of the polytope P depends on the choice of vertex, then all the vertex cuts of P are not rigid. Hence it is natural to ask whether $\mathrm{vc}(P)$ is rigid if the vertex cut of P is independent of the choice of vertex. In Section 6 we confirm this when P is a product of simplices.

Theorem E (Theorem 6.4). *If P is a finite product of simplices, then* $\mathrm{vc}(P)$ *is rigid.*

We can apply the above results to determine the rigidity of 3-dimensional simple convex polytopes with facet numbers up to nine. This result is given in Section 7. We also prove that a dodecahedron is rigid in Theorem 7.1.

The rigidity property for simple polytopes is closely related to the following interesting question on quasitoric manifolds.

Question 1.3. Suppose that M and N are two quasitoric manifolds such that $H^*(M) \cong H^*(N)$ as graded rings. Are M and N homeomorphic?

We can also consider the following slightly weaker question, which can be considered as an intermediate step to answering Question 1.3.

Question 1.4. Suppose that M and N are two quasitoric manifolds over the same simple convex polytope P such that $H^*(M) \cong H^*(N)$ as graded rings. Are M and N homeomorphic?

Question 1.3 for quasitoric manifolds whose cohomology rings are isomorphic to those of a product of copies of $\mathbb{C}P^1$ was considered in [9], and it was shown there that these manifolds are actually homeomorphic to a product of copies of $\mathbb{C}P^1$. This was done in two steps. First, the result was proved under the additional assumption that the quotient polytope is a cube I^n, and then the rigidity of I^n was established; see [9].

In [6] it was proved that, if M is a quasitoric manifold over a product of simplices $\prod_{i=1}^t \Delta^{n_i}$ such that $H^*(M) \cong H^*(\prod_{i=1}^t \mathbb{C}P^{n_i})$, then M is homeomorphic to $\prod_{i=1}^t \mathbb{C}P^{n_i}$. Since a product of simplices is rigid by Theorem 5.3, we have the following theorem.

Theorem 1.5. *Suppose that M is a quasitoric manifold such that $H^*(M) \cong H^*(\prod_{i=1}^t \mathbb{C}P^{n_i})$ as graded rings. Then M is homeomorphic to $\prod_{i=1}^t \mathbb{C}P^{n_i}$.*

The main technical ingredient for the proofs of the results in this paper is the following proposition. For a polytope P, let $\beta^{-i,2j}(P)$ be the bigraded Betti numbers of the Stanley–Reisner ring $\mathbb{Q}(P)$ of P; see Section 3 or [5] for details.

Proposition F (Proposition 3.8). *Let M and N be quasitoric manifolds over P and Q, respectively. If $H^*(M : \mathbb{Q}) \cong H^*(N : \mathbb{Q})$ as graded rings, then $\beta^{-i,2j}(P) = \beta^{-i,2j}(Q)$ for all i and j.*

2. *Rigidity and f-vectors*

For a convex n-dimensional polytope P, let f_i denote the number of codimension $i+1$ faces of P and let $f(P) = (f_0, \dots, f_{n-1})$ denote the f-vector of P. Note that, if P and Q are two 2-dimensional polytopes, then $f(P) = f(Q)$ implies that $P \approx Q$. Recall that the h-vector $h(P) = (h_0, \dots, h_n)$ of P is defined by

$$\sum_{i=0}^{n} h_i t^{n-i} = \sum_{j=0}^{n} f_{j-1}(t-1)^{n-j}.$$

The following theorem that was proved in [7] shows that the f-vector of the base polytope P is determined by the cohomology ring of the quasitoric manifold M over P.

THEOREM 2.1 [7]. *For a quasitoric manifold M over P, the $2i$th Betti number $b_{2i}(M)$ of M is equal to the ith component h_i of the h-vector of P.*

THEOREM 2.2. *Let P be a simple polytope supporting a quasitoric manifold. If there is no other simple polytope with the same numbers of i-faces as those of P for all i, then P is rigid.*

Proof. Now let P be a polytope and let M be a given quasitoric manifold over P. Suppose that N is another quasitoric manifold over Q such that $H^*(M) \cong H^*(N)$ as graded rings. Then the cohomology isomorphism implies that $b_{2i}(M) = b_{2i}(N)$ for all i. Hence $h(P) = h(Q)$ by Theorem 2.1, which implies that $f(P) = f(Q)$. Since there is no other simple convex polytope with the same face numbers of P, it follows that $P \approx Q$ and hence Theorem 2.2 is proved. $\square$

COROLLARY 2.3. *Every polygon, that is, 2-dimensional convex polytope, is rigid.*

Proof. Corollary 2.3 follows immediately from Theorem 2.2. $\square$

3. *Bigraded Betti numbers of polytopes*

Let $A = \mathbb{Q}[x_1, \ldots, x_m]$ be the polynomial graded ring in $x_1, \ldots, x_m$ over the rationals with $\deg x_i = 2$ for all i. A *free resolution* $[R, d]$ of a finitely generated A-module M is an exact sequence

$$0 \longrightarrow R^{-n} \xrightarrow{d} R^{-n+1} \xrightarrow{d} \cdots \xrightarrow{d} R^0 \xrightarrow{d} M \longrightarrow 0\ , \tag{3.1}$$

where R^{-i} are finitely generated free graded A-modules and d are degree-preserving homomorphisms. If we take R^{-i} to be the module generated by the minimal basis of $\operatorname{Ker}(d : R^{-i+1} \to R^{-i+2})$, then we get a *minimal resolution* of M. This also shows the existence of a resolution.

Dropping the last term M in the sequence (3.1) and tensoring it over A with another finitely generated A-module N, we obtain the following sequence:

$$0 \longrightarrow R^{-n} \otimes_A N \xrightarrow{d\otimes 1} R^{-n+1} \otimes_A N \xrightarrow{d\otimes 1} \cdots \xrightarrow{d\otimes 1} R^0 \otimes_A N \longrightarrow 0\ . \tag{3.2}$$

This sequence is not necessarily exact, and its cohomology is known as the *Tor-modules*, given by

$$\operatorname{Tor}_A^{-i}(M, N) := H^i(R^{-*} \otimes_A N).$$

Since everything is graded, we actually have the grading

$$\operatorname{Tor}_A^{-i}(M, N) = \bigoplus_j \operatorname{Tor}_A^{-i,j}(M, N).$$

The following proposition is well known, and we refer the reader to [5] for details.

PROPOSITION 3.1. *The above-defined Tor-modules satisfy the following properties*:

(i) $\operatorname{Tor}_A(M, N)$ *is independent of the choice of a resolution* $[R, d]$ *of* M;

(ii) $\operatorname{Tor}_A(M, N)$ *is functorial in all three arguments, that is, in* A, *in* M, *and in* N;

(iii) $\mathrm{Tor}^0_A(M,N) = M \otimes_A N$;
(iv) $\mathrm{Tor}^{-i}_A(M,N) = \mathrm{Tor}^{-i}_A(N,M)$.

We regard $\mathbb{Q}$ as an A-module via the ring map $A \to \mathbb{Q}$ sending each x_i to 0. Set $N = \mathbb{Q}$ and consider $\mathrm{Tor}_A(M,\mathbb{Q})$.

Definition 3.2. The *bigraded Betti numbers* of M are defined by

$$\beta^{-i}(M) = \dim_{\mathbb{Q}} \mathrm{Tor}^{-i}_A(M,\mathbb{Q}),$$
$$\beta^{-i,j}(M) = \dim_{\mathbb{Q}} \mathrm{Tor}^{-i,j}_A(M,\mathbb{Q}).$$

When $[R,d]$ is a minimal resolution of M, the map

$$d \otimes 1 : R^{-i} \otimes_A \mathbb{Q} \longrightarrow R^{-i+1} \otimes_A \mathbb{Q}$$

is the zero map for each i. Hence $\beta^{-i,j} = \mathrm{rank}_{\mathbb{Q}} R^{-i,j}$.

We now consider the case when M is the Stanley–Reisner ring $\mathbb{Q}(P)$ of a simple convex polytope P, that is,

$$\mathbb{Q}(P) = \mathbb{Q}[x_1,\dots,x_m]/I_P$$

where x_i are indeterminates corresponding to the facets F_i of P, m is the number of facets, and I_P is the homogeneous ideal generated by the monomials $x_{i_1}\dots x_{i_\ell}$ whenever $F_{i_1} \cap \dots \cap F_{i_\ell} = \emptyset$. This I_P is called the *Stanley–Reisner ideal* of P. Then $\mathbb{Q}(P)$ is a graded A-module with $\deg x_i = 2$ for all $i = 1,\dots,m$. The *bigraded Betti numbers* of P are defined to be $\beta^{-i,2j}(P) = \beta^{-i,2j}(\mathbb{Q}(P))$. Since $\deg x_i = 2$, we only have even index $2j$.

From the previous observation that $\beta^{-i,j} = \mathrm{rank}_{\mathbb{Q}} R^{-i,j}$ for a minimal resolution $[R,d]$, we can easily see that $\beta^{-1,2j}$ is equal to the number of degree $2j$ monomial elements in a minimal basis of the ideal I_P. For example, if $P = I^n$, then $x_i x_{n+i}$ for $i = 1,\dots,n$ form a minimal basis for the Stanley–Reisner ideal I_P of P (here we assume that x_i and x_{n+i} are the generators corresponding to the opposite facets F_i and F_{n+i} of I^n). Hence

$$\beta^{-1,2j}(I^n) = \begin{cases} n, & j = 2, \\ 0 & \text{otherwise.} \end{cases}$$

The following theorem of Hochster gives a nice formula for bigraded Betti numbers.

Theorem 3.3 [8]. *Let P be a simple convex polytope with facets $F_1,\dots,F_m$. For a subset $\sigma \subset \{1,\dots,m\}$, let $P_\sigma = \bigcup_{i\in\sigma} F_i \subset P$. Then we have*

$$\beta^{-i,2j}(P) = \sum_{|\sigma|=j} \dim \tilde{H}^{j-i-1}(P_\sigma).$$

Here $\dim \tilde{H}^{-1}(\emptyset) = 1$ by convention.

Bigraded Betti numbers also satisfy the following relations; see [5] for details.

Proposition 3.4. *Let P be an n-dimensional simple convex polytope with m facets, that is, $f_0(P) = m$. Then we have the following*:
(i) $\beta^{0,0}(P) = \beta^{-(m-n),2m}(P) = 1$;
(ii) (*Poincaré duality*) $\beta^{-i,2j}(P) = \beta^{-(m-n)+i,2(m-j)}(P)$;
(iii) $\beta^{-i,2j}(P_1 \times P_2) = \sum_{i'+i''=i,j'+j''=j} \beta^{-i',2j'}(P_1)\beta^{-i'',2j''}(P_2)$.

Definition 3.5. A sequence $\lambda_1, \ldots, \lambda_p$ of homogeneous elements in $\mathbb{Q}(P)$ is a *regular sequence* if it is algebraically independent and $\mathbb{Q}(P)$ is a free module over $\mathbb{Q}[\lambda_1, \ldots, \lambda_p]$.

Let J be an ideal of $\mathbb{Q}(P)$ generated by a regular sequence $\lambda_1, \ldots, \lambda_p$. Let $\pi : A \to \mathbb{Q}(P)$ be the projection. Choose homogeneous $t_i \in A$ such that $\pi(t_i) = \lambda_i$. Let J also denote the ideal of A generated by $t_1, \ldots, t_p$.

Lemma 3.6 [**5**, Lemma 3.35]. *Let J be an ideal generated by a regular sequence of $\mathbb{Q}(P)$. Then we have the following algebra isomorphism:*

$$\operatorname{Tor}_A^{*,*}(\mathbb{Q}(P), \mathbb{Q}) \cong \operatorname{Tor}_{A/J}^{*,*}(\mathbb{Q}(P)/J, \mathbb{Q}).$$

Lemma 3.7. *Let P and P' be two n-dimensional simple convex polytopes. Let $J = (\lambda_1, \ldots, \lambda_n)$ and $J' = (\lambda'_1, \ldots, \lambda'_n)$ be ideals of $\mathbb{Q}(P)$ and $\mathbb{Q}(P')$ generated by regular sequences of degree 2 elements λ_i and λ'_i, respectively. If there is a graded ring isomorphism $h\colon \mathbb{Q}(P)/J \xrightarrow{\cong} \mathbb{Q}(P')/J'$, then $f_0(P) = f_0(P')$ and*

$$\operatorname{Tor}_A^{*,*}\big(\mathbb{Q}(P), \mathbb{Q}\big) = \operatorname{Tor}_A^{*,*}\big(\mathbb{Q}(P'), \mathbb{Q}\big).$$

Proof. Note that the Stanley–Reisner ring $\mathbb{Q}(P)$ is generated by $f_0(P)$ elements of degree 2. Since J and J' are generated by degree 2 elements, the equality $f_0(P) = f_0(P')$ follows immediately from the isomorphism of degree 2 subgroups induced from $\mathbb{Q}(P)/J \cong \mathbb{Q}(P')/J'$. Thus we may assume that J and J' are both ideals of $A = \mathbb{Q}[x_1, \ldots, x_m]$ and h is an A-algebra isomorphism. By Lemma 3.6, we have $\operatorname{Tor}_A(\mathbb{Q}(P), \mathbb{Q}) = \operatorname{Tor}_{A/J}(\mathbb{Q}(P)/J, \mathbb{Q})$, and a similar equality holds for P'.

Now we claim that there is an A-algebra isomorphism $\bar{h}\colon A/J \to A/J'$ closing the commutative diagram

$$\begin{tikzcd} A/J \arrow[r, "\bar{h}"] \arrow[d] & A/J' \arrow[d] \\ \mathbb{Q}(P)/J \arrow[r, "h"] & \mathbb{Q}(P')/J'. \end{tikzcd} \tag{3.3}$$

Note that both A/J and A/J' are isomorphic to $\mathbb{Q}[x_1, \ldots, x_{m-n}]$, where $m = f_0(P) = f_0(P')$. Also note that the projection maps $A/J \to \mathbb{Q}(P)/J$ and $A/J' \to \mathbb{Q}(P')/J'$ induce isomorphisms $(A/J)_2 \to (\mathbb{Q}(P)/J)_2$ and $(A/J')_2 \to (\mathbb{Q}(P')/J')_2$ on degree 2 subgroups. Therefore we have an isomorphism $(A/J)_2 \to (\mathbb{Q}(P)/J)_2 \to (\mathbb{Q}(P')/J')_2 \to (A/J')_2$. Since A/J and A/J' are generated in degree 2, we obtain the isomorphism $\bar{h}\colon A/J \to A/J'$ as necessary.

Finally, the required isomorphism

$$\operatorname{Tor}_{A/J}^{*,*}\big(\mathbb{Q}(P)/J, \mathbb{Q}\big) \cong \operatorname{Tor}_{A/J'}^{*,*}\big(\mathbb{Q}(P')/J', \mathbb{Q}\big)$$

follows from (3.3) and the functoriality of Tor in Proposition 3.1(ii). □

We are now ready to prove the invariance of the bigraded Betti numbers.

Proposition 3.8. *Let M and N be quasitoric manifolds over P and Q, respectively. If $H^*(M : \mathbb{Q}) \cong H^*(N : \mathbb{Q})$ as graded rings, then $\beta^{-i,2j}(P) = \beta^{-i,2j}(Q)$ for all i and j.*

Proof. Recall that, if M is a quasitoric manifold over a simple convex polytope P, then $H^*(M\colon \mathbb{Q}) \cong \mathbb{Q}[x_1,\dots,x_m]/K$, where $K = I_P + J$ and I_P is the rational Stanley–Reisner ideal of P, and J is an ideal generated by some linear combinations $\lambda_{i1}x_1 + \ldots + \lambda_{im}x_m \in \mathbb{Q}[x_1,\dots,x_m]$ for $i = 1,\dots,n$ that project to a regular sequence $\theta_1,\dots,\theta_n$ in $\mathbb{Q}[x_1,\dots,x_m]/I_P$; see [**7**]. Here m is the number of facets in P. Therefore we have the isomorphism

$$\mathbb{Q}(P)/J \cong H^*(M : \mathbb{Q}) \cong H^*(N : \mathbb{Q}) \cong \mathbb{Q}(P')/J'.$$

Hence the proposition follows from Lemma 3.7. □

Since $\beta^{-i,j} = \operatorname{rank}_{\mathbb{Q}} R^{-i,j}$ for a minimal resolution $[R, d]$, and since the Betti numbers are independent of the choice of a resolution, it is convenient to calculate $\beta^{-i,j}$ using a particular minimal resolution. For this purpose, we consider the minimal resolution of $\mathbb{Q}(P)$ corresponding to the *canonical minimal basis* of the rational Stanley–Reisner ideal I_P, which we define below. The following procedure is explained in [**5**, Example 3.2]; we also reproduce it here for the reader's convenience.

In general, for a finitely generated graded A-module M the canonical minimal basis can be chosen as follows. Take the lowest degree, say d_1, elements in M that form a $\mathbb{Q}$-vector subspace of M, and choose its basis $\mathcal{B}_{d_1}$. Then span an A-submodule M_1 of M spanned by $\mathcal{B}_{d_1}$. Then take the lowest degree, say d_2, elements in $M \setminus M_1$ that form a $\mathbb{Q}$-subspace of M, and choose its basis $\mathcal{B}_{d_2}$. Span an A-submodule M_2 of M with $\mathcal{B}_{d_1} \cup \mathcal{B}_{d_2}$. Continue this process. Since M is finitely generated, this process must stop at some pth step, and we get $\mathbb{Q}$-subspace $\mathcal{B}_{d_p}$ for $M \setminus M_{p-1}$. Then M is generated by $\mathcal{B} = \bigcup_{i=1}^{p} \mathcal{B}_{d_i}$ as an A-module. The generator set $\mathcal{B}$ constructed in this way has the minimal possible number of elements, and we call it the canonical minimal basis of M.

In particular, for $M = I_P$, the canonical basis is $\mathcal{B} = \bigcup_{\ell \geqslant 1} \mathcal{B}_{2\ell}$, where $\mathcal{B}_{2\ell}$ are inductively defined as follows. The basis $\mathcal{B}_2$ consists of all monomials $x_i x_j$ such that $F_i \cap F_j = \emptyset$, where F_k is the facet of P corresponding to x_k. Assume that $\mathcal{B}_{2k}$ is defined for $k < \ell$. Then $\mathcal{B}_{2\ell}$ consists of the monomials $x_{i_1} \ldots x_{i_\ell}$ that are not divisible by the elements in $\bigcup_{i=1}^{\ell-1} \mathcal{B}_{2i}$ such that $\bigcap_{k=1}^{\ell} F_{i_k} = \emptyset$.

For a finitely generated A-module N, there is the following way of constructing a minimal resolution of N. Take a minimal basis $\mathcal{B}_N$ and define R^0 to be a free A-module generated by the elements of $\mathcal{B}_N$. There is an obvious epimorphism $R^0 \to N$. Take a minimal basis for $\ker(R^0 \to M)$ and define R^{-1} to be a free A module with these generators, and so on.

Example 3.9. (i) If $P = I^n$, the n-dimensional cube, then

$$\mathcal{B}(I_P) = \{x_i x_{n+i} \mid i = 1,\dots,m\}.$$

(ii) If $P = \prod_{i=1}^{t} \Delta^{n_i}$, a product of simplices, then

$$\mathcal{B}(I_P) = \{x_{i,0} \ldots x_{i,n_i} \mid i = 1,\dots,t\}.$$

We close this section by giving an algebraic version of rigidity. Recall that the rational Stanley–Reisner ring $\mathbb{Q}(K)$ of a simplicial complex K with m vertices $v_1,\dots,v_m$ is the quotient ring $\mathbb{Q}[x_1,\dots,x_m]/I_K$, where I_K is the ideal generated by the monomials $x_{i_1} \ldots x_{i_\ell}$, where the corresponding vertices $v_{i_1},\dots,v_{i_\ell}$ do not form a simplex on K. Then the rational Stanley–Reisner ring $\mathbb{Q}(P)$ of a simple convex polytope P is actually the rational Stanley–Reisner ring of the dual simplicial complex of ∂P, that is, $\mathbb{Q}(P) = \mathbb{Q}((\partial P)^*)$. Since P is simple, it follows that $(\partial P)^*$ is a simplicial complex.

The above-constructed minimal basis $\mathcal{B}$ of I_P coincides with the canonical minimal basis of the ideal I_K (see [**5**, § 3.4]) consisting of monomials corresponding to all *missing faces* of the

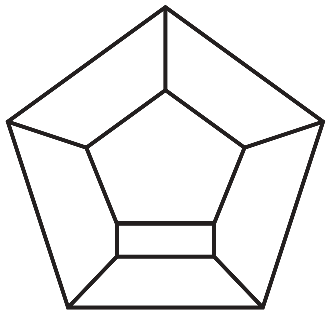


FIGURE 1. *Schlegel diagram of Q.*

simplicial complex K dual to the boundary of P (a missing face of a simplicial complex is its subset of vertices that does not span a simplex, but whose every proper subset does span a simplex).

A simplicial complex of dimension $n-1$ is called *Cohen–Macaulay* if there exists a length n regular sequence in $\mathbb{Q}(K)$. For any n-dimensional simple convex polytope P, its dual $(\partial P)^*$ is known to be Cohen–Macaulay. Therefore the definition of rigidity of a simple polytope can be generalized to that of a Cohen–Macaulay complex as follows.

DEFINITION 3.10. An $(n-1)$-dimensional Cohen–Macaulay complex K is *rigid* if, for any $(n-1)$-dimensional Cohen–Macaulay complex K' and for ideals $J \subset \mathbb{Q}(K)$ and $J' \subset \mathbb{Q}(K')$ generated by degree 2 regular sequences of length n, the isomorphism $\mathbb{Q}(K)/J \cong \mathbb{Q}(K')/J'$ implies that $\mathbb{Q}(K) \cong \mathbb{Q}(K')$.

4. *Rigidity of triangle-free simple polytopes*

It was shown in [1] that, if P is a triangle-free convex n-polytope, then $f_i(P) \geqslant f_i(I^n)$ for all $i = 0, \ldots, n-1$. Therefore the number of facets of P satisfies $f_0(P) \geqslant 2n$. Furthermore, it was shown in [2] that, if P is simple then the following hold:

(1) if $f_0(P) = 2n$, then $P \approx I^n$;
(2) if $f_0(P) = 2n+1$, then $P \approx P_5 \times I^{n-2}$, where P_5 is a pentagon;
(3) if $f_0 = 2n+2$, then $P \approx P_6 \times I^{n-2}$, $Q \times I^{n-3}$, or $P_5 \times P_5 \times I^{n-4}$, where P_6 is a hexagon and Q is the 3-dimensional simple convex polytope obtained from a pentagonal prism by cutting out one of the edges forming a pentagonal facet (see Figure 1).

LEMMA 4.1. *Let P be an n-dimensional simple polytope. If $\beta^{-1,2j}(P) = 0$ for all $j \geqslant 3$, then P is triangle-free.*

Proof. Suppose otherwise; namely, suppose that there exists a triangular 2-face T of P. Each edge e_i of T, for $i = 1, 2, 3$, is an intersection of $n-1$ facets of P. Thus there exists a unique facet, say F_i, that contains the edge e_i but not the triangle T for $i = 1, 2, 3$. Since P is simple, T is the intersection of exactly $n-2$ facets, and we may assume that $T = \bigcap_{i=4}^{n+1} F_i$. Then $\bigcap_{i=1}^{n+1} F_i = \emptyset$ because P is simple. This means that the monomial $\prod_{i=1}^{n+1} x_i$ is contained in I_P, where x_i is the degree 2 generating element of I_P corresponding to the facet F_i for $i = 1, \ldots, n+1$. Therefore there exists a minimal basis element $x_{n_1} \ldots x_{n_k}$ of I_P that divides $\prod_{i=1}^{n+1} x_i$. Now consider the set $\mathcal{S} = \{F_{n_1}, \ldots, F_{n_k}\}$ of facets corresponding to x_{n_i}

for $i = 1, \ldots, k$. Then the intersection of the elements of any proper subset of $\mathcal{S}$ is non-empty, but the intersection of the elements of $\mathcal{S}$ is empty. Note that $\bigcap_{i=1, i\neq j}^{n+1} F_i = v_j$ for $j = 1, 2, 3$, where v_j is the opposite vertex of T to the edge e_j. Therefore $\mathcal{S}$ must contain the facets F_1, F_2, and F_3. Therefore the minimal basis element $x_{n_1} \ldots x_{n_k}$ should divide $x_1x_2x_3$, and hence $x_{n_1} \ldots x_{n_k}$ is of degree greater than or equal to 6, which contradicts the hypothesis $\beta^{-1,2j}(P) = 0$ for all $j \geqslant 3$. □

Note that the condition $\beta^{-1,2j}(P) = 0$ for all $j \geqslant 3$ means that the Stanley–Reisner ideal I_P of P is generated by quadratic monomials of the form x_ix_j, and this is equivalent to saying that the simplicial complex $K = (\partial P)^*$ is flag.

If the number of facets of P is less than or equal to $2n + 2$, then the converse of Lemma 4.1 is true; namely, we have the following.

Lemma 4.2. *If P is a triangle-free n-dimensional simple convex polytope with $f_0(P) \leqslant 2n + 2$, then $\beta^{-1,2j}(P) = 0$ for all $j \geqslant 3$.*

Proof. Since $f_0(P) \leqslant 2n + 2$, we know that $P \approx I^n$, $P_5 \times I^{n-2}$, $P_6 \times I^{n-2}$, $Q \times I^{n-3}$, or $P_5 \times P_5 \times I^{n-4}$. Since $\beta^{-1,2j}$ is equal to the number of degree $2j$ monomial elements in a minimal basis of the Stanley–Reisner ideal of the polytope, we can see that

$$\beta^{-1,2j}(P_5) = \begin{cases} 5, & j = 2, \\ 0, & j \geqslant 3, \end{cases} \qquad \beta^{-1,2j}(P_6) = \begin{cases} 9, & j = 2, \\ 0, & j \geqslant 3, \end{cases}$$

$$\beta^{-1,2j}(Q) = \begin{cases} 10, & j = 2, \\ 0, & j \geqslant 3, \end{cases} \qquad \beta^{-1,2j}(I^k) = \begin{cases} k, & j = 2, \\ 0, & j \geqslant 3. \end{cases}$$

By Propositon 3.4(iii), we have $\beta^{-1,2j}(P' \times I^k) = 0$ for $j \geqslant 3$, where $P' \approx I^2$, P_5, P_6, Q, or $P_5 \times P_5$. □

We now prepare for the proof of Theorem 4.3. By Theorem 3.3, we have $\beta^{-2,8}(P) = \sum_{|\sigma|=4} \dim \widetilde{H}^1(P_\sigma)$. Therefore

$$\begin{gathered} \beta^{-2,8}(P_5) = \beta^{-2,8}(P_6) = 0, \\ \beta^{-2,8}(Q) = 5, \\ \beta^{-2,8}(P_5 \times P_5) = \beta^{-1,4}(P_5)\beta^{-1,4}(P_5) = 25 \end{gathered}$$

(note that, since Q does not have triangular faces, it follows that $\beta^{-2,8}(Q)$ equals the number of 4-facet 'belts' in Q). Hence we have

$$\begin{aligned} \beta^{-1,4}(P_6 \times I^{n-2}) &= \beta^{-1,4}(P_6) + \beta^{-1,4}(I^{n-2}) = n + 7, \\ \beta^{-2,8}(P_6 \times I^{n-2}) &= \beta^{-1,4}(P_6) \cdot \beta^{-1,4}(I^{n-2}) + \beta^{0,0}(P_6) \cdot \beta^{-2,8}(I^{n-2}) \\ &\quad + \beta^{-2,8}(P_6) \cdot \beta^{0,0}(I^{n-2}). \end{aligned}$$

On the other hand, by an inductive application of Propositon 3.4(iii), we can see easily that $\beta^{-2,8}(I^{n-2}) = (n - 2)(n - 3)/2$. Therefore we have

$$\beta^{-1,4}(P_6 \times I^{n-2}) = n + 7, \quad \beta^{-2,8}(P_6 \times I^{n-2}) = \frac{n^2 + 13n - 30}{2}.$$

By a similar computation, we have

$$\beta^{-1,4}(Q \times I^{n-3}) = n+7, \quad \beta^{-2,8}(Q \times I^{n-3}) = \frac{n^2+13n-38}{2},$$
$$\beta^{-1,4}(P_5 \times P_5 \times I^{n-4}) = n+6, \quad \beta^{-2,8}(P_5 \times P_5 \times I^{n-4}) = \frac{n^2+11n-10}{2}.$$

THEOREM 4.3. *Every triangle-free n-dimensional simple convex polytope with less than $2n+3$ facets is rigid.*

Proof. Let P be triangle-free with $f_0(P) \leqslant 2n+2$ and let M be a quasitoric manifold over P. Let P' be another simple convex polytope and let M' be a quasitoric manifold over P'. If $H^*(M:\mathbb{Q}) \cong H^*(M':\mathbb{Q})$ as graded rings, then, by Proposition 3.8, we have the equality $\beta^{-i,2j}(P) = \beta^{-i,2j}(P')$ for all $1 \leqslant i,j \leqslant m$. Since P is triangle-free with $f_0(P) \leqslant 2n+2$, it follows that $\beta^{-1,2j}(P) = 0$ for all $j \geqslant 3$ by Lemma 4.2. Hence $\beta^{-1,2j}(P') = 0$ for all $j \geqslant 3$, and Lemma 4.1 implies that P' is triangle-free. Furthermore, $H^*(M:\mathbb{Q}) \cong H^*(M':\mathbb{Q})$ implies, in particular, that $f_0(P) = f_0(P')$. If $f_0(P) = 2n$ or $2n+1$, then there is only one simple polytope with the given number of facets. Hence $P \approx P'$. When $f_0(P) = 2n+2$, there are three possible polytopes, but the above computation shows that $\beta^{-i,2j}$ are distinct for these three polytopes. This shows that $P \approx P'$, which proves the theorem. The existence of quasitoric manifolds over P is clear because we know the existence of quasitoric manifolds over any 2- or 3-dimensional simple convex polytope and any n-simplex as well as any finite product of these polytopes. □

5. *Rigidity of products of simplices*

We make use of the following invariant in this section and Section 6.

DEFINITION 5.1. The *sigma invariant* of P is defined as $\sigma(P) = \sum_{j \geqslant 2} j\beta^{-1,2j}(P)$.

Proposition 3.8 implies that $\sigma(P)$ is a cohomology invariant of quasitoric manifolds over P. As we observed in Section 3, the Betti number $\beta^{-1,2j}(P)$ is equal to the number of degree $2j$ elements in a minimal basis of the Stanley–Reisner ideal I_P of P. Therefore $2\sigma(P)$ is nothing but the sum of the degrees of all elements of a minimal basis of I_P.

LEMMA 5.2. *Let P be a simple polytope with m facets. Then the following conditions are equivalent:*

(i) $\sigma(P) = m$;
(ii) *the canonical minimal basis $\mathcal{B}$ of I_P forms a regular sequence;*
(iii) *P is combinatorially equivalent to a product of simplices.*

Proof. (iii) ⇒ (i): The proof is clear.

(i) ⇒ (ii): Let $\mathbb{Q}(P) = \mathbb{Q}[x_1,\dots,x_m]/I_P$, where x_i corresponds to a facet F_i of P. Let $\mathcal{B} = \{g_1,\dots,g_t\}$ be the canonical minimal basis of I_P. Since $\sigma(\mathbb{Q}(P)) = m$, each x_j must appear in exactly one element of $\mathcal{B}$ with exponent 1. It follows easily that $g_1,\dots,g_t$ is a regular sequence.

(ii) ⇒ (iii): Let $\mathbb{Q}(P) = \mathbb{Q}[x_1,\dots,x_m]/(g_1,\dots,g_t)$, where $g_1,\dots,g_t$ is a monomial regular sequence. It is well known [5, § 3.2] that $g_1,\dots,g_t$ is a regular sequence if and only if g_i is not a zero divisor in the quotient $\mathbb{Q}[x_1, \dots, x_m]/(g_1, \dots, g_{i-1})$ for $1 \leqslant i \leqslant t$ (this property is often taken as the definition of a regular sequence). Assume that some x_j appears in more than one of $g_1, \dots, g_t$, say in g_1 and g_2. Then g_2 is a zero divisor in $\mathbb{Q}[x_1, \dots, x_m]/(g_1)$, which leads to a contradiction. Therefore each x_j appears in at most one of the monomials $g_1, \dots, g_t$. Since every x_j must appear in at least one element in I_P, we obtain that every x_j enters in exactly one of $g_1, \dots, g_t$. Hence we can rename $x_1, \dots, x_m$ by $y_{1\,0}, \dots, y_{1\,n_1}, \dots, y_{t\,0}, \dots, y_{t\,n_t}$ such that $g_j = \prod_{k=0}^{n_j} y_{j\,k}$ for $j = 1, \dots, t$. Therefore we can see immediately that

$$\begin{aligned}
\mathbb{Q}(P) &\cong \mathbb{Q}[x_1, \dots, x_m]/I_P \\
&\cong \bigotimes_{i=1}^{t} \mathbb{Q}[y_{i\,0}, \dots, y_{i\,n_i}]/(g_i) \\
&\cong \bigotimes_{i=1}^{t} \mathbb{Q}(\Delta^{n_i}) \\
&\cong \mathbb{Q}\left(\prod_{i=1}^{t} \Delta^{n_i}\right).
\end{aligned}$$

Since the Stanley–Reisner ring with $\mathbb{Q}$-coefficients determines the combinatorial type of a simple polytope [3], we have $P \approx \prod_{i=1}^{t} \Delta^{n_i}$. □

Note that Lemma 5.2(ii) is equivalent to saying that $\mathbb{Q}(P)$ is a complete intersection ring.

Theorem 5.3. *A finite product of simplices is rigid.*

Proof. Let M be a $2n$-dimensional quasitoric manifold over $P = \prod_{i=1}^{t} \Delta^{n_i}$. Let N be another quasitoric manifold over a simple convex polytope Q such that $H^*(M : \mathbb{Z}) \cong H^*(N : \mathbb{Z})$. Then $H^*(M : \mathbb{Q}) \cong H^*(N : \mathbb{Q})$ and $f_i(P) = f_i(Q)$ for all i. In particular, $\sigma(\mathbb{Q}(P)) = f_0(P) = f_0(Q) = n + t$. Thus Q is a simple convex polytope with $\sigma(\mathbb{Q}(Q)) = f_0(Q)$. Therefore Q is also a product of simplices, that is, $Q \approx \prod_{j=1}^{s} \Delta^{m_j}$; but $H^*(M : \mathbb{Q}) \cong H^*(N : \mathbb{Q})$ implies that $\beta^{-1,2j}(P) = \beta^{-1,2j}(Q)$ for all J. This implies that $\{n_i\} = \{m_j\}$ and $t = s$. Thus $P \cong Q$. □

6. *Rigidity of vertex cuts*

The following proposition shows that certain Betti numbers and the sigma invariant of a vertex cut of P are independent of the choice of the cut vertex, whereas the combinatorial type of P may depend on this choice; see Example 1.1.

Proposition 6.1. *Let P be an n-dimensional simple convex polytope with m facets that is different from the n-simplex Δ^n. Then we have the following:*

(i) $\beta^{-1,2j}(\mathrm{vc}(P)) = \begin{cases} \beta^{-1,2j}(P) + m - n, & j = 2, \\ \beta^{-1,2j}(P), & 3 \leqslant j \leqslant n-1, \\ \beta^{-1,2j}(P) + 1, & j = n; \end{cases}$

(ii) $\sigma(\mathrm{vc}(P)) = \sigma(P) + 2m - n$.

Proof. Both statements follow easily from the interpretation of $\beta^{-1,2j}(P)$ as the number of degree $2j$ elements in the minimal basis of the ideal I_P. □

When $P = \prod_{i=1}^{t} \Delta^{n_i}$ with $t \neq 1$, we have $n = \sum_{i=1}^{t} n_i$, $m = n + t$, and $\sigma(P) = m$. Hence we have $\sigma(\mathrm{vc}(P)) = 3m - n$.

Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be the set of facets in P. Let x_i be the generator corresponding to F_i in $\mathbb{Q}(P) = \mathbb{Q}[x_1, \ldots, x_m]/I_P$. Let $\mathcal{B} = \{h_1, \ldots, h_\ell\}$ be the canonical minimal basis for I_P. For each x_i, the *frequency* $\mathfrak{f}(x_i)$ is the number of h_k in $\mathcal{B}$ divisible by x_i.

LEMMA 6.2. *Let P be an n-dimensional simple convex polytope. Let $\mathcal{B}$ be the canonical minimal basis for I_P. If $\mathfrak{f}(x_i) = 1$ for some i, then $P \approx \Delta^k \times P'$ for some polytope P' of dimension $n - k$ and $k = \deg h/2 - 1$, where h is the unique element in $\mathcal{B}$ such that $x_i \,|\, h$.*

Proof. Let $\mathcal{B} = \{h_1, \ldots, h_s\}$. Assume that $\mathfrak{f}(x_1) = 1$ and $h_1 = x_1 \ldots x_t$ for simplicity. Hence h_1 is the unique element of $\mathcal{B}$ that is divisible by x_1. We claim that $\mathfrak{f}(x_2) = \ldots = \mathfrak{f}(x_t) = 1$. Assume otherwise, say $\mathfrak{f}(x_2) \geqslant 2$. Without loss of generality, we may assume that $h_2 = x_2 x_{i_1} \ldots x_{i_k}$. Then $x_{i_j} \neq x_1$ and x_2 for all $j = 1, \ldots, k$ because h_1 is the only element of $\mathcal{B}$ divisible by x_1. Since $h_2 \in \mathcal{B}$, if we let $T := F_{i_1} \cap \ldots \cap F_{i_k}$, then $T \neq \emptyset$, but $F_2 \cap T = \emptyset$. On the other hand, since $x_1 \nmid h_2$, we have $F_1 \cap T \neq \emptyset$ (otherwise $x_1 x_{i_1} \ldots x_{i_k} \in I_P$, and so there would be another element in $\mathcal{B}$ divisible by x_1).

If $k \geqslant n$, then $F_1 \cap T \neq \emptyset$ implies that more than n facets of P are intersecting, which is impossible because P is simple. Therefore $\dim T = n - k \geqslant 1$. Since $\dim(F_1 \cap T) = \dim T - 1$, there exists a vertex v of T that does not belong to F_1. Let v be the intersection of n facets $F_{\ell_1}, \ldots, F_{\ell_n}$. Since $F_2 \cap T = \emptyset$, the vertex v does not belong to F_2, and hence $F_{\ell_j} \neq F_2$ for all $j = 1, \ldots, n$. Since v does not belong to F_1, we have $F_1 \cap F_{\ell_1} \cap \ldots \cap F_{\ell_n} = \emptyset$. Therefore there must exist an element $h \in \mathcal{B}$ that divides the monomial $x_1 x_{\ell_1} \ldots x_{\ell_n}$, but, since $F_{\ell_1} \cap \ldots \cap F_{\ell_n} = v \neq \emptyset$, the element h must be divisible by x_1. Since $F_{\ell_j} \neq F_2$ for all $j = 1, \ldots, n$, it follows that $x_2 \nmid h$. Thus $h_1 \nmid h$, which contradicts the condition that $\mathfrak{f}(x_1) = 1$. This shows that $\mathfrak{f}(x_2) = 1$, and by a similar argument we can see that $\mathfrak{f}(x_i) = 1$ for all $i = 1, \ldots, t$. Hence,

$$\mathbb{Q}(P) = \mathbb{Q}[x_1, \ldots, x_t]/h_1 \otimes \mathbb{Q}[x_{t+1}, \ldots, x_m]/I',$$

where I' is the ideal generated by $\mathcal{B} \setminus \{h_1\}$.

Since $\mathbb{Q}[x_1, \ldots, x_t]/(h_1) \cong \mathbb{Q}(\Delta^t)$, it is enough to prove that there is an isomorphism $\mathbb{Q}[x_{t+1}, \ldots, t_m]/I' \cong \mathbb{Q}(P')$ for some polytope P' of dimension $n - k$. (Indeed, then we instantly get $P \approx \Delta^k \times P'$ because the rational Stanley–Reisner ring determines the combinatorial type of a simple polytope [**3**].) Let $P' := F_2 \cap \ldots \cap F_t$. Then every facet except F_1 intersects with P'. Let $G_j = F_j \cap P'$ for $j = t+1, \ldots, m$. Then the G_j are facets of P'. This implies that the face poset structure of P' agrees with the face poset structure of $\{F_{t+1}, \ldots, F_m\}$. Thus $\mathcal{B} \setminus \{h_1\} = \{h_2, \ldots, h_s\}$ is the canonical minimal basis for $I_{P'}$. Hence $\mathbb{Q}[x_{t-1}, \ldots, x_n]/I' \cong \mathbb{Q}(P')$. □

THEOREM 6.3. *Let Q be an n-dimensional simple convex polytope with $m + 1$ facets. If $\sigma(Q) = 3m - n$ and $\beta^{-1,2n}(Q) \neq 0$, then Q is a vertex cut of a product of simplices.*

Proof. We claim that one of the facets of Q is an $(n-1)$-simplex. Then Q is a vertex cut of some simple convex polytope P. By Proposition 6.1, we have

$$\sigma(P) = \sigma(Q) - 2m + n = (3m - n) - (2m - n) = m.$$

Thus, by Lemma 5.2, we see that P is a product of simplices, and the proof is completed.

We now prove the claim. Let $F_1, \ldots, F_{m+1}$ be the facets of Q and let $x_1, \ldots, x_{m+1}$ be the associated generators of $\mathbb{Q}(Q)$. Let $\mathcal{B}$ be the canonical minimal basis for the ideal I_Q. Since $\beta^{-1,2n}(Q) \geqslant 1$, there exists $\tilde{h} \in \mathcal{B}$ with $\deg \tilde{h} = 2n$. Without loss of generality, we may assume that $\tilde{h} = x_1 \ldots x_n$. Then we can see easily that $F_1 \cup \ldots \cup F_n$ is homeomorphic to $S^{n-2} \times I$, while $F_1 \cup \ldots \cup F_{m+1} \cong S^{n-1}$. Thus $F_1 \cup \ldots \cup F_{m+1} \backslash F_1 \cup \ldots \cup F_n = F_{n+1} \cup \ldots \cup F_{m+1}$ has two connected components. For simplicity, let $F_{n+1} \cup \ldots \cup F_{n+k}$ and $F_{n+k+1} \cup \ldots \cup F_{m+1}$ be the two components. Then $F_{n+i} \cap F_{n+j} = \emptyset$ for $i = 1, \ldots, k$ and $j = k+1, \ldots, m+1-n$.

If $k = 1$ or $m - n$, then one of the components of $F_{n+1} \cup \ldots \cup F_{m+1}$ is a single facet of Q, and this facet is an $(n-1)$-simplex. This proves the claim. Assume otherwise, that is, suppose that $2 \leqslant k \leqslant [(m+1-n)/2]$. Let $\mathcal{B}_1 = \{x_{n+i}x_{n+j} \mid i = 1, \ldots, k \text{ and } j = k+1, \ldots, m+1-n\}$. Then we have

$$\sum_{h \in \mathcal{B}_1} \deg(h) = 4k(m+1-n-k) \geqslant 8(m-n-1) \tag{6.1}$$

because $k(m+1-n-k)$ is increasing for $2 \leqslant k \leqslant [(m+1-n)/2]$. Note that the frequencies satisfy $\mathfrak{f}(x_i) \geqslant 2$ for all $i = 1, \ldots, n$ since otherwise Lemma 6.2 would imply that $Q \approx \Delta^{n-1} \times \Delta^1$, but in this case $\sigma(Q) = n+2 \neq 3m-n = 2n+3$. Therefore, for each x_i, there exists $h_i \in \mathcal{B}$ such that $x_i \mid h_i$ and $h_i \neq \tilde{h}$ for $i = 1, \ldots, n$. Note that some of $h_1, \ldots, h_n$ may coincide. Hence we let $\tilde{h}_1, \ldots, \tilde{h}_s$ denote all distinct elements among the h_i. If $s = 1$, then $\tilde{h}_1$ is divisible by all x_i for $i = 1, \ldots, n$. Hence $\tilde{h} \mid \tilde{h}_1$ and therefore $\tilde{h} = \tilde{h}_1$, which is a contradiction. Therefore $s \geqslant 2$.

If $s \geqslant 3$, then

$$\begin{aligned}\sum_{h \in \mathcal{B} \backslash \mathcal{B}_1} \deg h &\geqslant \deg \tilde{h} + \sum_{i=1}^{s} \deg \tilde{h}_i \\ &\geqslant \deg \tilde{h} + 2n + 6 = 4n + 6,\end{aligned} \tag{6.2}$$

where the last inequality follows from the conditions $s \geqslant 3$, $\deg \tilde{h}_i \geqslant 4$, and $x_1 \ldots x_n \mid \tilde{h}_1 \ldots \tilde{h}_s$.

Suppose that $s = 2$. Then, without loss of generality, we may assume that $\tilde{h}_1 = g_1 x_1 \ldots x_\ell$ and $\tilde{h}_2 = g_2 x_{\ell+1} \ldots x_n$ with $1 \leqslant \ell \leqslant n-1$ for some monomials g_1 and g_2 in $x_{n+1}, \ldots, x_{m+1}$ of degree at least 2. If degree $g_2 \geqslant 4$, then

$$\deg \tilde{h}_1 + \deg \tilde{h}_2 = 2n + \deg g_1 + \deg g_2 \geqslant 2n + 6.$$

Therefore the inequality (6.2) holds in this case. Now suppose that degree $g_2 = 2$. Then $g_2 = x_i$ for $n+1 \leqslant i \leqslant n+k$ or $g_2 = x_{n+j}$ for $k+1 \leqslant j \leqslant m+1-n$. We only prove the case when $g_2 = x_{n+k+1}$. The other cases are similar. In this case consider the monomial $q = \prod_{j=2}^{n+2} x_j$. By the assumption, $\tilde{h}_i \nmid q$ for $i = 1, 2$. However, q must vanish in $\mathbb{Q}(Q)$ because any set of $n+1$ facets has an empty intersection in a simple polytope. Therefore there exists a monomial q' of degree at least 4 in $\mathcal{B} \setminus \mathcal{B}_1$, that divides q. Thus

$$\sum_{h \in \mathcal{B} \backslash \mathcal{B}_1} \deg h \geqslant \deg \tilde{h} + \deg q' + \deg \tilde{h}_1 + \deg \tilde{h}_2 \geqslant 4n + 8 > 4n + 6.$$

Thus we have proved that, in all cases, we have

$$2\sigma(Q) = \sum_{h \in \mathcal{B}} \deg h \geqslant 8(m-n-1) + 4n + 6 = 8m - 4n - 2. \tag{6.3}$$

On the other hand, by the assumption of the theorem, $\sigma(Q) = 3m - n$. Thus $3m - n \geqslant 4m - 2n - 1$, and hence $n + 2 \geqslant m + 1$. Therefore Q is combinatorially equivalent to either $\Delta^{n_1} \times \Delta^{n_2}$ or Δ^n; but $\beta^{-1,2n}(Q) \neq 0$ gives that $Q \approx \Delta^{n-1} \times \Delta^1$, which implies that $\sigma(Q) = m + 1 \neq 3m - n$. This is a contradiction. Thus we have $k = 1$ or $m - n$, which proves the theorem. □

THEOREM 6.4. *If P is a finite product of simplices, then $\mathrm{vc}(P)$ is rigid.*

Proof. If P is an n-simplex, then $\mathrm{vc}(P) = \Delta^{n-1} \times \Delta^1$, which is rigid by Theorem 5.3. Let us assume otherwise. Let $Q = \mathrm{vc}(P)$ and let M be a quasitoric manifold over Q. Suppose that N is a quasitoric manifold over another simple convex polytope Q' such that $H^*(M:\mathbb{Q}) \cong H^*(N:\mathbb{Q})$ as graded rings. Then $\beta^{-1,2j}(Q) = \beta^{-1,2j}(Q')$, and hence $\sigma(Q') = \sigma(Q) = 3m - n$ and $\beta^{-1,2n}(Q') = \beta^{-1,2n}(Q) \neq 0$. By Theorem 6.3, we have $Q' = \mathrm{vc}(P')$ for $P' = \prod \Delta^{n_i}$. By Proposition 6.1(i), we find that $\beta^{-1,2j}(Q) = \beta^{-1,2j}(Q')$ implies that $\beta^{-1,2j}(P) = \beta^{-1,2j}(P')$ for all j. Both P and P' are products of simplices, and thus $P \approx P'$. Hence $Q \approx Q'$, which proves the theorem. □

7. *Rigidity of* 3*-dimensional simple convex polytopes*

Since the rigidity of a 2-dimensional simple convex polytope is settled by Corollary 2.3, the rigidity of a 3-dimensional simple convex polytope is naturally the next target. Note that any 3-dimensional simple convex polytope supports a quasitoric manifold. The four colour problem gives an easy proof of this.

In [**11**, Appendix A.5, pp. 192–193] there is a list of 3-dimensional simple convex polytopes with at most nine facets. In the list the polytopes are labelled in the form $\alpha^x\beta^y\gamma^z$, which means that the polytope has x many α-gon facets, y many β-gon facets, and z many γ-gon facets. For example, the polytope 3^4 is the tetrahedron, and $3^2 4^3$ is the triangular prism.

Table 1 lists simple 3-polytopes with at most eight facets, their bigraded Betti numbers, and rigidity. Table 2 contains the same information about simple 3-polytopes with nine facets. In the tables $\mathrm{vc}^k(P)$ denotes a k-fold vertex cut of P. The Betti numbers are listed in the form

$$(\beta^{-1,4}, \ldots, \beta^{-(j-1),2j}, \ldots, \beta^{-(m-4),2(m-3)}).$$

Note that the numbers above completely determine all bigraded Betti numbers of a 3-dimensional polytope. Indeed, unless $(i,j) = (0,0)$ or $(m-3, m)$ the number $\beta^{-i,2j}$ is zero for $j - i \neq 1, 2$ by Theorem 3.3. By Propositon 3.4, we have $\beta^{0,0} = \beta^{-(m-3),2m} = 1$ and $\beta^{-(j-1),2j} = \beta^{-i',2j'}$, where $i' = (m-3) - (j-1)$ and $j' = m - j$. Note that $j' - i' = 2$. This implies that the whole set of Betti numbers is determined by the $\beta^{-(j-1),2j}$ for $j = 2, \ldots, m$. Moreover, for a subset $\sigma \subset \{1, \ldots, m\}$, if $|\sigma| > m - 3$, then P_σ is always connected. We therefore consider only $\beta^{-(j-1),2j}$ for $j = 2, \ldots, m-3$.

Both 3^4 and $3^2 4^3$ are rigid by Theorem 5.3.

There are exactly two polytopes $3^2 4^2 5^2$ and 4^6 with $f_0 = 6$. The polytope 4^6 is rigid because it is the cube I^3, and the polytope $3^2 4^2 5^2$ is the vertex cut of the triangular prism, and hence it is also rigid by Theorem 6.4.

There are five different polytopes with $f_0 = 7$, which are $3^2 4^3 6^2$, $3^3 5^3 6^1$, $3^2 4^2 5^2 6^1$, $3^1 4^3 5^3$, and $4^5 5^2$. The first three are the polytopes obtained from the triangular prism $\Delta^2 \times I$ by taking vertex cuts twice. Thus, by the argument of Example 1.1, they are all non-rigid. The polytope $3^1 4^3 5^3$ is the vertex cut of the cube I^3, and hence rigid by Theorem 6.4. The polytope $4^5 5^2$ is the pentagonal prism, which is rigid by Theorem 4.3.

There are fourteen different polytopes with $f_0 = 8$. Seven of them are obtained from the triangular prism by taking vertex cuts three times, and so they are all non-rigid. These are $3^2 4^4 7^2$, $3^3 4^1 5^2 6^1 7^1$, $3^2 4^3 5^1 6^1 7^1$, $3^2 4^2 5^3 7^1$, $3^4 6^4$, $3^3 4^1 5^1 6^3$, and $3^2 4^2 5^2 6^2$(i). There are four polytopes obtained from the cube by taking vertex cuts twice. They are $3^2 4^2 5^2 6^2$(ii), $3^1 4^5 5^1 6^2$, $3^2 4^1 5^4 6^1$, and $3^2 5^6$, and all of them are non-rigid. The remaining polytopes are as follows: $3^1 4^3 5^3 6^1$, which is the vertex cut of the pentagonal prism; $4^6 6^2$, which is the hexagonal prism $P_6 \times I$; and $4^4 5^4$, which is obtained from the pentagonal prism by cutting out a triangular prism-shaped neighbourhood of an edge. Since the Betti numbers $\beta^{-1,2j}$ of $3^1 4^3 5^3 6^1$ are different from those of the other two and also different from the previous groups, this polytope is rigid. The remaining polytopes are $4^6 6^2$ and $4^4 5^4$. These polytopes have $2n+2=8$ facets. Hence, by Theorem 4.3, they are rigid.

There are 50 different polytopes with $f_0 = 9$, and only six of them are rigid. Among them, five are triangle-free polytopes, namely, $4^6 6^3$, $4^5 5^2 6^2$, $4^4 5^4 6^1$, $4^3 5^6$, and $4^7 7^2$, and the sixth is the polytope $3^1 4^4 5^2 6^1 7^1$(ii), which is the vertex cut of $P_6 \times I$. In each case the rigidity is established by comparing the Betti numbers, and observing that these numbers for each of the six polytopes are different from those of the others.

Finally, we give a proof of the rigidity of a dodecahedron.

THEOREM 7.1. *A dodecahedron is rigid.*

Proof. A computation using Theorem 3.3 shows that the $(-2,8)$th Betti number of a dodecahedron is 0. Let P be a simple 3-polytope with twelve facets whose Betti numbers

TABLE 1. *Rigidity of simple* 3*-polytopes with* $f_0 \leqslant 8$.

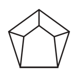
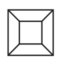
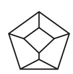
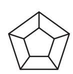
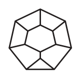
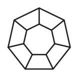
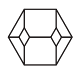
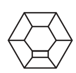
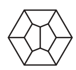
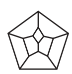
Type (Betti numbers)	Simple polytopes	
3^4 ( )	 3^4	Rigid
$\mathrm{vc}(3^4)$ (1)	 $3^2 4^3$	Rigid
$\mathrm{vc}^2(3^4)$ $(3,2)$	 $3^2 4^2 5^2$	Rigid
4^6 $(3,0)$	 4^6	Rigid
$\mathrm{vc}^3(3^4)$ $(6,8,3)$	$3^2 4^3 6^2, 3^3 5^3 6^1, 3^2 4^2 5^2 6^1$ (three polytopes)	Non-rigid
$\mathrm{vc}(4^6)$ $(6,6,1)$	 $3^1 4^3 5^3$	Rigid
$4^5 5^2$ $(6,5,0)$	 $4^5 5^2$	Rigid
$\mathrm{vc}^4(3^4)$ $(10,20,15,4)$	$3^2 4^4 7^2, 3^3 4^1 5^2 6^1 7^1, 3^2 4^3 5^1 6^1 7^1, 3^2 4^2 5^3 7^1, 3^4 6^4$ $3^3 4^1 5^1 6^3, 3^2 4^2 5^2 6^2$(i)(seven polytopes)	Non-rigid
$\mathrm{vc}^2(4^6)$ $(10,18,11,2)$	$3^2 4^2 5^2 6^2$(ii)$, 3^1 4^4 5^1 6^2, 3^2 4^1 5^4 6^1, 3^2 5^6$(four polytopes)	Non-rigid
$\mathrm{vc}(4^5 5^2)$ $(10,17,9,1)$	 $3^1 4^3 5^3 6^1$	Rigid
$4^6 6^2$ $(10,16,9,0)$	 $4^6 6^2$	Rigid
$4^4 5^4$ $(10,16,5,0)$	 $4^4 5^4$	Rigid

TABLE 2. *Rigidity of simple* 3*-polytopes with* $f_0 = 9$.

Type (Betti numbers)	Simple polytopes	
$\mathrm{vc}^5(3^4)$ $(15, 40, 45, 24, 5)$	$3^2 4^5 8^2, 3^3 4^2 5^2 7^1 8^1\,(\mathrm{i}), 3^3 4^2 5^2 7^1 8^1\,(\mathrm{ii}), 3^2 4^2 5^1 7^1 8^1, 3^4 5^2 6^2 8^1$ $3^3 4^2 5^1 6^2 8^1\,(\mathrm{i}), 3^3 4^2 5^1 6^2 8^1\,(\mathrm{ii}), 3^2 4^4 6^2 8^1, 3^3 4^1 5^3 6^1 8^1, 3^2 4^3 5^2 6^1 8^1\,(\mathrm{i})$ $3^2 4^3 5^2 6^1 8^1\,(\mathrm{ii}), 3^2 4^2 5^4 8^1, 3^4 4^1 6^2 7^2, 3^4 5^2 6^1 7^2, 3^3 4^2 5^1 6^1 7^2\,(\mathrm{i})$ $3^3 4^2 5^1 6^1 7^2\,(\mathrm{ii}), 3^3 4^1 5^3 7^2, 3^2 4^3 5^2 7^2\,(\mathrm{ii}), 3^3 4^2 6^3 7^1\,(\mathrm{i}), 3^3 4^2 6^3 7^1\,(\mathrm{ii})$ $3^3 4^1 5^2 6^2 7^1\,(\mathrm{ii}), 3^2 4^3 5^1 6^2 7^1\,(\mathrm{iii}), 3^2 4^2 5^3 6^1 7^1\,(\mathrm{iii}), 3^2 4^2 5^2 6^3\,(\mathrm{iii})$ (twenty-four polytopes)	Non-rigid
$\mathrm{vc}^3(4^6)$ $(15, 38, 39, 18, 3)$	$3^2 4^3 5^2 7^2\,(\mathrm{i}), 3^1 4^5 5^1 7^2, 3^3 4^1 5^2 6^2 7^1\,(\mathrm{i}), 3^2 4^3 5^1 6^2 7^1\,(\mathrm{i}), 3^2 4^3 5^1 6^2 7^1\,(\mathrm{ii})$ $3^2 4^2 5^3 6^1 7^1\,(\mathrm{ii}), 3^1 4^4 5^2 6^1 7^1\,(\mathrm{iii}), 3^3 5^3 6^3\,(\mathrm{i}), 3^3 5^3 6^3\,(\mathrm{ii}), 3^2 4^2 5^2 6^3\,(\mathrm{ii})$ $3^2 4^1 5^4 6^2\,(\mathrm{ii})$ (eleven polytopes)	Non-rigid
$\mathrm{vc}^2(4^5 5^2)$ $(15, 37, 36, 15, 2)$	$3^2 4^2 5^3 6^1 7^1\,(\mathrm{i}), 3^1 4^4 5^2 6^1 7^1\,(\mathrm{i}), 3^2 4^1 5^5 7^1, 3^2 4^3 6^4, 3^2 4^2 5^2 6^3\,(\mathrm{i})$ $3^1 4^4 5^1 6^3, 3^2 4^1 5^4 6^2\,(\mathrm{i})$ (seven polytopes)	Non-rigid
$\mathrm{vc}(4^6 6^2)$ $(15, 36, 35, 14, 1)$	 $3^1 4^4 5^2 6^1 7^1\,(\mathrm{ii})$	Rigid
$\mathrm{vc}(4^4 5^4)$ $(15, 36, 31, 10, 1)$	$3^1 4^3 5^3 6^2, 3^1 4^2 5^5 6^1$ (two polytopes)	Non-rigid
$4^7 7^2$ $(15, 35, 35, 14, 0)$	 $4^7 7^2$	Rigid
$4^6 \sharp 4^6$ $(15, 36, 33, 12, 1)$	 $4^6 6^3$	Rigid
$4^5 5^2 6^2$ $(15, 35, 29, 8, 0)$	 $4^5 5^2 6^2$	Rigid
$4^4 5^4 6^1$ $(15, 35, 27, 6, 0)$	 $4^4 5^4 6^1$	Rigid
$4^3 5^6$ $(15, 35, 24, 3, 0)$	 $4^3 5^6$	Rigid

are equal to those of a dodecahedron. Let x_k be the number of k-gonal facets of P. By the Euler equation, $\sum_{k\geqslant 3} x_k(6-k) = 12$. Since the number of facets $\sum_{k\geqslant 3} x_k$ is 12, we have $\sum_{k\geqslant 3} x_k(5-k) = 0$. If P has triangular or quadrangular facets, then $\beta^{-2,8}(P) \neq 0$ by Theorem 3.3. Therefore, $x_3 = x_4 = 0$. Now, if $x_k \neq 0$ for $k \geqslant 6$, then $\sum_{k\geqslant 3} x_k(5-k)$ must be negative. This implies that $x_5 = 12$. Hence P is a dodecahedron. $\square$

8. *Some variations of the definition of rigidity*

There are several variations of the definition of cohomological rigidity. As is mentioned in Section 1, cohomological rigidity was first introduced in [**10**] in terms of toric manifolds and simplicial complexes; namely, a simplicial complex Σ_X associated with a toric manifold X

is *rigid* if $\Sigma_X \approx \Sigma_Y$ whenever $H^*(X) \cong H^*(Y)$ as graded rings. Therefore our definition is a variation of the original definition of rigidity.

Moreover, we may consider the cohomological rigidity of simple convex polytopes in terms of small covers, which gives another variation of the definition; namely, we may replace 'quasitoric manifolds' by 'small covers' and 'integral cohomology rings' by 'mod 2 cohomology rings' in Definition 1.2. A small cover is a closed n-dimensional manifold with a locally standard mod 2 torus $(\mathbb{Z}_2)^n$-action over a simple convex polytope. It is therefore a mod 2 analogue of a quasitoric manifold. Small covers were introduced by Davis and Januszkiewicz [**7**].

In the proof of our rigidity results we have made essential use of bigraded Betti numbers, which are purely combinatorial invariants of the polytopes. Considering this, Buchstaber asked the following question in his lecture notes [**4**].

QUESTION 8.1. Let K and K' be simplicial complexes, and let $\mathcal{Z}_K$ and $\mathcal{Z}_{K'}$ be their respective moment angle complexes. When does a cohomology ring isomorphism $H^*(\mathcal{Z}_K : k) \cong H^*(\mathcal{Z}_{K'} : k)$ imply a combinatorial equivalence $K \approx K'$, where k is a field?

Let us call the simplicial complexes giving a positive answer to the question *B-rigid*. Note that $H^*(\mathcal{Z}_K : k) \cong \mathrm{Tor}(k(K), k)$; see [**5**]. Let $K = (\partial P)^*$ and $K' = (\partial P')^*$ be the duals of the boundaries of the simple convex polytopes P and P', respectively. Let M and M' be quasitoric manifolds over P and P', respectively, such that $H^*(M) \cong H^*(M)$. Then, by Lemma 3.7 and the ring isomorphism $H^*(\mathcal{Z}_K : k) \cong \mathrm{Tor}(k(K), k)$, we have the isomorphism $H^*(\mathcal{Z}_K : k) \cong H^*(\mathcal{Z}_{K'} : k)$. Hence, if P is cohomologically rigid, then K is B-rigid. Furthermore, Example 1.1 still gives non-B-rigid simplicial complexes. However, at this moment we do not know whether cohomological rigidity is equivalent to B-rigidity for simple convex polytopes supporting quasitoric manifolds.

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