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TORIC KÄHLER METRICS SEEN FROM INFINITY, QUANTIZATION AND COMPACT TROPICAL AMOEBAS

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Abstract

We consider the metric space of all toric Kähler metrics on a compact toric manifold; when "looking at it from infinity" (following Gromov), we obtain the tangent cone at infinity, which is parametrized by equivalence classes of complete geodesics. In the present paper, we study the associated limit for the family of metrics on the toric variety, its quantization, and degeneration of generic divisors.

The limits of the corresponding Kähler polarizations become degenerate along the Lagrangian fibration defined by the moment map. This allows us to interpolate continuously between geometric quantizations in the holomorphic and real polarizations and show that the monomial holomorphic sections of the prequantum bundle converge to Dirac delta distributions supported on Bohr-Sommerfeld fibers.

In the second part, we use these families of toric metric degenerations to study the limit of compact hypersurface amoebas and show that in Legendre transformed variables they are described by tropical amoebas. We believe that our approach gives a different, complementary, perspective on the relation between complex algebraic geometry and tropical geometry.

Contents

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1. Introduction and main results

Studying families of toric Kähler metrics on a smooth toric variety, we investigate limits corresponding to holomorphic Lagrangian distributions degenerating to the real Lagrangian torus fibration defined by the moment map. We use methods of Kähler geometry and geometric quantization, which permits us to consider degenerations even though the algebraic-geometric moduli space of complex structures associated to toric varieties consists of a point only. More precisely (see Section 2.3), we consider the space of all toric Kähler metrics on a fixed very ample toric line bundle, and the limits we take along complete geodesics are parametrized in a natural way by the tangent cone at infinity of this space. Below, we study the associated limit for the corresponding family of metrics on the toric variety, its quantization, and degeneration of generic divisors. While the metric limits are distinct across the tangent cone at infinity, the limit lagrangian foliation is the same for all points.

This approach permits us to obtain the following main results. Let P be a Delzant polytope and let X_P be the associated compact toric variety [**De**]. Let $\psi \in C^\infty(P)$ be a smooth function with positive definite Hessian on P. Such a ψ defines a complete geodesic in the space of toric Kähler metrics (see Section 2.3). Then,

1. In Theorem 1.1, we determine the weakly covariantly constant sections of the natural line bundle on X_P with respect to the (singular) real polarization defined by the Lagrangian fibration given by the moment map μ_P. In particular, we see that they are naturally indexed by the integer points in the polytope P.

2. We show in Theorem 1.2 that the family of Kähler polarizations corresponding to the mentioned geodesic converges to the real polarization, independently of the direction ψ of deformation.

3. Theorem 1.3 states that the holomorphic monomial sections of the natural line bundle converge to the Dirac delta distributions supported on the corresponding Bohr-Sommerfeld orbit of Theorem 1.1.

For the class of symplectic toric manifolds, this solves the important question, in the context of geometric quantization, on the explicit link between Kähler polarized Hilbert spaces and real polarized ones, in particular in a situation where the real polarization is singular. For a different, but related, recent result in this direction see [**BGU**].

4. We show that the compact amoebas [**GKZ**, **FPT**, **Mi**] of complex hypersurface varieties in X_P converge in the Hausdorff metric to tropical amoebas in the (ψ-dependent) variables defined from the symplectic ones via the Legendre transform

$$\begin{array}{rcl} \mathcal{L}_\psi \ : \ P & \to & \mathcal{L}_\psi P \subset \mathbb{R}^n \\ u & = & \mathcal{L}_\psi(x) = \dfrac{\partial \psi}{\partial x}(x). \end{array} \tag{1}$$

This framework gives a new way of obtaining tropical geometry from complex algebraic geometry by degenerating the ambient toric metric rather than taking a limit of deformations of the complex field [**Mi**, **EKL**].

Another significative difference is that the limit amoebas described above live inside the compact image $\mathcal{L}_\psi P$ and are tropical in the interior of $\mathcal{L}_\psi P$.

Let us describe these results in more detail.

1.1. Geometric quantization of toric varieties. Let P be a Delzant polytope with vertices in $\mathbb{Z}^n$ defining, via the Delzant construction [**De**], a compact symplectic toric manifold $(X_P, \omega, \mathbb{T}^n, \mu_P)$, with moment map μ_P. Let $\mathcal{P}_\mathbb{R} \subset (TX_P)_\mathbb{C}$ be the (singular) real polarization, in the sense of geometric quantization [**Wo**], corresponding to the orbits of the Hamiltonian $\mathbb{T}^n$ action. The Delzant construction also defines a complex structure J_P on X_P such that the pair (ω, J_P), is Kähler, with Kähler polarization $\mathcal{P}_\mathbb{C}$. In addition, the polytope P defines, canonically, an equivariant J_P-holomorphic line bundle, $L \to X_P$ with curvature $-i\omega$ [**Od**].

A result, usually attributed to Danilov and Atiyah [**Da**, **GGK**], states that the number of integer points in P, which are equal to the images under μ_P of the Bohr-Sommerfeld (BS) fibers of the real polarization

$\mathcal{P}_{\mathbb{R}}$, is equal to the number of holomorphic sections of L, i.e. to the dimensionality of $H^0(X_P, L)$.

An important general problem in geometric quantization is understanding the relation between quantizations associated to different polarizations and, in particular, between real and holomorphic quantizations. Hitchin [**Hi**] has shown that, in some general situations, the bundle of quantum Hilbert spaces over the space of deformations of the complex structure, is equipped with a (projectively) flat connection that provides the identification between holomorphic quantizations corresponding to different complex structures. These results do not, however, directly apply in the present situation as the complex structure on a toric variety is rigid. Concerning real polarizations, Śniatycki [**Sn**] has shown that for non-singular real polarizations of arbitrary (quantizable) symplectic manifolds, the set of BS fibers is in bijective correspondence with a generating set for the space of cohomological wave functions which define the quantum Hilbert space in the real polarization. Explicit geometro-analytic relations between real polarization wave functions and holomorphic ones via degenerating families of complex structures have been found for theta functions on abelian varieties (see [**FMN**, **BMN**] and references therein). Similar studies have been performed for cotangent bundles of Lie groups [**Hal**, **FMMN**]. Some of the results in this paper, in fact, are related to these results for the case $(T^*\mathbb{S}^1)^n = (\mathbb{C}^*)^n$, where in the present setting $(\mathbb{C}^*)^n$ becomes the open dense orbit in the toric variety X_P.

As opposed to all these cases, however, the real polarization of a compact toric variety always contains singular fibers. As was shown by Hamilton [**Ham**], the sheaf cohomology used by Śniatycki only detects the non-singular BS leaves. Also, a possible model for the real quantization that includes the singular fibers has recently been described in [**BGU**].

If, on one hand, it is natural to expect that by finding a family of (Kähler) complex structures degenerating to the real polarization, the holomorphic sections will converge to delta distributions supported at the BS fibers, on the other hand, it was unclear how to achieve such behavior from the simple monomial sections characteristic of holomorphic line bundles on toric manifolds (where the series characteristic of theta functions on Abelian varieties are absent).

The detailed study of the degenerating Kähler structures and their quantization is made possible by Abreu's description of toric complex structures [**Ab1**, **Ab2**], following Guillemin's characterization of a canonical toric Kähler metric on (X_P, ω) determined by a symplectic potential $g_P : P \to \mathbb{R}$ [**Gui**]. In particular, for any pair of smooth functions φ, ψ satisfying certain convexity conditions (see Section 2.1), the functions g_s

$$s \mapsto g_s = g_P + \varphi + s\psi, \tag{2}$$

are admissible as symplectic potentials, i.e. define toric Kähler metrics for all positive s.

The quantization of a compact symplectic manifold in the real polarization is given by distributional sections. In this case, conditions of covariant constancy of the wave functions have to be understood as local rather than pointwise (see, for instance, [**Ki**]) and the relevant piece of data is the sheaf of smooth local sections of a polarization $\mathcal{P}$, $C^\infty(\mathcal{P})$.

Applying this approach just outlined, in Section 3.1 we obtain a description of the quantum space of the real polarization $\mathcal{Q}_{\mathbb{R}}$. The main technical difference as compared with the techniques of [**Sn**] (applied to the present situation in [**Ham**]) consists in the fact that we do not only use the sheaf of sections in the kernel of covariant differentiation, but also the cokernel. It is therefore not surprising that our result differs from that of [**Ham**], where the dimension is given by the number of non-degenerate Bohr-Sommerfeld fibers only. In contrast to this, we find

Theorem 1.1. *For the singular real polarization $\mathcal{P}_{\mathbb{R}}$ defined by the moment map, the space of covariantly constant distributional sections of the prequantum line bundle L_ω is spanned by one section δ^m per Bohr-Sommerfeld fiber $\mu_P^{-1}(m), m \in P \cap \mathbb{Z}^n$, with*

$$\operatorname{supp} \delta^m = \mu_P^{-1}(m).$$

But not only does the result of the quantization in the real polarization change; actually, the weak equations of covariant constancy allow for a continuous passage from quantization in complex to real polarizations. The first step in this direction is to verify that the conditions imposed on distributional sections by the set of equations of covariant constancy converge in a suitable sense: if we denote by $\mathcal{P}_{\mathbb{C}}^s$ the holomorphic polarization corresponding to the complex structure defined by (2), our second main finding is

Theorem 1.2. *For any $\psi \in C^\infty_{Hess>0}(P)$, we have*

$$C^\infty(\lim_{s\to\infty} \mathcal{P}_{\mathbb{C}}^s) = C^\infty(\mathcal{P}_{\mathbb{R}}),$$

where the limit is taken in the positive Lagrangian Grassmannian of the complexified tangent space at each point in X_P.

Identifying holomorphic sections with distributional sections in the usual way (as in [**Gun**], but making use of the Liouville measure on the base) we may actually keep track of the monomial basis of holomorphic sections as s changes and show that they converge in the space of distributional sections.

Consider the prequantum bundle L_ω equipped with the holomorphic structure defined by the prequantum connection ∇, defined in (12), and by the complex structure J_s corresponding to g_s in (2). Let $\iota : C^\infty(L_\omega) \to (C_c^\infty(L_\omega^{-1}))'$ be the natural injection of the space of smooth into distributional sections defined in (13). For any lattice point $m \in P \cap \mathbb{Z}^n$, let $\sigma_s^m \in C^\infty(L_\omega)$ be the associated J_s-holomorphic section of L_ω and δ^m the delta distribution from the previous Theorem. Our third main result is the following:

Theorem 1.3. *For any ψ strictly convex in a neighborhood of P and $m \in P \cap \mathbb{Z}^n$, consider the family of L^1-normalized J_s-holomorphic sections*

$$\mathbb{R}^+ \ni s \mapsto \xi_s^m := \frac{\sigma_s^m}{\|\sigma_s^m\|_1} \in C^\infty(L_\omega) \overset{\iota}{\subset} (C_c^\infty(L_\omega^{-1}))' \quad .$$

Then, as $s \to \infty$, $\iota(\xi_s^m)$ converges to δ^m in $(C_c^\infty(L_\omega^{-1}))'$.

Remark 1.4. Note that the sections $\sigma_s^m, \sigma_s^{m'}$ are L^2-orthogonal for $m \neq m'$.

Remark 1.5. We note that the set up above can be easily generalized to a larger family of deformations given by symplectic potentials of the form $g_s = g_P + \varphi + \psi_s$, where ψ_s is a family of smooth strictly convex functions on P, such that $\frac{1}{s}\psi_s$ has a strictly convex limit in the C^2-norm in $C^\infty(P)$.

Remark 1.6. For non-compact symplectic toric manifolds X_P, the symplectic potentials in (2) still define compatible complex structures on X_P; however, Abreu's theorem no longer holds. Theorems 1.2 and 1.3 remain valid in the non-compact case, if one assumes uniform strict convexity of ψ for the latter.

As mentioned above, these results provide a setup for relating quantizations in different polarizations. In particular, Theorem 1.3 gives an explicit analytic relation between holomorphic and real wave functions by considering families of complex structures converging to a degenerate point.

1.2. Compact tropical amoebas. Let now Y_s denote the one-parameter family of hypersurfaces in (X_P, J_s) given by

$$Y_s = \left\{ p \in X_P \ : \ \sum_{m \in P\cap\mathbb{Z}^n} a_m \mathrm{e}^{-sv(m)} \sigma_s^m(p) = 0 \right\}, \tag{3}$$

where $a_m \in \mathbb{C}^*, v(m) \in \mathbb{R}, \forall m \in P \cap \mathbb{Z}^n$. The image of Y_s in P under the moment map μ_P is naturally called the compact amoeba of Y_s. Note that Y_s is a complex submanifold of X_P equipped with the Kähler structure $(J_s, \gamma_s = \omega(., J_s.))$. This is in contrast with the compact amoeba of [**GKZ**, **FPT**, **Mi**] where the Kähler structure is held fixed.

Using the family of Legendre transforms in (1) associated to the potentials g_s on the open orbit, we relate the intersection $\mu_P(Y_s) \cap \check{P}$ with the Log_t-amoeba of [**FPT**, **Mi**] for finite $t = \mathrm{e}^s$. For $s \to \infty$, the Hausdorff limit of the compact amoeba is then characterized by the tropical amoeba $\mathcal{A}_{\mathrm{trop}}$ defined as the support of non-differentiability, or corner locus, of the piecewise smooth continuous function $\mathbb{R}^n \to \mathbb{R}$,

$$u \quad \mapsto \quad \max_{m \in P \cap \mathbb{Z}^n} \left\{ {}^t m \cdot u - v(m) \right\}.$$

In Section 4, we show that, as $s \to \infty$, the amoebas $\mu_P(Y_s)$ converge in the Hausdorff metric to a limit amoeba $\mathcal{A}_{\mathrm{lim}}$. The relation between $\mathcal{A}_{\mathrm{trop}}$ and $\mathcal{A}_{\mathrm{lim}}$ is given by a projection $\pi : \mathbb{R}^n \to \mathcal{L}_\psi P$ (defined in Lemma 4.7, see Figure 3) determined by ψ and the combinatorics of the fan of P. The fourth main result is, then:

Theorem 1.7. *The limit amoeba is given by*

$$\mathcal{L}_\psi \mathcal{A}_{\mathrm{lim}} = \pi \mathcal{A}_{\mathrm{trop}}.$$

Remark 1.8. There is a set with non-empty interior of valuations $v(m)$ in (3), the convex projection $\pi\mathcal{A}_{\mathrm{trop}}$ coincides with the intersection of $\mathcal{A}_{\mathrm{trop}}$ with the image of the moment polytope $\mathcal{L}_\psi P$. In particular, if $\psi(x) = \frac{x^2}{2}$, then $\mathcal{L}_\psi = \mathrm{Id}_P$ and $\mathcal{A}_{\mathrm{lim}}$ is a (compact part) of a tropical amoeba, see Figure 5.

Remark 1.9. Note that for quadratic $\psi(x) = \frac{{}^t x G x}{2} + {}^t b x$, where ${}^t G = G > 0$, the limit amoeba $\mathcal{A}_{\mathrm{lim}} \subset P$ itself is piecewise linear.

Under certain conditions concerning ψ, this construction produces naturally a singular affine manifold $\mathcal{L}_\psi \mathcal{A}_{\mathrm{lim}}$, with a metric structure (induced from the inverse of the Hessian of ψ). In the last Section we comment on the possible relation of this result to the study of mirror symmetry from the SYZ viewpoint.

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2. Preliminaries and notation

Let us briefly review a few facts concerning compatible complex structures on toric symplectic manifolds and also fix some notation. For reviews on toric varieties, see [**Co**, **Da**, **dS**]. Consider a Delzant lattice polytope $P \subset \mathbb{R}^n$ given by

$$P = \{x \in \mathbb{R}^n \ : \ \ell_r(x) \geq 0, r = 1, \ldots, d\}, \tag{4}$$

where

$$\begin{array}{rcl} \ell_r : \mathbb{R}^n & \to & \mathbb{R} \\ \ell_r(x) & = & {}^t\nu_r x - \lambda_r, \end{array} \tag{5}$$

$\lambda_r \in \mathbb{Z}$ and ν_r are primitive vectors of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$, inward-pointing and normal to the r-th facet, i.e. codimension-1 face of P. We denote the interior of P by $\breve{P}$, and the convex hull of k points $v_1, \dots, v_k$ by $\langle v_1, \dots, v_k \rangle$.

Let X_P be the associated smooth toric variety, with moment map $\mu_P : X_P \to P$. On the open dense orbit $\breve{X}_P = \mu_P^{-1}(\breve{P}) \cong \breve{P} \times \mathbb{T}^n$, one considers symplectic, or action-angle, coordinates $(x, \theta) \in \breve{P} \times \mathbb{T}^n$ for which the symplectic form is the standard one, $\omega = \sum_{i=1}^n dx_i \wedge d\theta_i$ and $\mu_P(x, \theta) = x$.

2.1. Symplectic potentials for toric Kähler structures. Recall ([**Ab1**, **Ab2**]) that any compatible complex structure on X_P can be written via a symplectic potential $g = g_P + \varphi$, where $g_P \in C^\infty(\breve{P})$ is given by [**Gui**]

$$g_P(x) = \frac{1}{2} \sum_{r=1}^{d} \ell_r(x) \log \ell_r(x), \tag{6}$$

and φ belongs to the convex set, $C^\infty_{g_P}(P) \subset C^\infty(P)$, of functions φ such that $\mathrm{Hess}_x(g_P + \varphi)$ is positive definite on $\breve{P}$ and satisfies the regularity conditions

$$\det(\mathrm{Hess}_x(g_P + \varphi)) = \left[\alpha(x) \Pi_{r=1}^d \ell_r(x)\right]^{-1}, \tag{7}$$

for α smooth and strictly positive on P, as described in [**Ab1**, **Ab2**].

The complex structure J associated to a potential $g = g_P + \varphi$ and the Kähler metric $\gamma = \omega(\cdot, J\cdot)$ are given by

$$J = \begin{pmatrix} 0 & -G^{-1} \\ G & 0 \end{pmatrix} \; ; \; \gamma = \begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix}, \tag{8}$$

where $G = \mathrm{Hess}_x g > 0$ is the Hessian of g. For recent applications of this result see e.g. [**Do**, **MSY**, **SeD**].

The complex coordinates are related with the symplectic ones by a bijective Legendre transform

$$\breve{P} \ni x \mapsto y = \frac{\partial g}{\partial x} \in \mathbb{R}^n,$$

that is, g fixes an equivariant biholomorphism $\breve{P} \times \mathbb{T}^n \cong (\mathbb{C}^*)^n$,

$$\breve{P} \times \mathbb{T}^n \ni (x, \theta) \mapsto w = \mathrm{e}^{y + \mathrm{i}\theta} \in (\mathbb{C}^*)^n.$$

The inverse transformation is given by $x = \frac{\partial h}{\partial y}$, where

$$h(y) = {}^t x(y) y - g(x(y)). \tag{9}$$

Let us describe coordinate charts covering the rest of X_P, that is, the loci of compactification. Using the Legendre transform associated to the symplectic potential g, one can describe, in particular, holomorphic charts around the fixed points of the torus action. Consider, for any vertex v of P, any ordering of the n facets that contain v; upon reordering the indices, one may suppose that

$$\ell_1(v) = \cdots = \ell_n(v) = 0.$$

Consider the affine change of variables on P

$$l_i = \ell_i(x) = {}^t\nu_i x - \lambda_i, \quad \forall i = 1, \ldots, n, \text{ i.e.} \quad l = Ax - \lambda,$$

where $A = (A_{ij} = (\nu_i)_j) \in Gl(n, \mathbb{Z})$ and $\lambda \in \mathbb{Z}^n$. This induces a change of variables on the open orbit $\breve{X}_P$,

$$\breve{P} \times \mathbb{T}^n \ni (x, \theta) \mapsto (l = Ax - \lambda, \vartheta = {}^tA^{-1}\theta) \in (AP - \lambda) \times \mathbb{T}^n \subset (\mathbb{R}_0^+)^n \times \mathbb{T}^n$$

such that $\omega = \sum \mathrm{d}x_i \wedge \mathrm{d}\theta_i = \sum \mathrm{d}l_i \wedge \mathrm{d}\vartheta_i$.

Consider now the union

$$\breve{P}_v := \{v\} \cup \bigcup_{\substack{F \text{ face} \\ v \in \overline{F}}} \breve{F}$$

of the interior of all faces adjacent to v and set $V_v := \mu_P^{-1}(\breve{P}_v) \subset X_P$. This is an open neighborhood of the fixed point $\mu_P^{-1}(v)$, and it carries a smooth chart $V_v \to \mathbb{C}^n$ that glues to the chart on the open orbit as

$$(A\breve{P} - \lambda) \times \mathbb{T}^n \ni (l, \vartheta) \mapsto w_v = (w_{l_j} = \mathrm{e}^{y_{l_j} + \mathrm{i}\vartheta_j})_{j=1}^n \in \mathbb{C}^n, \tag{10}$$

where $y_{l_j} = \frac{\partial g}{\partial l_j}$. We will call this "the chart at v" for short, dropping any reference to the choice of ordering of the facets at v to disburden the notation; usually we will then write simply w_j for the components of w_v. It is easy to see that a change of coordinates between two such charts, at two vertices v and $\widetilde{v}$, is holomorphic. The complex manifold, W_P, obtained by taking the vertex complex charts and the transition functions between them, does not depend on the symplectic potential. It will be convenient for us to distinguish between X_P and W_P, noting that the g-dependent map $\chi_g : X_P \to W_P$ described locally by (10), introduces the g-dependent complex structure (8) on X_P, making χ_g a biholomorphism.

2.2. The prequantum line bundle. In the same way, the holomorphic line bundle L_P on W_P determined canonically by the polytope P [**Od**], induces, via the pull-back by χ_g, an holomorphic structure on the (smooth) prequantum line bundle L_ω on X_P, as will be described below:

$$\begin{tikzcd} L_\omega \arrow[r] \arrow[d] & L_P \arrow[d] \\ X_P \arrow[r, "\chi_g"] & W_P \end{tikzcd}.$$

Using the charts V_v at the fixed points, one can describe L_P by

$$L_P = \left(\coprod_v V_v \times \mathbb{C} \right) / \sim,$$

where the equivalence relation $\sim$ is given by the transition functions for the local trivializing holomorphic sections $\mathbb{1}_v(w_v) = (w_v, 1)$ for $p \in V_v$,

$$\mathbb{1}_v = w_{\tilde{v}}^{(\widetilde{A}A^{-1}\lambda - \widetilde{\lambda})} \mathbb{1}_{\tilde{v}},$$

on intersections of the domains. Here, we use the data of the affine changes of coordinates

$$l = \ell(x) = Ax - \lambda, \qquad \widetilde{l} = \widetilde{\ell}(x) = \widetilde{A}x - \widetilde{\lambda},$$

associated to the vertices v and $\widetilde{v}$ (and the order of the facets there).

Note that one also has a trivializing section on the open orbit, such that for a vertex $v \in P$, $\mathbb{1}_v = w_v \breve{\mathbb{1}}$ on $\breve{W}_P$. However, note that the sections $\mathbb{1}_v$ extend to global holomorphic sections on W_P while the extension of $\breve{\mathbb{1}}$ will, in general, be meromorphic and will be holomorphic iff $0 \in P \cap \mathbb{Z}^n$. Sections in the standard basis $\{\sigma^m\}_{m \in P \cap \mathbb{Z}^n} \subset H^0(W_P, L_P)$ read $\sigma^m = \sigma^m_{\breve{P}} \breve{\mathbb{1}} = \sigma^m_v \mathbb{1}_v$, with

$$\sigma^m_{\breve{P}}(w) = w^m, \qquad \sigma^m_v(w_v) = w_v^{\ell(m)},$$

in the respective domains.

The prequantum line bundle L_ω on the symplectic manifold X_P is, analogously, defined by unitary local trivializing sections $\mathbb{1}_v^{U(1)}$ and transition functions

$$\mathbb{1}_v^{U(1)} = \mathrm{e}^{\mathrm{i}^t(\widetilde{A}A^{-1}\lambda - \widetilde{\lambda})\tilde{\vartheta}} \mathbb{1}_{\tilde{v}}^{U(1)}. \tag{11}$$

L_ω is equipped with the compatible prequantum connection ∇, of curvature $-\mathrm{i}\omega$, defined by

$$\nabla \mathbb{1}_v^{U(1)} = -\mathrm{i}^t(x - v)\mathrm{d}\theta\, \mathbb{1}_v^{U(1)} = -\mathrm{i}^t l\, \mathrm{d}\vartheta\, \mathbb{1}_v^{U(1)}, \tag{12}$$

where we use $\ell(v) = 0$.

The bundle isomorphism relating L_ω and L_P is determined by

$$\mathbb{1}_v^{U(1)} = \mathrm{e}^{h_v \circ \mu_P} \chi_g^* \mathbb{1}_v, \qquad \breve{\mathbb{1}}^{U(1)} = \mathrm{e}^{h \circ \mu_P} \chi_g^* \breve{\mathbb{1}},$$

where, for $m \in \mathbb{Z}^n$, $h_m(x) = (x - m)\frac{\partial g}{\partial x} - g(x)$ and h is the function in (9) defining the inverse Legendre transform.

In these unitary local trivializations, the sections $\chi_g^*\sigma^m$ read

$$\chi_g^*\sigma^m = \mathrm{e}^{-h_m\circ\mu_P}\mathrm{e}^{\mathrm{i}^t m\theta}\breve{\mathbb{1}}^{U(1)} = \mathrm{e}^{-h_m\circ\mu_P}\mathrm{e}^{\mathrm{i}^t\ell(m)\vartheta}\mathbb{1}_v^{U(1)},$$

where, after an affine change of coordinates $x\mapsto\ell(x)$ on the moment polytope, as above, we get $h_m(l) = {}^t(l-\ell(m))\frac{\partial g}{\partial l} - g(l)$. Then, for all $\sigma\in H^0(W_P,L_P)$, $\chi_g^*\sigma\in C^\infty(L_\omega)$ is holomorphic, that is

$$\nabla_{\bar\xi}\chi_g^*\sigma = 0,$$

for any holomorphic vector field ξ. That is, such sections are polarized with respect to the distribution of holomorphic vector fields on X_P (see, for instance, [**Wo**]).

To treat the real polarization defined by the moment map, we will find it necessary to extend the operator of covariant differentiation from smooth to distributional sections: we consider the injection of smooth in distributional sections determined by Liouville measure,

$$\begin{array}{cccc} \iota: & C^\infty(L_\omega|_U) & \longrightarrow & C^{-\infty}(L_\omega|_U) = \left(C_c^\infty(L_\omega^{-1}|_U)\right)' \\ & s & \mapsto & \iota s(\phi) = \int\limits_U s\phi\frac{\omega^n}{n!} \end{array}, \tag{13}$$

where $U\subset X_P$ is any open set. To extend the operator ∇_ξ on smooth sections to an operator we denote ∇''_ξ on distributional sections we demand commutativity of the diagram

$$\begin{tikzcd} C^\infty(L_\omega|_U) \arrow[r, hook, "\iota"] \arrow[d, "\nabla_\xi"'] & C^{-\infty}(L_\omega|_U) \arrow[d, "\nabla''_\xi"] \\ C^\infty(L_\omega|_U) \arrow[r, hook, "\iota"] & C^{-\infty}(L_\omega|_U) \end{tikzcd} .$$

To determine $\nabla''_\xi\sigma$ for a general distributional section σ not of the form ιs, we establish what its transpose is by integrating the operator ∇_ξ by parts. This gives, for any smooth section $s\in C^\infty(L_\omega|_U)$ and smooth test section $\phi\in C_c^\infty(L_\omega^{-1}|_U)$,

$$\left(\nabla''_\xi\iota s\right)(\phi) = \int\limits_U\left(\nabla_\xi s\right)\phi\frac{\omega^n}{n!} = -\int\limits_U s\left(\mathrm{div}\xi\phi + \nabla_\xi^{-1}\phi\right)\frac{\omega^n}{n!}.$$

Here we use the fact that given a connection ∇ on L_ω, the inverse line bundle L_ω^{-1} (defined by the inverse cocycle in a trivialization) comes naturally equipped with a connection (defined by the negative of the connection one-forms); we will denote this connection by ∇^{-1}. Therefore, ∇''_ξ is characterized by its transpose,

$$\nabla''_\xi\sigma(\phi) = \sigma({}^t\nabla_\xi\phi), \quad \forall\phi\in C_c^\infty(L_\omega^{-1}|_U),$$

where

$${}^t\nabla_\xi\phi = -\left(\mathrm{div}\xi\phi + \nabla_\xi^{-1}\phi\right).$$

Remark 2.1. The formulae for the definition of weak covariant constancy would become more involved if we wrote them using the Hilbert space structure on sections of L given by the Hermitean structure. For example, we can extend the operator of covariant differentiation by use of the (restriction of the) adjoint of ∇_ξ as operator on the dense subspace $C_c^\infty(L_\omega|_U) \subset L^2(L_\omega|_U)$. Note that

$$\langle s, s' \rangle_{L^2} = (\iota s)(\overline{s'}h), \quad \forall s, s' \in C_c^\infty(L_\omega|_U),$$

where $h \in C^\infty\left((L_\omega \otimes \overline{L_\omega})^{-1}\right)$ is the Hermitean structure on the line bundle L_ω. This gives

$$\langle \nabla_\xi s, s' \rangle_{L^2} = \langle s, \nabla_\xi^* s' \rangle_{L^2} \iff (\iota s)^t \nabla_\xi(\overline{s'}h) = (\iota s)\left(\left(\overline{\nabla_\xi^* s'}\right) h\right),$$

or

$$\nabla_\xi^* s = \overline{{}^t\nabla_\xi\left(\overline{s}h\right) h^{-1}}.$$

2.3. The space of toric Kähler metrics. If we denote the set of toric Kähler metrics on (X_P, ω) by $\mathcal{M}_P$, it is parametrized by the convex set of functions $C_{g_P}^\infty(P)$. The space $\mathcal{M}_P$ carries a Riemannian metric $\gamma_{\mathcal{M}_P}$ introduced by Mabuchi, Semmes and Donaldson (see [**SZ**] and references therein), whose geodesic segments are linear in terms of the symplectic potential φ,

$$s \mapsto g_P + \varphi_0 + s\left(\varphi_1 - \varphi_0\right).$$

From this, it is clear that the tangent cone at infinity (introduced by Gromov, and which we think of as "the space seen from infinity", cf. [**Gr**, **JM**])

$$T_\infty \mathcal{M}_P := \lim_{t\to\infty}\left(\mathcal{M}_P, \frac{1}{t}\gamma_{\mathcal{M}_P}\right) \tag{14}$$

consists of all functions $\psi \in C_{g_P}^\infty(P)$ with non-negative definite Hessian on the whole interior of P, which is the necessary and sufficient condition for the geodesic ray

$$s \mapsto g_P + \varphi + s\psi \tag{15}$$

to be defined for all $s \geq 0$. Denoting this set by $C^\infty_{\mathrm{Hess}\geq 0}(P)$, we have therefore a natural identification

$$T_\infty \mathcal{M}_P \cong C^\infty_{\mathrm{Hess}\geq 0}(P). \tag{16}$$

Actually, for technical reasons we will restrict mostly to the subset

$$T_\infty^+ \mathcal{M}_P :\cong C^\infty_{\mathrm{Hess}>0}(P)$$

of strictly convex directions in $\mathcal{M}_P$, for the following reason: if we consider the family of Riemannian metrics on X_P over $\mathcal{M}_P$,

$$\begin{array}{c} \mathcal{X}_P \cong \mathcal{M}_P \times X_P \\ \big\downarrow \\ \mathcal{M}_P \end{array} \quad , \text{ where } \left(\mathcal{X}_P\right)_\varphi := \left(X_P, \gamma_\varphi\right),$$

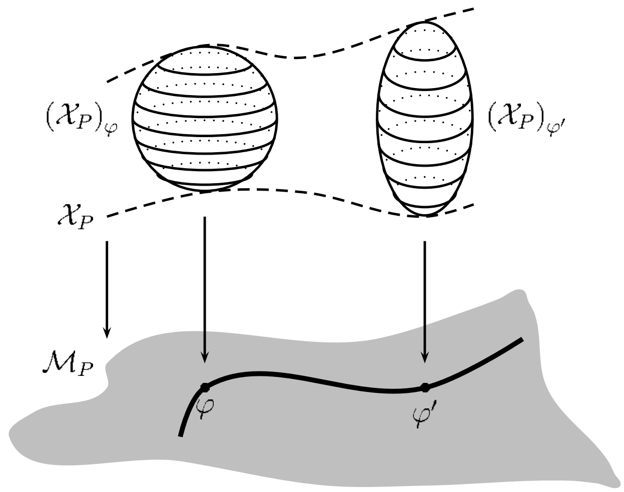


Figure 1. The family of toric Kähler metrics, schematically.

we can "lift the limit" (14) to the geodesic families. Indeed, substituting the potential $g_P + \varphi + s\psi$ in (8) and restricting to the diagonal $s = t$, we see that

$$\frac{1}{s}\gamma_{\varphi+s\psi} = \tag{17}$$

$$\begin{pmatrix} \frac{1}{s}\mathrm{Hess}_x\left(g_P+\varphi\right)+\mathrm{Hess}_x\psi & 0 \\ 0 & \frac{1}{s}\left(\mathrm{Hess}_x\left(g_P+\varphi\right)+s\mathrm{Hess}_x\psi\right)^{-1} \end{pmatrix},$$

that is,

$$\lim_{s\to\infty}\left(X_P, \frac{1}{s}\gamma_{\varphi+s\psi}\right) = \left(P, \mathrm{Hess}_x\psi\right).$$

It is in this sense that the family of toric Kähler metrics on X_P, when seen from infinity, collapses to a family of Hessian metrics on P over $T_\infty\mathcal{M}_P$ that we denote (with a slight abuse of notation) by $T_\infty\mathcal{X}_P$,

$$\begin{array}{c} T_\infty\mathcal{X}_P \cong P \times T_\infty\mathcal{M}_P \\ \big\downarrow \\ T_\infty\mathcal{M}_P \end{array}, \text{ where } \left(T_\infty\mathcal{X}_P\right)_\varphi := \left(P, \mathrm{Hess}_x\psi\right).$$

It is natural to turn the attention primarily to the limits which have a non-degenerate metric, that is, to $T_\infty^+\mathcal{M}_P$.

In Section 4 we will study the geometry (and how much of it can be recovered from the limit) of generic divisors in X_P along these lines.

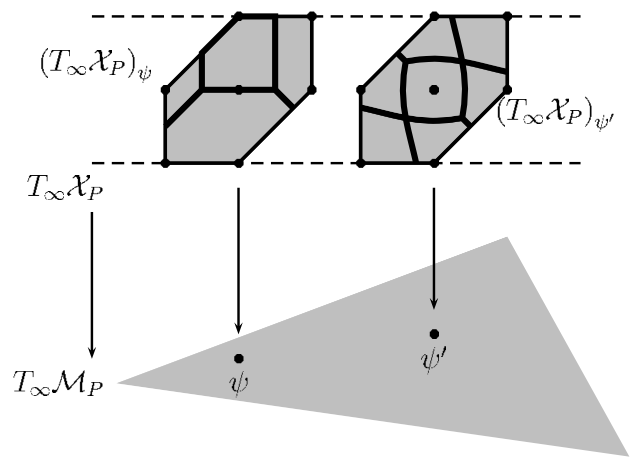


Figure 2. The family of Hessian metrics on a polytope with limit amoebas, schematically.

3. Quantization

In this Section, we study the geometric quantization of toric symplectic manifolds for the family $\mathcal{M}_P$ of toric Kähler metrics, and their degeneration.

3.1. Quantization in a real polarization. We begin by describing the quantization obtained using the definition outlined above for the singular real polarization $\mathcal{P}_\mathbb{R}$ defined by the moment map

$$\mathcal{P}_\mathbb{R} = \ker \mathrm{d}\mu_P.$$

This means that we consider the space of weakly covariant constant distributional sections, $\mathcal{Q}_\mathbb{R} \subset (C^\infty(L_\omega^{-1}))'$,

$$\mathcal{Q}_\mathbb{R} := \big\{\sigma \in (C^\infty(L_\omega^{-1}))' \mid \\ \forall\, W \subset X_P \text{ open}, \forall\, \xi \in C^\infty(\mathcal{P}_\mathbb{R}|_W),\, \nabla''_\xi(\sigma|_W) = 0\big\}\,.$$

In the present Section, "covariantly constant" is always understood to mean "covariantly constant with respect to the polarization $\mathcal{P}_\mathbb{R}$". We first give a result describing the local covariantly constant sections. Note that it only depends on the local structure of the polarization, and thus applies also in cases where globally one is not dealing with toric varieties.

Proposition 3.1. (i) *Any covariantly constant section on a $\mathbb{T}^n$-invariant open set $W \subset X_P$ is supported on the Bohr-Sommerfeld fibers in W.*

(ii) *The distribution*

$$\delta^m(\tau) = \int\limits_{\mu_P^{-1}(m)} \mathrm{e}^{\mathrm{i}\,{}^t\ell(m)\vartheta}\tau_v, \quad \forall \tau \in C_c^\infty(L_\omega^{-1}|_W),$$

is covariantly constant.

(iii) *The sections δ^m, $m \in \mu_P(W)\cap\mathbb{Z}^n$ span the space of covariantly constant sections on W.*

Proof. Note that all the statements are local in nature. Using the action of $Sl(n,\mathbb{Z})$ on polytopes, we can reduce without loss of generality an arbitrary Bohr-Sommerfeld fiber $\mu_P^{-1}(m)$ with $m \in P\cap\mathbb{Z}^n$ being contained in a codimension k (and no codimension $k+1$) face of P to the form

$$m = (\underbrace{0,\ldots,0}_{k},\widetilde{m}) \quad \text{ with } \widetilde{m}_j > 0 \quad \forall j = k+1,\ldots,n.$$

Furthermore, we can cover all of X_P by charts $\widetilde{W}$ of special neighborhoods of $\mu_P^{-1}(m)$ of the form

$$\widetilde{W} := B_\varepsilon(0)\times\left(\widetilde{m}+]-\varepsilon,\varepsilon[^{n-k}\right)\times\mathbb{T}^{n-k}$$

with $0<\varepsilon<1$ and $B_\varepsilon(0)\subset\mathbb{R}^{2k}$ being the ball of radius ε in $\mathbb{R}^{2k}$. This chart is glued onto the open orbit via the map,

$$\begin{gathered}\breve{P}\times\mathbb{T}^n \ni (x_1,\ldots,x_n,\vartheta_1,\ldots,\vartheta_n)\mapsto\\ \big(u_1=\sqrt{x_1}\cos\vartheta_1, v_1=\sqrt{x_1}\sin\vartheta_1,\ldots,u_k=\sqrt{x_k}\cos\vartheta_k, v_k=\sqrt{x_k}\sin\vartheta_k,\\ x_{k+1},\ldots,x_n,\vartheta_{k+1},\ldots,\vartheta_n\big)\in\widetilde{W}\end{gathered}$$

and therefore induces the standard symplectic form in the coordinates u, v, x, ϑ

$$\omega = \sum_{j=1}^k \mathrm{d}u_j\wedge\mathrm{d}v_j + \sum_{j=k+1}^n \mathrm{d}x_j\wedge\mathrm{d}\vartheta_j.$$

We then trivialize the line bundle with connection L_ω over $\widetilde{W}$ by setting

$$\nabla = \mathrm{d}-\mathrm{i}\left(\frac{1}{2}({}^tu\mathrm{d}v-{}^tv\mathrm{d}u)+{}^tx\mathrm{d}\vartheta\right).$$

Therefore, we are reduced to studying the equations of covariant constancy on the space of (usual) distributions $C^{-\infty}(\widetilde{W})$ on $\widetilde{W}$, using arbitrary test functions in $C_c^\infty(\widetilde{W})$ and vector fields $\xi\in C^\infty(\mathcal{P}_\mathbb{R}|_{\widetilde{W}})$. These can be written as

$$\xi_{(u,v,x,\vartheta)} = \sum_{j=1}^k \alpha_j\left(u_j\frac{\partial}{\partial v_j}-v_j\frac{\partial}{\partial u_j}\right)+\sum_{j=k+1}^n\beta_j\frac{\partial}{\partial\vartheta_j}, \tag{18}$$

where α_j,β_j are smooth functions on $\widetilde{W}$.

(i) First, note that if a distribution σ is covariantly constant, it is unchanged by parallel transport around a loop in $\mathbb{T}^n$, while on the other hand this parallel transport results in multiplication of all test sections by the holonomies of the respective leaves. Explicitly, for a loop specified by a vector $a \in \mathbb{Z}^n$, any test function $\tau \in C_c^\infty(\widetilde{W})$ is multiplied by the smooth function $\mathrm{e}^{2\pi \mathrm{i}\, {}^t a\, \mu_P(u,v,x,\vartheta)}$. Therefore,

$$\sigma = \mathrm{e}^{2\pi \mathrm{i}\, {}^t a\, \mu_P} \sigma, \quad \forall a \in \mathbb{Z}^n,$$

and σ must be supported in the set where all the exponentials equal 1, i.e. on the Bohr-Sommerfeld fibers, where μ_P takes integer values.

(ii) In the chart on $\widetilde{W}$, any test function τ can be written as a Fourier series

$$\tau(u,v,x,\vartheta) = \sum_{b \in \mathbb{Z}^{n-k}} \widehat{\tau}(u,v,x,b) \mathrm{e}^{\mathrm{i}\, {}^t b\, \vartheta}, \tag{19}$$

with coefficients that are smooth and compactly supported in the other variables,

$$\forall b \in \mathbb{Z}^{n-k}: \quad \widehat{\tau}(.,b) \in C_c^\infty\left(B_\varepsilon(0) \times \left(\widetilde{m} +]-\varepsilon, \varepsilon[^{n-k}\right)\right).$$

The local representative of the distributional section δ^m is calculated as

$$\begin{aligned} \delta^m(\tau) &= \int_{\mu_P^{-1}(m)} \mathrm{e}^{\mathrm{i}\, {}^t \widetilde{m} \vartheta} \tau \\ &= \int_{\mathbb{T}^{n-k}} \mathrm{e}^{\mathrm{i}\, {}^t \widetilde{m} \vartheta} \tau(u=0, v=0, x=\widetilde{m}, \vartheta) \mathrm{d}\vartheta \\ &= \widehat{\tau}(u=0, v=0, x=\widetilde{m}, b=-\widetilde{m}). \end{aligned}$$

Differentiating an arbitrary test function τ along any vector field

$$\xi_{(u,v,x,\vartheta)} = \sum_{j=1}^k \alpha_j \left(u_j \frac{\partial}{\partial v_j} - v_j \frac{\partial}{\partial u_j}\right) + \sum_{j=k+1}^n \beta_j \frac{\partial}{\partial \vartheta_j},$$

with constant coefficients α_j, β_j in the polarization $\mathcal{P}$ (which is generated by such vector fields over $C^\infty(W)$) gives

$$\begin{aligned} {}^t\nabla_\xi \tau &= \mathrm{d}\tau \xi + \mathrm{i}\left(\frac{1}{2}({}^t u \mathrm{d}v - {}^t v \mathrm{d}u) + {}^t x \mathrm{d}\vartheta\right) \xi \tau \\ &= \sum_{b \in \mathbb{Z}^{n-k}} \left(\sum_{j=1}^k \alpha_j \left(u_j \frac{\partial \widehat{\tau}}{\partial v_j} - v_j \frac{\partial \widehat{\tau}}{\partial u_j} + \frac{\mathrm{i}}{2}(u_j^2 + v_j^2)\widehat{\tau}\right) \right. \\ &\qquad \left. + \mathrm{i} \sum_{j=k+1}^n \beta_j (x_j + b_j) \widehat{\tau} \right) \mathrm{e}^{\mathrm{i}\, {}^t b\, \vartheta}, \end{aligned} \tag{20}$$

whence for arbitrary τ and any such ξ

$$\begin{aligned}\delta^m({}^t\nabla_\xi\tau) &= \left(\sum_{j=1}^k \alpha_j\left(u_j\frac{\partial\widehat{\tau}}{\partial v_j} - v_j\frac{\partial\widehat{\tau}}{\partial u_j} + \frac{\mathrm{i}}{2}(u_j^2+v_j^2)\widehat{\tau}\right)\right.\\ &\qquad \left. + \mathrm{i}\sum_{j=k+1}^n \beta_j(x_j+b_j)\widehat{\tau}\right)_{(u=0,v=0,x=\widetilde{m},b=-\widetilde{m})} = 0,\end{aligned}$$

so that δ^m is covariantly constant.

(iii) Consider an arbitrary distribution $\sigma \in C^{-\infty}(\widetilde{W})$. Using Fourier expansion of test functions along the non-degenerating directions of the polarization on $\widetilde{W}$, as in item (ii), to σ we associate a family of distributions $\widehat{\sigma}_b$ on $B_\varepsilon(0)\times\left(\widetilde{m}+]-\varepsilon,\varepsilon[^{n-k}\right)$ by setting

$$\widehat{\sigma}_b(\psi) := \sigma\left(\psi(u,v,x)\mathrm{e}^{\mathrm{i}\,{}^t b\vartheta}\right), \quad \forall\psi\in C_c^\infty\left(B_\varepsilon(0)\times\left(\widetilde{m}+]-\varepsilon,\varepsilon[^{n-k}\right)\right),$$

so that

$$\sigma(\tau) = \sum_{b\in\mathbb{Z}^{n-k}} \widehat{\sigma}_b(\widehat{\tau}(.,b)).$$

It is clear that the map $\sigma\mapsto(\widehat{\sigma}_b)_{b\in\mathbb{Z}^{n-k}}$ is injective, so we need to show that the condition of covariant constancy

$$\sigma({}^t\nabla_\xi\tau) = 0, \quad \forall\tau\in C_c^\infty(\widetilde{W}), \xi\in C^\infty(\mathcal{P}_\mathbb{R}|_{\widetilde{W}})$$

implies that

$$\widehat{\sigma}_b = \begin{cases} 0 & \text{if } b\neq-\widetilde{m}, \\ \lambda\delta(u,v,x-\widetilde{m}) & \text{if } b=-\widetilde{m}, \text{ where } \lambda\in\mathbb{C}. \end{cases}$$

From (i) we know that $\widehat{\sigma}_b$ has support at the point $(u,v,x)=(0,0,\widetilde{m})$ for each b, therefore it must be of the form (see, for instance, [**Tr**] Chapter 24; here j,k,l are multi-indices)

$$\widehat{\sigma}_b = \sum_{\text{finite}} \gamma^b_{jkl}\frac{\partial^{|j|+|k|+|l|}}{\partial u^j\partial v^k\partial x^l}\delta(u,v,x-\widetilde{m}).$$

Using for example a vector field ξ with $\alpha_j=\beta_j\equiv 1$ and explicit test functions τ that are polynomial near the point $(u,v,x)=(0,0,\widetilde{m})$, it is easy to give examples that show that if $\gamma^b_{jkl}\neq 0$ and either $b\neq-\widetilde{m}$ or $|j|+|k|+|l|>0$, then there exist ξ and τ such that

$$\sigma({}^t\nabla_\xi\tau)\neq 0,$$

thus producing a contradiction. q.e.d.

Immediately from this proposition follows the

Proof of Theorem 1.1. Since the open neighborhoods considered in the Proposition are $\mathbb{T}^n$-invariant and the covariantly constant sections are supported on closed subsets, each of them can evidently be extended by 0 to give a global covariantly constant section. q.e.d.

3.2. The degenerate limit of Kähler polarizations. As mentioned above, we will turn our attention to a family of compatible complex structures determined by the symplectic potentials $g_s = g_P + \varphi + s\psi$, being interested in the limit of holomorphic polarizations, in the sense of geometric quantization, and subsequently in the convergence of monomial sections to the delta distributions just described.

In the vertex coordinate charts V_v described above, the holomorphic polarizations are given by

$$\mathcal{P}^s_{\mathbb{C}} = \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial w^s_j} : j = 1, \dots, n \right\}.$$

Let $\mathcal{P}_{\mathbb{R}}$ stand for the vertical polarization, that is

$$\mathcal{P}_{\mathbb{R}} := \ker \mathrm{d}\mu_P,$$

which is real and singular above the boundary ∂P. Let now $\mathcal{P}^\infty := \lim_{s\to\infty} \mathcal{P}^s_{\mathbb{C}}$ where the limit is taken in the positive Lagrangian Grassmannian of the complexified tangent space at each point in X_P.

Lemma 3.2. *On the open orbit $\check{X}_P$, $\mathcal{P}^\infty = \mathcal{P}_{\mathbb{R}}$.*

Proof. By direct calculation,

$$\frac{\partial}{\partial y^s_{l_j}} = \sum_k (G_s^{-1})_{jk} \frac{\partial}{\partial l_k}$$

where

$$(G_s)_{jk} = \mathrm{Hess}\, g_s$$

is the Hessian of g_s. Since $G_s > s\mathrm{Hess}\,\psi > 0$, $(G_s^{-1})_{jk} \to 0$ and

$$\mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial w^s_j} \right\} = \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial y^s_{l_j}} - \mathrm{i}\frac{\partial}{\partial \vartheta_j} \right\} \to \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \vartheta_j} \right\}.$$

q.e.d.

At the points of X_P that do not lie in the open orbit, the holomorphic polarization “in the degenerate angular directions” is independent of g_s, in the following sense:

Lemma 3.3. *Consider two charts around a fixed point $v \in P$, $w_v, \widetilde{w}_v : V_v \to \mathbb{C}^n$, specified by symplectic potentials $g \neq \widetilde{g}$.*

Whenever $w_j = 0$, also $\widetilde{w}_j = 0$, and at these points

$$\mathbb{C} \cdot \frac{\partial}{\partial w_j} = \mathbb{C} \cdot \frac{\partial}{\partial \widetilde{w}_j}, \quad j = 1, \dots, n.$$

Proof. According to the description of the charts,

$$w_j = \widetilde{w}_j f,$$

where f is a real-valued function, smooth in P, that factorizes through μ_P, that is through $|\widetilde{w}_1|, \dots, |\widetilde{w}_n|$. Therefore,

$$\mathrm{d}w_j = \sum_k \left[\left(\delta_{j,k} f + \widetilde{w}_j \frac{\partial f}{\partial \widetilde{w}_k} \right) \mathrm{d}\widetilde{w}_k + \widetilde{w}_j \frac{\partial f}{\partial \overline{\widetilde{w}}_k} \mathrm{d}\overline{\widetilde{w}}_k \right]$$

and at point with $\widetilde{w}_j = 0$ one finds, in fact,

$$\mathbb{C} \cdot \frac{\partial}{\partial w_j} = \mathbb{C} \cdot \frac{\partial}{\partial \widetilde{w}_j}.$$

q.e.d.

The two lemmata together give

Theorem 3.4. *In any of the charts w_v, the limit polarization $\mathcal{P}^\infty$ is*

$$\mathcal{P}^\infty = \mathcal{P}_\mathbb{R} \oplus \mathrm{span}_\mathbb{C} \left\{ \frac{\partial}{\partial w_j} : w_j = 0 \right\}$$

Proof. It suffices to show the convergence

$$\mathrm{span}_\mathbb{C} \left\{ \frac{\partial}{\partial w_k^s} \right\} \to \mathrm{span}_\mathbb{C} \left\{ \frac{\partial}{\partial \theta_k} \right\}$$

whenever $w_k \neq 0$, which really is a small modification of Lemma 3.2. For any face F in the coordinate neighborhood, we write abusively $j \in F$ if $w_j = 0$ along F. For any such affine subspace we have then

$$\begin{aligned} \mathcal{P}_\mathbb{C}^s &= \mathrm{span}_\mathbb{C} \left\{ \frac{\partial}{\partial w_j^s} : j \in F \right\} \oplus \mathrm{span}_\mathbb{C} \left\{ \frac{\partial}{\partial y_{l_k}^s} - \mathrm{i} \frac{\partial}{\partial \vartheta_k} : k \notin F \right\} \\ &= \mathrm{span}_\mathbb{C} \left\{ \frac{\partial}{\partial w_j^s} : j \in F \right\} \oplus \mathrm{span}_\mathbb{C} \left\{ \sum_{k' \notin F} ((G_s)_F)^{-1}_{kk'} \frac{\partial}{\partial l_{k'}} - \mathrm{i} \frac{\partial}{\partial \vartheta_k} : k \notin F \right\} \end{aligned}$$

where $(G_s)_F$ is the minor of G_s specified by the variables that are unrestricted along F, which is well-defined and equals the Hessian of the restriction of g_s there. Since this tends to infinity, its inverse goes to zero and the statement follows. q.e.d.

This results in

Proof of Theorem 1.2. Given Theorem 3.4, we are reduced to proving that

$$C^\infty \left(\mathcal{P}_\mathbb{R} \oplus \mathrm{span}_\mathbb{C} \left\{ \frac{\partial}{\partial w_j} : w_j = 0 \right\} \right) = C^\infty(\mathcal{P}_\mathbb{R}).$$

This is clear since any smooth complexified vector field $\xi \in C^\infty(T^\mathbb{C} X)$ that restricts to a section of $\mathcal{P}_\mathbb{R}$ on an open dense subset must satisfy

$\overline{\xi} = \xi$ throughout. Such a vector field cannot have components along the directions of $\mathrm{span}_{\mathbb{C}} \{\frac{\partial}{\partial w_j} : w_j = 0\}$. q.e.d.

Remark 3.5. Note that the Cauchy-Riemann conditions hold for distributions (see e.g. [**Gun**] for the case $n = 1$, or Lemma 2 in [**KY**]), that is, for any complex polarization $\mathcal{P}_{\mathbb{C}}$, considering the intersection of the kernels of $\nabla''_{\frac{\partial}{\partial \bar{z}_j}}$ gives exactly the space

$$\bigcap_{\frac{\partial}{\partial \bar{z}_j} \in \overline{\mathcal{P}_{\mathbb{C}}}} \ker \nabla''_{\frac{\partial}{\partial \bar{z}_j}} = \iota H^0(X_P(\mathcal{P}_{\mathbb{C}}), L_\omega(\mathcal{P}_{\mathbb{C}})) \subset (C^\infty(L_\omega^{-1}))'$$

of holomorphic sections (viewed as distributions). Thus one can view the 1-parameter family of quantizations associated to g_s and the real quantization on equal footing, embedded in the space of distributional sections, $\mathcal{Q}_s \subset (C^\infty(L_\omega^{-1}))'$,

$$\mathcal{Q}_s := \left\{\sigma \in (C^\infty(L_\omega^{-1}))' \mid \forall\, W \subset X_P \text{ open}, \forall\, \xi \in C^\infty(\overline{\mathcal{P}^s}|_W), \nabla''_\xi(\sigma|_W) = 0\right\},$$

for $s \in [0, \infty]$, where $\mathcal{P}^\infty := \mathcal{P}_{\mathbb{R}}$, and $\mathcal{Q}_\infty = \mathcal{Q}_{\mathbb{R}}$ in the previous notation. In the next Section, we will see that the convergence of polarizations proved here translates eventually into a continuous variation of the subspace $\mathcal{Q}_s$ in the space of distributional sections as $s \to \infty$.

Remark 3.6. Notice that Theorem 1.2 in this and Proposition 3.1 in the previous Section are also valid for non-compact P, with some obvious changes such as taking test sections with compact support.

3.3. Degeneration of holomorphic sections and BS fibers. Here, we use convexity to show that, as $s \to \infty$, the holomorphic sections converge, when normalized properly, to the distributional sections δ^m, described in Proposition 3.1, and that are supported along the Bohr-Sommerfeld fibers of μ_P and covariantly constant along the real polarization.

First, we show an elementary lemma on certain Dirac sequences associated with the convex function ψ that will permit us to prove Theorem 1.3.

Lemma 3.7. *For any ψ strictly convex in a neighborhood of the moment polytope P and any $m \in P \cap \mathbb{Z}^n$, the function*

$$P \ni x \mapsto f_m(x) := {}^t(x - m)\frac{\partial \psi}{\partial x} - \psi(x) \in \mathbb{R}$$

has a unique minimum at $x = m$ and

$$\lim_{s\to\infty} \frac{\mathrm{e}^{-sf_m}}{\|\mathrm{e}^{-sf_m}\|_1} \to \delta(x - m),$$

in the sense of distributions.

Proof. For x in a convex neighborhood of P, using $f_m(m) = -\psi(m)$ and $\nabla f_m(x) = {}^t(x-m)\mathrm{Hess}_x\psi$, we have

$$\begin{aligned} f_m(x) &= f_m(m) + \int_0^1 \frac{d}{dt} f_m(m + t(x-m))\mathrm{d}t \\ &= -\psi(m) + \int_0^1 t\, {}^t(x-m)\left(\mathrm{Hess}_{m+t(x-m)}\psi\right)(x-m)\,\mathrm{d}t. \end{aligned}$$

Since ψ is strictly convex, with

$$\mathrm{Hess}_x\psi > 2cI,$$

for some positive c and all x in a neighborhood of P, it follows that

$$f_m(x) \geq -\psi(m) + c\|x-m\|^2.$$

Obviously, m is the unique absolute minimum of f_m in P. For the last assertion, we show that the functions

$$\zeta_s := \frac{\mathrm{e}^{-sf_m}}{\|\mathrm{e}^{-sf_m}\|_1}$$

form a Dirac sequence. (We actually show convergence as measures on P.) It is clear from the definition that $\zeta_s > 0$ and $\|\zeta_s\|_1 = 1$, so it remains to show that the norms concentrate around the minimum, that is, given any $\varepsilon, \varepsilon' > 0$ we have to find a s_0 such that

$$\forall s \geq s_0: \quad \int\limits_{B_\varepsilon(m)} \zeta_s(x)\mathrm{d}x \geq 1 - \varepsilon'.$$

Let $r_0 > 0$ be sufficiently small and let 2α be the maximum of $\mathrm{Hess}_x\psi$ in $B_{r_0}(m)$. Observe that

$$\|\mathrm{e}^{-sf_m}\|_1 = \int_P \mathrm{e}^{-sf_m(x)}\mathrm{d}x \geq \int_{B_r(m)} \mathrm{e}^{-sf_m(x)}\mathrm{d}x \geq d_n r^n \mathrm{e}^{s\psi(m) - s\alpha r^2},$$

for any $r > 0$ such that $r_0 > r$, and where $d_n r^n = \mathrm{Vol}(B_r(m))$. On the other hand,

(21)
$$\int\limits_{P\backslash B_\varepsilon(m)} \mathrm{e}^{-sf_m(x)}\mathrm{d}x \leq \int\limits_{P\backslash B_\varepsilon(m)} \mathrm{e}^{s\psi(m) - sc\|x-m\|^2}\,\mathrm{d}x \leq \mathrm{Vol}(P)\mathrm{e}^{s\psi(m)}\mathrm{e}^{-sc\varepsilon^2}.$$

Therefore,

$$\int\limits_{P\backslash B_\varepsilon(m)} \zeta_s(x)\mathrm{d}x \leq \frac{\mathrm{Vol}(P)\mathrm{e}^{-sc\varepsilon^2 + s\alpha r^2}}{d_n r^n}.$$

Choosing r sufficiently small, the right hand side goes to zero as $s \to \infty$ and the result follows. q.e.d.

Let now $\{\sigma_s^m\}_{m\in P\cap\mathbb{Z}^n}$ be the basis of holomorphic sections of L_ω, with respect to the holomorphic structure induced from the map χ_{g_s} in Section 2, that is $\sigma_s^m = \chi_{g_s}^*(\sigma^m)$, for $\sigma^m \in H^0(W_P, L_P)$. We then have

Proof of Theorem 1.3. Using a partition of unity $\{\rho_v\}$ subordinated to the covering by the vertex coordinate charts $\{\breve{P}_v\}$, the result can be checked chart by chart. Let $\tau \in C^\infty(L_\omega^{-1})$ be a test section and let

$$\begin{aligned} h_m^s(x) = {}^t(x-m)y - g_s &= \left[{}^t(x-m)\left(\frac{\partial g_P}{\partial x} + \frac{\partial \varphi}{\partial x}\right) - g_P - \varphi\right] \\ &\quad + s\left[{}^t(x-m)\frac{\partial \psi}{\partial x} - \psi\right] \\ &= h_m^0(x) + s f_m(x), \end{aligned}$$

with f_m as in Lemma 3.7. Then

$$\begin{aligned} (\iota(\xi_s^m))(\tau) &= \frac{1}{\|\sigma_s^m\|_1}\sum_v \int\limits_{V_v} \rho_v\circ\mu_P(w_v)\mathrm{e}^{-h_m^s\circ\mu_P(w_v)}\mathrm{e}^{\mathrm{i}\ell(m)\vartheta}\tau_v(w_v)\omega^n \\ &= \frac{1}{\|\sigma_s^m\|_1}\sum_v \int\limits_{\breve{P}_v} \rho_v(x)\mathrm{e}^{-h_m^s(x)} \\ &\quad \cdot \left(\int\limits_{\mu_P^{-1}(x)} \mathrm{e}^{2\pi\mathrm{i}\ell(m)u}\tau_v\left(\mathrm{e}^{\frac{\partial g_s}{\partial l}(\ell(x))+2\pi\mathrm{i}u}\right)\mathrm{d}u\right)\mathrm{d}x \\ &= \frac{1}{\|\sigma_s^m\|_1}\int\limits_P \mathrm{e}^{-h_m^s(x)}\widehat{\tau}(x,-m)\mathrm{d}x, \end{aligned}$$

where $\widehat{\tau}$ is the fiberwise Fourier transform from equation (19).

Now the L^1-norm in question calculates as

$$\|\sigma_s^m\|_{L^1} = \int\limits_{X_P} \mathrm{e}^{-h_m^s\circ\mu_P}\omega^n = (2\pi)^n\int\limits_P \mathrm{e}^{-h_m^s}\mathrm{d}x.$$

According to Lemma 3.7,

$$\frac{\|\mathrm{e}^{-h_m^0 - sf_m}\|_1}{\|\mathrm{e}^{-sf_m}\|_1} = \int\limits_P \frac{\mathrm{e}^{-sf_m}}{\|\mathrm{e}^{-sf_m}\|_1}\mathrm{e}^{-h_m^0}\mathrm{d}x \to \mathrm{e}^{-h_m^0(m)} \text{ as } s\to\infty,$$

and therefore

$$\iota(\xi_s^m)(\tau) = \int\limits_P \frac{\mathrm{e}^{-h_m^0+sf_m}}{\|\mathrm{e}^{-h_m^0+sf_m}\|_1}\widehat{\tau}(.,-m)\mathrm{d}x \to \widehat{\tau}(m,-m) = \delta^m(\tau)$$

which finishes the proof. q.e.d.

Corollary 3.8. *The results of Lemma* 3.7 *and Theorem* 1.3 *are valid for non-compact toric manifolds if one assumes uniform strict convexity of* ψ.

Proof. In Lemma 3.7, the estimate for $f_m(x)$ remains valid for any $x \in P$, so that the function e^{-sf_m} will be integrable even if P is not compact. As for the second part of the proof of Lemma 3.7, instead of (21) one can use

$$\begin{aligned} & \int_{P \setminus B_\varepsilon(m)} \mathrm{e}^{-sf_m} \mathrm{d}x \\ \leq \quad & d_n \int_\varepsilon^{+\infty} \mathrm{e}^{s\psi(m)} \mathrm{e}^{-scr^2} r^{n-1} \mathrm{d}r \leq M d_n \int_\varepsilon^{+\infty} \mathrm{e}^{s\psi(m)} \mathrm{e}^{-s\frac{c}{2}r^2} \mathrm{d}r \\ \leq \quad & M d_n \mathrm{e}^{s\psi(m)} \mathrm{e}^{-s\frac{c}{2}\varepsilon^2} \int_0^{+\infty} \mathrm{e}^{-s\frac{c}{2}u^2} \mathrm{d}u \leq M' \mathrm{e}^{s\psi(m)} \mathrm{e}^{-s\frac{c}{2}\varepsilon^2}, \end{aligned}$$

for appropriate constants $M, M' > 0$, where M, M' depend on s but are bounded from above as $s \to \infty$, so that the assertion follows. q.e.d.

4. Compact tropical amoebas

In this section, we undertake a detailed study of the behavior of the compact amoebas in P associated to the family of symplectic potentials in (2)

$$g_s = g_P + \varphi + s\psi,$$

which define the complex structure J_s on X_P, and of their relation to the Log_t amoebas in $\mathbb{R}^n$ [**GKZ**, **Mi**, **FPT**, **R**].

Let $\breve{Z}_s \subset (\mathbb{C}^*)^n$ be the complex hypersurface defined by the Laurent polynomial

$$\breve{Z}_s = \left\{ w \in (\mathbb{C}^*)^n : \sum_{m \in P \cap \mathbb{Z}^n} a_m \mathrm{e}^{-sv(m)} w^m = 0 \right\},$$

where $a_m \in \mathbb{C}^*, v(m) \in \mathbb{R}$. One natural thing to do in order to obtain the large Kähler structure limit, consists in introducing the complex structure on $(\mathbb{C}^*)^n$ defined by the complex coordinates $w = e^{z_s}$ where $z_s = sy + i\theta$, and taking the $s \to +\infty$ limit. Then, the map $w \mapsto y$ coincides with the Log_t map for $s = \log t$. However, this deformation of the complex structure, which is well defined for the open dense orbit $(\mathbb{C}^*)^n \subset X_P$ never extends to any (even partial) toric compactification of $(\mathbb{C}^*)^n$. Indeed, that would correspond to rescaling the original symplectic potential by s, which is incompatible with the correct behavior at the boundary of the polytope found by Guillemin and Abreu.

As we will describe below, for deformations in the direction of quadratic ψ in (2), in the limit we obtain the Log_t map amoeba intersected with the polytope P. The significative difference is that our limiting tropical amoebas are now compact and live inside P. For more general ψ, they live in the compact image of P by the Legendre transform $\mathcal{L}_\psi$ in (1) and are determined by the locus of non-differentiability of a piecewise linear function, namely as the tropical amoeba of [**GKZ**, **Mi**],

$$\mathcal{A}_{\text{trop}} := C^0 - \operatorname{loc}\left(u \mapsto \max_{m \in P \cap \mathbb{Z}^n} \{{}^t m u - v(m)\}\right).$$

4.1. Limit versus tropical amoebas. We are interested in the μ_P-image of the family of (complex) hypersurfaces

$$Y_s := \left\{p \in X_P : \sum_{m \in P \cap \mathbb{Z}^n} a_m \mathrm{e}^{-sv(m)} \sigma_s^m(p) = 0\right\} \subset (X_P, J_s)$$

where $a_m \in \mathbb{C}^*$ and $v(m) \in \mathbb{R}$ are parameters and $\sigma_s^m \in H^0\big((X_P, J_s), \chi_{g_s}^* L_P\big)$ is the canonical basis of holomorphic sections of the line bundle $\chi_{g_s}^* L_P$ associated with the polytope P and the symplectic potential (2), introduced in Section 2. We call the image $\mu_P(Y_s) \subset P$ the compact amoeba of Y_s in P.

Definition 4.1. The limit amoeba $\mathcal{A}_{\lim}$ is the subset

$$\mathcal{A}_{\lim} := \lim_{s \to \infty} \mu_P(Y_s)$$

of the moment polytope P, where the limit is to be understood as the Hausdorff limit of closed subsets of P.

The existence of this limit is shown in the proof of Theorem 1.7 below. We will relate this amoeba to the tropical amoeba of [**GKZ**, **Mi**] using a Legendre transform $\check{\chi}_s$ that is the restriction of the map χ_{g_s} described in Section 2 to the open orbit $\check{X}_P$:

$$\begin{tikzcd}
Y_s \arrow[r, hookleftarrow] \arrow[d, phantom, "\cap"] & \check{Y}_s \arrow[r, "\cong"] \arrow[d, phantom, "\cap"] & \check{Z}_s \arrow[r, hook] \arrow[d, phantom, "\cap"] & Z_s \arrow[d, phantom, "\cap"] \\
(X_P, \omega, J_s, G_s) \arrow[r, hookleftarrow] \arrow[d, "\mu_P"'] & \check{X}_P \arrow[r, "\cong", "\check{\chi}_s"'] \arrow[d, "\mu_P"'] & (\mathbb{C}^*)^n \arrow[r, hook] \arrow[d, "Log_t"] & (W_P, J) \\
P \arrow[r, hookleftarrow] & \check{P} \arrow[r, "\cong", "\kappa_s"'] & \mathbb{R}^n &
\end{tikzcd}$$

where κ_s is the family of rescaled Legendre transforms

$$\check{P} \ni x \mapsto \kappa_s(x) := \frac{1}{s}\mathcal{L}_{(g_P+\varphi)+s\psi} = \frac{\partial \psi}{\partial x} + \frac{1}{s}\frac{\partial(g_P+\varphi)}{\partial x} \in \mathbb{R}^n.$$

For any $s > 0$, this is a diffeomorphism $\check{P} \to \mathbb{R}^n$.

Let $\mathcal{A}_s := Log_t(\check{Z}_s)$ be the amoeba of [**GKZ**, **Mi**]. Recall that $\mathcal{A}_s \to \mathcal{A}_{\text{trop}}$ in the Hausdorff topology [**Mi**, **R**].

Proposition 4.2. *The family of rescaled Legendre transforms κ_s satisfies*

$$\kappa_s \circ \mu_P(\breve{Y}_s) = \mathcal{A}_s$$

Proof. Under the trivialization of L_P determined by g_s on the open orbit $\breve{X}_P$, the sections $\sigma_s^m(x,\theta)$ correspond to polynomial sections w^m, where

$$w = \mathrm{e}^{\frac{\partial(g_P+\varphi)}{\partial x} + s\frac{\partial\psi}{\partial x} + \mathrm{i}\theta}.$$

Combining this with the Log_t-map for $t = \mathrm{e}^s$ gives precisely

$$Log_t w = \frac{\partial\psi}{\partial x} + \frac{1}{s}\frac{\partial(g_P+\varphi)}{\partial x} = \kappa_s(x).$$

q.e.d.

Remark 4.3. Note that since Y_s is defined as the zero locus of a global section, one has $Y_s = \overline{\breve{Y}_s}$ and, in particular, $\mu_P(Y_s) = \overline{\kappa_s^{-1}\mathcal{A}_s}$. On each face F of the moment polytope P, this will consist exactly of the amoeba defined by the sum of monomials corresponding to integer points in F, cf. [**Mi**].

The family of inverse maps κ_s^{-1} will permit us to capture information not only concerning the open orbit but also the loci of compactification of X_P.

Lemma 4.4. *For any compact subset $C \subset \breve{P}$ and any ψ strictly convex on a neighborhood of P,*

$$\kappa_s \to \mathcal{L}_\psi \textit{ pointwise on } \breve{P} \textit{ and uniformly on } C$$

and

$$\kappa_s^{-1} \to \mathcal{L}_\psi^{-1} \textit{ uniformly on } \mathcal{L}_\psi C.$$

Proof. Since $\frac{\partial(g_P+\varphi)}{\partial x}$ is a smooth function on $\breve{P}$, $\kappa_s \to \mathcal{L}_\psi$ pointwise on $\breve{P}$ and uniformly on compact subsets $C \subset \breve{P}$. Furthermore,

$$\|\kappa_s(x) - \kappa_s(x')\| \geq c\|x - x'\|, \qquad \forall x, x' \in \breve{P}, \tag{22}$$

with a constant $c > 0$ uniform in s, since the derivative

$$\frac{\partial\kappa_s}{\partial x} = \mathrm{Hess}_x\psi + \frac{1}{s}\mathrm{Hess}_x(g_P+\varphi) > \mathrm{Hess}_x\psi > 0$$

is (uniformly) positive definite. Therefore, the family of inverse mappings κ_s^{-1} is uniformly Lipschitz (on $\mathbb{R}^n$). Thus the pointwise convergence $\kappa_s^{-1} \to \mathcal{L}_\psi^{-1}$ on $\mathcal{L}_\psi\breve{P}$ is uniform on any compact $\mathcal{L}_\psi C$. q.e.d.

Before proving the main theorem, we recall some facts about convex sets in $\mathbb{R}^n$ and also show two auxiliary lemmata on the behavior of the gradient of any toric symplectic potential near the boundary of the moment polytope. Consider any constant metric $G = {}^tG > 0$ on $\mathbb{R}^n$. For an arbitrary closed convex polyhedral set $P \subset \mathbb{R}^n$ and any point

$p \in \partial P$, denote by $\mathcal{C}_p^G$ the closed cone of directions that are "outward pointing at p" in the following sense,

$$\mathcal{C}_p^G(P) := \{c \in \mathbb{R}^n : {}^t c G(p - p') \geq 0, \quad \forall p' \in P\}.$$

Notice that for the Euclidean metric $G = I$, the cone of p is precisely the negative of the cone of the fan of P, corresponding to the face p lies in; in this case we will write $\mathcal{C}_p := \mathcal{C}_p^I$.

Lemma 4.5. *For any sequence $x_k \in \breve{P}$ that converges to a point in the boundary, $x_k \to p \in \partial P$, we have*

$$\frac{\partial(g_P + \varphi)}{\partial x}|_{x_k} \to \mathcal{C}_p(P),$$

in the sense that for any $c \notin \mathcal{C}_p(P)$ there is an open neighborhood $U \ni p$ such that

$$\mathbb{R}_0^+ \frac{\partial(g_P + \varphi)}{\partial x}|_x \neq \mathbb{R}_0^+ c \qquad \forall x \in U \cap \breve{P}.$$

Proof. Suppose (using an affine change of coordinates $l(x) = Ax - \lambda$) that p lies in the codimension k face, $k > 0$, where $l_1 = \cdots = l_k = 0$ and $l_j > 0$ for $j = k+1, \ldots, n$. We have

$$\mathcal{C}_p = \{c \in \mathbb{R}^n : {}^t c A^{-1}(l(p) - l(p')) \geq 0, \forall p' \in P\},$$

and $l_i(p - p') \leq 0$ for $i = 1, \ldots, k$, whereas there is no restriction on the sign of $l_j(p - p')$ for $j = k+1, \ldots, n$. Therefore,

$$c \in \mathcal{C}_p \iff c = {}^t A \widetilde{c} \text{ with } \widetilde{c} \in (\mathbb{R}_0^-)^k \times \{0\} \subset \mathbb{R}^n.$$

Since

$$\frac{\partial(g_P + \varphi)}{\partial x} = {}^t A \frac{\partial(g_P + \varphi)}{\partial l}$$

it is therefore sufficient to prove that $\frac{\partial(g_P+\varphi)}{\partial l}$ approaches $(\mathbb{R}_0^-)^k \times \{0\} \subset \mathbb{R}^n$ as we get near p. Indeed, we find that

$$\begin{aligned} \frac{\partial g_P}{\partial l} &= \frac{1}{2} \frac{\partial}{\partial l} \sum_{a=1}^d \ell_a \log \ell_a \\ &= \frac{1}{2} \frac{\partial}{\partial l} \left(\sum_{i=1}^k l_i \log l_i + \sum_{j=k+1}^n l_j \log l_j + \sum_{m=n+1}^d \ell_m \log \ell_m \right) \end{aligned}$$

and hence

$$\frac{\partial g_P}{\partial l_r} = \frac{1}{2} \left(1 + \log l_r + \sum_{m=n+1}^d \frac{\partial \ell_m}{\partial l_r} (1 + \log \ell_m) \right). \tag{23}$$

For $m > n$ (actually, for $m > k$), the sum is bounded in a neighborhood of p since $\frac{\partial \ell_m}{\partial l_r}$ is constant and $\ell_m > 0$ at p. Since $\frac{\partial \varphi}{\partial l_r}$ is also bounded for any r, $\frac{\partial(g_P+\varphi)}{\partial l_j}$ is bounded in a neighborhood of p for $j = k+1, \ldots, n$

and clearly $\frac{\partial(g_P+\varphi)}{\partial l_i} \to -\infty$ for $i = 1, \ldots, k$ as we approach p, which proves the lemma. q.e.d.

In the following lemma, we relate the Legendre transforms κ_s and $\mathcal{L}_\psi$ at large s, more precisely:

Lemma 4.6. *For any two points $p \neq p' \in P$, there exist $\varepsilon > 0$ and $s_0 \geq 0$ that depend only on p' and the distance $d(p,p')$, such that for all $s \geq s_0$*

$$\overline{\kappa_s(B_\varepsilon(p') \cap \breve{P})} \cap \left(\mathcal{L}_\psi(p) + \mathcal{C}_p(P)\right) = \emptyset.$$

Proof. We will show that there is a hyperplane separating $\kappa_s(B_\varepsilon(p') \cap \breve{P})$ and $\mathcal{L}_\psi(p) + \mathcal{C}_p(P)$; we continue to use a chart as in Lemma 4.5.

For any two points $p \neq p'$,

$$\begin{aligned} & {}^t(l(p) - l(p')) \left(\frac{\partial \psi}{\partial l}|_p - \frac{\partial \psi}{\partial l}|_{p'} \right) \\ = \;& {}^t(l(p) - l(p')) \int_0^1 (\mathrm{Hess}_{\,p'+\tau(p-p')}\psi) \mathrm{d}\tau (l(p) - l(p')) \\ \geq \;& c_\psi \|l(p) - l(p')\|^2 \;>\; 0 \end{aligned}$$

where $c_\psi > 0$ is a constant depending only ψ. This implies that there is at least one index j such that $l_j(p) \neq l_j(p')$,

$$\mathrm{sign}\left(\frac{\partial \psi}{\partial l_j}|_p - \frac{\partial \psi}{\partial l_j}|_{p'} \right) = \mathrm{sign}(l_j(p) - l_j(p')),$$

and also

$$\left| \frac{\partial \psi}{\partial l_j}|_p - \frac{\partial \psi}{\partial l_j}|_{p'} \right| \geq \frac{c_\psi}{n} |l_j(p) - l_j(p')|.$$

Choose $\varepsilon > 0$ small enough (this choice depends on ψ only) so that

$$\frac{\partial \psi}{\partial l_j}|_p \notin \frac{\partial \psi}{\partial l_j}(B_\varepsilon(p') \cap \breve{P})$$

and consider first the case that $0 \leq l_j(p) < l_j(p')$. For all $x \in B_\varepsilon(p') \cap \breve{P}$, from equation (23),

$$\frac{\partial g_P}{\partial l_j}|_x \geq c_1 \log^-(l_j(p') - \varepsilon) + c_2 = C,$$

where $\log^-$ denotes the negative part of the logarithm, and c_1, c_2, C are constants depending only on p' and ε. Then, for any $\delta > 0$ such that

$$\frac{\partial \psi}{\partial l_j}|_p + \delta \notin \frac{\partial \psi}{\partial l_j}(B_\varepsilon(p') \cap \breve{P}),$$

we find $s_0 = \frac{2|C|}{\delta}$ so that

$$\frac{\partial \psi}{\partial l_j}|_x + \frac{1}{s}\frac{\partial g_P}{\partial l_j}|_x \geq \frac{\partial \psi}{\partial l_j}(p) + \frac{\delta}{2}, \qquad \forall x \in B_\varepsilon(p') \cap \breve{P}.$$

Hence, also

$$\frac{\partial(\psi + \frac{1}{s}\varphi)}{\partial l_j}|_x + \frac{1}{s}\frac{\partial g_P}{\partial l_j}|_x \geq \frac{\partial \psi}{\partial l_j}(p) + \frac{\delta}{4}, \qquad \forall x \in B_\varepsilon(p') \cap \breve{P},$$

for s big enough (the additional condition depending only on φ, which is globally controlled on the whole polytope P), which proves our assertion.

If, on the other hand, $0 \leq l_j(p') < l_j(p)$, we see again from equation (23) that

$$\frac{\partial g_P}{\partial l_j} \leq c_1' \log^+ l_j + c_2'$$

on $B_\varepsilon(p') \cap \breve{P}$, where $\log^+$ stands for the positive part of the logarithm. Again,

$$\frac{\partial g_P}{\partial l_j}|_x \leq c_1' \log^+(l_j(p') + \varepsilon) + c_2' = C', \qquad \forall x \in B_\varepsilon(p') \cap \breve{P}$$

and the same argument applies. q.e.d.

We will now characterize the limit amoeba in terms of the tropical amoeba via a projection π that can be described as follows.

Lemma 4.7. *For any strictly convex ψ as above, there exists a partition of $\mathbb{R}^n$ indexed by P of the form*

$$\mathbb{R}^n = \coprod_{p \in P} \mathcal{L}_\psi(p) + \mathcal{C}_p(P). \tag{24}$$

In particular, there is a well-defined continuous projection $\pi : \mathbb{R}^n \to \mathcal{L}_\psi P$ given by

$$\pi(\mathcal{L}_\psi(p) + \mathcal{C}_p(P)) = \mathcal{L}_\psi(p).$$

Proof. It suffices to show that for $p \neq p'$,

$$\big(\mathcal{L}_\psi(p) + \mathcal{C}_p(P)\big) \cap \big(\mathcal{L}_\psi(p') + \mathcal{C}_{p'}(P)\big) = \emptyset.$$

To see this, assume that

$$\mathcal{L}_\psi(p) + c = \mathcal{L}_\psi(p') + c', \text{ with } c \in \mathcal{C}_p(P), c' \in \mathcal{C}_{p'}(P).$$

Then ${}^t(c - c')(p - p') \geq 0$, from the definition of the cones; on the other hand,

$$\begin{aligned} {}^t(p - p')(c - c') &= {}^t(p - p')(\mathcal{L}_\psi(p') - \mathcal{L}_\psi(p)) \\ &= {}^t(p - p') \int_0^1 (\mathrm{Hess}_{\,p+\tau(p'-p)}\psi)\mathrm{d}\tau(p' - p) < 0, \end{aligned}$$

which is a contradiction. q.e.d.

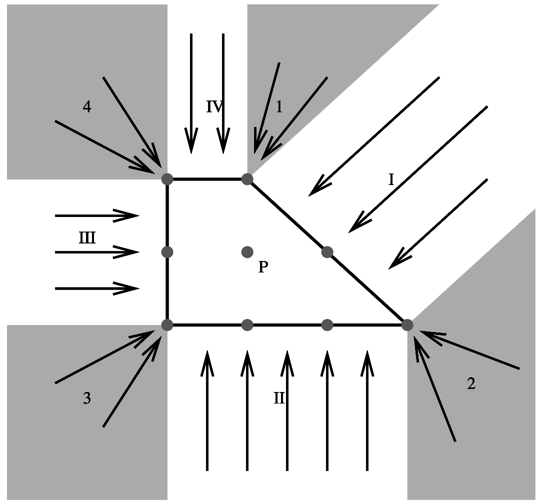


Figure 3. The map π.

Remark 4.8. For quadratic ψ with ${}^tG = G > 0$ symmetric and positive definite there is a more intrinsic description of the map π: it is given by the projection of $\mathbb{R}^n$ on the closed convex subset P under which each point projects onto its best approximation in the polytope $\mathcal{L}_\psi P$ with respect to the metric G^{-1} (see, for instance, chapter V of [**Bou**], and also Figure 3),

$$p = \pi(y) \iff p \in \mathcal{L}_\psi P \wedge \forall p' \in \mathcal{L}_\psi P \setminus \{p\} : \|y - p\|_{G^{-1}} < \|y - p'\|_{G^{-1}}.$$

Note also that

$$\forall y \in \mathbb{R}^n : y - \pi(y) \in \mathcal{C}^{G^{-1}}_{\pi(y)}(\mathcal{L}_\psi P)$$

and that, in fact, $\pi(y)$ is characterised by this property, i.e.

$$\forall p \in \mathcal{L}_\psi P : y - p \in \mathcal{C}^{G^{-1}}_p(\mathcal{L}_\psi P) \iff \pi(y) = p.$$

In this sense, $\mathcal{C}^{G^{-1}}_p(\mathcal{L}_\psi P)$ is a kind of "convex kernel at p" of the convex projection π. Note, by the way, that in this case $\mathcal{C}_x(P) = \mathcal{C}^{G^{-1}}_{\mathcal{L}_\psi(x)}(\mathcal{L}_\psi P)$.

Finally, we have

Proof of Theorem 1.7. We first show that $\mathcal{L}_\psi \circ \kappa_s^{-1} \to \pi$ pointwise on $\mathbb{R}^n$. For points in the interior of $\mathcal{L}_\psi P$, where $\pi|_{\breve{P}} = \mathrm{id}_{\breve{P}}$, this is clear from Lemma 4.4.

Consider, therefore, any point $y \notin \mathcal{L}_\psi \breve{P}$, its family of inverse images $x_s = \kappa_s^{-1}(y) \in \breve{P}$, and any convergent subsequence $x_{s_k} \to p$. Then the limit lies in the boundary, $p \in \partial P$. We need to show that $\mathcal{L}_\psi(p) = \pi(y)$, or, what is the same, that $y - \mathcal{L}_\psi(p) \in \mathcal{C}_p(P)$. This is guaranteed by

Lemma 4.5,

$$\frac{1}{s}\frac{\partial(g_P+\varphi)}{\partial x}|_{x_{s_k}} = (\kappa_s - \mathcal{L}_\psi)(x_{s_k}) = y - \mathcal{L}_\psi(x_{s_k}) \to y - \mathcal{L}_\psi(p) \in \mathcal{C}_p(P),$$

and proves pointwise convergence.

Now we can use compactness of P (and hence of the space of closed non-empty subsets of P with the Hausdorff metric) to show the result. Throughout the proof we will not distinguish between sets and their closures since the Hausdorff topology does not separate them.

Let us first show that

$$\kappa_s^{-1}\mathcal{A}_{\mathrm{trop}} \xrightarrow{\mathrm{H}} \mathcal{L}_\psi^{-1}\circ\pi\mathcal{A}_{\mathrm{trop}} \quad .$$

Take any convergent subsequence $\kappa_{s_k}^{-1}\mathcal{A}_{\mathrm{trop}} \xrightarrow{\mathrm{H}} K \subset P$; since $\kappa_s^{-1} \to \mathcal{L}_\psi^{-1}\circ\pi$ pointwise, it follows that

$$K \supset \mathcal{L}_\psi^{-1}\circ\pi\mathcal{A}_{\mathrm{trop}}.$$

For the other inclusion, consider any point $p' \notin \mathcal{L}_\psi^{-1}\circ\pi\mathcal{A}_{\mathrm{trop}}$; since the distance of p' to $\mathcal{L}_\psi^{-1}\circ\pi\mathcal{A}_{\mathrm{trop}}$ is strictly positive, by Lemma 4.6 there is a neighborhood U of p' in $\breve{P}$ and a s_0 such that for all $p \in \mathcal{L}_\psi^{-1}\circ\pi\mathcal{A}_{\mathrm{trop}}$ and $s \geq s_0$, the sets $\kappa_s(U)$ and $\mathcal{L}_\psi(p)+\mathcal{C}_p(P)$ not only have empty intersection but are actually separated by a hyperplane. But this implies, in particular, that for s large enough

$$U \cap \kappa_s^{-1}\mathcal{A}_{\mathrm{trop}} = \emptyset$$

and $p' \notin K$, as we wished to show.

In the last step, we prove that $\kappa_s^{-1}\mathcal{A}_s \xrightarrow{\mathrm{H}} \mathcal{L}_\psi^{-1}\circ\pi\mathcal{A}_{\mathrm{trop}}$. Again, using compactness, it is sufficient to consider any convergent subsequence $\kappa_{s_k}^{-1}\mathcal{A}_{s_k} \xrightarrow{\mathrm{H}} K'$.

To show that $\mathcal{L}_\psi^{-1}\circ\pi\mathcal{A}_{\mathrm{trop}} \subset K'$, it is sufficient to observe that $\mathcal{A}_{\mathrm{trop}} \subset \mathcal{A}_{s_k}$ (see [**GKZ**, **Mi**]) and hence $\kappa_{s_k}^{-1}\mathcal{A}_{\mathrm{trop}} \subset \kappa_{s_k}^{-1}\mathcal{A}_{s_k}$.

For the converse inclusion $K' \subset \mathcal{L}_\psi^{-1}\circ\pi\mathcal{A}_{\mathrm{trop}}$, consider the constant c from inequality (22) above, and set

$$\varepsilon_k := \frac{1}{c}\mathrm{dist}(\mathcal{A}_{s_k}, \mathcal{A}_{\mathrm{trop}}).$$

This sequence converges to zero, and therefore the closed ε_k-neighborhoods $(\kappa_{s_k}^{-1}\mathcal{A}_{\mathrm{trop}})_{\varepsilon_k} \supset \kappa_{s_k}^{-1}\mathcal{A}_{\mathrm{trop}}$ still converge to $\mathcal{L}_\psi^{-1}\circ\pi\mathcal{A}_{\mathrm{trop}}$. But as κ_{s_k} satisfies the uniform bound in (22),

$$\kappa_{s_k}((\kappa_{s_k}^{-1}\mathcal{A}_{\mathrm{trop}})_{\varepsilon_k}) \supset (\mathcal{A}_{\mathrm{trop}})_{c\varepsilon_k} \supset \mathcal{A}_{s_k}$$

and hence

$$(\kappa_{s_k}^{-1}\mathcal{A}_{\mathrm{trop}})_{\varepsilon_k} \supset \kappa_{s_k}^{-1}\mathcal{A}_{s_k}$$

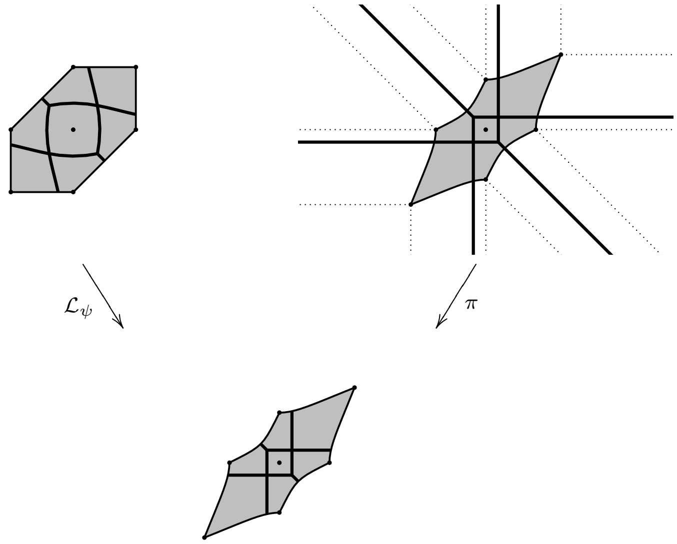


Figure 4. The situation of the main theorem (for ψ not quadratic): $P \supset \mathcal{A}_{\mathrm{lim}}$ (top left), $\mathcal{L}_\psi P$ with the cones of projection $\mathcal{C}_p$ (dotted) and the tropical amoeba $\mathcal{A}_{\mathrm{trop}}$ (top right) and $\mathcal{L}_\psi P \supset \pi\mathcal{A}_{\mathrm{trop}} = \mathcal{L}_\psi \mathcal{A}_{\mathrm{lim}}$ (bottom).

which proves the second inclusion. q.e.d.

Low-dimensional examples of the relation between tropical and limit amoebas are illustrated in Figures 4 to 9 below. In the following remarks, we collect basic facts about limit amoebas and their relation to their tropical counterparts.

Remark 4.9. (i) The first fact to catch the eye about the limit amoebas is that they depend on more parameters than the tropical amoebas: while the latter are determined by the valuation $v(m)$, the former vary heavily, depending on the direction ψ of the geodesic ray $g_P + \varphi + s\psi$ we follow. This reflects the fact that we look at the family of hypersurfaces in different categories: while the complex biholomorphism class of the hypersurface $Y_s \subset X_P$ is independent of the Kähler metric we put on X_P, the Hausdorff limit of $\mu_P(Y_s) \subset P$ does vary substantially. This is illustrated for the simplest possible example, $\mathbb{P}^2$, with moment polytope the standard simplex in $\mathbb{R}^2$ and valuation $v(0,0) = 0$, $v(1,0) = \frac{1}{2}$, $v(0,1) = \frac{1}{4}$, in Figure 5.

(ii) It is clear from this example already that it is not, in general, possible to recover the tropical amoeba from the limit amoeba,

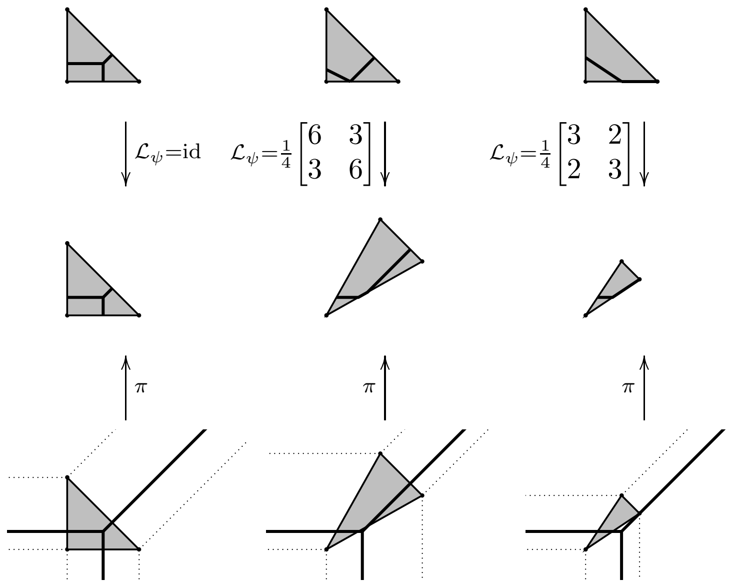


Figure 5. Limit amoebas ($\mathcal{A}_{\mathrm{lim}} \subset P$, top row), their image under $\mathcal{L}_\psi$, and tropical amoebas ($\mathcal{A}_{\mathrm{trop}}$, bottom row) for different quadratic ψ and fixed valuation v.

although we have (from the proof of the theorem) that always

$$\mathcal{L}_\psi \left(\mathcal{A}_{\mathrm{lim}} \cap \breve{P} \right) = \mathcal{A}_{\mathrm{trop}} \cap \mathcal{L}_\psi \breve{P}.$$

(For example, if ψ is quadratic (thus $\mathcal{L}_\psi$ linear), the limit amoeba itself will be piecewise linear).

(iii) There is, however, an open set of valuations v and directions ψ such that the projection $\pi \mathcal{A}_{\mathrm{trop}}$ will coincide with $\mathcal{L}_\psi P \cap \mathcal{A}_{\mathrm{trop}}$; this happens whenever the "nucleus" (i.e. the complement of all unbounded hyperplane pieces) of the tropical amoeba $\mathcal{A}_{\mathrm{trop}}$ lies inside $\mathcal{L}_\psi P$, since the infinite legs run off to infinity along directions in the cone of the relevant faces. This situation is depicted in Figure 4 for a limit of elliptic curves in a del Pezzo surface,

$$\begin{aligned} P &= \langle (1,0), (1,1), (0,1), (-1,0), (-1,-1), (0,-1) \rangle \\ \psi(x) &= \frac{x^2}{2} + \frac{\|x\|^4}{4} \\ v(m) &= \frac{m^2}{2}. \end{aligned}$$

At this point, naturally the question arises how much of the information encoded by tropical amoebas can be recovered from the compact limit amoebas, which we turn to now.

4.2. Compact amoebas and enumerative information. Even without establishing a precise criterion for when the limit amoeba permits the recovery of the tropical amoeba, we can address these questions qualitatively.

Proposition 4.10. (i) *For a fixed potential ψ, there is a set (with non-empty interior) of valuations such that the tropical amoeba can be recovered from the limit amoeba.*

(ii) *Conversely, for a fixed valuation v, there is a set (with non-empty interior) of potentials ψ such that the tropical amoeba can be recovered from the limit amoeba.*

Proof. Both assertions follow from the fact that scaling ψ and v (separately), the image of the polytope $\mathcal{L}_\psi P$ can be made arbitrarily big in relation to the tropical amoeba, since the tropical amoeba furthermore is determined once we reach its "tentacles" (the unbounded parts of hyperplanes).

In particular, the set of valuations in (i) contains all valuations such that $\mathcal{L}_\psi \mathcal{A}_{\mathrm{lim}} = \mathcal{A}_{\mathrm{trop}} \cap P$. q.e.d.

It is evident that for a fixed potential, only a bounded set of valuations will permit recovery of the tropical amoeba; for a fixed valuation or, actually, *any bounded set* of valuations, any potential that is "large enough" will do. Applying this, for example, to the enumerative problem studied in [**Mi2**, **GM**], we immediately obtain:

Corollary 4.11. *Let $P = \langle [0,0], [d,0], [0,d] \rangle$. For a fixed set S of $3d-1+g$ points in the plane in tropically generic position, there is a set (with non-empty interior) of potentials ψ on P such that the set of tropical curves of genus g through S is in bijective correspondence with the set of limit amoebas of genus g through $\mathcal{L}_\psi^{-1} S$ in P.*

4.3. Implosion of polytopes versus explosion of fans. For simplicity, in the present subsection, we restrict ourselves to the case $\psi(x) = \frac{1}{2}x^2$. We remark that while the map π projects onto P, the map $id - \pi$ is injective in the interior of the cones $v + \mathcal{C}_v$ for all vertices $v \in P$ (regions $1-4$ in figure 6). For a face F of dimension $k > 0$ of P, the region $\breve{F} + \mathcal{C}_p$, for any $p \in \breve{F}$, implodes to the cone $\mathcal{C}_p$ of codimension k. In particular, the polytope P implodes to the origin.

Dually, the map $id - \pi$ explodes the fan along positive codimension cones. In particular, the origin is exploded to P. In Figure 6, we consider the non-generic Laurent polynomial

$$a_1 + a_2 t^{-0.6} x + a_3 t^{-0.4} x^2 + a_4 t^{1.8} \frac{y^2}{x},$$

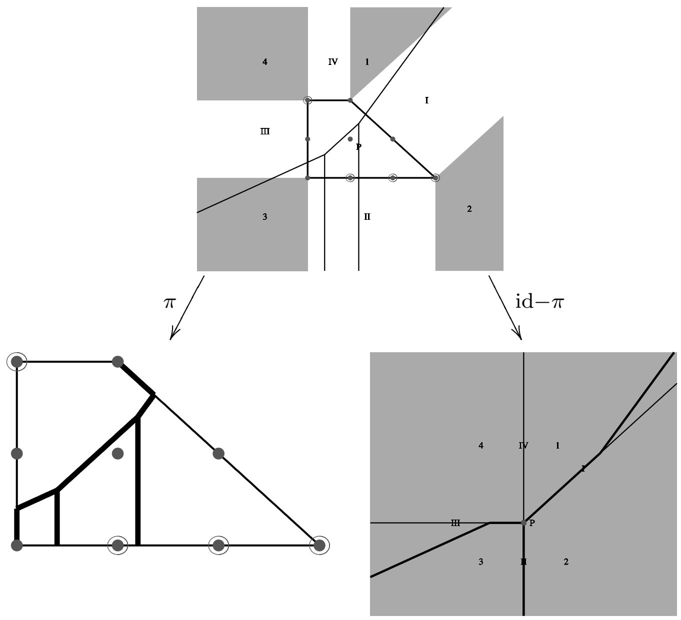


Figure 6. Explosion of the fan and implosion of the polytope. The marked points correspond to non-zero coefficients a_m.

its tropical amoeba for $\mathbb{P}^2$ blown-up at one point and the corresponding images under the maps π and $id - \pi$. Note that, for this non generic polynomial, part of the compact amoeba $\pi\mathcal{A}_{\text{trop}}$ lies in the boundary of P. Only in $\breve{P}$ does $\pi\mathcal{A}_{\text{trop}}$ coincide with the tropical non-Archimedean amoeba.

4.4. Amoebas associated to geometric quantization. As we could observe in Remark 4.9, the behavior of the limit amoeba defined via a fixed valuation is rather unstable. Actually, it does not only depend on the choice of ψ, but also behaves badly under integer translations of the moment polytope P. These shortcomings are somehow overcome by a specific choice of valuation that is associated with a toric variety and a large Kähler structure limit with quadratic ψ. This construction provides the natural link between the convergence of sections to delta distributions considered in Section 3.3 that comes out of geometric quantization, and the consideration of the image of hypersurfaces defined by the zero locus of generic sections.

Geometric quantization motivates considering the hypersurfaces defined by

$$\breve{Y}_s^{\mathrm{GQ}} = \left\{ p \in X_P : \sum_{m \in P \cap \mathbb{Z}^n} a_m \xi_s^m(p) = 0 \right\},$$

where $a_m \in \mathbb{C}^*$, and ξ_s^m are the L^1-normalized holomorphic sections converging to delta distributions, as in Section 3.3. A simple estimate of the order of decay of $\|\sigma_s^m\|_1$ as $s \to \infty$ gives, in the notation of Lemma 3.7,

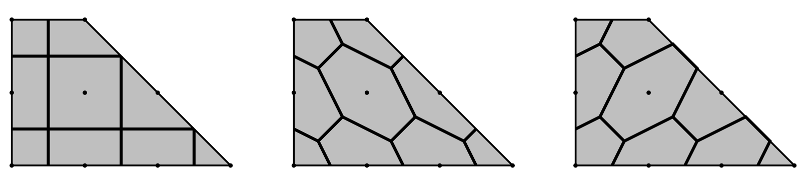
$$\begin{aligned}\|\sigma_s^m\|_1^{-1}\frac{\mathrm{d}}{\mathrm{d}s}\|\sigma_s^m\|_1 &= \|\sigma_s^m\|_1^{-1}\frac{\mathrm{d}}{\mathrm{d}s}\int_P \mathrm{e}^{-h_m^s(x)}\mathrm{d}x \\ &= \|\mathrm{e}^{-h_m^s}\|_1^{-1}\frac{\mathrm{d}}{\mathrm{d}s}\int_P \mathrm{e}^{-h_m^0(x)-sf_m(x)}\mathrm{d}x \\ &= -\int_P f_m(x)\frac{\mathrm{e}^{-h_m^s(x)}}{\|\mathrm{e}^{-h_m^s}\|_1}\mathrm{d}x \to -f_m(m) = \psi(m).\end{aligned}$$

Therefore, we call the limit of the family of amoebas $\mu_P(Y_s)$, where

$$\breve{Y}_s = \left\{ w \in (\mathbb{C}^*)^n : \sum_{m\in P\cap\mathbb{Z}^n} a_m \mathrm{e}^{-s\psi(m)} w^m = 0 \right\},$$

the GQ limit amoeba, $\mathcal{A}_{\mathrm{lim}}^{\mathrm{GQ}}$. The fact that for this choice of valuations, inspired by geometric quantization, the limit amoeba keeps away from integral points in P is consistent with the convergence of the holomorphic sections ξ_s^m to delta distributions supported on the Bohr-Sommerfeld fibers corresponding to those integral points.

The behavior of this "natural" (from the point of view of geometric quantization) choice of valuation is illustrated in Figure 7 for different quadratic ψ. Note, in particular, that in the case G_2 there are parts of the tropical amoeba $\mathcal{A}_{\mathrm{trop}}$ that lie outside $\mathcal{L}_\psi P$ and get projected onto subsets of faces (with non-empty interior in the relative topology). Below, we will give a complete characterization of the GQ limit amoeba in this situation.



$G_0 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ $\quad G_1 = \frac{1}{4}\begin{bmatrix}6 & 3\\ 3 & 6\end{bmatrix}$ $\quad G_2 = G_1^{-1}$

Figure 7. GQ amoebas associated to different quadratic ψ's.

We start by observing that from the point of view of equivalence of toric varietes described by different Delzant polytopes, the GQ limit amoeba is well behaved:

Proposition 4.12. *Let* $\psi \in C^\infty_{Hess>0}(P)$.

a. *For any integer vector* $k \in \mathbb{Z}^n$*, setting* $\widetilde{P} = P + k$ *and* $\widetilde{\psi}(\widetilde{x}) = \psi(\widetilde{x} - k)$*, we have*

$$\widetilde{\mathcal{A}}^{\mathrm{GQ}}_{\mathrm{lim}} = \mathcal{A}^{\mathrm{GQ}}_{\mathrm{lim}} + k,$$

b. *For any base change* $A \in Sl(n, \mathbb{Z})$ *of the lattice* $\mathbb{Z}^n$*, setting* $\widetilde{P} = AP$ *and* $\widetilde{\psi}(\widetilde{x}) = \psi(A^{-1}\widetilde{x})$*, we have*

$$\widetilde{\mathcal{A}}^{\mathrm{GQ}}_{\mathrm{lim}} = A\mathcal{A}^{\mathrm{GQ}}_{\mathrm{lim}}.$$

Proof. a.: since $\widetilde{\psi}(\widetilde{m} = m + k) = \psi(m)$, $\widetilde{\mathcal{A}}_{\mathrm{trop}} = \mathcal{A}_{\mathrm{trop}}$; on the other hand,

$$\frac{\partial \widetilde{\psi}}{\partial x}|_{\widetilde{x}=x+k} = \frac{\partial \psi}{\partial x}|_x$$

thus $\mathcal{L}_\psi P = \mathcal{L}_{\widetilde{\psi}} \widetilde{P}$ and

$$\mathcal{L}_\psi P \cap \mathcal{A}_{\mathrm{trop}} = \mathcal{L}_{\widetilde{\psi}} \widetilde{P} \cap \widetilde{\mathcal{A}}_{\mathrm{trop}}.$$

b.: similarly, since $\widetilde{\psi}(\widetilde{m} = Am) = \psi(m)$ and the tropical amoeba $\widetilde{\mathcal{A}}_{\mathrm{trop}}$ is defined via the functions

$$\widetilde{u} \mapsto {}^t\widetilde{m}\widetilde{u} - \widetilde{\psi}(\widetilde{m}) = {}^t m {}^t A \widetilde{u} - \psi(m)$$

it follows that $\widetilde{\mathcal{A}}_{\mathrm{trop}} = {}^t A^{-1} \mathcal{A}_{\mathrm{trop}}$; for the Legendre transforms one finds

$$\frac{\partial \widetilde{\psi}}{\partial x}|_{\widetilde{x}=Ax} = {}^t A^{-1} \frac{\partial \psi}{\partial x}|_x$$

which proves the second claim. q.e.d.

Actually, as is to be expected from the convergence of the sections defining $\mathcal{A}^{\mathrm{GQ}}_{\mathrm{lim}}$ to delta distributions, these amoebas never intersect lattice points:

Proposition 4.13. *For any strictly convex* $\psi \in C^\infty_{Hess>0}(P)$*, the GQ amoeba* $\mathcal{A}^{\mathrm{GQ}}_{\mathrm{lim}}$ *stays away from lattice points in the interior of* P*, that is,*

$$\mathcal{A}^{\mathrm{GQ}}_{\mathrm{lim}} \cap \breve{P} \cap \mathbb{Z}^n = \emptyset.$$

Proof. We consider the functions that were used to define the tropical amoeba,

$$\eta_m(u) = {}^t m u - \psi(m), \qquad \forall m \in P \cap \mathbb{Z}^n$$

and observe that for $u = u_m = \mathcal{L}_\psi m = \frac{\partial \psi}{\partial x}|_m$ this is the value of the Legendre transform h at u_m,

$$\eta_m(u_m) = h(u_m)$$

where $h(u) = {}^t x(u)u - \psi(x(u))$. To show that

$$\eta_m(u_m) > \eta_{\widetilde{m}}(u_m) \qquad \forall \widetilde{m} \in P \cap \mathbb{Z}^n,\ \widetilde{m} \neq m$$

we use convexity of the Legendre transform $u \mapsto h(u)$, namely

$$\begin{aligned} \eta_m(u_m) = h(u_m) &> h(u_{\widetilde{m}}) + {}^t(u_m - u_{\widetilde{m}}) \frac{\partial h}{\partial u}|_{u_{\widetilde{m}}} \\ &= {}^t\widetilde{m}u_{\widetilde{m}} - \psi(\widetilde{m}) + {}^t(u_m - u_{\widetilde{m}})\widetilde{m} = \eta_{\widetilde{m}}(u_m) \end{aligned}$$

where we used the fact that

$$\frac{\partial h}{\partial u}|_{u_{\widetilde{m}}} = \frac{\partial h}{\partial u}|_{\frac{\partial \psi}{\partial x}|_{\widetilde{m}}} = \widetilde{m}.$$

q.e.d.

When ψ is quadratic it is possible to characterize the GQ limit amoeba completely in terms of the limit metric on P only. Let F_p denote the minimal face containing any given point $p \in \partial P$. Note that for any point $x \in P$,

$$x \in F_p \iff (x - p) \perp_G \mathcal{C}_p^G$$

and, more generally, for any $x \in P$ and $c \in \mathcal{C}_p^G$

$$\|x + c - p\|_G^2 = \|x - p\|_G^2 + \|c\|_G^2 - 2\|x - p\|_G \|c\|_G \cos\alpha \tag{25}$$

where $\alpha = \angle_G(x-p, c) \geq \frac{\pi}{2}$ since ${}^t cG(p-x) \geq 0$ by definition of $\mathcal{C}_p^G(P)$.

Proposition 4.14. *Let $\psi(x) = \frac{{}^t xGx}{2} + {}^t bx$ with ${}^t G = G > 0$; then a point $p \in P$ belongs to $\mathcal{A}_{\mathrm{lim}}^{\mathrm{GQ}}$ if and only if one of the following conditions holds:*

- **a.** *There are (at least) two lattice points $m_1 \neq m_2 \in P \cap \mathbb{Z}^n$ such that*

$$\|p - m_1\|_G = \|p - m_2\|_G = \min_{m \in P \cap \mathbb{Z}^n} \{\|p - m\|_G\}.$$

- **b.** *$p \in \partial P$ and the unique closest lattice point does not lie in the face F_p.*

Remark 4.15. The two conditions are, evidently, mutually exclusive: from the description of the map π it is evident that the inverse image of the intersection of the tropical amoeba $\mathcal{A}_{\mathrm{trop}}$ with $\mathcal{L}_\psi P$ is always a subset of $\mathcal{A}_{\mathrm{lim}}$. This is taken care of by a., while b. describes the parts of the GQ limit amoeba that arise from parts of $\mathcal{A}_{\mathrm{trop}}$ that are "smashed on the boundary" by the convex projection. In particular, for points in the interior condition b. is irrelevant.

Proof. a.: For quadratic ψ and $u = \mathcal{L}_\psi x = Gx + b$, take

$$\begin{aligned} \eta_m(u) &= {}^t m u - \psi(m) = {}^t m u - \left(\frac{{}^t m G m}{2} + {}^t b m\right) \\ &= \left({}^t(\mathcal{L}_\psi m) G^{-1} u - {}^t b G^{-1} u\right) - \frac{1}{2}\left(\|\mathcal{L}_\psi m\|^2_{G^{-1}} - \|b\|^2_{G^{-1}}\right) \\ &= \frac{1}{2}\left(\|u-b\|^2_{G^{-1}} - \|u - \mathcal{L}_\psi m\|^2_{G^{-1}}\right). \end{aligned}$$

Since the first term is independent of m, it is irrelevant for the locus of non-differentiability that defines the tropical amoeba,

$$\begin{aligned} \mathcal{A}_{\mathrm{trop}} &= C^0 - \mathrm{loc}\left(u \mapsto \max_{m \in P\cap\mathbb{Z}^n} \{\eta_m(u)\}\right) \\ &= C^0 - \mathrm{loc}\left(u \mapsto \frac{1}{2}\|u-b\|^2_{G^{-1}} - \min_{m\in P\cap\mathbb{Z}^n}\left\{\frac{1}{2}\|u - \mathcal{L}_\psi m\|^2_{G^{-1}}\right\}\right). \end{aligned}$$

Therefore, u lies in the tropical amoeba if and only if there are two distinct lattice points $m_1 \neq m_2$ in P such that

$$\|u - \mathcal{L}_\psi m_1\|_{G^{-1}} = \|u - \mathcal{L}_\psi m_2\|_{G^{-1}} \iff \|p - m_1\|_G = \|p - m_2\|_G$$

for $u = \mathcal{L}_\psi p \in \mathcal{L}_\psi P$. Taking into account that $\mathcal{L}_\psi \mathcal{A}_{\mathrm{lim}} \supset \mathcal{A}_{\mathrm{trop}} \cap \mathcal{L}_\psi P$, this proves that condition a. is necessary and sufficient for u to belong to this intersection. Thus, either p satisfies a. or if it belongs to $\mathcal{A}_{\mathrm{lim}}$ then it belongs to $\mathcal{A}_{\mathrm{lim}} \setminus \mathcal{A}_{\mathrm{trop}}$.

b.: Fix $p \in \partial P$. First we show that if there is a unique nearest lattice point, say m_p, that lies in the face of p, F_p, then $p \notin \mathcal{A}_{\mathrm{lim}}$. By the definition of $\mathcal{A}_{\mathrm{lim}}$, we have to show that

$$\forall c \in \mathcal{C}^G_p(P): \qquad \mathcal{L}_\psi p + c \notin \mathcal{A}_{\mathrm{trop}},$$

which follows in particular if $\eta_{m_p}(\mathcal{L}_\psi p + c) > \eta_m(\mathcal{L}_\psi p + c)$ for any lattice point $m \neq m_p$. By the reasoning above, this is equivalent to

$$\|\mathcal{L}_\psi p + c - \mathcal{L}_\psi m_p\|^2_{G^{-1}} < \|\mathcal{L}_\psi p + c - \mathcal{L}_\psi m\|^2_{G^{-1}},$$

which follows straight from equation (25) (with $\cos\alpha = 0$).

For the other implication, assume $m_p \notin F_p$. Then,

$$\eta_{m_p}(\mathcal{L}_\psi p) > \eta_m(\mathcal{L}_\psi p), \qquad \forall m \in P \cap \mathbb{Z}^n, m \neq m_p$$

and it suffices to show that for any $m \in F_p \cap \mathbb{Z}^n$ and $c \in \mathcal{C}^G_p(P) \setminus \{0\}$

$$\|\mathcal{L}_\psi p + \tau c - \mathcal{L}_\psi m\|^2_{G^{-1}} < \|\mathcal{L}_\psi p + \tau c - \mathcal{L}_\psi m_p\|^2_{G^{-1}} \tag{26}$$

for some $\tau > 0$ large enough. The left hand side equals

$$\|\mathcal{L}_\psi p + \tau c - \mathcal{L}_\psi m\|^2_{G^{-1}} = \|p - m\|^2_G + \tau^2 \|c\|^2_{G^{-1}}$$

whereas the right hand side gives

$$\|\mathcal{L}_\psi p + \tau c - \mathcal{L}_\psi m_p\|^2_{G^{-1}} = \|p - m_p\|^2_G + \tau^2\|c\|^2_{G^{-1}} - 2\tau\|p - m_p\|_G \|c\|_{G^{-1}} \cos\alpha$$

where $\cos\alpha < 0$. Subtracting therefore the left hand side of inequality (26) from the right hand side, we are left with the expression

$$\begin{aligned} & \|\mathcal{L}_\psi p + \tau c - \mathcal{L}_\psi m_p\|^2_{G^{-1}} - \|\mathcal{L}_\psi p + \tau c - \mathcal{L}_\psi m\|^2_{G^{-1}} \\ = \quad & -2\tau \|p - m_p\|_G \|c\|_{G^{-1}} \cos\alpha \to +\infty, \end{aligned}$$

as $\tau \to \infty$, which finishes the proof. q.e.d.

4.5. Relation to other aspects of degeneration of Kähler structures. Degenerating families of Kähler structures have been studied from a variety of viewpoints. In this section we would like to briefly address aspects of the relation of the present work to some of those.

The first link is to the occurence of torus fibrations in mirror symmetry, following [**SYZ**] (see also [**Au**]). As described in Section 2.3, the Kähler metrics along a geodesic ray $g_P + \varphi + s\psi$ collapse, when rescaled appropriately, to a Hessian metric on the moment polytope P. The metric and/or affine structure the limit amoeba $\mathcal{A}_{\lim}$ inherits for certain combinations of valuation $v(m)$ and direction ψ could be of interest to the SYZ approach to mirror symmetry [**GW**, **KS**], in the case when P is reflexive. Even though the induced metric on the complex hypersurfaces $Y_s \subset X_P$ will not in general be Ricci flat, it is not inconceivable that by carefully choosing the available parameters one might obtain the desired asymptotic behaviour.

Example 4.16. Consider the tropical version of a quartic surface in $\mathbb{P}^3$, with moment polytope given by the tetrahedron

$$P = \langle [-1,-1,-1], [3,-1,-1], [-1,3,-1], [-1,-1,3] \rangle.$$

If we set the valuation to be 1 on these vertices, 0 on the origin, and sufficiently negative on the other lattice points in P, we obtain a tropical amoeba $\mathcal{A}_{\mathrm{trop}}$ whose "nucleus" is a tetrahedron Q with vertices

$$Q := \langle [1,1,1], [-1,0,0], [0,-1,0], [0,0,-1] \rangle.$$

Choosing ψ quadratic corresponding to the matrix

$$\psi \sim \frac{1}{4}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

we obtain a transformed polyhedron $\mathcal{L}_\psi P = -Q$, and the image of the projection $\pi\mathcal{A}_{\mathrm{trop}} \subset \mathcal{L}_\psi P$ is the octahedron O with vertices

$$O := \langle \pm\left[\tfrac{1}{2},\tfrac{1}{2},0\right], \pm\left[\tfrac{1}{2},0,\tfrac{1}{2}\right], \pm\left[0,\tfrac{1}{2},\tfrac{1}{2}\right] \rangle.$$

The situation is depicted in Figure 8. Note that from the construction and the drawing it is clear that the compact amoeba $\pi\mathcal{A}_{\mathrm{trop}}$ inherits an affine structure (from the Legendre transformed coordinates on the moment polyhedron). It is, however, nonsingular even around the vertices.

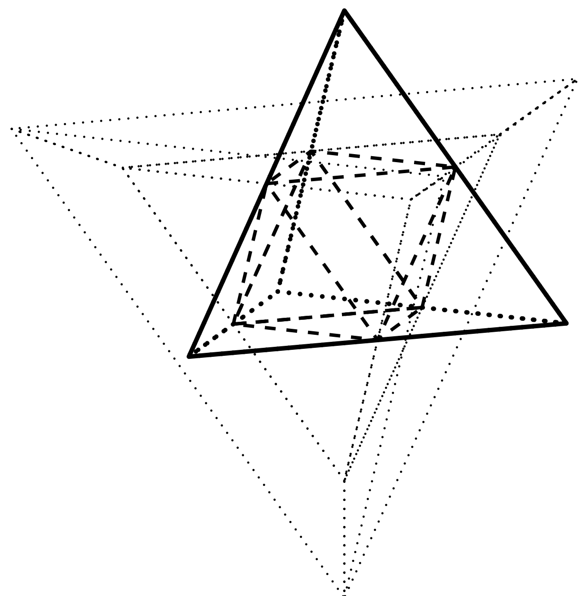


Figure 8. Example of the projection of a tropical amoeba for a quartic surface in $\mathbb{P}^3$, with $\mathcal{A}_{\text{trop}}$ dotted, $\mathcal{L}_\psi P$ solid, and $\pi\mathcal{A}_{\text{trop}}$ dashed.

There exists an entirely analogous example for the quintic in $\mathbb{P}^4$, which however is more difficult to draw; the tropical amoeba's "nucleus" is

$$\begin{aligned} Q &= \left\langle [1,1,1,1], [-1,0,0,0], [0,-1,0,0], [0,0,-1,0]\ [0,0,0,-1] \right\rangle \\ &= -\mathcal{L}_\psi P, \end{aligned}$$

which again is symmetric to the Legendre transformed moment polyhedron. The (projection to the first three dimensions of the) intersection of $\pi\mathcal{A}_{\text{trop}}$ with a facet F of $\mathcal{L}_\psi P$ is shown in Figure 9. The affine structure apparently has singularities along the line segments (twenty, overall) where $\pi\mathcal{A}_{\text{trop}}$ meets the edges of $\mathcal{L}_\psi P$.

In a different context, in [**Pa1**, **Pa2**] Parker considers degenerating families of almost complex structures in an extension of the smooth category constructed using symplectic field theory, to study holomorphic curve invariants. The typical behaviour of his families of almost complex structures, depicted for the moment polytope in the case of $\mathbb{P}^2$ in the introduction of [**Pa1**], is, in the case of toric manifolds, remarkably reproduced in our approach. In fact, in the notation of sections 2 and

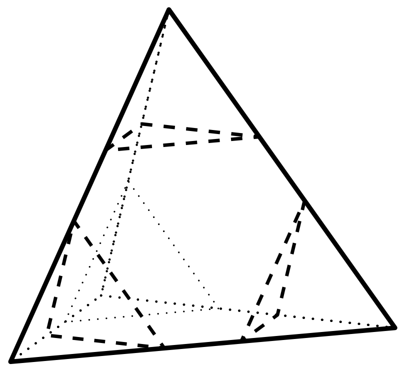


Figure 9. The interior of the dashed polyhedron corresponds to the intersection of $\pi\mathcal{A}_{\text{trop}}$ with a facet F of $\mathcal{L}_\psi P$ for a quintic in $\mathbb{P}^4$.

3.2,

$$\sum_{l,k=1}^{n} (G_0)_{jk}(G_0 + s\text{Hess}\,\psi)^{-1}_{kl}\frac{\partial}{\partial y_l} = \frac{\partial}{\partial y^s_j},$$

where $J_s(\frac{\partial}{\partial\theta_j}) = \frac{\partial}{\partial y^s_j}$. Therefore, in interior regions of P, where as $s \to \infty$ the term with $\text{Hess}\,\psi$ dominates, we have that the coordinates y^s appear stretched relative to the coordinates y by an y-dependent transformation that scales with s. On the other hand, as we approach a face F of P where some coordinates l_j vanish, the derivatives of g_P with respect to these l_j's will dominate, so that the corresponding $y^s_{l_j}$'s do not scale with respect to the y_{l_j}'s. In the directions parallel to F, however, we still have the term with $\text{Hess}\,\psi$ dominating and for these directions the scaling with s will occur. This is exactly the qualitative behaviour described in [**Pa1**]. We note, however, that in our approach this behaviour is implemented by deforming integrable toric complex structures.

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