Toric Mathematics from semigroup viewpoint

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Introduction

Toric geometry is a subject of increasing activity. Toric varieties are objects on which one usually can check explicitly properties and compute invariants from algebraic geometry. This happens for the so-called normal toric varieties, i.e. algebraic varieties which are constructed from rational fans in an euclidean space. In the last 10 years the theory of non normal toric varieties has also been developed providing a very different and new scope as well as interesting and beautiful new applications.

Normal toric geometry mainly uses techniques from convex geometry, as it is technically founded on the concepts of fan and cone. Fans are sets of polyhedral cones in such a way that each cone provides an affine chart of the toric variety. Namely, those charts have, as coordinate algebra, the algebra of the semigroup of lattice points lying inside the dual cone of the corresponding cone of the fan.

To study non normal toric geometry one needs to be more precise than considers only cones. In fact, what one needs is to consider affine charts where coordinate algebras are semigroup ones for more general classes of semigroups. Thus, convex geometry should be used only as a tool by taking into account that nice semigroup generate concrete polyhedral cones.

The purpose of this paper is to show how mathematics in toric geometry can be understood as the theory of appropriate classes of commutative semigroups with given generators. This viewpoint involves the description of various kinds of derived objects as abelian groups and lattices, algebras and binomial ideals, cones and fans, affine and projective algebraic varieties, simplicial and cellular complexes, polytopes, and arithmetics.

Our approach consists in showing the mathematical relations among above objects and clarifying their possibilities for future developments in the area. For that purpose, we will survey some recent results and concrete applications.

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1 Semigroup and generators of toric geometry

The central object we will consider along the paper should be finitely generated cancellative commutative semigroups with a specified system of generators.

For a commutative semigroup we understand here a set S endowed with an internal commutative operation denoted by $+$ having a zero element denoted 0. Semigroup homomorphisms are maps preserving the operation $+$ and the element 0. Thus, one has the category of semigroups.

Cancellative for S means that S is isomorphic to a subsemigroup of an abelian group, or in other words that the semigroup homomorphism $S \to G(S)$, where $G(S)$ is the abelian group generated by S, is injective. Here $G(S)$ is the abelian group of classes of elements pairs $(m, n) \in S \times S$ for the relationship $(m, n) \sim (m', n')$ iff $m + n' = m' + n$.

Thus, our central object should be the data of a semigroup S as above plus a surjective semigroup homomorphism

$$\pi_0 : \mathbf{N}^h \to S,$$

where $\mathbf{N}$ is the semigroup of nonnegative integers. Notice that π_0 is just the same data than the choice of a generator system of the semigroup S, namely the generator system $n_1, \ldots, n_h$, where $n_j = \pi_0(e_j)$, e_j being the h-upla with j-coordinate 1 and other coordinates 0.

Since toric geometry is a subject providing explicit computations and results, one can think that toric mathematics essentially consists in the detailed study of maps π_0 of above type. Along the paper it will be shown how above statement stands when dealing with affine or projective toric objects.

For studying such a map π_0 one needs to understand the structure and behavior of its fibers $\pi_0^{-1}(m)$ for $m \in S$. This is an elementary and difficult problem which, for many purposes, becomes the key problem of toric geometry.

First remark is that one should consider some kind of finiteness hypothesis, namely requiring that the fibers $\pi_0^{-1}(m)$ be finite for every m. The following result gives some distinct characterizations of that hypothesis.

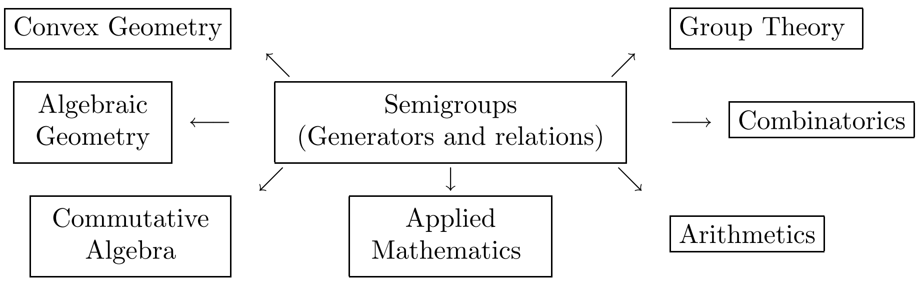
Proposition 1.1 : *(see [4]) Let $\pi_0 : \mathbf{N}^h \to S$ be a surjective semigroup homomorphism where S is a cancellative commutative semigroup. Then, the following conditions are equivalent:*

1. *$\pi_0^{-1}(m)$ is finite for every m.*
2. *There is no infinite sequence $m \in S$, $m_1, \ldots, m_i, \ldots \in S - \{0\}$, such that $m - m_1 - \cdots - m_i \in S$ for every i.*
3. *$S \cap (-S) = \{0\}$.*
4. *There exists a semigroup homomorphism $\lambda : S \to \mathbf{N}$ such that $\lambda(m) = 0$ iff $m = 0$.*

Semigroups satisfying conditions in Proposition 1.1 are called, in the literature, of different forms according as the property one wants to emphasize. Thus, they are said to be combinatorially finite in view of (1) (see [5]), Nakayama in view of (2) (see [18] and [24]), strongly convex in view of (3) (see [11]), or positive in view of (4). The terminology which will be used along this paper is that of "positive".

The description of the fibers is related to the study of relations among the chosen generators of S. Since the "kernel" of π_0 does not exist in the category of semigroups, to describe the relations one needs a different object, the congruence Γ of π_0, to define those relations. The congruence Γ is the binary relation on $\mathbf{N}^h$ consisting of those pairs $(\mathbf{u}, \mathbf{v}) \in \mathbf{N}^h \times \mathbf{N}^h$ such that $\mathbf{u}$, $\mathbf{v}$ belong to the same fiber $\pi_0^{-1}(m)$ for some $m \in S$. Congruences are binary equivalence relations on semigroups allowing to give a semigroup structure on the quotient, i.e. with the property that $(\mathbf{u}, \mathbf{v}) \in \Gamma$ and $\mathbf{w}$ is in the semigroup (i.e. $\mathbf{N}^h$ in our case) then $(\mathbf{u}+\mathbf{w}, \mathbf{v}+\mathbf{w}) \in \Gamma$. Since S is a finitely generated semigroup, by [13, 1.6], one has that the congruence Γ is finitely generated, i.e. that Γ is the least congruence containing one finite set of elements in it. In other words, one can say that S is a finitely presented semigroup.

In the rest of the paper we will show to treat and exploit the information in a semigroup with their generators and relations. This will involve several fields of mathematics on each of which one will derive concrete perspectives and consequences. Figure below shows the scheme of the spirit of our discussions.



2 Abelian groups and lattices

Consider a map $\pi_0 : \mathbf{N}^h \to S$ as in section 1. Since the assignment to a semigroup of the group generated by the semigroup is functorial, one has an induced exact sequence of abelian groups given by

$$0 \to L \to G(\mathbf{N}^h) = \mathbf{Z}^h \to G(S) \to 0,$$

where L is a subgroup of $\mathbf{Z}^h$, so L is finitely generated and torsion free. We will refer to L as the lattice associated to the data π_0. Notice that L is nothing but the kernel of surjective induced group homomorphism $\pi : \mathbf{Z}^h \to G(S)$, so that L is the object keeping the information of the group theoretical relations among the semigroup generators $n_1, \ldots, n_h$.

The relation between the congruence Γ and the lattice L can be described easily. In fact, if $(\mathbf{u}, \mathbf{v}) \in \Gamma$ then there is a unique element $\mathbf{w} \in \mathbf{N}^h$ such that $(\mathbf{u}-\mathbf{w}, \mathbf{v}-\mathbf{w}) \in \Gamma$ and that the supports of $\mathbf{u}-\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ are disjoint. Here for support of an element of $\mathbf{Z}^h$ we mean the set of indices whose coordinates are nonzero for such element. Notice that if $\leq$ denotes the componentwise product ordering on $\mathbf{Z}^h$, then $\mathbf{w}$ is nothing but the infimum of $\mathbf{u}$, $\mathbf{v}$ for that ordering. Thus, one has a well defined map

$$b : \Gamma \to \mathbf{N}^h \times L,$$

given by $b(\mathbf{u}, \mathbf{v}) = (\mathbf{w}, \mathbf{u}-\mathbf{v})$.

If $(\mathbf{w}, \mathbf{l}) \in \mathbf{N}^h \times L$, set $\mathbf{l} = \mathbf{l}^+ - \mathbf{l}^-$, where $\mathbf{l}^+ = sup(\mathbf{l}, \mathbf{0})$, $\mathbf{l}^- = sup(-\mathbf{l}, \mathbf{0})$ and $sup(,)$ stands for the supremum relative to the ordering $\leq$. Then the assignment to the element $(\mathbf{w}, \mathbf{l})$ of the couple $(\mathbf{l}^+ + \mathbf{w}, \mathbf{l}^- + \mathbf{w})$ is a map $\mathbf{N}^h \times L \to \Gamma$ which is, by construction, inverse of b. So one has the following result.

Proposition 2.1 : *The map b is a bijection.*

It follows from Proposition 2.1 that the information in Γ is just the same than in L and how one can get one from the other.

Moreover, from free abelian groups and their sublattices one can study the semigroups we are interested in. In fact, if a lattice $L \subset \mathbf{Z}^h$ is given, then from the obvious exact sequence

$$0 \to L \to \mathbf{Z}^h \to \mathbf{Z}^h/L \to 0,$$

one can consider the subsemigroup S of the group $\mathbf{Z}^h/L$ given by the image of $\mathbf{N}^h$ and generators given by the images of the elements $e_1, \ldots, e_h$. Also notice that the condition on S to be positive is equivalent to the condition $L \cap \mathbf{N}^h = (\mathbf{0})$.

Notice that, in general, the abelian group $G(S) = \mathbf{Z}^h/L$ can have torsion, so the semigroup S can also have torsion in the sense that it can contain elements $m, n \in S$, $m \neq n$ and integers $a \in \mathbf{N}$ such that $am = an$. If T is the torsion subgroup of $G(S)$, the image of S in $G(S)/T$ is a new semigroup $\overline{S}$ of the same kind than S. Notice that S is a positive iff $\overline{S}$ is so. This follows from the fact that $L \cap \mathbf{N}^h = (0)$ iff $\overline{L} \cap \mathbf{N}^h = (0)$, $\overline{L}$ being the lattice for the induced map $\overline{\pi}_0 : \mathbf{N}^h \to \overline{S}$.

Finally, we remark that since S is not only the image of $\mathbf{N}^h$ by π, but S is also the image of others subsets of $\mathbf{Z}^h$, in particular of the set $\mathbf{N}^h + L$. This new set is also a semigroup which has an obvious structure of $\mathbf{N}^h$-module. As semigroup it is not positive (unless the trivial case $L = 0$ for which $S = (0)$), however if S is positive then $\mathbf{N}^h + L$ has the property analogous to (2) in Proposition 1.1, i.e. there is no an infinite sequence of elements $m = m_0 > m_1 > \ldots > m_i > \ldots$ in $\mathbf{N}^h + L$. In other words, if S is positive then $\mathbf{N}^h + L$ is generated by its minimal elements for the ordering $\leq$. Note that such minimal elements are nothing but the primitive elements of the set $\mathbf{N}^h + L$, i.e. those elements which are not a sum of a nonzero element of $\mathbf{N}^h$ with another element of $\mathbf{N}^h + L$.

3 Semigroup ideals and algebras

In this section we will fix a commutative field k. Then, one has a functor from the category of semigroups to that of k-algebras taking each semigroup to its semigroup k-algebra. Notice that, for any semigroup S, the semigroup k-algebra $k[S]$ consists of the vector space generated (freely) by the symbols χ^m, one for each $m \in S$, endowed with a multiplication given on symbols by the rule $\chi^m \cdot \chi^n = \chi^{m+n}$, for $m,n \in S$.

Now, consider a map $\pi_0 : \mathbf{N}^h \to S$ as in section 1, and apply above funtor to it. One gets an exact sequence

$$0 \to I \to A = k[\mathbf{N}^h] \overset{\varphi_0}{\to} R = k[S] \to 0,$$

where I is the kernel of the k-algebra homomorphism φ_0 associated to π_0, which is called the ideal of the semigroup relative to the generators $n_1, \ldots, n_h$. Notice that, if $X_1, \ldots, X_h$ are variables corresponding to the coordinates in $\mathbf{N}^h$, one has a canonical identification $A = k[X_1, \ldots, X_h]$. Moreover, both R and A are graded over the semigroup S (say, S-graded) by giving the obvious degree m to the symbol χ^m and degree n_i to the variable X_i. In particular, one has decomposition into homogeneous components $A = \bigoplus_{m\in S} A_m$, $R = \bigoplus_{m\in S} k\chi^m$. Here A_m is the vector space generated by all the monomials of degree m, i.e. $\mathbf{X}^u = X_1^{u_1} \cdots X_h^{u_h}$ with $\sum_{i=1}^h u_i n_i = m$.

The homomorphism φ becomes S-graded of degree 0 and, therefore, the semigroup ideal is S-homogeneous, i.e. one has $I = \bigoplus_{m\in S} I_m$ with $I_m = I \cap A_m$ for every $m \in S$.

Notice that R is generated, as k-algebra, by the symbols $\chi^{n_1}, \ldots, \chi^{n_h}$, so that I can be understood as the ideal of polynomial relations of such symbols. The ideal I is binomial as it is generated by the binomials $\mathbf{X}^u - \mathbf{X}^v$ for $(\mathbf{u}, \mathbf{v})$ ranging over the congruence Γ. Using Proposition 2.1 one sees that it is also generated by $\mathbf{X}^{l^+} - \mathbf{X}^{l^-}$ where $\mathbf{l}$ ranges over the lattice L. Anyway, notice that to generate I it is enough to take a finite number of binomials $\mathbf{X}^u - \mathbf{X}^v$, where the couples $(\mathbf{u}, \mathbf{v})$ generate the congruence Γ.

Now, assume that S is positive. Then nice properties occur. First, by 3) in Proposition 1.1, one has that the irrelevant $M_R = \bigoplus_{m\neq 0} k\chi^m$ and $M_A = \bigoplus_{m\neq 0} A_m$ are ideals of R and A respectively. Second, by 1) one has that each A_m is a finitely dimensional vector space. Third, by 2), Nakayama lemma holds for S-graded modules; in particular one can speak about minimal systems of homogeneous generators for I which are nothing but those inducing a basis of the vector space $I/M_A I$. It is clear that one can consider minimal sets of binomial generators for I.

In fact, one can consider S-graded free homogeneous resolutions of R as A-module. If S is positive, Nakayama's lemma shows that one can consider the minimal free resolution (which is unique up to isomorphism) which is one of type

$$0 \to F_p \stackrel{\varphi_p}{\to} \cdots \to F_2 \stackrel{\varphi_2}{\to} F_1 \stackrel{\varphi_1}{\to} F_0 = A \stackrel{\varphi_0}{\to} R \to 0,$$

where each F_i is a free S-graded finite A-module, the φ_i are graded of degree 0 and p the projective dimension of R as A-module, i.e. the least integer p such that $F_p \neq 0$. Auslander-Buchsbaum theorem shows the relation $p+r = h$, where r is the depth of R. The integer r ranges on the values $0 \leq r \leq d$, where d is the Krull dimension of R. Notice that the Krull dimension of R coincides with the rank of the abelian group $G(S)$. Last statement follows from the computation of dimensions in terms of trascendence degrees. It implies, in particular that the dimension of the k-algebra $k[S]$ does not depend on the field k. This is not the case for the integer r which could depend on k.

Commutative algebra brings interesting particular cases. First, when $r = d$, the ring $k[S]$ is said to be Cohen-Macaulay. This property depends on S and k but not on the map π_0. If $k[S]$ is Cohen-Macaulay and, moreover, F_p has rank 1 as A-module, then $k[S]$ is said to be Gorenstein. This is a case in which the minimal resolution is self-dual, i.e., by applying the functor $\mathrm{Hom}(-, A)$ and considering the natural grading, the induced exact sequence

$$0 \to \mathrm{Hom}(F_0, A) \to \mathrm{Hom}(F_1, A) \to \cdots \to \mathrm{Hom}(F_p, A) \to \mathrm{Coker}(\varphi_p^t) \to 0,$$

is S-graded isomorphic to the minimal resolution of R. Again Gorensteiness depends on S and k, not on π_0. Finally, $k[S]$ is said to be complete intersection if I can be generated by $h-d$ homogeneous elements (in fact binomials). Equivalently, complete intersection means that the congruence Γ can be generated by $h-d$ pairs. Complete intersection is a property which only depends on S and which implies Gorensteiness.

4 Cones and fans

With assumptions as above, next object one can associate to a semigroup S is the cone $C(S)$ generated by S, i.e. the cone generated by the image of S in the $\mathbf{Q}$-vector space $V_{\mathbf{Q}} := G(S) \otimes_{\mathbf{Z}} \mathbf{Q}$. Since the base ring extension from $\mathbf{Z}$ to $\mathbf{Q}$ kills the torsion, the cone $C(S)$ obviously coincides with that of its image $\overline{S}$ in $G(S)/T$.

If S is not positive, then $C(S)$ is equal to the whole $V_{\mathbf{Q}}$, so it contains a trivial information. Thus, the interesting case turns out to be the case in which S is positive. Note that S is positive if and only if $C(S)$ is a strongly convex cone (i.e., if one has $C(S) \cap -C(S) = 0$). This fact justifies the terminology in (3) section 1.

Now, if one takes into account the generators of S, then one has that the cone $C(S)$ is the rational polyhedral one generated by (i.e. it is the convex hull of) the images in $V_{\mathbf{Q}}$ of the generators $n_1, ..., n_h$. Thus, convex geometry occurs as a useful technique of toric mathematics.

There is a very important case, in which the cone $C(S)$ determines the semigroup S. In fact, a semigroup is said to be normal if it is torsion free and if, moreover, one has $S = C(S) \cap G(S)$. It is well known that S is a normal semigroup if and only if the semigroup k-algebra $k[S]$ is an integrally closed domain, i.e. a normal ring. Hoschter theorem [14] shows that if S is normal then, in fact, $k[S]$ is Cohen-Macaulay.

A trivial example of normal semigroups are the free semigroups, i.e. those which are isomorphic to $\mathbf{N}^t$ for some integer t. In fact, free semigroups are the only ones such that the k-algebra $k[S]$ is a regular ring. The terminology "regular" is coherently used also in convex geometry, being it applied to a cone on $\mathbf{Q}^t$ which is generated by a basis of the lattice $\mathbf{Z}^{\mathbf{t}}$. Notice that a semigroup is free if and only if it is normal and if the cone it generates is regular.

Toric geometry appears initially as the study of normal toric varieties. Thus, the development of normal toric geometry is settled on convex geometry and, therefore, one can says that normal toric mathematics are convex geometry mathematics.

Coming back to the general case, the convex cone $C(S)$ provides a new interesting invariant for a semigroup S, namely the number of edges e of $C(S)$. Comparing with the dimension, one has $e \geq d$, and the equality holds whenever the cone $C(S)$ is simplicial. Thus, semigroups for which $e = d$ will be called simplicial along the paper. Free semigroups are a very special case of simplicial semigroups.

Toric varieties also include non affine ones. Affine toric varieties are nothing but the affine varieties X with coordinate k-algebra equal to a semigroup k-algebra with the assumptions of section 1. General toric varietes are algebraic varieties which can be covered by affine toric varieties with overlapings which are also affine toric.

Normal toric varieties are usually given in terms of convex geometry. The data consists in a fan Φ of rational polyhedral cones in $\mathbf{Q}^n$, i.e, a set $\{\sigma\}_{\sigma \in \Phi}$, where Φ is a finite set, each σ a strongly convex polyhedral cone in $\mathbf{Q}^n$, the faces of each σ in Φ is also in Φ, and the intersection of every couple of two cones in Φ is a common face of both of them. The variety is built in the following way. For each σ in Φ, consider the semigroup S_σ of integer coordinates points which lie inside the dual cone of σ, and let X_σ be the affine toric variety given by S_σ. Then the toric variety X is the join of the affine varieties X_σ, the intersection of any two X_σ, X_τ of those affine charts being the toric variety $X_{\sigma \cap \tau}$.

Thus, for a normal toric variety, the fan Φ not only determines the variety but it represents and exhibits its geometry. In fact, cones in the fan correspond to affine charts in such a way that intersection of cones correspond to the overlapings of the corresponding charts.

For non normal toric varieties one can proceed in a similar way, but taking a further precision on the semigroups. Thus, one needs a fan Φ as above plus, for each cone σ, a subsemigroup S'_σ of S_σ generating the same cone as S_σ and in such a way that the intersection of two charts with respective coordinate algebras $k[S'_\sigma]$ and $k[S'_\tau]$ is the affine chart with coordinate algebra $k[S'_{\sigma\cap\tau}]$. Thus, one sees that, also in the global case, toric mathematics are not only convex geometry but again they involve finitely generated cancellative semigroups.

The support of a fan Φ is defined to be the union of the supports of the cones in the fan. The fan is said to be complete if its support is $\mathbf{Q}^n$. Toric varieties built from complete fans are complete algebraic varieties. Next section is devoted to the particular case of projective varieties, a subclass of complete toric varieties.

5 Affine and projective toric varieties

Toric varieties are algebraic ones, so algebraic geometry is naturally related to toric mathematics. Particularly interesting algebraic varieties are affine and projective ones. When a data $\pi_0 : \mathbf{N}^h \to S$, is given, the semigroup S gives rise to the (abstract) affine toric variety $X = Spec(k[S])$, whereas the choice of generators provided by π_0 gives rise to an embedding of X into the affine space $\mathbf{A}^h$. The dimension of X is just the rank d of the abelian group $G(S)$. Below, we discuss and emphasize how abstract and embedded projective toric varieties can be also described in nice terms.

Let S be a finitely generated cancellative commutative semigroup. Assume that S is endowed with a semigroup map $\lambda : S \to \mathbf{N}$ such that the semigroup is generated by the elements in the set $S_1 = \lambda^{-1}(1)$. Then, for any choice of the field k, the couple (S, λ) gives rise to an (abstract) $(d-1)$-dimensional projective algebraic scheme, namely $Z = Proj(k[S])$, where now $k[S]$ is seen as an $\mathbf{N}$-graded algebra by relaxing its natural S-grading via the map λ (in other words, degree $i \in \mathbf{N}$ homogeneous elements are the sums of homogeneous elements of S-degrees in $\lambda^{-1}(i)$). Along the paper couples (S, λ) as above will be related as polarized semigroups.

For a polarized semigroup (S, λ), one has the property that $m \in S$ is a sum of $i \geq 0$ elements of S_1 if and only if one has $\lambda(m) = i$. This property has two immediate consequences. First, the set S_1 (and hence any fiber $\lambda^{-1}(i)$) is finite, as S_1 is nothing but the set of irreducible elements in S. Second, the semigroup S is, a fortiori, positive. For the last statement, notice that to prove positiveness, when a map λ as above already exists, one only needs to check $\lambda^{-1}(0) = 0$ which follows from the aforesaid property.

Now, assume that S is torsion free. Then, since $k[S]$ is a domain, one has that the projective algebraic scheme Z is, in fact, a projective algebraic variety.

Proposition 5.1 : *Let (S, λ) be a polarized semigroup such that S is torsion free. Then $Z = Proj(k[S])$ is a projective toric variety.*

In fact, since S is torsion free, it can be viewed as a subset of $V_{\mathbf{Q}}$. On the other hand, the map λ extends to a group homomorphism $\lambda_{\mathbf{Z}} : G(S) \to \mathbf{Z}$ and to a $\mathbf{R}$-linear map $\lambda_{\mathbf{R}} : V_{\mathbf{R}} \to \mathbf{R}$, where $V_{\mathbf{R}} = G(S) \otimes_{\mathbf{Z}} \mathbf{R}$. Now, let Ω_1 be the convex hull of the set S_1 in $V_{\mathbf{R}}$, and let $S_1^0 \subset S_1$ the vertex set of S_1. Notice, that S_1^0,S_1 and Ω_1 lie in the affine hyperplane in $V_{\mathbf{R}}$ given by $\lambda_{\mathbf{R}}^{-1}(1)$.

Fix $m^0 \in S_1^0$. Then, the semigroup $S(m^0)$ generated by the set of elements of type $m - m^0$ with $m \in S_1$ is a new positive finitely generated semigroup whose associated group is $\lambda_{\mathbf{Z}}^{-1}(0)$. In particular, it follows that the dimension of the affine toric variety $X(m^0)$ given by $S(m^0)$ is $d-1$ where $d = rank G(S)$, i.e. the dimension of the projective variety Z. Moreover, $X = Spec(k[S])$ being the projecting cone of Z, the construction shows that the affine toric varieties $X(m^0)$, when m^0 ranges over S_1^0, form a covering of Z as affine charts, making of Z a projective toric variety. This shows the proposition.

For projective normal varieties it is possible to describe which Cartier divisors are ample and very ample ones. For a polarization of a projective variety one means to pick a very ample Cartier divisor class. It provides an embedding of the variety in a projective space. When the variety is toric, one sees that the polarization produces a polarized semigroup (S, λ) in such a way that the variety is isomorphic to the one given by the couple (S, λ). See [11] for details.

Thus, it is equivalent to give an embedded projective toric variety than to give a polarized semigroup. Notice that, for a polarized semigroup (S, λ) given, the set S_1 is the set of irreducible elements of S, so it is the only generator set contained in S_1 which gives the embedding of the affine toric variety $X = Spec(k[S])$ which is the projecting cone of Z. Thus, a polarized semigroup provides a canonical embedding of the projective toric variety into $\mathbf{P}^{h-1}$ where h is the cardinality of S_1.

We remark that the fan giving rise to the projective variety Z lies in the dual space of the hyperplane $\lambda_{\mathbf{Q}}^{-1}(0)$. Namely, the cones of the fan are exactly the duals of the cones generated by the semigroups $S(m^0)$. By construction, it is easy to see that the such a fan is a complete one which correspond to the algebraic geometric fact that any projective variety is a complete one.

Finally, as it occurs for affine toric varieties, main algebraic geometric characteristics of projective toric varieties are recognized in terms of the polarized semigroup (S, λ). Thus, $Z = Proj(k[S])$ is said to be arithmetically Cohen-Macaulay (resp. Gorenstein) if and only if the algebra $k[S]$ is Cohen-Macaulay (resp. Gorenstein). In the same way, Z is projectively normal if and only if $k[S]$ is normal, i.e., if the semigroup S is normal. Finally the variety Z is normal (resp. regular) if and only if each semigroup $S(m^0)$ is normal (resp. free).

Notice that to be projectively normal means that $S_i = \overline{S}_i$, where $S_i = \lambda^{-1}(i)$ and $\overline{S}_i = C(S) \cap \lambda^{-1}(i)$, i.e. if every element in $\overline{S}_i$ is a sum of i elements of S_1 for all i's. Normalness can be characterized in rather similar terms, using the Ehrhart and Hilbert functions. Ehrhart (resp. Hilbert) function is the map $E(resp.H) : \mathbf{N} \to \mathbf{N}$ given by $E(i) = card(\overline{S}_i)$ (resp. $H(i) = card(S_i)$), which coincides with a polynomial map of degree $d-1$ with coefficients in $\mathbf{Q}$ for i big enough. Then, under the most general conditions, the leader terms of the polynomials for E and H are equal, and the variety Z is normal exactly when both polynomials are equal. Obviously, in those terms, projective normalness is characterized by the property $E = H$.

6 Polytopes, simplicial and cellular complexes

Once one has an embedded affine or projective toric variety, one looks for describing and computing, when possible, equations and syzygies for the embedding. Most of results in this direction are recent and they use combinatorial objects such as simplicial and cellular complexes or polytopes. Note that, from section 5, the projective case is reduced to the affine one, as for a given polarized semigroup, the equations (and sygygies) of the embedded projective variety it defines are the same than the equations (and syzygies) for its projecting cone affine variety. Such affine variety is nothing but the (affine) toric variety given by the semigroup S of the polarization with S_1 as chosen system of generators.

Along this section, we will assume that a map $\pi_0 : \mathbf{N}^h \to S$, as in section 1, is fixed, and that S is a positive semigroup. Denote by Λ the generator system of S given by π_0, by Π the set of primitive elements of the $\mathbf{N}^h$-module $M = \mathbf{N}^h + L$, and, for every $m \in S$, by Υ_m the set of monomials of S-degree equal to m. Notice that the set Υ_m can be identified to the fiber $\pi_0^{-1}(m)$. Recall that the fact that S is positive implies that M is generated by Π and that each Υ_m is finite. Then, there are several combinatorial objects with vertex set one of Λ,Π or Υ_m which are naturally associated to π_0 as described below.

Associated to any fixed element m in S one has the simplicial complexes Δ_m, Θ_m and the polytope Ω_m defined, respectively as follows. First, Δ_m is the simplicial subcomplex of parts F of Λ such that $m - n_F \in S$, where $n_F = \sum_{n\in F} n$. Second, Θ_m is the simplicial subcomplex of parts G of Υ_m such that all the monomials of G have a non unit greatest common divisor (i.e. those monomials share at least one variable). Third, Ω_m is the polytope in $V_{\mathbf{R}} = \mathbf{Z}^h \otimes_{\mathbf{Z}} \mathbf{R}$ given by the convex hull of the set $\Upsilon_m = \pi_0^{-1}(m)$.

Notice that on the set S one can consider an ordering $\preceq$ defined by $m' \preceq m$ if and only if $m - m' \in S$, and that, if $m' \preceq m$ then one has $\Delta_{m'} \subset \Delta_m$, $\Theta_{m'}$ times a monomial of degree $m - m'$ is a subcomplex of Θ_m and, finally, the translation of $\Omega_{m'}$ by any vector in the fiber $\pi_0^{-1}(m - m')$ is a subset of Ω_m.

Associated to the whole S, one has two useful regular cellular subcomplexes of parts of Π. Namely, on one side one has the so-called Taylor complex Ξ which is nothing but the (simplicial) complex of all parts of Π, and, on the other hand, the hull complex Σ which is the subcomplex whose faces are the subsets of Π which correspond with some unbounded face of the convex hull of the set of points of $V_{\mathbf{R}}$ of type $t^a = (t^{a_1}, ..., t^{a_h})$ for $a = (a_1, ..., a_h) \in M$ where t is any real number big enough. The mentioned correspondence is the obvious one taking into account that any vertex of above convex hull is necessary one of type t^b with $b \in \Pi$. See [2] for details on the construction and properties of the hull cellular complex. Sometimes, a subcomplex of Σ, the so-called Scarf complex is considered. It is, in fact, a simplicial complex which is defined to be the set of parts H of Π satisfying the property $\mathbf{a}_H \neq \mathbf{a}_{H'}$ for every $H' \neq H$ where, $\mathbf{a}_H$ stands for the supremum of the elements in H for the ordering $\leq$ of section 2. The hull and the Scarf complexes coincide when the data π_0 is generic, i.e. when the congruence Γ can be generated by couples $(\mathbf{u}, \mathbf{v})$ such that the unions of the supports of $\mathbf{u}$ and $\mathbf{v}$ is the set $\{1, 2, \ldots, h\}$.

In the sequel, we will often use reduced homology with values in the field k for simplicial and cellular complexes. The corresponding i-th reduced homology vector spaces will be denoted by $\tilde{H}_i$.

The description of equations has to do, in practice, with the determination of sets of binomial generators of the semigroup ideal I (section 3) which are either a minimal set of generators or a Gröbner basis. For each monomial ordering (i.e. a total order on the set of monomials for which the monomial 1 is the minimum and which is closed under multiplication for constant monomials) one has a well defined reduced Gröbner basis with respect to such ordering, which happens to be also generated by binomials (see [22] for details). Thus, each such reduced Gröbner basis can be understood either as a subset of the congruence Γ or of the lattice L (sections 1 and 2). The union of reduced Gröbner bases for all the possible monomial orderings is called the universal Gröbner basis, and it has the property of being, simultaneously, a Gröbner basis for all monomial orderings. Again the universal Gröbner basis can be seen as a subset of Γ or L. A reduced Gröbner basis with respect to a concrete ordering can be computed from any other generator system by means of the well known Buchberger algorithm. The description of the universal Gröbner basis becomes more difficult and it will be stated precisely just below.

To find the universal Gröbner basis, consider the subset U of S consisting of those elements $m \in S$ such that the polytope Ω_m has an edge which is not parallel to some edge of some $\Omega_{m'}$ for some $m' \prec m$. Then, for each $m \in U$ consider the binomials of type $X^{\mathbf{u}} - X^{\mathbf{v}}$, where the coordinates of $\mathbf{u} - \mathbf{v}$ are relatively prime and the segment $[\mathbf{u},\mathbf{v}]$ is an edge of Ω_m. A result by Sturmfels, Weismantel and Ziegler [21] shows that the set of all binomials one obtains in this way when m ranges over U is exactly the universal Gröbner basis of I. Such universal basis is finite as, one can see, that it is contained into the so-called Graver basis which is itself finite. The Graver basis consists of the binomials corresponding to the primitive elements of the lattice L, i.e. those elements $\mathbf{l} = \mathbf{l}^+ - \mathbf{l}^-$ in L for which there are not other $\mathbf{l}' = \mathbf{l}'^+ - \mathbf{l}'^-$ in L such that $\mathbf{l} \neq \mathbf{l}'$ and $\mathbf{l}'^+ \leq \mathbf{l}^+$ and $\mathbf{l}'^- \leq \mathbf{l}^-$.

To find minimal sets of homogeneous generators of I one can proceed as follows. Consider the set C of elements $m \in S$ such that $\tilde{H}_0(\Theta_m) \neq 0$, i.e. those elements for which the complex Θ_m is not connected. The set C is finite. For each $m \in C$ pick a monomial $X^{\mathbf{u}}$ in each connected component of Θ_m and distinguish the monomial $X^{\mathbf{v}}$ picket for one concrete of the components. Then the binomials $X^{\mathbf{u}} - X^{\mathbf{v}}$, where $X^{\mathbf{u}}$ ranges over the picket monomials for the other components, are the degree m terms of a minimal system of homogeneous generators of I. Thus, when m ranges over the set C, the whole set of obtained binomials $X^{\mathbf{u}} - X^{\mathbf{v}}$ is a minimal set of homogeneous generators for the ideal.

A different way to find homogeneous generators for I, which involves the complexes Δ_m, is also available for higher order syzygies, and it will be next discussed in this higher order context. We do not know if such discussion could be also reasonably done in terms of the complexes Θ_m as well as if the complexes Δ_m or Θ_m could be used to describe the universal Gröbner basis.

The description of syzygies consists in obtaining either the minimal S-graded resolution (section 3) or concrete resolutions which other special properties, for example, the property of preserving the symmetries relative to the action of the lattice L.

With notations as in section 3, the i-th order syzygy module is the S-graded module $N_i = ker(\varphi_i)$. Notice that one has $N_0 = I$. For each degree $m \in S$, the number of generators of degree m in any minimal set of generators for N_i is, by Nakayama's lemma, the dimension of the k-vector space $V_i(m) = (N_i)_m/(M_A N_i)_m$. A first and key connection between syzygies and toric geometry is a result due initially to Hoschter, [15], and considered again by several authors in [7], [1], [5], which asserts that one has an explicit and natural vector space identification of type

$$V_i(m) = \tilde{H}_i(\Delta_m),$$

where $\tilde{H}_i$ stands for the reduced simplicial homology with coefficients in the field k. Moreover, computations of direct and inverse images by the isomorphisms given rise to above identification are available.

This result illustrates how combinatorics play a natural role also for describing syzygies, and, therefore, how one has many reasons to include combinatorics among toric mathematics. The first direct applications of above result are given by Briales, Campillo, Marijuán and Pisón in [4] to give an effective algorithmic to compute minimal systems binomial generators of the ideal I.

To apply for $i \geq 1$ above natural isomorphisms, the main difficulties which arise are first to compute those value of m such that $\tilde{H}_i(\Delta_m)$ is nonzero, and, second, to determine the homology. If one is able to avoid these two difficulties in concrete cases, then from the fact that the isomorphisms are explicit one can derive successive methodic constructions of minimal sets of generators for the syzygy modules in the minimal resolution of R (see [5] for details).

To approach the first difficulty, we will mention that, recently, Briales, Pisón and Vigneron [6] ([19] for the case $i = 1$) determine appropriate finite subsets C_i of S with the property that $m \notin C_i$ implies $\tilde{H}_i(\Delta_m) \neq 0$. As a consequence, they obtain an algorithm for computing the minimal resolution (see [6] for details), because the second difficulty is quite well understood from a computational viewpoint, as concrete homologies can be calculated by means of linear algebra and integer linear programming as pointed out in [18], [19] and [6].

However, integer programming being also a technique to which development toric geometry also is contributing (as we will show to the end of the paper), it is convenient to try to better understand the explicit structure of the homologies $\tilde{H}_{\cdot}(\Delta_m)$. This is treated by Campillo and Gimenez in [8]. For it, one considers a partition $\Lambda = \mathcal{E} \cup \mathcal{C}$, where $\mathcal{E}$ is a subset of generators whose image in $V_{\mathbf{Q}}$ generates minimally the cone $C(S)$, in the sense that, for each edge of $C(S)$, $\mathcal{E}$ contains exactly one element whose image generates such edge. Notice, that one has $e = card(\mathcal{E})$, e being nothing but the invariant of S in section 4. Algebraically, one deduces that $k[S]$ becomes a finite extension of $k[\mathcal{E}]$. Thus, the minimal graded resolution of $k[S]$ as A-module can be compared with its minimal resolution as B-module, where, now, $B = k[\mathbf{N}^e]$ corresponds, as in section 1, to the semigroup generated by the set $\mathcal{E}$.

This situation makes in evidence two kind of objects. First, one has the Apery set relative to $\mathcal{E}$, which is nothing but the set Q of elements of $q \in S$ such that $q - n \notin S$ for every $n \in \mathcal{E}$. In other words, the Apery set is nothing but the set of exponents whose corresponding symbols generate minimally $k[S]$ as $k[\mathcal{E}]$-module, and, therefore, it is a finite set. Second, for each $m \in S$ one has the analogous of Δ_m for this relative situation, namely the simplicial subcomplex $\mathcal{T}_m$ of parts J of $\mathcal{E}$ such that $m - n_J \in S$. Thus, one can see that the dimension of $\tilde{H}_i(\mathcal{T}_m)$ is exactly the number of degree m elements in a minimal set of $\mathcal{E}$-homogeneous generators of the $i-th$-order syzygy module in above minimal resolution of the B-module $k[S]$.

Now, for a fixed $m \in S$, one has a key long exact sequence of type

$$\ldots \to H_{i+1}(Q_m) \to K_i \to \tilde{H}_i(\Delta_m) \to H_i(Q_m) \to K_{i-1} \to \ldots$$

where $H_{\cdot}(Q_m)$ and $K_{\cdot}$ are appropriated vector spaces of the following nature. First, $H_{\cdot}(Q_m)$ is the homology of a complex associated to the vertex m of a graph $\mathcal{G}_{\mathcal{Q}}$ with coloured edges constructed from the knowledge of Q, which has $\mathcal{C}$ as colour set and which is called the Apery graph. The vertex set of $\mathcal{G}_{\mathcal{Q}}$ consists of the elements m of type $q + n_I$ where $q \in Q$ and $I \subset \mathcal{C}$. Edges of colour $n \in \mathcal{C}$ join a vertex m' to another m whenever $m - m' = n$. The complex associated to m has as i-th chain space that freely generated by the subsets $I \subset \mathcal{C}$ of cardinality $i + 1$ such that $m - n_I \in Q$, the boundary map being the projection of the usual simplicial boundary. Second, the spaces $K_{\cdot}$ are much more difficult to describe and we avoid the details. However, they can be computed in successive steps in two different and complementary ways. One, in terms of new graphs of exactly the same type than $\mathcal{G}_{\mathcal{Q}}$ but with other concrete sets instead Q. Other in terms of homologies of type $\tilde{H}_{\cdot}({\mathcal{T}_m}')$ where the elements m' are of type $m - n_I$ with $I \subset \mathcal{C}$. See [8] for the details and some applications.

An extra consequence of the construction of the complexes $\mathcal{T}_m$ is that one has a way to characterize the depth r of the ring $k[S]$. Recall that, the three integers r,d and e associated to a positive semigroup are such that $r \leq d \leq e$. The integers d and e are easily recognized from S. To recognize r, in [8] it is proved that if r_0 is a integer with $1 \leq r_0 \leq d$, then one has that the inequality $r \geq r_0$ is equivalent to the fact $\tilde{H}_{e-r_0}(\mathcal{T}_m) = 0$ for every $m \in S$. In particular, for $r_0 = d$ one gets a characterization of Cohen-Macaulayness by the property

$$\tilde{H}_{e-d}(\mathcal{T}_m) = 0$$

for every $m \in S$, which, for the simplicial case $e = d$ means that all the complexes $\mathcal{T}_m$ are connected. From this, it is easy to recover the well known characterization due to Goto [12] with asserts that, for simplicial semigroups, Cohen-Macaulay is equivalent to the property

$$m \in G(S), n, n' \in \mathcal{E}, n \neq n', m + n \in S, m + n' \in S \Rightarrow m \in S.$$

Other characterizations of Cohen-Macaulayness for nonsimplicial cases are given in [23] and [20].

A standard application which illustrates the using of the technique of above long exact sequences is to the case of simplicial Cohen-Macaulay semigroups (i.e. those for which $r = d = e$). Since, in that case, the complexes $\mathcal{T}_m$ are connected, one can deduce that $K_i = 0$ for every i, so that one gets

$$\tilde{H}_i(\Delta_m) = H_i(Q_m)$$

for every m and i. Thus, the minimal resolution for simplicial Cohen-Macaulay semigroups can be derived from a unique combinatorial object, the Apery graph.

For the general case, there are other ways to derive free resolutions for R from a unique combinatorial object. Namely, as shown in [2] this can be done either from the Taylor or from the hull complexes, Ξ and Σ respectively.

Let us explain how it works. For it, consider for each one of above cellular complexes an associated complex of A-modules given as follows. The i-th order chains are the elements of the free A-module generated by the i-dimensional faces of the considered cellular complex, and the boundary map is given on any such a face H by

$$\delta(H) = \sum_{H'} \epsilon(H, H') \frac{\mathbf{a}_H}{\mathbf{a}_{H'}} H'$$

where the sum ranges over all the faces H' of the considered cellular complex, $\epsilon(H, H') \in \{0, 1, -1\}$ denotes the incidence index for the cellular complex, and $\mathbf{a}_H$, $\mathbf{a}'_H$ are the elements defined above. Recall that, from the definition of regular cellular complex, the incidence index satisfies the properties $\epsilon(H, H') = 0$ unless H' be a facet of H, so above sum is extended only to facets of H in the cellular complex.

Because of the properties of the Taylor and hull complexes, one has that what one actually gets are A-module free resolutions of $k[M] = k[\mathbf{N}^h + L]$. Moreover, the resolution, which is $\mathbf{Z}^h$-graded by construction, it is in fact invariant by the action of the lattice L induced from its action on Π. This means, that each one of the chain A-modules is in fact also a free $A[L]$-module, where $A[L]$ is the algebra of the group L on the coefficient ring A. Notice that one has $A[L] = k[\mathbf{N}^h \times L]$, so the first projection $\mathbf{N}^h \times L \to \mathbf{N}^h$, seen as a $\mathbf{N}^h$-module homomorphism, gives rise to a surjective A-linear map $A[L] \to A$. Now, by extending scalars via the map $A[L] \to A$ and taking into account that

$$k[M] \otimes_{A[L]} A = k[S] = R,$$

one gets a complex which is in fact S-graded and exact. Thus, according to the considered cellular complex, one gets two S- graded resolutions of R, which are respectively called the Taylor and the hull resolution.

Both resolutions are, in general, far from being minimal; however, if the data π_0 is generic, then the hull resolution is (isomorphic) to the minimal one. However, they are interesting and useful since they keep the action of the lattice L. Notice that, as commented before, the hull complex is equal to the Scarf complex for the generic case, so, in that situation the Scarf complex can be directly used instead of the hull one for constructing above resolution, which is besides minimal. Nevertheless, we remark that the generic case is combinatorially characterized by the fact that the simplicial complexes Δ_m which are not connected have connected components which are full simplices. This is a strong assumption from the combinatorial point of view, so, in general, if one wants to know about the minimal resolution the only (by the moment) available description is that discussed before based on the study of the simplicial complexes Δ_m.

In the general (non generic) situation the hull resolution has, nevertheless, another nice property, as in fact it is a finite one, i.e. the involved free A-modules are or finite rank and the number of them is finite . This is a non obvious statement which follows from the fact that the Graver basis is finite. See [2] for details.

7 Multinumerical semigroups

In practice, the toric data π_0 (of a semigroup with a signaled generator set) is often given in arithmetical terms. In fact, the group $G(S)$ being finitely generated, it is nothing but, up to isomorphism, one of type

$$\mathbf{Z}^d \times \mathbf{Z}/g_1\mathbf{Z} \times \ldots \times \mathbf{Z}/g_l\mathbf{Z}$$

for convenient integers d,l,$g_1, \ldots, g_l$.

Thus, if such an isomorphism is, a priori, considered, then π_0 becomes equivalent to the specification of the coordinate $(d+l)$-tuples (in above product group) of the generators $n_1, \ldots, n_h$ of S. A semigroup given by such a specification is called a multinumerical semigroup. For the simplest case $d = 1$ and $l = 0$, they are usually refered as numerical semigroups in the literature.

What one would need, therefore, is to study toric varieties within arithmetics from multinumerical semigroups. It means to deduce the behaviour and geometrical properties of those varieties from arithmetical properties of the $(d+l)$-tuples of integers or modular integers given by the semigroup generators.

Such an arithmetical study becomes, nevertheless, difficult and it is an open problem except in rather few cases. The arising difficulties can be explained if one looks to the discussion in above section on how combinatorics are involved in the development of toric geometry. In fact, using objects such as polytopes or simplicial or cellular complexes avoids to deal with delicate relations among numbers.

However, mathematically speaking, once that combinatorial methods have grow up and produce nice results, one can hope to try to interpret them in the framework of arithmetics. This strategy is used in [8] for affine and projective toric curves and in [5] for affine and simplicial projective toric surfaces. For the general case, good computational results dealing with equations are also derived by Vigneron in [18]. To show the possibilities of above strategy, we will discuss, here, such results for curves.

An affine toric curve is given by the numerical semigroup S given by a set Λ of h nonnegative integers. One has $r = d = 1$ and, since the cone $C(S)$ has only one edge, also $e = 1$. Thus, this case is a simplicial Cohen-Macaulay one. Then, pick a partition of Λ in a set $\mathcal{E}$ consisting of any single element $s \in \Lambda$ and as complementary set $\mathcal{C}$ the set of the $h-1$ remaining elements. Now, consider the Apery set Q consisting of those integers $q \in S$ such that $q - s \notin S$, and from it construct the coloured graph $\mathcal{G}_\mathcal{Q}$. It is not difficult to translate the graph structure in arithmetical relations, so that the homologies $\tilde{H}_i(\Delta_m) = H_i(Q_m)$ for the vertices m of $\mathcal{G}_\mathcal{Q}$ can be derived from such relations. One concludes that the minimal resolution for affine toric varieties can be obtained in complete arithmetical terms from the set of generators of the given numerical semigroup. See [8] for details.

A projective toric curve of degree s is given by a subsemigroup of $\mathbf{N}^2$ generated by a set $\Lambda = \mathcal{E} \cup \mathcal{C}$, where $\mathcal{E}$ is the set consisting of the two elements $(s, 0)$ and $(0, s)$ and $\mathcal{C}$ consists of elements $(c_1, s - c_1), \ldots, (c_{h-2}, s - c_{h-2})$ for different values c_i with $0 < c_i < s$. The semigroup S can be polarized by the function λ given by $\lambda(c, c') = (c + c')/s$, so that S defines an embedding of the projective toric curve in $\mathbf{P}^{h-1}$. Notice that one has $d = e = 2$ and that either $r = 2$ or $r = 1$, according as the projective curve be or not be arithmetically Cohen-Macaulay.

Let S_1 be the numerical semigroup generated by $c_1, \ldots, c_{h-2}, s$, and for each $c \in S_1$ denote by $\mu(c)$ the least number of the generators the above generators of S_1 needed the achieve the sum c. Notice, that the function μ satisfies the property $\mu(c) \leq \mu(c - s) + 1$ for every $c \in S$ whenever $c - s \in S$. By translating into arithmetics the methods in [8], in [9] it is shown that the projective toric curve is arithmetically Cohen-Macaulay if and only if one has $\mu(c) = \mu(c-s)+1$ for every $s \in S_1$ such that $c - s \in S$.

In general, from the knowledge of the function μ it is easy to find Apery set Q relative to above partition $\Lambda = \mathcal{E} \cup \mathcal{C}$ as well as the set D consisting of those elements m in S such that $m - (s,0) \in S$, $m - (0,s) \in S$, $m - (s,s) \notin S$. One can consider a coloured graph $\mathcal{G}_{\mathcal{D}}$ in an identical way that $\mathcal{G}_{\mathcal{Q}}$ but replacing Q by D. In [8] it is shown that the vector space K_i in the long exact sequence of the precedent section can be identified to the homology $H_i(D_m)$, where this last homology has also an identical construction than that for the case of the set Q. Thus, one deduces the long exact sequence

$$\ldots \to H_{i+1}(Q_m) \to H_i(D_m) \to \tilde{H}_i(\Delta_m) \to H_i(Q_m) \to \ldots.$$

The involved homogies as well as the images maps in this exact sequence can be given in aritmetical terms from the given data $s, c_1, ..., c_{h-2}$. From here, this is so for the reduced homologies $\tilde{H}_.(\Delta_m)$, therefore, the minimal resolution of the projective toric curve is obtained from arithmetics.

8 Applications

The development of toric geometry has provided applications to many problems in geometry. This is related to the fact that, quite often, toric varieties are objects on which one can determine and describe the main involved ingredients in the considered problems. Applications also occur to some external problems to geometry and algebra, in such a way that, toric geometry is becoming also to be an interesting topic of applied mathematics. Those external applications are mainly related to applied combinatorics or to applied optimization. We will end this paper by illustrating this situation with two examples of current research.

The first one is to the coin exchange problem, a classical problem of applied combinatorics. The approach and results are recently obtained by Campillo and Revilla in the paper [9]. Assume one has a coin system with coins of values $c_1 < c_2 < \ldots < c_{h-1}$. Then, setting $s = c_{h-1}$ one has a projective toric curve Z, namely that of degree s given by the (polarized) subsemigroup of $\mathbf{N}^2$ generated by the elements $(0,s), (c_1, s-c_1), \ldots, (c_{h-1}, s-c_{h-1}) = (s,0)$.

The exchange problem looks for achieving a value c in the semigroup S_1 generated by the coin values in an appropriate way. One wants, in particular, to achieve the value c with the minimum number of coins, namely the integer $\mu(c)$ introduced in above section. The problem can be, therefore, formulated as to give good ways or algorithms to achieve the value c with $\mu(c)$ coins in practice. Comments in above section show how, this problem is mathematically closed to that of the determination of equations and syzygies for projective toric curves.

Usually considered coin systems have a strong property, namely for they the greedy algorithm to achieve the values c with $\mu(c)$ coins works. The greedy algorithm achieves a values $c \in S_1$ by , first, taking the largest coin c_j such that $c_j \leq c$ and, then, restart with the value $c - c_j$ and continue in the same way. If, for every $c \in S_1$, the greedy algorithm uses $\mu(c)$ coins then one says that it works for the system. From the discussion at the end of last section, one deduces that if the greedy algorithm works then Z should be arithmetically Cohen-Macaulay.

From that, one shows how toric geometry brings an interesting new class of coin systems with nice properties, namely the Cohen-Macaulay ones, i.e. those such that the associated projective toric curve Z is arithmetically Cohen-Macaulay. For them, in general, the greedy algorithm to achieve values with minimum number of coins is not available, but one has an alternative new and good algorithm to do so (see [9] for details).

The second application is to integer linear programming, also a classical problem, this time of applied optimization. Integer linear programming is related to multinumerical subsemigroups of $\mathbf{Z}^d$, which, for the sake of simplicity, will be assumed to be positive. Let S be such a subsemigroup and assume that it is generated by the elements $n_1, \ldots, n_h \in \mathbf{Z}^d$. The integer linear programming problem consists in finding the optimal solution with non negative integral coordinates to one of type

$$\sum_{i=1}^{h} x_i n_i = m$$

which minimizes a linear map (the cost map)

$$\rho(x_1, \ldots, x_h) = \sum_{i=1}^{h} x_i \rho_i.$$

Here, the coefficients ρ_i are real numbers in general.

An integer linear program can be seen, therefore, as the specification of type (π_0, ρ) where π_0 is the data of a semigroup and generators as above and ρ the cost function. For each $m \in S$ the solutions of the integer linear programming problem for m are among the elements in the fiber $\pi_0^{-1}(m)$ and, moreover, among the vertices of the polytope Ω_m.

Now, notice that, once one fixes any monomial ordering on the variables $x_1, \ldots, x_h$ (for instance the reverse lexicographic ordering), the cost function gives rise to another monomial ordering by comparing two monomials, first, by the value of ρ on the exponents and, second, in case of equal values of ρ by the previous fixed ordering (i.e. the weighted ordering corresponding to the above one) . Then, one can prove that the reduced Gröbner basis of the ideal I given by π_0 relative to this new ordering provides a minimal test set for the integer programming as described in the sequel.

In fact, the reduced Gröbner basis is generated by binomials, therefore, it can be understood as a subset U_ρ of the lattice L. On the other hand, one has the property that if $\mathbf{x} = (x_1, \ldots, x_h) \in \mathbf{N}^h$ is in a fiber and $\mathbf{l} \in \mathbf{L}$ (i.e. a feasible solution) then $\mathbf{x} - \mathbf{l}$ is again a feasible solution whenever $\mathbf{x} - \mathbf{l} \in \mathbf{N}^h$. Then, the set $\mathbf{U}_\rho$ is a test set for the program as it satisfies the following two conditions. First, if $\mathbf{x}$ is a feasible solution which is not optimal, then there exists $\mathbf{l} \in \mathbf{U}_\rho$ such that $\mathbf{x} - \mathbf{l}$ is also a feasible solution. Second, if $\mathbf{x}$ is an optimal solution to a program, then $\mathbf{x} - \mathbf{l}$ is not a feasible solution for every $\mathbf{l} \in \mathbf{U}_\rho$. The condition on the Gröbner basis to be reduced implies that $\mathbf{U}_\rho$ is minimal among the subsets satisfying above two conditions. Test sets provide nice algorithms, in the obvious way suggested by both conditions, to solve the integer linear programming problem.

Non reduced Gröbner bases provide non minimal test sets. In particular, the set U giving the universal Gröbner basis in section 6, which is finite and the union of all U_ρ for all cost functions, is a test set for all programs when ρ varies, i.e. it is a data which only depends on π_0. See [16] and [17] for details. For algorithms involving cases of non positive semigroups see [3] or consider the *Lawrence lifting* (see for example [22]).

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