



Toric Quasifolds Elisa Prato

Quasifolds are a class of highly singular spaces. They are locally modeled by manifolds modulo the smooth action of countable groups. If all the countable groups happen to be finite, then quasifolds are orbifolds, and if they all happen to be equal to the identity, they are manifolds. They were first introduced in [25] in order to address, from the symplectic viewpoint, the longstanding open problem of extending the classical constructions of toric geometry to convex polytopes that are not rational.

In order to clarify this last statement, let us begin by recalling what it means for a convex polytope to be rational. It is well known that every convex polytope in $(\mathbb{R}^n)^*$ can be written as the bounded intersection of finitely many closed half-spaces:

$$\Delta = \bigcap_{j=1}^d \{ \mu \in (\mathbb{R}^n)^* \mid \langle \mu, X_j \rangle \geq \lambda_j \}, \quad (1)$$

where $X_1, \dots, X_d \in \mathbb{R}^n$, $\lambda_1, \dots, \lambda_d \in \mathbb{R}$, and d is the number of facets (codimension-one faces) of Δ [31, Theorem 1.1]. It is not restrictive to assume that Δ has full dimension n . We remark that the vectors X_1, \dots, X_d are orthogonal to the facets of Δ and inward-pointing. For brevity, we will refer to these vectors as normals for Δ . The polytope is then said to be rational if the normals can be chosen inside of a lattice $L \subset \mathbb{R}^n$. Rationality is a rather restrictive condition, and in fact, many interesting convex polytopes are not rational, for instance the regular pentagon and the regular dodecahedron.

Toric geometry, initiated by Michel Demazure in [12], sets out to associate with each rational convex polytope a beautiful geometric space with special torus symmetries. One of the remarkable consequences of doing so is that the geometry of the space can be used to deduce combinatorial information on the polytope, and conversely. The construction of toric spaces can be done from different geometric perspectives: algebraic [13], complex [1, 10], and symplectic [11].¹

The crucial fact to recall here is that these constructions always rely on the lattice L and on a set of primitive normals in L . Evidently, for nonrational polytopes this setup is missing. The first step in generalizing toric geometry to this case (see [25]) consists in replacing the lattice with a similar

enough object that allows sufficient freedom to contain a set of normals for the polytope. The optimal choice turns out to be a quasilattice Q , namely the \mathbb{Z} -span of a set of \mathbb{R} -spanning vectors of \mathbb{R}^n .

In the case of the regular pentagon, for example, one considers the \mathbb{Z} -span of the fifth roots of unity (see Figure 1). We thus have a new framework given by the triple

$$(\Delta, Q, \text{normals in } Q), \quad (2)$$

and once this has been fixed, the standard toric constructions can be extended. For polytopes that are simple (meaning that each vertex is the intersection of exactly n facets), these constructions give rise to what we call toric quasifolds. This was done first in the symplectic category [25], and then, jointly with Fiammetta Battaglia, in the complex/Kähler category [5]. The torus symmetries of the rational case are replaced by the symmetries of a quasitorus: it is the abelian group \mathbb{R}^n/Q , which is itself a quasifold. Though not Hausdorff in general, toric quasifolds have beautiful atlases that generalize the standard toric atlases of the rational case: each chart is the quotient of an open subset of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ modulo the smooth action of a countable subgroup of the standard torus $\mathbb{R}^n/\mathbb{Z}^n$.

Battaglia has extended both the symplectic and complex/Kähler constructions to completely general convex polytopes, no longer necessarily simple; the resulting toric spaces are even more singular, but they turn out to be stratified by toric quasifolds [3, 4].

It is interesting that quasilattices are also crucial in the theories of aperiodic tilings (see [22] and [28, Chapter 2]). The pentagonal quasilattice above, for example, arises in relation to Penrose tilings.

It is our goal here to illustrate toric quasifolds, and their atlases, by describing a number of examples. We do so in the symplectic category, but of course everything can be reformulated in the complex one. We begin with a 2-dimensional example that displays all of the main characteristics of quasifolds: the quasisphere. We pass on to considering examples of dimensions 4 and 6 that came about by exploring the natural connection with Penrose and Ammann tilings. We then briefly address the toric spaces corresponding to the regular convex polyhedra. We conclude with a number of considerations.

For the formal definition of quasifold, we refer the reader to [25, 5]. The complex and symplectic

¹The starting point in the algebraic and complex category is actually, more generally, a fan instead of a polytope, but the basic idea that follows applies verbatim (see [6]).

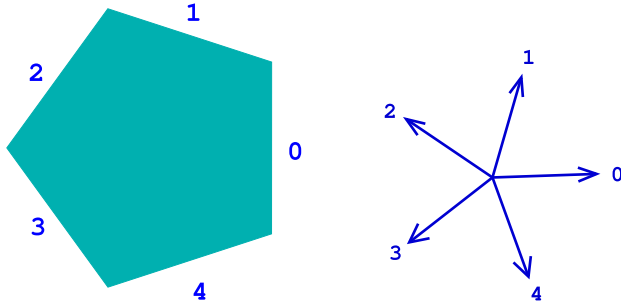


Figure 1. The regular pentagon and the fifth roots of unity.

atlases for toric quasifolds are explicitly described in [5, proofs of Theorems 2.2 and 3.2].

From Sphere to Orbisphere to Quasisphere

Quasispheres, introduced in [25], are generalizations of spheres and orbispheres, so we will begin by recalling some relevant facts about those two objects.

For a positive real number r , let $B(r) \subset \mathbb{C}$ be the open ball with center the origin and radius \sqrt{r} . Consider, for every positive real number α , the group

$$\Gamma_\alpha = \{ e^{2\pi i k \alpha} \in S^1 \mid k \in \mathbb{Z} \}.$$

Notice that Γ_α is the identity when α is an integer, it is finite for α rational, and it is countable for α irrational. The group Γ_α acts on the ball $B(r)$ by complex multiplication. For $z \in B(r)$, we will denote by $[z] \in B(r)/\Gamma_\alpha$ the corresponding orbit.

The Sphere

Let us write the 2- and 3-dimensional unit spheres as follows:

$$S^2 = \{ (z, x) \in \mathbb{C} \times \mathbb{R} \mid |z|^2 + x^2 = 1 \},$$

$$S^3 = \{ (z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1 \}.$$

The surjective mapping

$$f : S^3 \longrightarrow S^2,$$

$$(z, w) \longmapsto (2z\bar{w}, |z|^2 - |w|^2),$$

is known as the Hopf fibration. It is easily seen that the fibers of this mapping are given by the orbits of the circle group

$$S^1 = \{ e^{2\pi i \theta} \mid \theta \in \mathbb{R} \}$$

acting on S^3 by complex multiplication as follows:

$$e^{2\pi i \theta} \cdot (z, w) = (e^{2\pi i \theta} z, e^{2\pi i \theta} w).$$

Therefore, S^2 can be identified with the space of orbits S^3/S^1 . Notice that the S^1 -orbits through the points $(0, 1)$ and $(1, 0)$ of S^3 correspond, respectively, to the south pole, $S = (0, -1)$, and north pole, $N = (0, 1)$, of S^2 .

For each $(z, w) \in S^3$, we denote by $[z : w] \in S^3/S^1 \simeq S^2$ the corresponding orbit. Let us describe the standard atlas of S^2 . Consider the covering given by the open subsets

$$U_S = \{ [z : w] \in S^2 \mid w \neq 0 \},$$

$$U_N = \{ [z : w] \in S^2 \mid z \neq 0 \}.$$

As the notation suggests, the first of these subsets is a neighborhood of the south pole $S = [0 : 1]$, while the second is a neighborhood of the north pole $N = [1 : 0]$. Finally, we have the homeomorphisms

$$B(1) \longrightarrow U_S, \quad z \longmapsto [z : \sqrt{1 - |z|^2}],$$

and

$$B(1) \longrightarrow U_N, \quad w \longmapsto [\sqrt{1 - |w|^2} : w].$$

The Orbisphere

This simple quotient construction can be extended to the orbifold setting as follows. Let p, q be two relatively prime positive integers and consider the 3-dimensional ellipsoid

$$S_{p,q}^3 = \{ (z, w) \in \mathbb{C}^2 \mid p|z|^2 + q|w|^2 = p \}.$$

The circle group S^1 acts on $S_{p,q}^3$ as follows:

$$e^{2\pi i \theta} \cdot (z, w) = (e^{2\pi i p \theta} z, e^{2\pi i q \theta} w). \quad (3)$$

Taking the space of orbits in this case yields the 2-dimensional orbifold $S_{p,q}^2 = S_{p,q}^3/S^1$, called the orbisphere. It admits the two singular points $S = [0 : \sqrt{p}]$ and $N = [\sqrt{q} : 0]$.

Similarly to what we have done for the sphere, for each $(z, w) \in S_{p,q}^3$, we denote by $[z : w] \in S_{p,q}^2$ the corresponding orbit. We then consider the covering given by the two open subsets

$$U_S = \{ [z : w] \in S_{p,q}^2 \mid w \neq 0 \},$$

$$U_N = \{ [z : w] \in S_{p,q}^2 \mid z \neq 0 \}.$$

The first is a neighborhood of the point $S = [0 : \sqrt{p}]$, while the second is a neighborhood of the point $N = [\sqrt{q} : 0]$. The mappings

$$B(q)/\Gamma_{\frac{1}{q}} \longrightarrow U_S, \quad [z] \longmapsto \left[z : \sqrt{p - \frac{p}{q}|z|^2} \right],$$

and

$$B(p)/\Gamma_{\frac{1}{p}} \longrightarrow U_N, \quad [w] \longmapsto \left[\sqrt{q - \frac{q}{p}|w|^2} : w \right],$$

are homeomorphisms, turning U_S and U_N into orbifold charts.

The Quasisphere

We now extend the construction even further. Let s, t be two positive real numbers with $s/t \notin \mathbb{Q}$ and consider the 3-dimensional ellipsoid

$$S_{s,t}^3 = \{ (z, w) \in \mathbb{C}^2 \mid s|z|^2 + t|w|^2 = st \}.$$

Simply substituting s, t for p, q in (3) does not define an S^1 -action on $S_{s,t}^3$. In fact, if you replace θ by $\theta + h$, where h is a nonzero integer, you get $e^{2\pi i(\theta+h)} = e^{2\pi i\theta}$, but $(e^{2\pi i s(\theta+h)}, e^{2\pi i t(\theta+h)}) \neq (e^{2\pi i s\theta}, e^{2\pi i t\theta})$. The idea is to consider the irrational wrap on the standard two-torus instead:

$$N = \{ (e^{2\pi i s\theta}, e^{2\pi i t\theta}) \in \mathbb{R}^2/\mathbb{Z}^2 \mid \theta \in \mathbb{R} \}.$$

The standard action of N on $S_{s,t}^3$ is now well defined, and we take our quasisphere to be the space of orbits $S_{s,t}^2 = S_{s,t}^3/N$. This quotient is the simplest example of a quasifold. It is wilder than the sphere and orbisphere in that it is not a Hausdorff topological space. However, quasisphere charts are a straightforward and very natural generalization of the standard sphere and orbisphere charts.

Exactly as done above, for each $(z, w) \in S_{s,t}^3$, we denote by $[z : w]$ the corresponding orbit. We then consider the covering of $S_{s,t}^2$ given by the open subsets

$$U_S = \left\{ [z : w] \in S_{s,t}^2 \mid w \neq 0 \right\},$$

$$U_N = \left\{ [z : w] \in S_{s,t}^2 \mid z \neq 0 \right\}.$$

The first is a neighborhood of the point $S = [0 : \sqrt{s}]$, while the second is a neighborhood of the point $N = [\sqrt{t} : 0]$. They are each homeomorphic to the quotient of an open subset of \mathbb{C} modulo the action of a countable group. In fact, the mappings

$$B(t)/\Gamma_{s/t} \longrightarrow U_S, \quad [z] \longmapsto \left[z : \sqrt{s - \frac{s}{t}|z|^2} \right],$$

and

$$B(s)/\Gamma_{t/s} \longrightarrow U_N, \quad [w] \longmapsto \left[\sqrt{t - \frac{t}{s}|w|^2} : w \right],$$

are homeomorphisms.

Remark 1. The sphere is the symplectic toric manifold corresponding to the unit interval with lattice \mathbb{Z} and primitive normals $X_1 = 1, X_2 = -1$. The orbisphere, on the other hand,

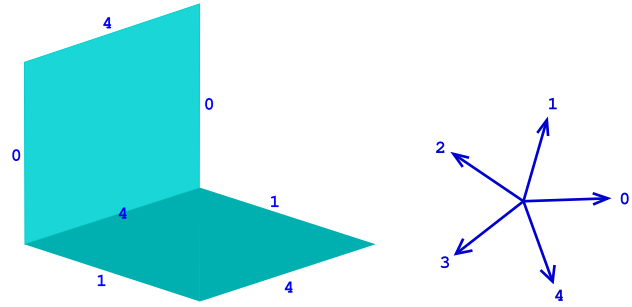


Figure 2. The Penrose rhombuses.

is the symplectic toric orbifold corresponding to the same interval with same lattice and normals $X_1 = q, X_2 = -p$. Finally, the quasisphere is the symplectic toric quasifold corresponding to the same interval with quasilattice $Q = s\mathbb{Z} + t\mathbb{Z}$ and normals $X_1 = t, X_2 = -s$. Wanting to consider a rational polytope, such as the unit interval, in a nonrational setting may seem strange at first sight, but in fact it is quite useful. We will see other instances of this in the next section. Also, the sphere and orbisphere provide the simplest examples showing that the same polytope and (quasi)lattice yield different symplectic toric spaces if the normals are changed. The choice of normals within the same quasilattice is in fact totally free, but sometimes a natural choice is dictated by the context. This is actually the case for all of the examples that follow.

Quasifolds and Aperiodic Tilings

Quasifolds Corresponding to Penrose and Ammann Tilings

The fact that quasilattices appear naturally in aperiodic tilings led us to explore, jointly with Battaglia, the connection between toric quasifolds and Penrose and Ammann tilings.

Penrose rhombus tilings are aperiodic tilings that are composed of two different types of rhombuses, thick and thin [24]. These rhombuses are simple convex polytopes, and it is natural to want to compute the corresponding toric quasifolds. The normals of each rhombus taken separately actually span a lattice, so each of them is rational in its own right. However, if we want to treat simultaneously all of the rhombuses of a given tiling, we need to consider a quasilattice. The natural choice here is the pentagonal quasilattice that we introduced earlier, with normals the relevant fifth roots of unity (see Figure 2). The generalized toric construction then yields a pair of four-dimensional toric quasifolds, one for each type of rhombus. They are both given by a quotient of the type $(S^2(r) \times S^2(r))/\Gamma$, where $S^2(r)$ denotes the 2-sphere of radius r and Γ is a countable subgroup of the standard 2-torus. The radius r is $\left(\frac{1}{2}\sqrt{2+\phi}\right)^{1/2}$ for the thick rhombus and $\left(\frac{1}{2\phi}\sqrt{2+\phi}\right)^{1/2}$ for the thin one, where

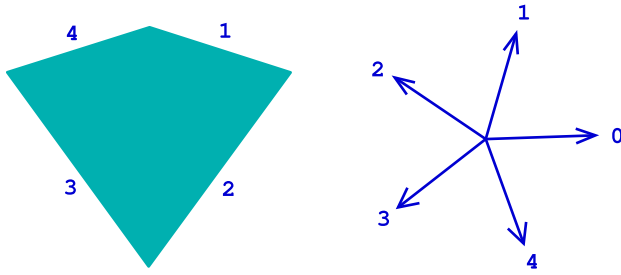


Figure 3. The Penrose kite.

$\phi = \frac{1+\sqrt{5}}{2}$ is the *golden ratio*. The two quasifolds are diffeomorphic but not symplectomorphic.

Something analogous happens for the three-dimensional generalization of this tiling due to Ammann, which is composed of two different types of rhombohedrons, prolate and oblate [29]. Again, each rhombohedron is rational, but to treat all of them simultaneously, we need to consider a quasilattice, known in crystallography as the face-centered icosahedral lattice. As normals we choose the relevant generators. One then obtains a pair of six-dimensional symplectic toric quasifolds, one for each type of rhombohedron. Similarly to what happens with the rhombus tiling, they are given by $(S^2(r) \times S^2(r) \times S^2(r))/\Gamma$, where Γ is a countable subgroup of the standard 3-torus. The radius r here is $[2\phi^2(3-\phi)]^{-1/4}$ for the oblate rhombohedron and $[2(3-\phi)]^{-1/4}$ for the prolate one. Again, the two spaces here are diffeomorphic but not symplectomorphic.

As we have seen, the quasifolds for both Penrose rhombus tilings and Ammann tilings are global, namely the quotient of a manifold modulo the action of a countable group.

Something entirely different happens with the kite-and-dart tiling [24]. First of all, the only tile here that is convex, and therefore relevant to our discussion, is the kite. Moreover, the kite, unlike the rhombuses and rhombohedrons, is actually nonrational. So there is no choice but to consider a quasilattice, and the natural one happens to be, again, the pentagonal quasilattice; the normals are, up to sign, the relevant fifth roots of unity (see Figure 3). Then the resulting toric quasifold is not global. It is the four-dimensional quasifold given by

$$M = \left\{ (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \mid \phi|z_1|^2 + |z_2|^2 + \phi|z_3|^2 = \phi|z_2|^2 + |z_3|^2 + \phi|z_4|^2 = \phi \right\} / \left\{ \exp(-s + \phi t, s, t, -t + \phi s) \in \mathbb{R}^4 / \mathbb{Z}^4 \mid s, t \in \mathbb{R} \right\}.$$

Let us describe one of its charts. Consider the open subset $\tilde{U} = \{ (z_2, z_3) \in \mathbb{C}^2 \mid |z_2|^2 + \phi|z_3|^2 < \phi, \phi|z_2|^2 + |z_3|^2 < \phi \}$ and the countable group

$$\Gamma = \left\{ (e^{2\pi i \phi h}, e^{2\pi i \phi k}) \in \mathbb{R}^2 / \mathbb{Z}^2 \mid h, k \in \mathbb{Z} \right\}.$$

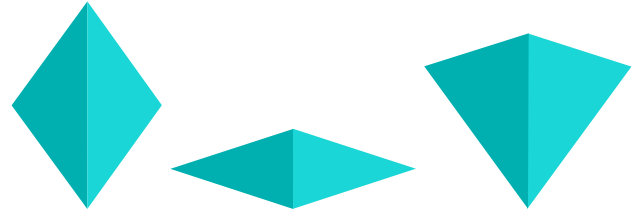


Figure 4. Cutting Penrose tiles.

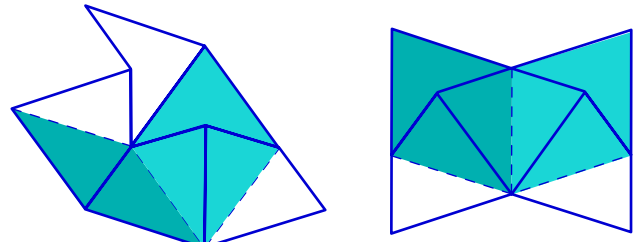


Figure 5. From a kite-and-dart tiling to a rhombus tiling.

Then the mapping

$$\begin{aligned} \tilde{U}/\Gamma &\longrightarrow \{ [z_1 : z_2 : z_3 : z_4] \in M \mid z_1 \neq 0, z_4 \neq 0 \}, \\ [z_2 : z_3] &\longmapsto \left[\sqrt{\phi - |z_2|^2 - \phi|z_3|^2} : z_2 : z_3 : \sqrt{\phi - \phi|z_2|^2 - |z_3|^2} \right], \end{aligned}$$

is a homeomorphism.

Decomposing Penrose Tiles and Symplectic Cutting

Decomposing Penrose tiles by cutting them in half, yielding isosceles triangles as in Figure 4, is a very simple geometric operation that has important repercussions.

First of all, it is the first step in both the inflation and deflation procedures. In the case of inflation, the triangles are appropriately combined to form a new tiling, whose tiles are rescaled by a factor ϕ . In the case of deflation, the triangles are further decomposed into smaller ones to yield the half-tiles of another tiling that is rescaled by a factor $1/\phi$. It is easy to see that these operations are inverses of each other. We refer the reader to [2] for a detailed description in the case of rhombus tilings.

Cutting kites in half can also be used to transform a kite-and-dart tiling into a rhombus tiling. The triangles are appropriately combined with each other and possibly a dart to form thick and thin rhombuses (see [28] and Figure 5).

The process of subdividing a simple convex polytope into two smaller ones corresponds, at the (smooth) symplectic level, to the symplectic cutting operation, which was introduced by Lerman [20]. In the toric setting, the original manifold decomposes into two new ones, each corresponding to one of the subdivided polytopes. The decomposition of Penrose tiles motivated us to extend this

operation to the simple nonrational toric case. We find, for example, that the toric quasifold corresponding to each half-kite is given by

$$\frac{\{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid |z_1|^2 + \phi|z_2|^2 + \phi|z_3|^2 = 1\}}{\{(e^{2\pi i s}, e^{2\pi i \phi s}, e^{2\pi i \phi(s+k)}) \in \mathbb{R}^3 / \mathbb{Z}^3 \mid s \in \mathbb{R}, k \in \mathbb{Z}\}}.$$

The Regular Convex Polyhedra

The regular convex polyhedra are notable examples of convex polytopes, and it is thus only natural to want to understand what the corresponding toric spaces look like. The cube and the regular tetrahedron are rational and simple, and they yield smooth manifolds given, respectively, by $S^2 \times S^2 \times S^2$ and $\mathbb{C}\mathbb{P}^3$. The other three each present their complexities. The regular octahedron is rational but not simple, the regular dodecahedron is simple but not rational, while the regular icosahedron is neither rational nor simple. The first yields a space that is stratified by manifolds, the second yields a quasifold, while the third yields a space that is stratified by quasifolds; they are described explicitly in joint work with Battaglia. The quasilattice for the dodecahedron is known in physics as the simple icosahedral lattice, while the one for the icosahedron is known as the body-centered icosahedral lattice. Here, too, the normals are chosen among the quasilattice generators.

Final Considerations

Quasifolds, Aperiodic Tilings, and Quasicrystals

As we have shown, a number of interesting examples of toric quasifolds arise in connection with aperiodic tilings. There also appears to be a correspondence between some of the fundamental operations in the two theories. We have seen, in fact, that decomposing convex Penrose tiles into halves corresponds to cutting the associated symplectic toric quasifolds. We expect, moreover, that recombining these half-tiles, as needed for the inflation and deflation procedures, will correspond to a nonrational generalization of the inverse operation of symplectic cutting, which is given, in the smooth case, by the symplectic sum [14]. We believe that it would be interesting to pursue the study of these connections even further. As a matter of fact, certain aperiodic tilings have been used as mathematical models for the theory of quasicrystals [28]; these are special materials that were experimentally discovered by Dan Shechtman et al. [27] that have discrete nonperiodic diffraction patterns. Indeed, the very existence of these materials had been predicted by Alan Mackay in connection with his studies of Penrose and Ammann tilings [22, 23]. Ultimately, it is quite possible that toric quasifolds might contribute to their theoretical understanding. A first step would consist in analyzing from the toric viewpoint other tilings (and their operations) that are relevant in this respect. Significant (though not the only) examples would be Socolar's octagonal and dodecagonal tilings, which are used as a basis for a treatment of the elasticity of octagonal and dodecagonal quasicrystals [30].

Combinatorial Equivalence in Toric Geometry

By slightly perturbing the hyperplanes in (1), it can be shown that every simple or simplicial polytope can be perturbed to a rational one that is combinatorially equivalent [31, Proposition 2.17]. In the simple case, these perturbations yield, at the toric level, interesting families of quasifolds. For example, jointly with Battaglia and Zaffran, we have used one such perturbation to construct a one-parameter family of toric quasifolds that generalize and contain Hirzebruch surfaces. This perturbation property also holds true for every three-dimensional polytope, not necessarily simple or simplicial [31, Corollary 4.8]. In higher dimensions, there are examples of polytopes for which this does not happen. The first was found by Perles in the 1960s and has dimension 8 (see [31, Example 6.21] and [32]). As we have seen, from the toric viewpoint, these polytopes, being necessarily nonsimple, yield spaces that are stratified by quasifolds. We believe it would be interesting to study these stratified spaces and understand how their geometry is affected by the fact that the corresponding polytopes cannot be deformed to rational ones within their combinatorial class.

Recent Alternative Approaches to Nonrational Toric Geometry

In recent years, there has been a surge of interest in nonrational toric geometry, and several alternative approaches to this subject have been introduced. It should be said, first of all, that toric quasifolds can be thought of as examples of both stacks and diffeological spaces. The stack approach to nonrational toric geometry was espoused first by Hoffman–Sjamaar [16, 15] and then by Katzarkov et al. [19]. Diffeological quasifolds, on the other hand, were studied jointly with Iglesias–Zemmour in [17], providing an explicit link to noncommutative geometry [9]; applications of this viewpoint to the toric setting are a work in progress. Other recent points of view, due to Battaglia–Zaffran [7, 8], Lin–Sjamaar [21], Ratiu–Zung [26], and Ishida et al. [18], involve foliations of smooth manifolds, either in the complex or presymplectic setting. We would like to point out that most of the alternative nonrational toric viewpoints are founded on variations on the theme of the fundamental triple (2), beginning, first and foremost, with the quasilattice Q . In joint work with Battaglia [6], we describe in detail how many of these different perspectives connect with each other and with ours; a dictionary is provided, in the hope that it will offer clarity and facilitate future interaction in the field.

Acknowledgments

This work was partially supported by the PRIN Project “Real and Complex Manifolds: Topology, Geometry and Holomorphic Dynamics” (MIUR, Italy) and by GNSAGA (INdAM, Italy).

Funding Open access funding provided by Università degli Studi di Firenze within the CRUI-CARE Agreement.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

- [1] M. Audin. *The Topology of Torus Actions on Symplectic Manifolds*, Progress in Mathematics vol. 93. Birkhäuser, 1991.
- [2] D. Austin. Penrose tilings tied up in ribbons. Available at <http://www.ams.org/featurecolumn/archive/ribbons.html> (accessed March 28, 2022).
- [3] F. Battaglia. Convex polytopes and quasilattices from the symplectic viewpoint. *Comm. Math. Phys.* 269 (2007), 283–310.
- [4] F. Battaglia. Geometric spaces from arbitrary convex polytopes. *Internat. J. Math.* 23 (2012), 39 pp.
- [5] F. Battaglia and E. Prato. Generalized toric varieties for simple nonrational convex polytopes. *Int. Math. Res. Not.* 24 (2001), 1315–1337.
- [6] F. Battaglia and E. Prato. Nonrational polytopes and fans in toric geometry. [arXiv:2205.00417](https://arxiv.org/abs/2205.00417), 2022, 17 pp.
- [7] F. Battaglia and D. Zaffran. Foliations modeling nonrational simplicial toric varieties. *Int. Math. Res. Not.* 2015, 11785–11815.
- [8] F. Battaglia and D. Zaffran. Simplicial toric varieties as leaf spaces. In *Special Metrics and Group Actions in Geometry*, Springer INdAM Ser. 23, 21 pages. Springer, 2017.
- [9] A. Connes. *Noncommutative Geometry*. Acad. Press, 1994.
- [10] D. Cox, J. Little, and H. Schenck. *Toric Varieties*, Graduate Studies in Mathematics 124. American Mathematical Society, 2011.
- [11] T. Delzant. Hamiltoniens périodiques et images convexes de l'application moment. *Bull. Soc. Math. France* 116 (1988), 315–339.
- [12] M. Demazure. Sous-groupes algébriques de rang maximum du groupe de Cremona. *Ann. Sci. Éc. Norm. Supér.* 3 (1970), 507–588.
- [13] W. Fulton. *Introduction to Toric Varieties*. Princeton University Press, 1993.
- [14] R. Gompf. A new construction of symplectic manifolds. *Ann. Math.* 142 (1995), 527–595.
- [15] B. Hoffman. Toric symplectic stacks. *Adv. Math.* 368 (2020), 43 pp.
- [16] B. Hoffman and R. Sjamaar. Stacky Hamiltonian actions and symplectic reduction. *Int. Math. Res. Not.* (2020), 15209–15300.
- [17] P. Iglesias-Zemmour and E. Prato. Quasifolds, diffeology and noncommutative geometry. *J. Noncommut. Geom.* 15 (2021), 735–759.
- [18] H. Ishida, R. Krutowski, and T. Panov. Basic cohomology of canonical holomorphic foliations on complex moment-angle manifolds. *Int. Math. Res. Not.* (2022), 5541–5563.
- [19] L. Katzarkov, E. Lupercio, L. Meersseman, and A. Verjovsky. Quantum (non-commutative) toric geometry: foundations. *Adv. Math.* 391 (2021), 110 pp.
- [20] E. Lerman. Symplectic cuts. *Math. Res. Lett.* 2 (1995), 247–258.
- [21] Y. Lin and R. Sjamaar. Convexity properties of pre-symplectic moment maps. *J. Symplectic Geom.* 17 (2019), 1159–1200.
- [22] A. L. Mackay. De nive quinquangula—On the pentagonal snowflake. *Sov. Phys. Crystallogr.* 26 (1981), 517–522.
- [23] A. L. Mackay. Crystallography and the Penrose pattern. *Phys. A: Stat. Mech. Appl.* 114 (1982), 609–613.
- [24] R. Penrose. Pentaplexity: a class of non-periodic tilings of the plane. *Math. Intelligencer* 2 (1979), 32–37.
- [25] E. Prato. Simple non-rational convex polytopes via symplectic geometry. *Topology* 40 (2001), 961–975.
- [26] T. Ratiu and T. N. Zung. Presymplectic convexity and (ir)rational polytopes. *J. Symplectic Geom.* 17 (2019), 1479–1511.
- [27] D. Shechtman, I. Blech, D. Gratias, and J. W. Cahn. Metallic phase with long-range orientational order and no translational symmetry. *Phys. Rev. Lett.* 53 (1984), 1951–1953.
- [28] M. Senechal. *Quasicrystals and Geometry*. Cambridge University Press, 1995.
- [29] M. Senechal. The mysterious Mr. Ammann. *Math. Intelligencer* 26 (2004), 10–21.
- [30] J. E. S. Socolar. Simple octagonal and dodecagonal quasicrystals. *Phys. Rev. B* 39 (1989), 10519–10551.
- [31] G. Ziegler. *Lectures on Polytopes*, Graduate Texts in Mathematics 152. Springer, 1995.
- [32] G. Ziegler. Non-rational configurations, polytopes, and surfaces. *Math. Intelligencer* 30 (2008), 36–42.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Elisa Prato, Dipartimento di Matematica e Informatica
U. Dini, Università di Firenze, Piazza Ghiberti 27,
50122 Florence, Italy. E-mail: elisa.prato@unifi.it