Invent. math. 158, 453–495 (2004)
DOI: 10.1007/s00222-004-0375-2

Inventiones mathematicae

Toric reduction and a conjecture of Batyrev and Materov

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Oblatum 21-VII-2003 & 11-III-2004
Published online: 9 June 2004 –

0. Introduction

This paper grew out of our efforts to understand the Toric Residue Mirror Conjecture formulated by Batyrev and Materov in [2]. This conjecture has its origin in Physics and is based on a work by Morrison and Plesser [14]. According to the philosophy of *mirror symmetry*, to every manifold in a certain class one can associate a dual manifold, the so-called *mirror*, so that the intersection numbers of the moduli spaces of holomorphic curves in one of these manifolds are related to integrals of certain special differential forms on the other. While at the moment this mirror manifold is only partially understood, there is an explicit construction due to Victor Batyrev [1], in which the two manifolds are toric varieties whose defining data are related by a natural duality notion for polytopes.

Let us recall the setting of the conjecture of Batyrev and Materov. Let $\mathfrak{t}$ be a d-dimensional real vector space endowed with an integral structure: a lattice $\Gamma_{\mathfrak{t}} \subset \mathfrak{t}$ of full rank. We denote by $\Gamma_{\mathfrak{t}}^*$ the embedded dual lattice $\{v \in \mathfrak{t}^*;\ \langle v, \gamma\rangle \in \mathbb{Z} \text{ for all } \gamma \in \Gamma_{\mathfrak{t}}\}$.

Consider two convex polytopes, $\Pi \subset \mathfrak{t}$ and $\check{\Pi} \subset \mathfrak{t}^*$, containing the origin in their respective interiors, and related by the duality

$$\check{\Pi} = \{v \in \mathfrak{t}^*;\ \langle v, b\rangle \geq -1 \text{ for all } b \in \Pi\}.$$

To simplify the exposition in this introduction, we assume that both polytopes are simplicial and have integral vertices. Then the correspondence between convex and toric geometry associates to this data a pair of d-dimensional polarized toric varieties: $V(\Pi)$ and $V(\check{\Pi})$. Under our present assumptions the polarizing line bundles, which we denote by L_Π and $L_{\check{\Pi}}$, are the anticanonical bundles of the respective varieties. In the paper, we work in a more general setting which is described in detail in Sect. 1.

In the framework of the Batyrev-Materov conjecture, mirror symmetry has two "sides": A and B. The B-side is characterized by a certain function associated to the variety $V(\Pi)$ as follows. Each point γ of $\Pi \cap \Gamma_{\mathfrak{t}}$ gives rise to a holomorphic section of L_Π. In particular, denote by s_0 the section corresponding to the origin and by $\{s_i;\ i = 1, \dots, n\}$ the sections corresponding to the set of vertices $\{\beta_i;\ i = 1, \dots, n\}$ of Π.

Then, for a generic value of the complex vector parameter $z = (z_1, \dots, z_n)$, the function $F_z = s_0 - \sum_{i=1}^n z_i s_i$ is a holomorphic section of L_Π, and the equation $F_z = 0$ defines a family of Calabi-Yau hypersurfaces in $V(\Pi)$ as z varies. For each $v \in \mathfrak{t}^*$, let $F_{v,z} = \sum_{i=1}^n \langle v, \beta_i \rangle z_i s_i$, and consider the ideal $I(z)$ generated by the sections F_z and $\{F_{v,z},\ v \in \mathfrak{t}^*\}$ in the homogeneous coordinate ring of $V(\Pi)$. Then the toric residue introduced by Cox [7] defines a functional $\mathrm{TorRes}_{I(z)}$ on the space of sections of L_Π^d, which vanishes on the subspace $H^0(V(\Pi), L_\Pi^d) \cap I(z)$. Every homogeneous polynomial P of degree d in n variables gives rise to a section $S(P, z) = P(z_1 s_1, \dots, z_n s_n) \in H^0(V(\Pi), L_\Pi^d)$, and thus we obtain a function

$$\langle P \rangle_{\mathfrak{B}}(z) = \mathrm{TorRes}_{I(z)} S(P, z), \tag{0.1}$$

which is known to depend rationally on z.

Now we turn to the A-side of mirror symmetry, which is characterized by the solution to an enumerative problem on the variety $V(\check{\Pi})$. Introduce the notation $\mathfrak{a} = H_2(V(\check{\Pi}), \mathbb{R})$, $\Gamma_{\mathfrak{a}} = H_2(V(\check{\Pi}), \mathbb{Z})$ and also $\mathfrak{a}^* = H^2(V(\check{\Pi}), \mathbb{R})$, $\Gamma_{\mathfrak{a}}^* = H^2(V(\check{\Pi}), \mathbb{Z})$. Recall from the theory of toric varieties that to each vertex β_i of Π, and thus to each facet of $\check{\Pi}$, one can associate an integral element α_i of the second cohomology group $\Gamma_{\mathfrak{a}}^*$, which serves as the Poincaré dual of a particular torus-invariant divisor in $V(\check{\Pi})$. The class $\kappa = \sum_{i=1}^n \alpha_i$ is the Chern class of the anticanonical bundle of the variety; it plays an important role in the subject.

Let $\mathfrak{a}_{\mathrm{eff}} \subset \mathfrak{a}$ be the cone of effective curves. For each $\lambda \in \Gamma_{\mathfrak{a}} \cap \mathfrak{a}_{\mathrm{eff}}$, Morrison and Plesser introduced a simplicial toric variety MP_λ, which is a compactification of the space of holomorphic maps

$$\left\{\iota : \mathbb{P}^1 \to V(\check{\Pi});\ \iota_*(\phi) = \lambda\right\},$$

where ϕ is the fundamental class of $\mathbb{P}^1$ in $H_2(\mathbb{P}^1, \mathbb{Z})$. Their construction also produces a top-degree cohomology class Φ_λ^P of MP_λ, whose construction is similar to that of the class $P(\alpha_1, \alpha_2, \dots, \alpha_n)$. Then we can form the generating series

$$\langle P \rangle_{\mathfrak{A}}(z) = \sum_{\lambda \in \Gamma_{\mathfrak{a}} \cap \mathfrak{a}_{\mathrm{eff}}} \int_{\mathrm{MP}_\lambda} \Phi_\lambda^P \prod_{i=1}^n z_i^{\langle \alpha_i, \lambda \rangle}.$$

The *toric residue mirror conjecture* of Batyrev-Materov states that this generating series is an expansion of the rational function $\langle P\rangle_{\mathfrak{B}}(z)$ in a certain domain of values of the parameter z. The precise statement of the conjecture in our general framework is given in Theorem 4.1 after the preparations of Sects. 1, 2, 3.

The main goal of the present paper is the proof of this theorem, however, we feel that along the way we found a few results which are interesting on their own right. Below we sketch these results, and, at the same time, describe the structure of the paper and the highlights of the proof.

After describing our setup and recalling the necessary facts from the theory of toric varieties in Sect. 1, we turn to the intersection theory of toric varieties in Sect. 2. We approach the problem from the point of view of intersection numbers on symplectic quotients initiated by Witten [20] and Jeffrey and Kirwan [13].

Let us consider an arbitrary simplicial toric variety V of dimension d. We maintain the notation we introduced for $V(\check{\Pi})$: α_i, $i = 1, \ldots, n$, for the Poincaré duals of torus-invariant divisors, and $\mathfrak{a}$, $\Gamma_{\mathfrak{a}}$, $\mathfrak{a}^*$, $\Gamma^*_{\mathfrak{a}}$ for the appropriate second homology/cohomology groups. We denote by r the dimension of $\mathfrak{a}$. A polynomial P of degree d in n variables defines a top cohomology class of V and one can pose the problem of computing $\int_V P(\alpha_1, \ldots, \alpha_n)$. Witten [20] and Jeffrey-Kirwan [13] gave rather complicated analytic formulas for this quantity, involving some version of a multidimensional inverse Laplace transform. In [4] an algebraic residue technique was given to compute these numbers; this algebraic operation was named the Jeffrey-Kirwan residue. Our first theorem, Theorem 2.6, is a new iterated residue formula for the Jeffrey-Kirwan residue, which maybe given the following homological form.

Let $U = \{u \in \mathfrak{a} \otimes \mathbb{C};\ \prod_{i=1}^n \alpha_i(u) \neq 0\}$ be the complement of the complex hyperplane arrangement formed by the zero-sets of the complexifications of the αs in $\mathfrak{a}\otimes\mathbb{C}$, and denote by $\mathfrak{c}$ the ample cone of V in $\mathfrak{a}^*$. For generic $\xi \in \mathfrak{c}$ and a vector of auxiliary constants $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_r)$, we construct a cycle $Z(\xi, \boldsymbol{\epsilon}) \subset U(\mathfrak{A})$, which is a disjoint union $\cup_{F\in\mathcal{FL}(\xi)} T_F(\boldsymbol{\epsilon})$ of oriented r-dimensional real tori in U indexed by a subset of flags of our hyperplane arrangement depending on ξ. Fix an appropriately normalized holomorphic volume form $d\mu^{\mathfrak{a}}_{\Gamma}$ on $\mathfrak{a} \otimes \mathbb{C}$. The integration $f \mapsto \int_{T_F(\boldsymbol{\epsilon})} f\, d\mu^{\mathfrak{a}}_{\Gamma}$ along one of these tori is called an *iterated residue*; it is a simple algebraic functional on holomorphic functions on $U(\mathfrak{A})$. Our integral formula (Theorem 2.6) then takes the form

$$\int_V P(\alpha_1, \ldots, \alpha_n) = \int_{Z(\xi,\boldsymbol{\epsilon})} \frac{P(\alpha_1, \ldots, \alpha_n)\ d\mu^{\mathfrak{a}}_{\Gamma}}{\alpha_1 \ldots \alpha_n}.$$

Here, on the left hand side, we think of the αs as cohomology classes, while on the right hand side we consider them to be linear functionals on $\mathfrak{a} \otimes \mathbb{C}$.

Next, in Sect. 3, we study the moduli spaces MP_λ, $\lambda \in \Gamma_{\mathfrak{a}} \cap \mathfrak{a}_{\mathrm{eff}}$, which are toric varieties themselves. Using the results of Sect. 2, we derive an integral formula (3.12) for the generating function $\langle P\rangle_{\mathfrak{A}}(z)$ of the form $\int_{Z(\xi,\epsilon)} P(u)\Lambda(u)$, where $\Lambda(u)$ is a meromorphic top form in $U(\mathfrak{A})$. Here the constants ϵ need to be chosen appropriately, in order to make sure $Z(\xi,\epsilon)$ avoids the poles of Λ.

We turn to the B-side in Sect. 4. We use a localized formula [5,6,2] for the toric residue, which has the form of a sum of the values of a certain rational function over a finite set $O_{\mathfrak{B}}(z) \subset V(\check{\Pi})$. We make a key observation (Proposition 4.2 and Lemma 4.3) that this finite set is naturally embedded into U as the set of solutions of the system of equations:

$$\left\{\prod_{i=1}^{n} \alpha_i(u)^{\langle\alpha_i,\lambda\rangle} = \prod_{i=1}^{n} z_i^{\langle\alpha_i,\lambda\rangle}, \quad \lambda \in \Gamma_{\mathfrak{a}}\right\}.$$

This infinite system of equations in the variable $u \in U$ easily reduces to r independent equations. This presentation of $O_{\mathfrak{B}}(z)$ allows us to write down an integral formula for $\langle P\rangle_{\mathfrak{B}}(z)$ in Proposition 4.7, which has the form of $\int_{Z'} P(u)\Lambda(u)$, where Z' is a another cycle in U avoiding the poles of Λ. This way, we essentially reduce the Batyrev-Materov conjecture to a topological problem of comparing cycles.

The cycle Z' is closely related to a real algebraic subvariety $\widehat{Z}(\xi)$ of U given by the set of equations

$$\widehat{Z}(\xi) = \left\{u \in U(\mathfrak{A});\ \prod_{i=1}^{n} |\alpha_i(u)|^{\langle\alpha_i,\lambda\rangle} = e^{-\langle\xi,\lambda\rangle} \text{ for all } \lambda \in \Gamma_{\mathfrak{a}}\right\}.$$

In Sect. 5 we prove the central result of the paper, Theorem 5.1, in which we compute the homology class of the cycle $\widehat{Z}(\xi)$ in U for any generic ξ. The proof uses certain type of degenerations reminiscent of the methods of tropical geometry in real algebraic geometry (cf. [19,18]).

In Sect. 6 we specialize this result to the case when the generic vector ξ is near $\kappa = \sum_{i=1}^{n} \alpha_i$, and combining it with Theorem 2.6, we arrive at the statement that for such ξ the cycle $\widehat{Z}(\xi)$ is contained in a small neighborhood of the origin in $\mathfrak{a} \otimes \mathbb{C}$, and it is a small deformation of the cycle $Z(\xi,\epsilon)$ which represents the Jeffrey-Kirwan residue (Theorem 6.2). Armed with this result, the proof of the conjecture is quickly completed.

We would like to end this introduction with a remark on the conditions of our main result. Although here for simplicity we assumed that the polytope Π is simplicial and reflexive, neither of these conditions are necessary. In the paper, we prove our result for an arbitrary polytope with integral vertices, which contains the origin in its interior.

Finally, we note that after this work was substantially completed, we were informed by Lev A. Borisov that he had also obtained a proof of the Toric Residue Mirror Conjecture by a completely different method.

Acknowledgments. We would like to thank the Mathematisches Forschungsinstitut Oberwolfach for wonderful working conditions. We are thankful to Alicia Dickenstein for discussions on toric residues, to Victor Batyrev and Eugene Materov for explaining their conjecture to us, and to the referees for their careful reading of the manuscript and useful suggestions. The first author would like to express his gratitude for the hospitality of Ecole Polytechnique and acknowledge the support of OTKA.

1. Preliminaries: Toric varieties

In this section, we describe standard facts from projective toric geometry. The proofs will be mostly omitted (cf. [8–10, 12]).

1.1. Polytopes and toric varieties. For a real vector space $\mathfrak{v}$ endowed with a lattice of full rank $\Gamma_{\mathfrak{v}}$, denote by $\mathfrak{v}_{\mathbb{C}}$ the complexification $\mathfrak{v} \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathfrak{v}$, by $T_{\mathfrak{v}}$ the compact torus $\mathfrak{v}/\Gamma_{\mathfrak{v}}$ and by $T_{\mathbb{C}\mathfrak{v}}$ the complexified torus $\mathfrak{v}_{\mathbb{C}}/\Gamma_{\mathfrak{v}}$; finally, for $\gamma \in \Gamma_{\mathfrak{v}}^*$, where

$$\Gamma_{\mathfrak{v}}^* = \{\gamma \in \mathfrak{v}^*;\ \langle \gamma, v\rangle \in \mathbb{Z} \text{ for all } v \in \Gamma_{\mathfrak{v}}\},$$

denote by e_γ the character $v \mapsto e^{2\pi i \gamma(v)}$ of $T_{\mathfrak{v}}$. We will keep the notation e_γ for the holomorphic extension of this character to the complexified torus $\mathfrak{v}_{\mathbb{C}}/\Gamma_{\mathfrak{v}}$.

Given a polytope $\Pi \subset \mathfrak{v}^*$ with integral vertices, one can construct a polarized toric variety with action of the complex torus $T_{\mathbb{C}\mathfrak{v}}$ as follows. Consider the graded algebra

$$\oplus \mathbb{C} e_\gamma g^k,\ k = 1, \dots;\ \gamma \in k\Pi \cap \Gamma_{\mathfrak{v}}^*, \tag{1.1}$$

where the multiplication among the basis elements comes from addition in $\mathfrak{v}^*$, and g is an auxiliary variable marking the grading. This algebra is the homogeneous ring of a polarized projective toric variety $V(\Pi)$ endowed with a line bundle L_Π and an action of the torus $T_{\mathbb{C}\mathfrak{v}}$, which lifts to an action on L_Π. Each lattice point $\gamma \in \Pi \cap \Gamma_{\mathfrak{v}}^*$ gives rise to a section s_γ of the line bundle L_Π, and the set $\{s_\gamma, \gamma \in \Pi \cap \Gamma_{\mathfrak{v}}^*\}$ forms a basis of $H^0(V(\Pi), L_\Pi)$. Note that the toric variety $V(\Pi + t)$ corresponding to the polytope Π translated by an element $t \in \Gamma_{\mathfrak{v}}^*$ is isomorphic to the variety $V(\Pi)$, and the line bundle L_Π is equivariantly isomorphic to $L_\Pi \otimes \mathbb{C}_t$, where $\mathbb{C}_t$ is the one-dimensional representation of $T_{\mathbb{C}\mathfrak{v}}$ corresponding to the character e_t. Thus, starting from an integral polytope in an affine space endowed with a lattice, one can construct a well-defined polarized toric variety.

1.2. The quotient construction. Now we give a more concrete description of toric varieties. Let $\mathfrak{g} = \oplus_{i=1}^n \mathbb{R}\omega_i$ be an n-dimensional real vector space with a fixed ordered basis, and let

$$0 \to \mathfrak{a} \to \mathfrak{g} \xrightarrow{\pi} \mathfrak{t} \to 0 \tag{1.2}$$

be an exact sequence of finite dimensional real vector spaces of dimensions r, n and d, respectively. We assume that the lattice $\Gamma_{\mathfrak{g}} = \oplus_{i=1}^{n}\mathbb{Z}\omega_i$ intersects $\mathfrak{a}$ in a lattice $\Gamma_{\mathfrak{a}}$ of full rank, and we denote the image $\pi(\Gamma_{\mathfrak{g}})$ in $\mathfrak{t}$ by $\Gamma_{\mathfrak{t}}$. This means that the sequence restricted to the lattices is also exact. In this case the dual sequence

$$0 \to \mathfrak{t}^* \to \mathfrak{g}^* \xrightarrow{\mu} \mathfrak{a}^* \to 0 \tag{1.3}$$

restricted to the dual lattices $\Gamma_{\mathfrak{t}}^*$, $\Gamma_{\mathfrak{g}}^*$ and $\Gamma_{\mathfrak{a}}^*$, respectively, is also exact.

Denoting the elements of the dual basis by ω^i, $i = 1, \dots, n$, we have $\mathfrak{g}^* = \oplus_{i=1}^{n}\mathbb{R}\omega^i$; in particular, $\Gamma_{\mathfrak{g}}^* = \oplus_{i=1}^{n}\mathbb{Z}\omega^i$. Now introduce the notation α_i for the image vector $\mu(\omega^i)$ in $\Gamma_{\mathfrak{a}}^*$, $i = 1, \dots, n$, and consider the sequence $\mathfrak{A} := [\alpha_1, \alpha_2, \dots, \alpha_n]$. We emphasize that some of the αs may coincide. The order of the elements of this sequence will be immaterial, however.

Definition 1.1. *We call a sequence* $\mathfrak{A}$ *in* $\mathfrak{a}^*$ projective *if it lies in an open half space of the vector space* $\mathfrak{a}^*$.

The relevance of this condition will be explained below. Note that according to our assumptions, the elements of $\mathfrak{A}$ generate $\Gamma_{\mathfrak{a}}^*$ over $\mathbb{Z}$, and this will *always* be tacitly assumed in this paper.

Definition 1.2. *Let* $\mathfrak{A}$ *be a not necessarily projective sequence in* $\Gamma_{\mathfrak{a}}^*$. *Denote by* BInd$(\mathfrak{A})$ *the set of* basis index sets, *that is the set of those index subsets* $\sigma \subset \{1, \dots, n\}$ *for which the set* $\{\alpha_i;\ i \in \sigma\}$ *is a basis of* $\mathfrak{a}^*$. *We will also use the notation*

$$\boldsymbol{\gamma}^\sigma = \left(\gamma_1^\sigma, \dots, \gamma_r^\sigma\right)$$

for the basis associated to $\sigma \in$ BInd$(\mathfrak{A})$*; here a certain ordering of the basis elements, say the one induced by the natural ordering of* σ*, has been fixed.*

Definition 1.3. *For any set or sequence* S *of vectors in a real vector space, denote by* Cone(S) *the closed cone spanned by the elements of* S. *Let us consider the case of a projective sequence* $\mathfrak{A}$ *in* $\mathfrak{a}^*$. *We denote by* Cone$_{\text{sing}}(\mathfrak{A})$ *the union of the boundaries of the simplicial cones* Cone$(\boldsymbol{\gamma}^\sigma)$*,* $\sigma \in$ BInd$(\mathfrak{A})$. *Elements of* Cone$_{\text{sing}}(\mathfrak{A})$ *will be called* singular*, the others,* regular. *A connected component of* Cone$(\mathfrak{A}) \setminus$ Cone$_{\text{sing}}(\mathfrak{A})$ *is called a* chamber. *Then for a chamber* $\mathfrak{c}$*, we can define* BInd$(\mathfrak{A}, \mathfrak{c})$ *to be the set of those* $\sigma \in$ BInd$(\mathfrak{A})$ *for which* Cone$(\boldsymbol{\gamma}^\sigma) \supset \mathfrak{c}$. □

Now we assume that $\mathfrak{A}$ is projective. Then we can proceed to construct the toric variety $V_{\mathfrak{A}}(\mathfrak{c})$ as a quotient of the open set

$$U_{\mathfrak{c}} = \bigcup_{\sigma \in \mathrm{BInd}(\mathfrak{A}, \mathfrak{c})} \left\{ (z_1, \dots, z_n) \,\middle|\, \prod_{i \in \sigma} z_i \neq 0 \right\} \subset \mathbb{C}^n$$

by the action of the complexified torus $T_{\mathbb{C}\mathfrak{a}}$, where we let $T_{\mathbb{C}\mathfrak{a}}$ act on $\mathbb{C}^n$ diagonally with weights $(\alpha_1, \ldots, \alpha_n)$. If $\mathfrak{A}$ is projective then the quotient

$$V_{\mathfrak{A}}(\mathfrak{c}) = U_{\mathfrak{c}}/T_{\mathbb{C}\mathfrak{a}} \tag{1.4}$$

is a compact orbifold of dimension d.

To compare this construction to the one in Sect. 1.1, let θ be an integral point in $\mathfrak{c}$, and assume that the *partition polytope*

$$\Pi_\theta = \mu^{-1}(\theta) \bigcap \sum_{i=1}^{n} \mathbb{R}^{\geq 0}\omega^i,$$

which lies in an affine subspace of $\mathfrak{g}^*$ parallel to $\mathfrak{t}^*$, has integral vertices. The toric variety $V_{\mathfrak{A}}(\mathfrak{c})$ is isomorphic to the variety $V(\Pi_\theta)$ described before by its homogeneous ring. The polarizing line bundle may be defined as $L_\theta = U_{\mathfrak{c}} \times_{T_{\mathbb{C}\mathfrak{a}}} \mathbb{C}_\theta$, where $\mathbb{C}_\theta$ is the one-dimensional representation of $T_{\mathbb{C}\mathfrak{a}}$ corresponding to the character e_θ. If

$$\gamma = \sum_{i=1}^{n} \gamma_i \omega^i \in \Pi_\theta \cap \Gamma^*_{\mathfrak{g}}, \text{ with } \gamma_i \in \mathbb{Z}^{\geq 0},\ i = 1, \ldots, n,$$

then the holomorphic function $\tilde{s}_\gamma : U_{\mathfrak{c}} \to \mathbb{C}$ given by $\tilde{s}_\gamma(z) = \prod_{i=1}^{n} z_i^{\gamma_i}$ descends to a section s_γ of the line bundle L_θ.

Note that under this quotient construction, for every $\eta \in \Gamma^*_{\mathfrak{a}}$, even for those not necessarily in $\mathfrak{c}$, a line bundle L_η may be defined by $L_\eta = U_{\mathfrak{c}} \times_{T_{\mathfrak{a}}} \mathbb{C}_\eta$. (In fact, L_η is usually only an orbi-bundle). The chamber $\mathfrak{c}$ is called the *ample cone* of the variety $V_{\mathfrak{A}}(\mathfrak{c})$ as the line bundles corresponding to lattice points in $\mathfrak{c}$ are ample.

1.3. Gale duality. Let $\mathfrak{B}$ be the sequence of vectors $\beta_i = \pi(\omega_i) \in \Gamma_{\mathfrak{t}}$, where π is the map in the exact sequence (1.2). The sequence $\mathfrak{B}$ is called the *Gale dual* sequence to the sequence $\mathfrak{A}$. It immediately follows that taking the Gale dual of a sequence twice, one recovers the original sequence.

The following lemma describes the fundamental relation between Gale dual vector configurations.

Lemma 1.1. *A linear combination $\sum_{i=1}^{n} m_i\alpha_i$ vanishes if and only if there is a linear functional $l \in \mathfrak{t}^*$ such that $l(\beta_i) = m_i$.*

This relation allows one to translate statements in the $\mathfrak{A}$-language into those in the Gale dual $\mathfrak{B}$-language and vice versa.

Proposition 1.2. *Let $\mathfrak{A}$ be a projective sequence in $\mathfrak{a}^*$ and $\mathfrak{c}$ be chamber. Then*

(1) *The Gale dual configuration $\mathfrak{B}$ does not lie in any closed half space of $\mathfrak{t}$, that is $\sum_{i=1}^{n} \mathbb{R}^{\geq 0}\beta_i = \mathfrak{t}$.*

(2) *If* $\sigma \in \mathrm{BInd}(\mathfrak{A})$*, then the complement* $\bar{\sigma} = \{1, \ldots, n\} \setminus \sigma$ *is an element of* $\mathrm{BInd}(\mathfrak{B})$.
(3) *Denote by* $\bar{\gamma}$ *the basis of* $\mathfrak{t}$ *corresponding to* $\bar{\sigma} \in \mathrm{BInd}(\mathfrak{B})$*. The set of cones* $\mathrm{Cone}(\bar{\gamma}^{\bar{\sigma}})$*,* $\sigma \in \mathrm{BInd}(\mathfrak{A}, \mathfrak{c})$ *forms a simplicial conic decomposition* $\mathrm{SCD}(\mathfrak{c})$ *of* $\mathfrak{t}$.
(4) *The simplicial conic decomposition associated to the partition polytope* $\Pi(\theta)$ *coincides with* $\mathrm{SCD}(\mathfrak{c})$ *for any* $\theta \in \mathfrak{c}$.

Remark 1.1. A simplicial conic decomposition is also called a complete simplicial *fan*.

Now we prove a quantitative version of statement (2) of Proposition 1.2. Endow the vector spaces $\mathfrak{g}$, $\mathfrak{t}$, $\mathfrak{a}$ with orientations compatible with the sequence (1.2). Observe that a vector space $\mathfrak{v}$ endowed with a lattice of full rank and an *orientation* has a natural translation-invariant volume form, that is an element of $\wedge^{\dim \mathfrak{v}}\mathfrak{v}^*$, such that the signed volume of a unit parallelepiped of the lattice is ± 1. Accordingly, we have a volume form on each vector space $\mathfrak{g}$, $\mathfrak{t}$, $\mathfrak{a}$; denote the volume form on $\mathfrak{a}$ by $d\mu_\Gamma^{\mathfrak{a}}$. Next, for $\sigma \in \mathrm{BInd}(\mathfrak{A})$, introduce the notation $\mathrm{vol}_{\mathfrak{a}^*}(\sigma)$ for the signed volume of the parallelepiped $\sum_{i\in\sigma}[0, 1]\gamma_i^\sigma$. This means that we take the volume of the parallelepiped measured in the units of the volume of a basic parallelepiped of $\Gamma_\mathfrak{a}^*$, and set the sign to $+1$ if the basis γ^σ is positively oriented, and to -1 otherwise; $\mathrm{vol}_\mathfrak{t}(\bar{\sigma})$ is defined similarly.

Now we can formulate our first duality statement.

Lemma 1.3. *For* $\sigma \in \mathrm{BInd}(\mathfrak{A})$ *we have* $\bar{\sigma} \in \mathrm{BInd}(\mathfrak{B})$*, and* $|\,\mathrm{vol}_{\mathfrak{a}^*}(\sigma)| = |\,\mathrm{vol}_\mathfrak{t}(\bar{\sigma})|$.

Proof. The exact sequence (1.2) gives rise to an isomorphism

$$I : \Lambda^d\mathfrak{t} \mapsto \Lambda^n\mathfrak{g} \otimes \Lambda^r\mathfrak{a}^*$$

as follows. For $y_1, y_2, \ldots, y_d \in \mathfrak{t}$ with representatives $Y_1, Y_2, \ldots, Y_d$ in $\mathfrak{g}$, and $u_1, u_2, \ldots, u_r \in \mathfrak{a}$, let

$$\langle I(y_1\wedge y_2\wedge\cdots\wedge y_d), u_1\wedge u_2\wedge\cdots\wedge u_r\rangle = Y_1\wedge Y_2\wedge\cdots\wedge Y_d\wedge u_1\wedge u_2\wedge\cdots\wedge u_r.$$

Denote by $\wedge\gamma^\sigma$ the form $\gamma_1^\sigma \wedge \gamma_2^\sigma \wedge \cdots \wedge \gamma_r^\sigma$. Define $\wedge\bar{\gamma}^{\bar{\sigma}}$ similarly. Then it is easy to verify that $I(\wedge\bar{\gamma}^{\bar{\sigma}}) = \pm(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n) \otimes \wedge\gamma^\sigma$. This implies the statement of the lemma. □

Now we translate a few important properties of vector configurations into Gale dual language.

Definition 1.4. *Given a projective sequence* $\mathfrak{A} = [\alpha_i]_{i=1}^n$ *in* $\Gamma_\mathfrak{a}^*$*, introduce the notation* $\kappa = \sum_{i=1}^n \alpha_i$*. We call the sequence* $\mathfrak{A}$ spanning *if for every* $k \in \{1, \ldots, n\}$ *the vector* κ *may be written as a non-negative linear combination* $\kappa = \sum_{i=1}^n t_i\alpha_i$ *with* $t_k = 0$*, and* $t_i \neq 0$*, for* $i = 1, 2, \ldots, k-1, k+1, \ldots, n$.

Lemma 1.4. *A sequence $\mathfrak{B}$ is the set of vertices of a convex polytope containing the origin in its interior if and only if the Gale dual sequence $\mathfrak{A}$ is projective and spanning.*

Proof. Indeed, for $\mathfrak{B} = \{\beta_1, \dots \beta_n\}$ to be the set of vertices of a convex polytope is equivalent to the existence of linear functionals $h_k \in \mathfrak{t}^*$ for $k = 1, \dots, n$, such that $\langle h_k, \beta_k\rangle = -1$ and $\langle h_k, \beta_i\rangle > -1$ for $i \neq k$. Then according to Lemma 1.1, we have

$$\sum_{i=1}^{n} (\langle h_k, \beta_i\rangle + 1)\alpha_i = \kappa,$$

and this is exactly the spanning property for $\mathfrak{A}$. □

Remark 1.2. It is easy to see that if $\mathfrak{A}$ is spanning, then the property described for κ extends to any θ which is in a chamber $\mathfrak{c}$ containing κ in its closure, i.e. for every such θ and for each $k \in \{1, \dots, n\}$ one can find a non-negative integral linear combination $\theta = \sum_{i=1}^{n} t_i\alpha_i$ with $t_k = 0$.

Now we formulate two consequences of the spanning property. Recall that according to statement (3) of Proposition 1.2, the set of one-dimensional faces of the fan $\mathrm{SCD}(\mathfrak{c})$ is a *subset* of the set of rays $\{\mathbb{R}^{\geq 0}\beta_i;\ i = 1, \dots, n\}$. Also, note that there is a natural map $\chi_{\mathfrak{c}} : \Gamma^*_{\mathfrak{a}} \to H^2(V_{\mathfrak{A}}(\mathfrak{c}), \mathbb{Q})$ which associates to each lattice point θ the first Chern class of the orbi-line-bundle L_θ.

Proposition 1.5. *Assume that $\mathfrak{A} \subset \mathfrak{a}^*$ is a spanning, projective sequence, and let $\mathfrak{c}$ be a chamber which contains κ in its closure. Then*

(1) *the set of one-dimensional faces of the fan $\mathrm{SCD}(\mathfrak{c})$ is the set of rays $\{\mathbb{R}^{\geq 0}\beta_i;\ i = 1, \dots, n\}$,*
(2) *the characteristic map $\chi_{\mathfrak{c}}$ is an isomorphism over $\mathbb{Q}$.*

The first statement follows from statement (4) of Proposition 1.2. Indeed, according to the above remark, for any $\theta \in \mathfrak{c}$, the partition polytope Π_θ has exactly n facets. The kth facet, which corresponds to the linear combinations mentioned in the definition, is perpendicular to the Gale dual vector β_k. In the non-spanning case such a facet may disappear.

Now we can describe Batyrev's mirror dual toric varieties, which have respective actions of the tori $T_{\mathbb{C}\mathfrak{t}}$ and $T_{\mathbb{C}\mathfrak{t}^*}$.

Let $\mathfrak{B} \subset \Gamma_{\mathfrak{t}}$ be the set of vertices of a convex polytope $\Pi^{\mathfrak{B}}$ containing the origin in its interior. Then, on the one hand, we can use this polytope to construct a projective toric variety $V(\Pi^{\mathfrak{B}})$. On the other hand, consider *star-like* triangulations of $\Pi^{\mathfrak{B}}$, that is triangulations τ of $\Pi^{\mathfrak{B}}$ with vertices at $\mathfrak{B} \cup \{0\}$ such that every simplex contains the origin. Clearly, such a triangulation τ gives rise to a simplicial fan whose cones are the cones of the simplices of τ based at the origin.

Proposition 1.6. *Let $\mathfrak{B}$ be a sequence of vectors whose elements serve as the vertices of an integral polytope $\Pi^{\mathfrak{B}}$. Then the fan* $\mathrm{SCD}(\mathfrak{c})$ *of a chamber* $\mathfrak{c}$ *of the Gale dual configuration* $\mathfrak{A}$ *induces a star-like triangulation of* $\Pi^{\mathfrak{B}}$ *if and only if* $\mathfrak{c}$ *contains the vector* $\kappa = \sum_{i=1}^{n} \alpha_i$ *in its closure.*

To summarize: the polytope $\Pi^{\mathfrak{B}}$ corresponds to a toric variety $V(\Pi^{\mathfrak{B}})$ on the one hand. On the other, it gives rise to a family of "mirror dual" toric varieties $V_{\mathfrak{A}}(\mathfrak{c})$ corresponding to those chambers $\mathfrak{c}$ of the Gale dual sequence $\mathfrak{A}$ which contain κ in their closure; the sequence $\mathfrak{A}$ is spanning.

Finally, we recall the following definitions from [2]. Let $\check{\Pi}^{\mathfrak{B}}$ be the *dual polytope* of $\Pi^{\mathfrak{B}}$ defined by

$$\check{\Pi}^{\mathfrak{B}} = \{l \in \mathfrak{t}^*;\ \langle l, b\rangle \geq -1,\ b \in \Pi^{\mathfrak{B}}\}.$$

Lemma 1.7. *The dual polytope* $\check{\Pi}^{\mathfrak{B}}$ *is a translate of the partition polytope* Π_κ *associated to the Gale dual configuration* $\mathfrak{A}$.

Proof. The point $t = \sum_{i=1}^{n} \omega^i$ is such that $\mu(t) = \kappa$. It is easy to see that $y \in \mathfrak{g}^*$ belongs to Π_κ if and only if $y - t \in \check{\Pi}^{\mathfrak{B}} \subset \mathfrak{t}^*$. □

Definition 1.5. *The polytope* $\Pi^{\mathfrak{B}}$ *is called* reflexive *if the dual polytope* $\check{\Pi}^{\mathfrak{B}}$ *has integral vertices.*

Batyrev and Materov consider dual pairs of reflexive polytopes. This has the advantage of putting the toric variety and its mirror dual on the same footing. In this paper, we will consider a more general framework: we assume that $\mathfrak{B}$ is the set of vertices of a polytope with the origin in its interior, but no condition on the dual polytope is imposed.

2. Intersection numbers of toric quotients and the Jeffrey-Kirwan residue

In this section, $\mathfrak{A}$ is any projective sequence in $\Gamma_{\mathfrak{a}}^*$. Recall that we have chosen an orientation of $\mathfrak{a}$, and that this, together with the lattice $\Gamma_{\mathfrak{a}}$ induces a volume form $d\mu_{\Gamma}^{\mathfrak{a}}$ on $\mathfrak{a}$.

Pick a chamber $\mathfrak{c} \subset \mathrm{Cone}(\mathfrak{A})$ and consider the orbifold toric variety $V_{\mathfrak{A}}(\mathfrak{c})$. Since $V_{\mathfrak{A}}(\mathfrak{c})$ is a quotient $U_{\mathfrak{c}}/T_{\mathbb{C}\mathfrak{a}}$, by the Chern-Weil construction, every polynomial Q on $\mathfrak{a}$ gives rise to a characteristic class $\chi(Q)$ of $V_{\mathfrak{A}}(\mathfrak{c})$. Thus we have a Chern-Weil map $\chi : \mathrm{Sym}(\mathfrak{a}^*) \to H^*(V_{\mathfrak{A}}(\mathfrak{c}), \mathbb{C})$ from the polynomials on $\mathfrak{a}$ to the cohomology of $V_{\mathfrak{A}}(\mathfrak{c})$. In particular, for $\eta \in \mathfrak{a}^*$ the Chern class of the orbi-line-bundle L_η is $\chi(\eta)$.

It is natural to look for formulas for the intersection numbers $\int_{V_{\mathfrak{A}}(\mathfrak{c})} \chi(Q)$, where, of course, only the degree d component of Q contributes. To write down a formula, we recall the notion of the Jeffrey-Kirwan residue in a form suggested by Brion and Vergne [4]. Define $U(\mathfrak{A})$ to be the complement in $\mathfrak{a}_{\mathbb{C}}$ of the complex hyperplane arrangement determined by $\mathfrak{A}$:

$$U(\mathfrak{A}) = \{u \in \mathfrak{a}_{\mathbb{C}};\ \alpha(u) \neq 0 \text{ for all } \alpha \in \mathfrak{A}\},$$

where we extended the functionals α_i from $\mathfrak{a}$ to $\mathfrak{a}_{\mathbb{C}}$.

Remark 2.1. 1. Note that $\alpha(u)$ and $\langle\alpha, u\rangle$ stand for the same thing; we use one form or the other depending on whether we consider u a variable or a constant.
2. The constructions of this section depend on the set of elements of $\mathfrak{A}$, and do not depend on the multiplicities. We are not going to reflect this in the notation, however.

Denote by $\mathbb{C}_{\mathfrak{A}}[\mathfrak{a}]$ the linear space of rational functions on $\mathfrak{a}_{\mathbb{C}}$ whose denominators are products of powers of elements of $\mathfrak{A}$. The space $\mathbb{C}_{\mathfrak{A}}[\mathfrak{a}]$ is $\mathbb{Z}$-graded by degree; the functions in $\mathbb{C}_{\mathfrak{A}}[\mathfrak{a}]$ are regular on $U(\mathfrak{A})$.

Of particular importance will be certain functions in $\mathbb{C}_{\mathfrak{A}}[\mathfrak{a}]$ of degree $-r$: for every $\sigma \in \mathrm{BInd}(\mathfrak{A})$ denote by f_σ the fraction $1/\prod_{i\in\sigma}\alpha_i$. We will call such fractions *basic*. Every function in $\mathbb{C}_{\mathfrak{A}}[\mathfrak{a}]$ of degree $-r$ may be decomposed into a sum of basic fractions and degenerate fractions; degenerate fractions are those for which the linear forms in the denominator do not span $\mathfrak{a}^*$. Now having fixed a chamber $\mathfrak{c}$, we define a functional $\mathrm{JK}_{\mathfrak{c}}$ on $\mathbb{C}_{\mathfrak{A}}[\mathfrak{a}]$ called the Jeffrey-Kirwan residue (or *JK-residue*) as follows. Let

$$\mathrm{JK}_{\mathfrak{c}}(f_\sigma) = \begin{cases} |\operatorname{vol}_{\mathfrak{a}^*}(\gamma^\sigma)|^{-1}, & \text{if } \mathfrak{c} \subset \operatorname{Cone}(\gamma^\sigma), \\ 0, & \text{if } \mathfrak{c} \cap \operatorname{Cone}(\gamma^\sigma) = \emptyset. \end{cases} \tag{2.1}$$

Also, set the value of the JK-residue of a degenerate fraction or that of a rational function of pure degree different from $-r$ equal to zero.

The definition of the functional $\mathrm{JK}_{\mathfrak{c}}(\cdot)$ is vastly over-determined, as there are many linear relations among the basic fractions f_σ, $\sigma \in \mathrm{BInd}(\mathfrak{A})$.

Proposition 2.1 ([4]). *The definition in* (2.1) *is consistent and defines a functional* $\mathrm{JK}_{\mathfrak{c}}$ *on* $\mathbb{C}_{\mathfrak{A}}[\mathfrak{a}]$.

One can give a homological interpretation to the JK-residue as follows.

Lemma 2.2. *For each chamber* $\mathfrak{c}$ *there is an homology class* $h(\mathfrak{c}) \in H_r(U(\mathfrak{A}), \mathbb{R})$ *such that*

$$\mathrm{JK}_{\mathfrak{c}}(f) = \frac{1}{\left(2\pi\sqrt{-1}\right)^r}\int_{h(\mathfrak{c})} f\, d\mu_\Gamma^{\mathfrak{a}} \quad \textit{for every} \quad f \in \mathbb{C}_{\mathfrak{A}}[\mathfrak{a}],$$

where $d\mu_\Gamma^{\mathfrak{a}}$ *is the translation invariant holomorphic volume form defined above.*

Proof. The integral on the right hand side is well-defined since the form $f\, d\mu_\Gamma^{\mathfrak{a}}$ is closed. The statement follows from Poincaré duality and the fact that $H^r(U(\mathfrak{A}), \mathbb{R})$ is spanned by holomorphic differential forms of the form $f_\sigma\, d\mu_\Gamma^{\mathfrak{a}}$, $\sigma \in \mathrm{BInd}(\mathfrak{A})$ (cf. [15, 17]). □

The integration over $V_{\mathfrak{A}}(\mathfrak{c})$ may be written in terms of the Jeffrey-Kirwan residue as follows.

Proposition 2.3 ([4]). *Let $\mathfrak{A}$ be a projective sequence in $\Gamma^*_{\mathfrak{a}}$, $\mathfrak{c}$ be a chamber and Q be a polynomial on $\mathfrak{a}$. Then we have*

$$\int_{V_{\mathfrak{A}}(\mathfrak{c})} \chi(Q) = \mathrm{JK}_{\mathfrak{c}}\left(\frac{Q}{\prod_{i=1}^n \alpha_i}\right). \tag{2.2}$$

Combining this with Lemma 2.2 we obtain the formula

$$\int_{V_{\mathfrak{A}}(\mathfrak{c})} \chi(Q) = \frac{1}{\left(2\pi\sqrt{-1}\right)^r} \int_{h(\mathfrak{c})} \frac{Q\ d\mu_\Gamma^{\mathfrak{a}}}{\prod_{i=1}^n \alpha_i}. \tag{2.3}$$

The main result of this section, Theorem 2.6 may be interpreted as a natural construction of a smooth cycle in $U(\mathfrak{A})$, which represents the class $h(\mathfrak{c})$.

We start with a few important notations and definitions related to our hyperplane arrangement.

Let $\mathcal{FL}(\mathfrak{A})$ be the finite set of flags

$$F = [F_0 = \{0\} \subset F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r = \mathfrak{a}^*],\ \dim F_j = j,$$

such that $\mathfrak{A}$ contains a basis of F_j for each $j = 1, \ldots, r$. For each $F \in \mathcal{FL}(\mathfrak{A})$, we choose, once and for all, an ordered basis $\gamma^F = (\gamma_1^F, \ldots, \gamma_r^F)$ of $\mathfrak{a}^*$ with the following properties:

(1) $\gamma_j^F \in \Gamma^*_{\mathfrak{a}} \otimes \mathbb{Q}$, for $j = 1, \ldots, r$,
(2) $\{\gamma_m^F\}_{m=1}^j$ is a basis of F_j for $j = 1, \ldots, r$,
(3) the basis γ^F is positively oriented,
(4) $d\gamma_1^F \wedge \cdots \wedge d\gamma_r^F = d\mu_\Gamma^{\mathfrak{a}}$.

To each flag $F \in \mathcal{FL}(\mathfrak{A})$, one can associate a linear functional Res_F on $\mathbb{C}_{\mathfrak{A}}[\mathfrak{a}]$, called an *iterated residue*. We consider the elements of the basis γ^F as coordinates on $\mathfrak{a}$ and we use the simplified notation $u_j = \gamma_j^F(u)$ for $u \in \mathfrak{a}_{\mathbb{C}}$. Then any rational function $\phi \in \mathbb{C}_{\mathfrak{A}}[\mathfrak{a}]$ on $\mathfrak{a}_{\mathbb{C}}$ may be written as a rational function ϕ^F of these coordinates:

$$\phi(u) = \phi^F(u_1, \ldots, u_r).$$

We define the iterated residue associated to the flag $F \in \mathcal{FL}(\mathfrak{A})$ as the functional $\mathrm{Res}_F : \mathbb{C}_{\mathfrak{A}}[\mathfrak{a}] \to \mathbb{C}$ given by the formula

$$\operatorname*{Res}_F \phi = \operatorname*{Res}_{u_r=0} du_r \operatorname*{Res}_{u_{r-1}=0} \cdots \operatorname*{Res}_{u_1=0} du_1\ \phi^F(u_1, u_2, \ldots, u_r),$$

where each residue is taken assuming that the variables with higher indices have a fixed, nonzero value.

It is easy to see that this linear form on $\mathbb{C}_{\mathfrak{A}}[\mathfrak{a}]$ depends only on the flag F and the volume form $d\mu_\Gamma^{\mathfrak{a}}$, and not on the particular choice of the ordered basis γ^F. In fact, this operation has a homological interpretation which is given below.

Let N be a positive real number. Denote by $U(F, N) \subset \mathfrak{a}_{\mathbb{C}}$ the open subset of $\mathfrak{a}_{\mathbb{C}}$ defined by

$$U(F, N) = \left\{u \in \mathfrak{a}_{\mathbb{C}}\,;\; 0 < N\left|\gamma_j^F(u)\right| < \left|\gamma_{j+1}^F(u)\right|,\, j = 1, 2, \ldots, r-1\right\}.$$

The following lemma is straightforward and its proof will be omitted.

Lemma 2.4. *There exist positive constants N_0 and c_0 such that for $N > N_0$ we have*

(1) *$U(F, N) \subset U(\mathfrak{A})$ for all $F \in \mathcal{FL}(\mathfrak{A})$, and*
(2) *the sets $U(F, N)$, $F \in \mathcal{FL}(\mathfrak{A})$ are disjoint.*
(3) *If $\alpha_i \in F_j$ and $\alpha_k \in F_{j+1} \setminus F_j$ for some $F \in \mathcal{FL}(\mathfrak{A})$ and $j < r$, then for every $u \in U(F, N)$ the inequality $|\alpha_k(u)/\alpha_i(u)| > c_0 N$ holds.*

From now on, when using the constant N, we will assume that $N > N_0$. Note that the set $U(F, N)$ depends on the choice of the basis γ^F made above, but we will not reflect this dependence in the notation explicitly. When F is fixed, we use as before the simplified notation $u_j = \gamma_j^F(u)$ for $u \in \mathfrak{a}_{\mathbb{C}}$, so we can write

$$U(F, N) = \{u \in \mathfrak{a}_{\mathbb{C}}\,;\; 0 < N|u_j| < |u_{j+1}|,\, j = 1, 2, \ldots, r-1\}.$$

Observe that the set $U(F, N)$ is diffeomorphic to $\mathbb{R}^r_{>0} \times (S^1)^r$, thus the rth homology of $U(F, N)$ is 1-dimensional. Choose a sequence of real numbers $\epsilon : 0 < \epsilon_1 \ll \epsilon_2 \ll \cdots \ll \epsilon_r$, where $\epsilon \ll \delta$ means $N\epsilon < \delta$. Define the torus

$$T_F(\epsilon) = \{u \in \mathfrak{a}_{\mathbb{C}};\; |u_j| = \epsilon_j,\; j = 1, \ldots, r\} \subset U(F, N) \subset U(\mathfrak{A}), \tag{2.4}$$

oriented by the form $d \arg u_1 \wedge \cdots \wedge d \arg u_r$. It is easy to see that this cycle is a representative of a generator of the homology $H_r(U(F, N), \mathbb{Z})$.

The homology class of this cycle in $U(\mathfrak{A})$ depends only on the flag F and not on the chosen positively oriented basis γ^F of F.

Definition 2.1. *Denote the homology class of the cycle $T_F(\epsilon)$ in $H_r(U(\mathfrak{A}), \mathbb{Z})$ by $h(F)$. This produces a map $h : \mathcal{FL}(\mathfrak{A}) \to H_r(U(\mathfrak{A}), \mathbb{Z})$.*

Lemma 2.5 ([16]). *For any $\phi \in \mathbb{C}_{\mathfrak{A}}[\mathfrak{a}]$ we have*

$$\frac{1}{\left(2\pi\sqrt{-1}\right)^r} \int_{h(F)} \phi \; d\mu_\Gamma^{\mathfrak{a}} = \operatorname*{Res}_F \phi,$$

where by integration over $h(F)$ we mean integration over any cycle representing it.

Our goal is to write the functional $\mathrm{JK}_{\mathfrak{c}}$ as a signed sum of iterated residues Res_F. This will allow us to write $\mathrm{JK}_{\mathfrak{c}}(\phi)$ as an integral of $\phi\, d\mu_\Gamma^{\mathfrak{a}}$ over the union of corresponding cycles. The flags entering our formula will depend on the choice of an element $\xi \in \mathfrak{c}$. This element will have to satisfy additional conditions of regularity that we formulate below.

Definition 2.2. *Denote by $\Sigma\mathfrak{A}$ the set of elements of $\mathfrak{a}^*$ obtained by partial sums of elements of $\mathfrak{A}$:*

$$\Sigma\mathfrak{A} = \left\{ \sum_{i\in\eta} \alpha_i;\ \eta \subset \{1, \dots, n\} \right\}.$$

For each subset $\boldsymbol{\rho} \subset \Sigma\mathfrak{A}$ which forms a basis of $\mathfrak{a}^*$, write ξ in this basis: $\xi = \sum_{\gamma\in\boldsymbol{\rho}} u_\gamma^{\boldsymbol{\rho}}(\xi)\,\gamma$. Then introduce the quantity

$$\min{}^{\Sigma\mathfrak{A}}(\xi) = \min\left\{ \left|u_\gamma^{\boldsymbol{\rho}}(\xi)\right|;\ \boldsymbol{\rho} \subset \Sigma\mathfrak{A},\ \boldsymbol{\rho} \text{ basis of } \mathfrak{a}^*,\ \gamma \in \boldsymbol{\rho} \right\}.$$

An element $\xi \in \mathfrak{a}^*$ will be called regular with respect to $\Sigma\mathfrak{A}$ *if* $\min^{\Sigma\mathfrak{A}}(\xi) > 0$. *For $\tau > 0$ we say that ξ is τ*-regular with respect to $\Sigma\mathfrak{A}$, *if* $\min^{\Sigma\mathfrak{A}}(\xi) > \tau$.

One could also say that a vector $\xi \in \mathfrak{a}^*$ is regular with respect to $\Sigma\mathfrak{A}$ if ξ does not belong to any hyperplane generated by elements of $\Sigma\mathfrak{A}$. Sometimes, we will use the term *sum-regular* for such vectors. Clearly, sum-regular vectors form a dense open subset in $\mathfrak{a}^*$.

Each flag $F \in \mathcal{FL}(\mathfrak{A})$ introduces a partition of the elements of the sequence $\mathfrak{A} = (\alpha_1, \dots, \alpha_n)$ induced by the representation of the space $\mathfrak{a}^*$ as a disjoint union $\cup_{j=1}^r F_j \setminus F_{j-1}$. For $j = 1, \dots, r$, introduce the vectors

$$\kappa_j^F = \sum\{\alpha_i;\ i = 1, \dots, n,\ \alpha_i \in F_j\}. \tag{2.5}$$

Note that the vectors κ_j^F are in $\Sigma\mathfrak{A}$, and that $\kappa_r^F = \kappa = \sum_{i=1}^n \alpha_i$ independently from F.

Definition 2.3. *1. A flag F in $\mathcal{FL}(\mathfrak{A})$ will be called* proper *if the elements κ_j^F, $j = 1, \dots, r$ are linearly independent.*

2. For each $F \in \mathcal{FL}(\mathfrak{A})$, define a number $\nu(F) \in \{0, \pm 1\}$ as follows:
- *set $\nu(F) = 0$ if F is not a proper flag;*
- *if F is a proper flag, then $\nu(F)$ is equal to 1 or -1 depending on whether the ordered basis $(\kappa_1^F, \kappa_2^F, \dots, \kappa_r^F)$ of $\mathfrak{a}^*$ is positively or negatively oriented.*

3. For a proper flag $F \in \mathcal{FL}(\mathfrak{A})$, introduce the closed simplicial cone $\mathfrak{s}^+(F, \mathfrak{A})$ generated by the non-negative linear combinations of the elements $\{\kappa_j^F, j = 1, \dots, r\}$:

$$\mathfrak{s}^+(F, \mathfrak{A}) = \sum_{j=1}^r \mathbb{R}^{\geq 0} \kappa_j^F.$$

Then for $\xi \in \mathfrak{a}^*$, denote by $\mathcal{FL}^+(\mathfrak{A}, \xi)$ the set of flags F such that ξ belongs to the cone $\mathfrak{s}^+(F, \mathfrak{A})$. □

Observe that if ξ is sum-regular, then every flag $F \in \mathcal{FL}^+(\mathfrak{A}, \xi)$ is proper, and thus for such F we have $\nu(F) = \pm 1$.

Now we are ready to formulate the main result of this section.

Theorem 2.6. *Let $\mathfrak{c}$ be any chamber of the projective sequence $\mathfrak{A}$, and let ξ be a vector in $\mathfrak{c}$ which is regular with respect to $\Sigma\mathfrak{A}$. Then for every $\phi \in \mathbb{C}_{\mathfrak{A}}[\mathfrak{a}]$*

$$\mathrm{JK}_{\mathfrak{c}}(\phi) = \sum_{F \in \mathcal{FL}^+(\mathfrak{A},\xi)} \nu(F) \operatorname*{Res}_F \phi. \tag{2.6}$$

Proof. Let $\sigma \in \mathrm{BInd}(\mathfrak{A})$, and consider the basic fraction

$$f_\sigma = \frac{1}{\prod_{j=1}^r \gamma_j^\sigma}.$$

First, observe that it is sufficient to prove the theorem for these basic fractions: $\phi = f_\sigma$ for $\sigma \in \mathrm{BInd}(\mathfrak{A})$. Indeed, both the Jeffrey-Kirwan residue and the iterated residues are degree $-r$ operations on $\mathbb{C}_{\mathfrak{A}}[\mathfrak{a}]$. This allows us to restrict ϕ to be of degree $-r$. Now it is easy to check that the iterated residues vanish on degenerate fractions, i.e. on fractions whose denominators do not contain linear forms spanning $\mathfrak{a}^*$. The JK-residue vanishes on degenerate fractions by definition.

Thus we can assume $\phi = f_\sigma$. By the definition of the chambers, the condition $\mathfrak{c} \subset \mathrm{Cone}(\gamma^\sigma)$ is equivalent to the condition $\xi \in \mathrm{Cone}(\gamma^\sigma)$. Then according to the definition of the JK-residue given in (2.1), we have

$$\mathrm{JK}_{\mathfrak{c}}(f_\sigma) = \begin{cases} \dfrac{1}{|\operatorname{vol}_{\mathfrak{a}^*}(\gamma^\sigma)|}, & \text{if } \mathfrak{c} \subset \mathrm{Cone}(\gamma^\sigma) \\ 0, & \text{otherwise.} \end{cases}.$$

Now we compute the right hand side of (2.6) for $\phi = f_\sigma$. It is not hard to see that $\mathrm{Res}_F(f_\sigma)$ is equal to 0 unless the flag F is such that its j-dimensional component F_j is spanned by elements of γ^σ. In other words, we have $\mathrm{Res}_F\, f_\sigma \neq 0$ for some $F \in \mathcal{FL}(\mathfrak{A})$ if and only if F is of the form

$$F^\pi(\sigma) = \left(F_1^\pi(\sigma), \ldots, F_r^\pi(\sigma)\right) \text{ with } F_j^\pi(\sigma) = \Sigma_{k=1}^j \mathbb{C}\gamma_{\pi(j)}^\sigma,$$

where π is an element of Σ_r, the group of permutations of r indices. We will simply write $F(\sigma)$ in the case when π is the identity permutation.

One can easily compute the appropriate iterated residue:

$$\operatorname*{Res}_{F^\pi(\sigma)} f_\sigma = \frac{(-1)^\pi}{\operatorname{vol}_{\mathfrak{a}^*}(\gamma^\sigma)},$$

where we denoted by $(-1)^\pi$ the value of the alternating character of Σ_r on π.

Given a closed cone C, denote by $\chi[C]$ its characteristic function. Using the above remarks, we can rewrite (2.6) as follows:

$$\sum_{\pi \in \Sigma_r} (-1)^\pi \nu(F^\pi(\sigma))\chi[\mathfrak{s}^+(F^\pi(\sigma), \mathfrak{A})](\xi) = \chi[\mathrm{Cone}(\gamma^\sigma)](\xi) \tag{2.7}$$

for any vector ξ regular with respect to $\Sigma\mathfrak{A}$.

As we will explain below in detail, this equality simply reflects the subdivision into cones of Cone (γ^σ) based on the rays $\mathbb{R}^{\geq 0}\kappa_j^{F^\pi(\sigma)}$, for $\pi \in \Sigma_r$, and $j = 1, \dots, r$. Denote by $I(\sigma, \xi)$ the expression on the left hand side of (2.7). We prove that $I(\sigma, \xi) = \chi[\mathrm{Cone}(\gamma^\sigma)](\xi)$ by induction on the dimension of $\mathfrak{a}$.

Consider the $(r-1)$-dimensional space $F_{r-1}(\sigma)$, the sequence $\mathfrak{A}' = \mathfrak{A} \cap F_{r-1}(\sigma)$, and the index set σ' obtained from σ by omitting its largest element. For $\pi \in \Sigma_{r-1}$, again we denote by $F^\pi(\sigma')$ the flag associated to the permuted basis. To compute $I(\sigma, \xi)$, we first study $I_r(\sigma, \xi)$, the sum over the set Σ_{r-1} of permutations of the first $(r-1)$ indices:

$$I_r(\sigma, \xi) = \sum_{\pi \in \Sigma_{r-1}} (-1)^\pi \nu(F^\pi(\sigma))\chi[\mathfrak{s}^+(F^\pi(\sigma), \mathfrak{A})](\xi).$$

Recall that for any $F \in \mathcal{FL}(\mathfrak{A})$ we have $\kappa_r^F = \kappa \stackrel{\text{def}}{=} \sum_{i=1}^n \alpha_i$. There are two cases:

(1) The element $\kappa \in F_{r-1}$.
(2) $\mathfrak{a}^* = F_{r-1} \oplus \mathbb{R}\kappa$.

We define $\nu_r \in \{-1, 0, 1\}$ as follows. In the first case $\nu_r = 0$. In the second case, we write $\nu_r = \pm 1$ depending on the orientation of $(\gamma_1^F, \gamma_2^F, \dots, \gamma_{r-1}^F, \kappa)$. Then in case (1), the sum $I_r(\sigma, \xi)$ is equal to 0, while in case (2), the cone $\mathfrak{s}^+(F^\pi(\sigma), \mathfrak{A})$ is equal to $\mathfrak{s}^+(F^\pi(\sigma), \mathfrak{A}') + \mathbb{R}^+\kappa$. Writing $\xi = \xi' + s\kappa$, we have

$$\sum_{\pi \in \Sigma_{r-1}} (-1)^\pi \nu(F^\pi(\sigma))\chi[\mathfrak{s}^+(F^\pi(\sigma), \mathfrak{A})](\xi) =$$

$$\begin{cases} \nu_r \sum_{\pi \in \Sigma_{r-1}} (-1)^\pi \nu(F^\pi(\sigma'))\chi[\mathfrak{s}^+(F^\pi(\sigma'), \mathfrak{A}')](\xi'), & \text{if } s > 0, \\ 0, & \text{if } s < 0. \end{cases}$$

As ξ is sum-regular, we cannot have $s = 0$. Thus if $s > 0$, then the point ξ' is sum-regular with respect to $\mathfrak{A}'$, and by the induction hypothesis we conclude that $I_r(\sigma, \xi) = \nu_r \chi[\mathrm{Cone}(\sigma')](\xi')$; if $s < 0$, then we have $I_r(\sigma, \xi) = 0$.

Consider the closed cone $\mathrm{Cone}(\gamma^{\sigma'} \cup \{\kappa\})$. The preceding relation reads as

$$I_r(\sigma, \xi) = \nu_r \chi[\mathrm{Cone}(\gamma^{\sigma'} \cup \{\kappa\})](\xi).$$

It remains to sum over all circular permutations. Taking care of the signs of the circular permutation and of orientations, we obtain this formula for the full sum:

$$I(\sigma, \xi) = \sum_{i=1}^r (-1)^i \nu_i \chi[C_i](\xi)$$

where C_i is the cone generated by $\gamma^\sigma \setminus \{\gamma_i^\sigma\}$ and κ, and $\nu_i = \pm 1$ depending on the orientation of this basis: $(\gamma_1^\sigma, \gamma_2^\sigma, \dots, \gamma_{i-1}^\sigma, \gamma_{i+1}^\sigma, \dots, \gamma_r^\sigma, \kappa)$. The fact that this sum equals $\chi[\mathrm{Cone}(\gamma^\sigma)](\xi)$ is a straightforward exercise. This completes the proof of our theorem. □

Remark 2.2. Using the results of [16,4], one can obtain a formula for the Jeffrey-Kirwan residue via iterated residues, using the concept of diagonal bases introduced in [16]. Our present formula is quite different; it is more symmetric and seems to be computationally more efficient as well.

Theorem 2.6 has the following

Corollary 2.7. *1. The equality*

$$h(\mathfrak{c}) = \sum_{F \in \mathcal{FL}^+(\mathfrak{A},\xi)} \nu(F) h(F)$$

holds in $H_r(U(\mathfrak{A}), \mathbb{Z})$.

2. The class $h(\mathfrak{c}) \in H_r(U(\mathfrak{A}), \mathbb{R})$ is integral; it may be represented by a disjoint union of embedded oriented tori.

The first statement is a homological rewriting of Theorem 2.6, while the second follows from the fact that $h(F)$ is represented by the torus $T_F(\epsilon)$. Thus we can reformulate Theorem 2.6 in a third, integral form as follows. Define the cycle

$$Z(\xi) = \cup_{F \in \mathcal{FL}^+(\mathfrak{A},\xi)} \nu(F) T_F(\epsilon), \tag{2.8}$$

where ϵ is a vector of appropriate positive constants. Then $Z(\xi)$ is an embedded oriented submanifold of $U(\mathfrak{A})$ depending on a set of auxiliary constants, and we have

$$\mathrm{JK}_{\mathfrak{c}}(\phi) = \int_{Z(\xi)} \phi \; d\mu_\Gamma^{\mathfrak{a}}. \tag{2.9}$$

3. The Morrison-Plesser moduli spaces

In this section we assume that $\mathfrak{A}$ is projective and spanning, and the chamber $\mathfrak{c}$ contains κ in its closure.

Then, according to Proposition 1.5, we have a natural isomorphism $H_2(V_{\mathfrak{A}}(\mathfrak{c}), \mathbb{Q}) \cong \Gamma_{\mathfrak{a}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Introduce the *cone of effective curves*

$$\bar{\mathfrak{c}}^\perp = \{\lambda \in \mathfrak{a};\ \langle \xi, \lambda \rangle \geq 0,\ \text{for all } \xi \in \mathfrak{c}\}.$$

Following Morrison and Plesser, we associate to each integral point $\lambda \in \bar{\mathfrak{c}}^\perp \cap \Gamma_{\mathfrak{a}}$ of the cone of effective curves a toric variety MP_λ together with a cohomology class $\Phi_\lambda \in H^*(\mathrm{MP}_\lambda, \mathbb{Z})$, called the *Morrison-Plesser moduli space* and its *fundamental class*. This is a variant of the space of holomorphic maps of $\mathbb{P}^1$ into a fixed Calabi-Yau subvariety of $V_{\mathfrak{A}}(\mathfrak{c})$ with a fixed image λ of the fundamental class of $\mathbb{P}^1$ (cf. [2]).

Assume first that $\lambda \in \Gamma_{\mathfrak{a}}$ is such that $\langle \alpha_i, \lambda \rangle \geq 0,\ i = 1, \ldots, n$. Then the Morrison-Plesser toric variety MP_λ is the toric variety represented by the data $\mathfrak{A}^\lambda$, consisting of repetitions of the linear forms α_i in $\mathfrak{a}^*$: each α_i is repeated $\langle \alpha_i, \lambda \rangle + 1$ times. Thus the total number of elements of $\mathfrak{A}^\lambda$ is $\langle \kappa, \lambda \rangle + n$, and the dimension of the resulting toric variety is $\langle \kappa, \lambda \rangle + d$. The polarizing chamber is the same one, $\mathfrak{c}$, as that of the original toric variety. Thus we have $\mathrm{MP}_\lambda = V_{\mathfrak{A}^\lambda}(\mathfrak{c})$. The fundamental class is given by $\Phi_\lambda = \chi(\kappa^{\langle \kappa, \lambda \rangle})$, where we used the notation of the previous section. We are interested in intersection numbers of the variety MP_λ of the following form.

Fix a polynomial P of degree d in n variables, and think of it as a function on $\mathfrak{g}$. Denote the restriction of P to $\mathfrak{a}$ by $P|\mathfrak{a}$; effectively, this means substituting α_i for the ith argument of P. Then having fixed a sequence $\mathfrak{A}$ and a chamber $\mathfrak{c}$ in $\mathfrak{a}^*$, define

$$\langle P \rangle_{\lambda, \mathfrak{A}, \mathfrak{c}} = \int_{\mathrm{MP}_\lambda} \Phi_\lambda \chi(P|\mathfrak{a}). \tag{3.1}$$

Using (2.3), we can write

$$\langle P \rangle_{\lambda, \mathfrak{A}, \mathfrak{c}} = \frac{1}{\left(2\pi\sqrt{-1}\right)^r} \int_{h(\mathfrak{c})} \frac{P(\alpha_1, \ldots, \alpha_n)\, \kappa^{\langle \kappa, \lambda \rangle}\ d\mu_\Gamma^{\mathfrak{a}}}{\prod_{i=1}^n \alpha_i^{\langle \alpha_i, \lambda \rangle + 1}}, \tag{3.2}$$

where $h(\mathfrak{c})$ is the homology class representing the JK-residue.

Pick a cycle $Z[\mathfrak{c}]$ representing the homology class $h(\mathfrak{c})$, which satisfies the condition

$$Z[\mathfrak{c}] \subset U(\mathfrak{A}) \cap \{u \in \mathfrak{a}_{\mathbb{C}};\ |\kappa(u)| < 1\}. \tag{3.3}$$

The cycle $Z(\xi)$ introduced in (2.8) will be suitable if the auxiliary constants $\epsilon_1, \ldots, \epsilon_r$ are chosen sufficiently small.

Now note that the rational function under the integral sign in (3.2) has exactly the correct degree: $-r$. This implies that we can replace $\kappa^{\langle \kappa, \lambda \rangle}$ in the formula by $(1 - \kappa)^{-1}$ as follows:

$$\langle P \rangle_{\lambda, \mathfrak{A}, \mathfrak{c}} = \frac{1}{\left(2\pi\sqrt{-1}\right)^r} \int_{Z[\mathfrak{c}]} \frac{P(\alpha_1, \ldots, \alpha_n)\ d\mu_\Gamma^{\mathfrak{a}}}{(1 - \kappa) \prod_{i=1}^n \alpha_i^{\langle \alpha_i, \lambda \rangle + 1}}, \tag{3.4}$$

Indeed to compute the right hand side of (3.4) on such cycle $Z[\mathfrak{c}]$, we can replace $1/(1 - \kappa)$ by its absolutely convergent expansion $\sum_{l=0}^\infty \kappa^l$. Then only the power $\kappa^{\langle \kappa, \lambda \rangle}$ gives a nonzero contribution to the integral.

Further, observe that the right hand side of (3.4) is meaningful for any $\lambda \in \Gamma_{\mathfrak{a}}$. Thus we can use it as a *definition* of the left hand side even for the cases when the condition $\langle \alpha, \lambda \rangle \geq 0$ does not hold for all $\alpha \in \mathfrak{A}$.

Definition 3.1. *For any $\lambda \in \Gamma_{\mathfrak{a}}$, define the rational function $p_\lambda \in \mathbb{C}_{\mathfrak{A}}[\mathfrak{a}]$ of homogeneous degree $\langle \kappa, \lambda \rangle$ by*

$$p_\lambda = \prod_{i=1}^{n} \alpha_i^{\langle \alpha_i, \lambda \rangle}.$$

One can write this function as quotient of two polynomials: $p_\lambda = p_\lambda^+ / p_\lambda^-$, where

$$p_\lambda^+ = \prod_{\langle \alpha_i, \lambda \rangle > 0} \alpha_i^{\langle \alpha_i, \lambda \rangle}, \quad p_\lambda^- = \prod_{\langle \alpha_i, \lambda \rangle < 0} \alpha_i^{-\langle \alpha_i, \lambda \rangle}.$$

The functions p_λ satisfy the relation $p_{\lambda_1} p_{\lambda_2} = p_{\lambda_1 + \lambda_2}$ for any $\lambda_1, \lambda_2 \in \Gamma_{\mathfrak{a}}$.

Definition 3.2. *For any $\lambda \in \Gamma_{\mathfrak{a}}$ and degree d polynomial P, we define*

$$\langle P \rangle_{\lambda, \mathfrak{A}, \mathfrak{c}} = \frac{1}{\left(2\pi\sqrt{-1}\right)^r} \int_{Z[\mathfrak{c}]} \frac{P(\alpha_1, \dots, \alpha_n) \;\; d\mu_\Gamma^{\mathfrak{a}}}{(1-\kappa) p_\lambda \prod_{i=1}^n \alpha_i}, \tag{3.5}$$

where $Z[\mathfrak{c}]$ is any cycle satisfying (3.3).

In the case when the condition $\langle \alpha, \lambda \rangle \geq 0$ does not hold for all $\alpha \in \mathfrak{A}$, the numbers $\langle P \rangle_{\lambda, \mathfrak{A}, \mathfrak{c}}$ may be interpreted as intersection numbers on a modified version of the pair $(\mathrm{MP}_\lambda, \Phi_\lambda)$. Our convention for $\langle P \rangle_{\lambda, \mathfrak{A}, \mathfrak{c}}$ induces a definition of the fundamental class Φ_λ in this general case, which coincides with the one given by Morrison and Plesser. For details cf. [2, 14].

The next observation is central for our computations.

Proposition 3.1. *For $\lambda \in \Gamma_{\mathfrak{a}} \setminus \bar{\mathfrak{c}}^\perp$, one has $\langle P \rangle_{\lambda, \mathfrak{A}, \mathfrak{c}} = 0$.*

Proof. We can assume $\langle \kappa, \lambda \rangle \geq 0$, since $\langle P \rangle_{\lambda, \mathfrak{A}, \mathfrak{c}}$ vanishes for $\langle \kappa, \lambda \rangle < 0$ by degree considerations. Then (3.4) may be rewritten as

$$\langle P \rangle_{\lambda, \mathfrak{A}, \mathfrak{c}} = \mathrm{JK}_{\mathfrak{c}} \left(\frac{P(\alpha_1, \dots, \alpha_n) p_\lambda^- \kappa^{\langle \kappa, \lambda \rangle}}{p_\lambda^+ \prod_{i=1}^n \alpha_i} \right) \tag{3.6}$$

Observe that the expression in (3.6) is a JK-residue of a rational function, denote it by ϕ_λ, whose poles lie on the hyperplanes $\alpha_i = 0$, *with* $\langle \alpha_i, \lambda \rangle \geq 0$. Indeed, if $\langle \alpha_i, \lambda \rangle < 0$, then α_i occurs in the denominator $p_\lambda^+ \prod_{i=1}^n \alpha_i$ with multiplicity 1, thus it is canceled by a factor in p_λ^- in the numerator.

Now comparing (3.6) to the definition of the Jeffrey-Kirwan residue in (2.1), we see that $\mathrm{JK}_{\mathfrak{c}}(\phi_\lambda) \neq 0$ implies that $\mathfrak{c}$ is contained in the cone generated by those α_i which satisfy $\langle \alpha_i, \lambda \rangle \geq 0$. Consequently, λ, as a linear functional on $\mathfrak{a}^*$ is positive on $\mathfrak{c}$, which is exactly the condition $\lambda \in \bar{\mathfrak{c}}^\perp$. □

Next, following [2], we write down a generating series of the numbers $\langle P\rangle_{\lambda,\mathfrak{A},\mathfrak{c}}$ for λ varying in the dual cone $\bar{\mathfrak{c}}^{\perp}$. To this end, introduce the notation z^{λ} for the Laurent monomial $\prod_{i=1}^{n} z_i^{\langle\alpha_i,\lambda\rangle}$ for any element $\lambda \in \Gamma_{\mathfrak{a}}$ and $z = \sum_{i=1}^{n} z_i\omega_i \in \mathfrak{g}$. Note that the restriction of the function z^{λ} to $\mathfrak{a}$ is exactly the rational function $p_{\lambda} = p_{\lambda}^{+}/p_{\lambda}^{-}$.

Then the generating series of intersection numbers in which we are interested has the form

$$\langle P\rangle_{\mathfrak{A},\mathfrak{c}}(z) = \sum_{\lambda\in\Gamma_{\mathfrak{a}}\cap\bar{\mathfrak{c}}^{\perp}} \langle P\rangle_{\lambda,\mathfrak{A},\mathfrak{c}} z^{\lambda}. \tag{3.7}$$

The chamber $\mathfrak{c}$, and thus $\bar{\mathfrak{c}}^{\perp}$, might be quite complicated, but using Proposition 3.1, we can rewrite the generating function (3.7) very simply.

First an auxiliary statement:

Lemma 3.2. *Let C be a closed rational polyhedral cone in a half-space of a real vector space $\mathfrak{v}$ of dimension r endowed with a lattice Γ of full rank, and let κ be a nonzero vector in C. Then there exist vectors $v_1, v_2, \ldots, v_r$ in Γ with the properties*

- $\sum_{j=1}^{r} \mathbb{Z}v_i = \Gamma$,
- $v_j \in C$, *for* $j = 1, \ldots, r$,
- $\kappa \in \sum_{j=1}^{r} \mathbb{R}^{\geq 0} v_j$.

Proof. Indeed, the cone C has a decomposition into simplicial cones generated by $\mathbb{Z}$-bases of Γ.

Now we return to our setup.

Definition 3.3. *Given a chamber $\mathfrak{c}$, we will call a set of vectors $\{\lambda_1, \ldots, \lambda_r\} \subset \Gamma_{\mathfrak{a}}$ a* $\mathfrak{c}$-positive basis *if the following conditions are satisfied:*

- $\sum_{j=1}^{r} \mathbb{Z}\lambda_j = \Gamma_{\mathfrak{a}}$,
- $\sum_{j=1}^{r} \mathbb{R}_{\geq 0}\lambda_j \supset \bar{\mathfrak{c}}^{\perp}$,
- $\langle\kappa, \lambda_j\rangle \geq 0$ *for* $j = 1, \ldots, r$.

Apply Lemma 3.2 to the pair $\kappa \in \bar{\mathfrak{c}}$. Then taking the dual basis guarantees the existence of a $\mathfrak{c}$-positive basis. We fix such a basis and denote it by $\boldsymbol{\lambda}$.

Now observe that according to Proposition 3.1 and the second property of a $\mathfrak{c}$-positive basis, we can replace the sum in the definition (3.7) by the sum over a simplicial cone:

$$\langle P\rangle_{\mathfrak{A},\mathfrak{c}}(z) = \sum \langle P\rangle_{\lambda,\mathfrak{A},\mathfrak{c}}\, z^{\lambda}, \quad \lambda \in \sum_{j=1}^{r} \mathbb{Z}^{\geq 0}\lambda_j.$$

Notation. Assume that a basis $\boldsymbol{\lambda}$ has been fixed, and let $z \in \mathfrak{g}_{\mathbb{C}}$ such that $z_i \neq 0$ for $i = 1, \ldots, n$. Then we introduce the simplified notation p_j, $p_j^{\pm}$ for p_{λ_j}, $p_{\lambda_j}^{\pm}$, respectively, and denote z^{λ_j} by q_j.

Using the integral definition of (3.5), we can write

$$\text{(3.8)}\quad \langle P\rangle_{\mathfrak{A},\mathfrak{c}}(z) = \frac{1}{\left(2\pi\sqrt{-1}\right)^r}\sum\int_{Z[\mathfrak{c}]}\prod_{j=1}^{r}\frac{q_j^{l_j}}{p_j^{l_j}}\cdot\frac{P(\alpha_1,\ldots,\alpha_n)\ d\mu_\Gamma^{\mathfrak{a}}}{(1-\kappa)\prod_{i=1}^{n}\alpha_i},$$

where the sum runs over $l_j \in \mathbb{Z}^{\geq 0},\ j = 1,\ldots,r.$

If the condition

$$\text{(3.9)}\qquad |q_j| < \max_{u\in Z[\mathfrak{c}]}|p_j(u)|,\qquad j = 1,\ldots,r$$

is satisfied, then the series is absolutely convergent.

In fact, together with Proposition 3.1, this integral representation allows us to determine the domain of convergence of $\langle P\rangle_{\mathfrak{A},\mathfrak{c}}(z)$ more precisely. For $\lambda \in \Gamma_{\mathfrak{a}}$, define

$$\text{(3.10)}\qquad \epsilon_\lambda = \max_{u\in Z[\mathfrak{c}]}|p_\lambda(u)|,$$

and consider the set

$$\text{(3.11)}\qquad W(Z[\mathfrak{c}]) = \{z \in (\mathbb{C}^*)^n;\ |z^\lambda| < \epsilon_\lambda \text{ for every } \lambda \in \Gamma_{\mathfrak{a}} \cap \bar{\mathfrak{c}}^\perp\}.$$

Since both z^λ and ϵ_λ are multiplicative in λ, the set $W(Z[\mathfrak{c}])$ is already defined by a set of inequalities of the form $|z^\lambda| < \epsilon_\lambda$, where λ runs through a finite subset of $\Gamma_{\mathfrak{a}} \cap \bar{\mathfrak{c}}^\perp$ which generates it as a semigroup. In particular, $W(Z[\mathfrak{c}])$ is open.

Lemma 3.3. *The series $\langle P\rangle_{\mathfrak{A},\mathfrak{c}}(z)$ converges for all $z \in W(Z[\mathfrak{c}])$.*

Proof. We can decompose $\bar{\mathfrak{c}}^\perp$ into simplicial cones generated by $\mathbb{Z}$-bases of $\Gamma_{\mathfrak{a}}$. The sum in each such cone will be a convergent geometric series thanks to the inequalities defining $W(Z[\mathfrak{c}])$. □

Now return to the fact that if the conditions (3.9) hold, then the series (3.8) converges absolutely. As a consequence, we can exchange the order of summation and integration in (3.8). Then we can sum the resulting geometric series under the integral sign and arrive at the following statement.

Proposition 3.4. *Let $Z[\mathfrak{c}]$ be a cycle in $U(\mathfrak{A})$ representing $h(\mathfrak{c})$ and satisfying (3.3). Fix a $\mathfrak{c}$-positive basis $\lambda_1, \lambda_2, \ldots, \lambda_r \in \Gamma_{\mathfrak{a}}$, and assume that $z \in (\mathbb{C}^*)^n$ is such that the inequalities $|q_j| < \max_{u\in Z[\mathfrak{c}]}|p_j(u)|$ hold, where $q_j = z^{\lambda_j}$. Then we have*

$$\text{(3.12)}\quad \langle P\rangle_{\mathfrak{A},\mathfrak{c}}(z) = \frac{1}{\left(2\pi\sqrt{-1}\right)^r}\int_{Z[\mathfrak{c}]}\frac{P(\alpha_1,\ldots,\alpha_n)\prod_{j=1}^{r}p_j\ d\mu_\Gamma^{\mathfrak{a}}}{(1-\kappa)\prod_{i=1}^{n}\alpha_i\prod_{j=1}^{r}\left(p_j-q_j\right)}.$$

4. An integral formula for toric residues

We start with the data considered so far: the exact sequences (1.2) and (1.3), the resulting sequence $\mathfrak{A}$, which we assume to be projective and spanning, a chamber $\mathfrak{c} \subset \mathfrak{a}^*$ containing κ in its closure, a polynomial P in n variables, and a point $z \in \mathfrak{g}$. Using these, we defined a series $\langle P\rangle_{\mathfrak{A},\mathfrak{c}}(z)$ in the previous section, and analyzed its domain of convergence. This series is the object that the Batyrev-Materov conjecture associates to the A-side of mirror symmetry.

Now we look at the same data in the Gale dual picture. According to Lemma 1.4, the Gale dual sequence $\mathfrak{B} \subset \Gamma_{\mathfrak{t}}$ serves as the set of vertices of a convex polytope $\Pi^{\mathfrak{B}}$ containing the origin. The object on the B-side of the Batyrev-Materov conjecture is the rational function $\langle P\rangle_{\mathfrak{B}}(z)$ defined in (0.1) of the introduction which uses the *toric residue* of Cox (cf. [7, 2]). As suggested in [2], rather than applying the original definition, we will use a localized formula for toric residues [5,6] which we recall below in (4.1). The applicability of this localization formula in our case was kindly explained to us by Alicia Dickenstein.

Consider the function $f = 1 - \sum_{i=1}^n z_i e_{\beta_i}$, parameterized by our chosen point $z = \sum_{i=1}^n z_i\omega_i \in \mathfrak{g}$. Pick a $\mathbb{Z}$-basis $(h_1, \dots, h_d)$ of $\Gamma_{\mathfrak{t}}^*$ and form the toric partial derivatives

$$f_k = -\sum_{i=1}^n z_i \langle h_k, \beta_i\rangle e_{\beta_i}, \quad k = 1, \dots, d,$$

which assemble into the toric gradient $\nabla f = (f_1, \dots, f_d)$. We can go on and define the toric Hessian of the function f as

$$H_f = \det\left(\sum_{i=1}^n \langle h_j, \beta_i\rangle\langle h_k, \beta_i\rangle z_i e_{\beta_i}\right)_{j,k=1}^d .$$

Now denote by $O_{\mathfrak{B}}(z)$ the set of toric critical points of f, i.e. the set

$$O_{\mathfrak{B}}(z) = \{\nabla f = 0\} \subset T_{\mathbb{C}\mathfrak{t}^*}.$$

For generic z, this set is discrete, and the toric critical points are non-degenerate. We take the following version of toric residues localized at the toric critical points to be the definition of $\langle P\rangle_{\mathfrak{B}}(z)$:

$$\langle P\rangle_{\mathfrak{B}}(z) = \sum \frac{\tilde{P}(w)}{f(w)H_f(w)}, \quad w \in O_{\mathfrak{B}}(z), \tag{4.1}$$

where the function $\tilde{P}$ is obtained by substituting $z_i e_{\beta_i}$ for x_i in our degree d polynomial $P(x_1, \dots, x_n)$.

In our setup, the conjecture of Batyrev and Materov generalizes to the following statement.

Theorem 4.1. *Let $\mathfrak{A}$ be a projective, spanning sequence, and $\mathfrak{c}$ be a chamber whose closure contains κ. Choose a cycle $Z[\mathfrak{c}]$ in $U(\mathfrak{A})$ representing $h(\mathfrak{c})$ and satisfying (3.3), and let $z \in W(Z[\mathfrak{c}])$, where $W(Z[\mathfrak{c}])$ is defined in (3.11). Then the series $\langle P\rangle_{\mathfrak{A},\mathfrak{c}}(z)$ converges absolutely, moreover, we have*

$$\langle P\rangle_{\mathfrak{A},\mathfrak{c}}(z) = \langle P\rangle_{\mathfrak{B}}(z). \tag{4.2}$$

The proof of this theorem is given at the end of the paper in Sect. 6.2. Its main ingredients are Propositions 3.4 and 4.7, and Theorem 6.2, which, in turn, follows from Theorems 2.6 and 5.1.

Remark 4.1. 1. In the course of the proof, we will construct an explicit cycle $Z[\mathfrak{c}]$, thus the domain of convergence of $\langle P\rangle_{\mathfrak{A},\mathfrak{c}}(z)$ will also be given explicitly.
2. We think of the right hand side here as a rational function of z given by the toric residue. Note that, in particular, the right hand side does not depend on the choice of the chamber $\mathfrak{c}$. This dependence is encoded in the domain of convergence.
3. The conjecture in [2] is formulated for the case of toric varieties corresponding to reflexive polytopes. As explained at the end of Sect. 1, this corresponds to the special case of the partition polytope Π_κ having integral vertices.

The key observation that begins relating the two seemingly unrelated expressions in (4.2) is the following. Take a point $z \in \mathfrak{g}_{\mathbb{C}}$ with all its coordinates $z_i \neq 0$, and embed $T_{\mathbb{C}\mathfrak{t}^*}$ into $\mathfrak{g}_{\mathbb{C}}$ via the formula

$$w \mapsto (z_1 e_{\beta_1}(w), \ldots, z_n e_{\beta_n}(w)). \tag{4.3}$$

This means that we consider the natural action of $T_{\mathbb{C}\mathfrak{t}^*}$ on $\mathfrak{g}_{\mathbb{C}}$ given by the set of weights $\mathfrak{B}$, and look at the orbit $\mathrm{Orb}_{\mathfrak{B}}(z)$ of the point $z \in \mathfrak{g}_{\mathbb{C}}$. Note that the coordinates of a point in $\mathrm{Orb}_{\mathfrak{B}}(z)$ are also all nonzero.

Proposition 4.2. *Under the embedding* (4.3)*, the set of critical points $O_{\mathfrak{B}}(z)$ corresponds to the intersection of the orbit* $\mathrm{Orb}_{\mathfrak{B}}(z)$ *with the linear subspace $\mathfrak{a}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$.*

Proof. We need to show that if $\nabla f = 0$, then $(z_1 e_{\beta_1}, \ldots, z_n e_{\beta_n}) \in \mathfrak{a}_{\mathbb{C}}$. Since $\nabla f = 0$ exactly when $\sum_{i=1}^n z_i e_{\beta_i} \beta_i = 0$, this immediately follows from Lemma 1.1. □

This proposition makes contact between the dual toric variety and the second homology of the original toric variety. What is more, clearly, the functions that appear in the definition of $\langle P\rangle_{\mathfrak{B}}(z)$ in (4.1) all come as restrictions of functions from the ambient space $\mathfrak{g}_{\mathbb{C}}$ identified with $\mathbb{C}^n$ by $(x_1, \ldots, x_n) \mapsto \sum_{i=1}^n x_i \omega_i$. Indeed, P was a polynomial in n variables, f is the restriction of the function $1 - \sum_{i=1}^n x_i$ and the Hessian may be considered as the restriction of the function

$$D^{\mathfrak{B}}(x) = \det\left(\sum_{i=1}^{n} \langle h_j, \beta_i\rangle\langle h_k, \beta_i\rangle x_i\right)_{j,k=1}^{d}, \tag{4.4}$$

which is a degree d polynomial on $\mathfrak{g}$.

Now that we may think of $O_{\mathfrak{B}}(z)$ as a finite subset of $\mathfrak{a}_{\mathbb{C}}$, we would like to know something about its geometry. Fix a $\mathbb{Z}$-basis $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$ of $\Gamma_{\mathfrak{a}}$, not necessarily a $\mathfrak{c}$-positive basis, and recall the notation

$$p_j(u) = p_j^+(u)/p_j^-(u) = \prod_{i=1}^{n} \alpha_i(u)^{\langle\alpha_i,\lambda_j\rangle}, \quad \text{for } j = 1, \ldots, r.$$

Also, having fixed an appropriate $z \in \mathfrak{g}_{\mathbb{C}}$, with $z_i \neq 0$ for all i, again denote by q_j the number z^{λ_j}. As $\mathfrak{a}_{\mathbb{C}}$ is embedded in $\mathfrak{g}_{\mathbb{C}}$ by $u \mapsto (\alpha_1(u), \ldots, \alpha_n(u))$, the set $O_{\mathfrak{B}}(z)$ is contained in $U(\mathfrak{A})$. Thus our set of critical points of the function f on $T_{\mathbb{C}\mathfrak{t}^*}$ becomes a finite subset of $U(\mathfrak{A})$. As such, it is cut out from $U(\mathfrak{A})$ by some equations.

Lemma 4.3. *We have*

$$O_{\mathfrak{B}}(z) = \{u \in U(\mathfrak{A});\ p_j(u) = q_j,\ j = 1, \ldots, r\}.$$

Proof. Considering Proposition 4.2, the statement follows if we show that the torus $T_{\mathbb{C}\mathfrak{t}^*}$ embedded via (4.3) is cut out from $\mathfrak{g}_{\mathbb{C}}$ by the equations

$$\prod_{i=1}^{n} x_i^{\langle\alpha_i,\lambda_j\rangle} = q_j,\ j = 1, \ldots, r.$$

Thus what we need to show is that if $\prod_{i=1}^{n} x_i^{\langle\alpha_i,\lambda_j\rangle} = 1$, for $j = 1, \ldots, r$, then for some $h \in \mathfrak{t}_{\mathbb{C}}^*$ we have $x_i = e^{2\pi\sqrt{-1}\langle h,\beta_i\rangle}$. Representing x_i as $e^{2\pi\sqrt{-1}l_i}$ and using the fact that $\boldsymbol{\lambda}$ is a basis of $\Gamma_{\mathfrak{a}}$ over $\mathbb{Z}$, we see that $\sum_{i=1}^{n} l_i\alpha_i \in \Gamma_{\mathfrak{a}}^*$. According to our assumptions, the α_is generate the lattice $\Gamma_{\mathfrak{a}}^*$ over $\mathbb{Z}$. Since the l_is are defined only up to integers, by choosing them appropriately, we may assume that $\sum_{i=1}^{n} l_i\alpha_i = 0$. Then the basic property of Gale duality, Lemma 1.1, completes the proof. □

We can summarize our results so far as follows: we have

$$\langle P\rangle_{\mathfrak{B}}(z) = \sum \frac{P(\alpha_1(u), \ldots, \alpha_n(u))}{(1-\kappa(u))D^{\mathfrak{B}}(\alpha_1(u), \ldots, \alpha_n(u))}, \tag{4.5}$$

where, as usual, $\kappa = \sum_{i=1}^{n} \alpha_i$, and the sum runs over the finite set $\{u \in U(\mathfrak{A});\ p_j(u) = q_j,\ j = 1, \ldots, r\}$.

The statement of Theorem 4.1 is thus reduced to showing that the integral in (3.12) of a rational differential form, which we denote by Λ, over the cycle $Z[\mathfrak{c}] \subset U(\mathfrak{A}) \subset \mathfrak{a}_{\mathbb{C}}$ is equal to the expression in (4.5): a finite sum of the values of a rational function over a finite set of common zeros of r other rational functions. The first step of the proof, completed in this section, will be showing that this finite sum also has a representation as an integral of the same form Λ over a different cycle. The second step, which will take up the rest of the paper, will be showing the equivalence of the two cycles.

First, we compute the coefficients of the polynomial $D^{\mathfrak{B}}(x)$ defined in (4.4) explicitly.

Lemma 4.4. *We have*

$$D^{\mathfrak{B}}(x) = \sum_{\bar{\sigma}\in \mathrm{BInd}(\mathfrak{B})} \mathrm{vol}_{\mathfrak{t}}(\bar{\gamma}^{\bar{\sigma}})^2 \prod_{i\in\bar{\sigma}} x_i. \tag{4.6}$$

Proof. Thinking of the vectors β_i, as d-component column vectors written in the basis $\{h_k\}_{k=1}^d$, we can write the matrix $M(x)$ the determinant of which is $D^{\mathfrak{B}}(x)$ as

$$M(x) = \sum_{i=1}^n x_i\beta_i\beta_i^T,$$

where β^T is the transposed matrix: a row vector. Using the fact that each of the terms in this sum is a rank-1 matrix, we can expand $\det(M(x))$ as

$$\det(M(x)) = \sum_{\bar{\sigma}\in \mathrm{BInd}(\mathfrak{B})} \det\left(\sum_{i\in\bar{\sigma}} x_i\beta_i\beta_i^T\right).$$

The term of this sum corresponding to the basis $\bar{\sigma} \in \mathrm{BInd}(\mathfrak{B})$, written in the basis $\bar{\sigma}$ itself, is simply a diagonal matrix with entries $\{x_i;\ i \in \bar{\sigma}\}$ on the diagonal. This immediately implies (4.6). □

Next we compute the Jacobian matrix of the vector valued function

$$p = (p_1, \ldots, p_r) : U(\mathfrak{A}) \to \mathbb{C}^{*r}. \tag{4.7}$$

Proposition 4.5. *Define the rational function $D_{\mathfrak{A}}$ on $\mathfrak{g}_{\mathbb{C}}$ as*

$$D_{\mathfrak{A}}(x) = \det\left(\sum_{i=1}^n \frac{\langle\alpha_i,\lambda_l\rangle\langle\alpha_i,\lambda_m\rangle}{x_i}\right)_{l,m=1}^r.$$

Then we have

(1) $D_{\mathfrak{A}}(x) = \sum_{\sigma\in \mathrm{BInd}(\mathfrak{A})} \mathrm{vol}_{\mathfrak{a}^*}(\gamma^{\sigma})^2 \prod_{i\in\sigma} \frac{1}{x_i},$

(2) $D_{\mathfrak{A}}(x) \prod_{i=1}^n x_i = D^{\mathfrak{B}}(x).$

(3) $\frac{dp_1}{p_1} \wedge \cdots \wedge \frac{dp_r}{p_r}(u) = D_{\mathfrak{A}}(\alpha_1(u), \ldots, \alpha_n(u))\ d\mu_{\Gamma}^{\mathfrak{a}}.$

Proof. The proof of (1) is exactly the same as that of Lemma 4.4. Then (1) and Lemma 4.4 together with Lemma 1.3 imply (2). Finally, (3) is a simple calculation: Taking the partial derivative of p_l with respect to λ_m is exactly p_l times the corresponding entry of the matrix in the definition of $D_{\mathfrak{A}}(x)$. □

Corollary 4.6. *The map $p : U(\mathfrak{A}) \to \mathbb{C}^{*r}$ is generically nonsingular.*

Indeed, statements (1) and (3) of Proposition 4.5 compute the Jacobian of this map explicitly. Since $\mathfrak{A}$ is projective, there is a $u \in U(\mathfrak{A})$ such that $\alpha_i(u) > 0$, $i = 1, \ldots, n$, and at such u the sum in statement (1) is clearly positive. □

Now we are ready to present our residue formula for $\langle P \rangle_{\mathfrak{B}}(z)$. For $z \in (\mathbb{C}^*)^n$ and $\delta > 0$ let

$$Z_\delta(\boldsymbol{\lambda}, q) = \{u \in U(\mathfrak{A});\ |p_j(u) - q_j| = \delta,\ j = 1, \ldots, r\}$$

oriented by the form $d\arg(p_1 - q_1) \wedge \cdots \wedge d\arg(p_r - q_r)$.

Proposition 4.7. *Let $z \in \mathbb{C}^{*r}$ be such that the set $O_{\mathfrak{B}}(z) \subset U(\mathfrak{A})$ is finite and the function $(1-\kappa)D^{\mathfrak{B}}(\alpha_1, \ldots, \alpha_n)$ does not vanish on it, and let $U(z)$ be a small neighborhood of $O_{\mathfrak{B}}(z)$ in $U(\mathfrak{A})$. Then for sufficiently small $\delta > 0$, we have*

(4.8)

$$\langle P \rangle_{\mathfrak{B}}(z) = \frac{1}{\left(2\pi\sqrt{-1}\right)^r} \int_{Z_\delta(\boldsymbol{\lambda}, q) \cap U(z)} \frac{P(\alpha_1, \ldots, \alpha_n) \prod_{j=1}^r p_j\ d\mu_\Gamma^{\mathfrak{a}}}{(1-\kappa) \prod_{i=1}^n \alpha_i \prod_{j=1}^r (p_j - q_j)}.$$

Proof. Consider the function

$$R = \frac{P(\alpha_1, \ldots, \alpha_n)}{D^{\mathfrak{B}}(\alpha_1, \ldots, \alpha_n)(1-\kappa)}$$

and the differential form

$$\omega = \frac{dp_1}{p_1 - q_1} \wedge \cdots \wedge \frac{dp_r}{p_r - q_r}$$

on $U(\mathfrak{A})$. Because of our assumptions, ω has a simple pole with residue equal 1 at each of the points of the finite set $O_{\mathfrak{B}}(z)$, and the function R is regular at these points. Our computation of the Jacobian of the map p shows that the divisors $\{u;\ p_j(u) = q_j\}$, $j = 1, \ldots, r$ intersect transversally at these points, and thus for small δ the set $Z_\delta(\boldsymbol{\lambda}, q)$ consists of tiny tori, one for each point of $O_{\mathfrak{B}}(z)$ plus, possibly, some additional components which we eliminate using the neighborhood $U(z)$ of $O_{\mathfrak{B}}(z)$.

Then according to the usual integral representation of residues, we have

$$\langle P \rangle_{\mathfrak{B}}(z) = \frac{1}{\left(2\pi\sqrt{-1}\right)^r} \times \int_{Z_\delta(\boldsymbol{\lambda}, q) \cap U(z)} \frac{P(\alpha_1, \ldots, \alpha_n) \bigwedge_{j=1}^r dp_j}{(1-\kappa) D^{\mathfrak{B}}(\alpha_1, \ldots, \alpha_n) \prod_{j=1}^r (p_j - q_j)}.$$

Substituting the expressions from (2) and (3) of Proposition 4.5 into this formula, we obtain (4.8). $\square$

Remark 4.2. 1. Note that, "miraculously", the differential form under the integral sign here coincides with that in Proposition 3.4.

2. We will show later that for z in a certain domain, the conditions of the proposition hold, moreover, $Z_\delta(\boldsymbol{\lambda}, q)$ is a genuine cycle, i.e. it is localized in a small neighborhood of $O_{\mathfrak{B}}(z)$ and has no non-compact components. This last statement is equivalent to the *properness* of the map p defined in (4.7), which, as we will see, turns out to be a subtle question.

5. Tropical calculations

In this section we only assume that $\mathfrak{A}$ is a projective sequence in $\Gamma^*_{\mathfrak{a}}$. Recall that for each $\lambda \in \Gamma_{\mathfrak{a}}$ we defined a rational function $p_\lambda(u) = \prod_{i=1}^n \alpha_i(u)^{\langle\alpha_i,\lambda\rangle}$ on $\mathfrak{a}_{\mathbb{C}}$, which is regular on $U(\mathfrak{A})$; these functions p_λ satisfy the relation

$$p_{\lambda_1+\lambda_2}(u) = p_{\lambda_1}(u)p_{\lambda_2}(u). \tag{5.1}$$

Definition 5.1. *Let $\xi \in \mathfrak{a}^*$. Define the set*

$$\widehat{Z}(\xi) = \{u \in U(\mathfrak{A});\ \ |p_\lambda(u)| = e^{-\langle\xi,\lambda\rangle} \textit{ for all } \lambda \in \Gamma_{\mathfrak{a}}\}.$$

Fix a $\mathbb{Z}$-basis $(\lambda_1, \lambda_2, \dots, \lambda_r)$ of the lattice $\Gamma_{\mathfrak{a}}$, which is a positively oriented relative to our chosen orientation of $\mathfrak{a}$, and introduce the notation p_j for p_{λ_j}. Then it follows from (5.1) that the set $\widehat{Z}(\xi)$ is the subset of $U(\mathfrak{A})$ defined by the r analytic equations:

$$|p_j(u)| = e^{-\langle\xi,\lambda_j\rangle},\ j = 1, \dots, r. \tag{5.2}$$

In particular, $\widehat{Z}(\xi)$ is an r-dimensional analytic subset of $U(\mathfrak{A})$. We can orient $\widehat{Z}(\xi)$ by the form $d\arg p_1 \wedge \cdots \wedge d\arg p_r$. It is easy to see that this orientation does not depend on the chosen positively oriented basis.

The aim of this section is to prove that under some mild conditions the cycle $\widehat{Z}(\xi)$ is smooth, and compute its homology class in $U(\mathfrak{A})$.

Recall from Sect. 2, that to each flag F in $\mathcal{FL}(\mathfrak{A})$, one can associate a homology class $h(F) \in H_r(U(\mathfrak{A}), \mathbb{Z})$ (Definition 2.1), a sign $\nu(F)$ given in Definition 2.3, and a sequence of vectors

$$\kappa_j^F = \sum\{\alpha_i;\ i = 1, \dots, n,\ \alpha_i \in F_j\},$$

given in (2.5), which one can collect into a sequence of r vectors denoted by $\boldsymbol{\kappa}^F$. By convention, we set $\kappa_0^F = 0$. Then for a flag $F \in \mathcal{FL}(\mathfrak{A})$, introduce the non-acute cone $\mathfrak{s}(F, \mathfrak{A})$ generated by the non-negative linear combinations of the elements $\{\kappa_j^F,\ j = 1, \dots, r-1\}$ and the line $\mathbb{R}\kappa$:

$$\mathfrak{s}(F, \mathfrak{A}) = \sum_{j=1}^{r-1} \mathbb{R}^{\geq 0}\kappa_j + \mathbb{R}\kappa.$$

Definition 5.2. *Let $\xi \in \mathfrak{a}^*$. We denote by $\mathcal{FL}(\mathfrak{A}, \xi)$ the set of flags $F \in \mathcal{FL}(\mathfrak{A})$ such that $\xi \in \mathfrak{s}(F, \mathfrak{A})$.*

Finally, recall from Sect. 2 that if ξ is sum-regular, then every flag $F \in \mathcal{FL}(\mathfrak{A}, \xi)$ is proper, and thus has $\nu(F) \neq 0$. The aim of this section is to prove the following theorem.

Theorem 5.1. *Let $\mathfrak{A}$ be a projective sequence and let ξ be a τ-regular element in $\mathfrak{a}^*$ with τ sufficiently large. Then the set $\widehat{Z}(\xi)$ is a smooth, compact r-dimensional submanifold in $U(\mathfrak{A})$. When oriented by the form $d\arg p_1 \wedge d\arg p_2 \wedge \cdots \wedge d\arg p_r$, it defines a cycle in $U(\mathfrak{A})$ whose homology class $[\widehat{Z}(\xi)]$ is given by*

$$[\widehat{Z}(\xi)] = \sum \nu(F)h(F), \qquad F \in \mathcal{FL}(\mathfrak{A}, \xi).$$

The proof of the theorem will start in Sect. 5.2 and will end with Proposition 5.15. The compactness is contained in Corollary 5.8, the smoothness in Corollary 5.11 and the computation of the homology class follows from Lemma 5.14 and Proposition 5.15.

5.1. The tropical equations. Our main tool of study of the set $\widehat{Z}(\xi)$ is considering the logarithm of the equations (5.2), which can be written as

$$\prod_{i=1}^{n} |\alpha_i(u)|^{\langle \alpha_i, \lambda_j \rangle} = e^{-\langle \xi, \lambda_j \rangle}, \; j = 1, \ldots, r.$$

This idea is related to tropical geometry [19, 18]. The logarithmic equations take the form

$$\sum_{i=1}^{n} \log |\alpha_i(u)| \langle \alpha_i, \lambda_j \rangle = -\langle \xi, \lambda_j \rangle, \; j = 1, \ldots, r.$$

This, in turn, can be written as a single vector equation:

$$-\sum_{i=1}^{n} \log |\alpha_i(u)| \, \alpha_i = \xi. \tag{5.3}$$

Define the map $L : U(\mathfrak{A}) \to \mathfrak{g}^*$ by

$$L(u) = -\sum_{i=1}^{n} \log |\alpha_i(u)| \omega^i.$$

Thus L is a map from a real $2r$-dimensional space to an n-dimensional one. If u tends to 0, then the point $L(u)$ tends to ∞.

Recall from Sect. 1 that we denoted by μ the linear map $\mu : \mathfrak{g}^* \to \mathfrak{a}^*$, which sends ω^i to α_i. Then we clearly have

$$\left| p_j(u) \right| = e^{-\langle \mu(L(u)), \lambda_j \rangle}, \tag{5.4}$$

and thus $u \in \widehat{Z}(\xi)$ if and only if $\mu(L(u)) = \xi$. Another way to write this is that $\widehat{Z}(\xi) = (\mu \circ L)^{-1}(\xi)$.

Our strategy is to separate the solution of (5.3) into two parts, using that solutions to (5.3) arise when the affine linear subspace $\mu^{-1}(\xi) \subset \mathfrak{g}^*$ of codimension r intersects the image $\mathrm{im}(L)$ of the map L.

Our next move is to give more precise information about the map L and its image $\mathrm{im}(L)$. Roughly, the idea is as follows. Since

$$\log|\alpha_1(u) + \alpha_2(u)| \sim \max(\log|\alpha_1(u)|, \log|\alpha_2(u)|)$$

if $\alpha_1(u)$ and $\alpha_2(u)$ have different orders of magnitude, we will be able to approximate the map L with a piecewise linear map from the r-dimensional space $U(\mathfrak{A})$ to $\mathfrak{g}^*$. Thus we will show that under some conditions the image $\mathrm{im}(L) \subset \mathfrak{g}^*$ is confined in a small neighborhood of a finite set of affine linear subspaces of dimension r which are transversal to $\mu^{-1}(\xi)$. This will allow us to describe the set $\widehat{Z}(\xi)$ rather precisely.

Thus consider the affine subspace $\mu^{-1}(\xi) \subset \mathfrak{g}^*$. It consists of the solutions $(t_1, t_2, \dots, t_n)$ of the equation

$$\mathrm{Part}(\xi) := \left\{ \sum_{i=1}^{n} t_i\alpha_i = \xi \right\}. \tag{5.5}$$

Often we will speak about the solutions of $\mathrm{Part}(\xi)$ rather than about the set $\mu^{-1}(\xi)$. Also, we will identify $\mathfrak{g}^*$ with $\mathbb{R}^n$ whenever convenient, using the basis ω^i. Motivated by Proposition 5.5 below, we will be interested in a special type of solutions of $\mathrm{Part}(\xi)$ for which several of the coordinates will be set equal.

Definition 5.3. *Let F be a flag in $\mathcal{FL}(\mathfrak{A})$ and $B = (B_1, B_2, \dots, B_r)$ be a sequence of r real numbers. Define the point $t(F, B) = \sum_{i=1}^{n} t_i\omega^i \in \mathfrak{g}^*$ by the condition $t_i = B_j$ if $\alpha_i \in F_j \setminus F_{j-1}$. We will say that a solution $t \in \mathfrak{g}^*$ is an F*-solution *of* $\mathrm{Part}(\xi)$ (5.5) *if t is of the form $t(F, B)$ for some sequence B of real numbers.*

Thus we see that $t(F, B)$ is a solution of $\mathrm{Part}(\xi)$ if and only if

$$\sum_{j=1}^{r} B_j\left(\kappa_j^F - \kappa_{j-1}^F\right) = \xi;$$

this can be rewritten as :

$$B_r\kappa + \sum_{j=1}^{r-1}(B_j - B_{j+1})\kappa_j^F = \xi. \tag{5.6}$$

Recall that a flag is proper, if the elements κ_j^F are linearly independent. The following statement then clearly follows:

Lemma 5.2. *For a proper flag F, there is exactly one solution of* Part(ξ) *of the form $t(F, B)$.*

We denote this solution by $\mathrm{sol}(F, \xi)$.

For any flag F, consider the system

$$\mathrm{Eq}(F) := \{t_b = t_c;\ \alpha_b, \alpha_c \in F_j \setminus F_{j-1} \text{ for some } j \le r\}. \tag{5.7}$$

The solution set $G(F)$ of $\mathrm{Eq}(F)$ is a linear subspace of $\mathfrak{g}^*$ of dimension r spanned by the vectors

$$s^{F,j} = \sum \left\{\omega^i;\ i = 1, \dots, n, \alpha_i \in F_j \setminus F_{j-1}\right\}, \quad j = 1, \dots, r.$$

This subspace is transversal to the subspace $\mu^{-1}(0)$ of $\mathfrak{g}^*$ if and only F is a proper flag. Indeed, the images of the vectors $s^{F,j}$ under μ are equal to $\kappa_j^F - \kappa_{j-1}^F$, so they span $\mathfrak{a}^*$ if and only if the vectors κ_j^F do so. Another way to state Lemma 5.2 is to say that whenever F is proper, then $G(F) \cap \mu^{-1}(\xi)$ is non-empty and consists of the single point $\mathrm{sol}(F, \xi)$. Also note that if ξ is regular with respect to $\Sigma\mathfrak{A}$ and there is an F-solution of Part(ξ), then F is necessarily proper. Indeed the equation (5.6) implies that ξ is in the span of the vectors κ_j^F belonging to $\Sigma\mathfrak{A}$.

Now we introduce a particular kind of F-solutions, which arise naturally in our study of the set $\widehat{Z}(\xi)$.

Definition 5.4. *Fix ξ, let F be a flag in $\mathcal{FL}(\mathfrak{A})$. We will say that a solution $t \in \mathfrak{g}^*$ is a* tropical F-solution *of* Part(ξ) (5.5) *if t is of the form $t(F, B)$ for some* **decreasing** *sequence $B = (B_1, B_2, \dots, B_r)$ of r real numbers, that is with $B_1 \ge \cdots \ge B_r$. A solution of the equation* Part(ξ) *of the form $t(F, B)$ for some flag F and some decreasing sequence B will be called a* tropical solution *of* Part(ξ). *Finally, denote by $\mathcal{FL}(\mathfrak{A}, \xi)$ the set of flags $F \in \mathcal{FL}(\mathfrak{A})$ for which* Part(ξ) *has a tropical F-solution.*

The following statement clearly follows from (5.6).

Lemma 5.3. *The flag F belongs to $\mathcal{FL}(\mathfrak{A}, \xi)$ if and only if $\xi \in \mathfrak{s}(F, \mathfrak{A})$.*

From now on, we will always assume that ξ is regular with respect to $\Sigma\mathfrak{A}$. In particular, this implies that all flags $F \in \mathcal{FL}(\mathfrak{A}, \xi)$ are proper.

5.2. Compactness. Now we are ready to start the *Proof of Theorem 5.1.* We will show that when τ is sufficiently large, then the cycle $\widehat{Z}(\xi)$ is a disjoint union of compact smooth components $\widehat{Z}^F(\xi)$ associated to flags $F \in \mathcal{FL}(\mathfrak{A}, \xi)$. The component $\widehat{Z}^F(\xi)$ will lie in an open set $U(F, N(\tau))$, where $N(\tau)$ is increasing exponentially with τ. The sets $U(F, N)$ were defined before Lemma 2.4. The homology class of $\widehat{Z}^F(\xi)$ will be a generator of the rth homology of this set.

The first idea is that if ξ is τ-regular, then for every $u \in \widehat{Z}(\xi)$, there exists a flag $F \in \mathcal{FL}(\mathfrak{A}, \xi)$ such that $L(u)$ is close to the tropical solution $\mathrm{sol}(F, \xi)$. In fact, there is a better approximation, as we show below.

In order to obtain a more precise estimate, we need to modify the system $\mathrm{Eq}(F)$. As the space F_j/F_{j-1} is 1-dimensional, for two vectors $\alpha_b, \alpha_c \in F_j \setminus F_{j-1}$ there exists a unique nonzero rational number m_{bc} such that $\alpha_b - m_{bc}\alpha_c \in F_{j-1}$. Then let

$$\widetilde{\mathrm{Eq}}(F) := \{t_c - t_b = \log|m_{bc}|;\ \alpha_b, \alpha_c \in F_j \setminus F_{j-1} \text{ for some } j \leq r\}. \tag{5.8}$$

The solution set $\widetilde{G}(F)$ of $\widetilde{\mathrm{Eq}}(F)$ in $\mathfrak{g}^*$ is an affine r-dimensional subspace parallel to the solution set of $\mathrm{Eq}(F)$. To be more specific, we construct concrete solutions of $\widetilde{\mathrm{Eq}}(F)$ as follows. Consider the ordered basis $\boldsymbol{\gamma}^F = (\gamma_1^F, \ldots, \gamma_r^F)$ of $\mathfrak{a}^*$ that we introduced earlier. It is such that $\{\gamma_m^F\}_{m=1}^j$ is a basis of F_j for $j = 1, \ldots, r$. For $\alpha_b \in F_j \setminus F_{j-1}$, define the rational number m_b such that $\alpha_b - m_b\gamma_j \in F_{j-1}$. Then the point $\sum_{i=1}^n -\log|m_i|\,\omega^i$ belongs to $\widetilde{G}(F)$, and $\widetilde{G}(F)$ is the affine space parallel to $G(F)$ through this point. This implies

Lemma 5.4. *Let F be a proper flag. Then the solution spaces of the systems of linear equations* $\mathrm{Part}(\xi)$ *and* $\widetilde{\mathrm{Eq}}(F)$*,* $\mu^{-1}(\xi)$ *and* $\widetilde{G}(F)$ *respectively, are transversal and of complementary dimensions.*

It follows from this lemma that there is a unique common solution of these equations; we denote this solution by $\widetilde{\mathrm{sol}}(\xi, F)$.

The following is a key technical statement of this paper, which justifies the validity of our tropical approximation.

Given a vector $t = (t_1, \ldots, t_n) \in \mathfrak{g}^*$, let $\|t\| = \max_{i=1}^n |t_i|$ be the maximum norm of t.

Proposition 5.5. *There exist positive constants τ_0, c_0 and c_1, which depend on $\mathfrak{A}$ only, such that if $\tau \geq \tau_0$ and ξ is τ-regular, then for every $u \in \widehat{Z}(\xi)$ there exists a flag $F \in \mathcal{FL}(\mathfrak{A}, \xi)$ such that*

$$\|L(u) - \widetilde{\mathrm{sol}}(F, \xi)\| \leq c_0 e^{-c_1\tau}.$$

We start the proof with two lemmas.

Lemma 5.6. *Let $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_r)$ be an ordered basis of a real vector space V^* of dimension r. Let $u \in V_{\mathbb{C}}$ be such that $\gamma_j(u) \neq 0$ for $j = 1, \ldots, r$, and set $B_j = -\log|\gamma_j(u)|$. There are positive constants λ_0, c_0 such that if $\lambda \geq \lambda_0$ and $B_j - B_{j+1} > \lambda$ for $j = 1, \ldots, r-1$, then the following holds: let $1 \leq a \leq r$ and $\alpha = m_1\gamma_1 + \cdots + m_a\gamma_a \in V^*$ a vector with $m_a \neq 0$. Then $\alpha(u) \neq 0$, and furthermore,*

$$\big|\log|\alpha(u)/m_a| - \log|\gamma_a(u)|\big| \leq c_0 e^{-\lambda}. \tag{5.9}$$

The constants λ_0, c_0 depend only on the data $(\boldsymbol{\gamma}, \alpha)$ and not on the element u satisfying the hypothesis of the lemma.

Remark 5.1. Consider λ a positive constant. Introduce the set

$$U[\lambda] = \{u \in V_{\mathbb{C}}\ ;\ |\log|\gamma_j(u)| - \log|\gamma_k(u)|| \geq \lambda,\ \text{for all } j \neq k\}.$$

The lemma above implies that, provided λ is sufficiently large, the distance between $\log|\alpha(u)|$ and the finite set $\{\log|\gamma_j(u)|, j = 1, \dots, r\}$ remains bounded as u varies in the open set $U[\lambda]$.

Proof of Lemma 5.6. For $u \in V_{\mathbb{C}}$, with $\gamma_a(u) \neq 0$ we can write

$$\alpha(u) = m_a\gamma_a(u)\left(1 + \sum_{k=1}^{a-1} \frac{m_k}{m_a}\frac{\gamma_k(u)}{\gamma_a(u)}\right).$$

This gives

$$\left(\frac{\alpha(u)}{m_a\gamma_a(u)} - 1\right) = \sum_{k=1}^{a-1} \frac{m_k}{m_a}\frac{\gamma_k(u)}{\gamma_a(u)}.$$

Now assume $B_j - B_{j+1} \geq \lambda$ for $j = 1, 2, \dots, r-1$. Then for $k < a$, we have $|\gamma_k(u)/\gamma_a(u)| = e^{-(B_k - B_a)} \leq e^{-\lambda}$.

Define $\delta = \sum_{k=1}^{a-1} |m_k/m_a|$. We obtain $|\sum_{k=1}^{a-1} m_k\gamma_k(u)/m_a\gamma_a(u)| \leq \delta e^{-\lambda}$ and thus

$$1 - \delta e^{-\lambda} \leq \left|\frac{\alpha(u)}{m_a\gamma_a(u)}\right| \leq 1 + \delta e^{-\lambda}.$$

Let λ_0 be such that $\delta e^{-\lambda_0} \leq \frac{1}{2}$, and assume that $\lambda \geq \lambda_0$. Then $\alpha(u) \neq 0$, and taking logarithms and using the inequalities $\log(1+x) \leq x$, $\log(1-x) \geq -2x$ for $0 \leq x \leq \frac{1}{2}$, we obtain

$$-2\delta e^{-\lambda} \leq \log|\alpha(u)/m_a| - \log|\gamma_a(u)| \leq \delta e^{-\lambda}.$$

Thus the estimate of the lemma holds with $c_0 = 2\delta$, $\lambda_0 = \log(2\delta)$. $\square$

For $\alpha \in \mathfrak{a}^*$ and a basis $\boldsymbol{\rho} \subset \Sigma\mathfrak{A}$ of $\mathfrak{a}^*$, we can write

$$\alpha = \sum_{\gamma \in \boldsymbol{\rho}} u_\gamma^{\boldsymbol{\rho}}(\alpha)\gamma.$$

Introduce the constant

$$M(\mathfrak{A}) = \max\{|u_\gamma^{\boldsymbol{\rho}}(\alpha_i)|;\ i = 1, \dots, n,\ \boldsymbol{\rho} \text{ basis of } \mathfrak{a}^*,\ \boldsymbol{\rho} \subset \Sigma\mathfrak{A},\ \gamma \in \boldsymbol{\rho}\}.$$

Lemma 5.7. *Let $\xi \in \mathfrak{a}^*$ be a τ-regular vector and $t = (t_1, t_2, \dots, t_n)$ a solution of* Part(ξ). *Then there exists an r-element subset $\sigma \subset \{1, \dots, n\}$, such that*

$$|t_i - t_j| \geq c_1\tau \text{ for } i, j \in \sigma,\ i \neq j,\ \text{where } c_1 = \frac{1}{nM(\mathfrak{A})}.$$

Proof. Let $\sigma \subset \{1, 2, \dots, n\}$ be a maximal subset satisfying the condition $|t_i - t_j| \geq c_1\tau$ for all $i, j \in \sigma$, $i \neq j$. We will arrive at a contradiction, assuming that the cardinality of σ is strictly less than r. For every $k \notin \sigma$, there exists $a(k) \in \sigma$ such that $|t_k - t_{a(k)}| < c_1\tau$. We can write

$$\xi = \sum_{i\in\sigma} t_i\alpha_i + \sum_{k\notin\sigma} t_k\alpha_k = \sum_{i\in\sigma} t_i\alpha_i + \sum_{k\notin\sigma}(t_k - t_{a(k)})\alpha_k + t_{a(k)}\alpha_k$$
$$= \sum_{i\in\sigma}\sum_{k\notin\sigma,a(k)=i} t_i\left(\alpha_i + \sum_{k\notin\sigma,a(k)=i}\alpha_k\right) + \sum_{k\notin\sigma}(t_k - t_{a(k)})\alpha_k.$$

Consider the set $\boldsymbol{\rho}_\sigma = \{\alpha_i + \sum_{k\notin\sigma,a(k)=i}\alpha_k;\ i \in \sigma\}$. This set is a subset of $\Sigma\mathfrak{A}$, and it spans a vector space of dimension strictly less than r. By passing to a subset if necessary, we can assume that the set $\boldsymbol{\rho}_\sigma \subset \Sigma\mathfrak{A}$ is linearly independent. Let $\boldsymbol{\rho} \subset \Sigma\mathfrak{A}$ be a basis of $\mathfrak{a}^*$ containing $\boldsymbol{\rho}_\sigma$. For an element $\gamma \in \boldsymbol{\rho} \setminus \boldsymbol{\rho}_\sigma$ we can write the γ-coordinate of ξ as

$$u_\gamma^{\boldsymbol{\rho}}(\xi) = \sum_{k\notin\sigma}(t_k - t_{a(k)})u_\gamma^{\boldsymbol{\rho}}(\alpha_k).$$

Each number $|u_\gamma^{\boldsymbol{\rho}}(\alpha_k)|$ is less or equal than $M(\mathfrak{A})$, and there are at most n terms of this kind. Thus $|u_\gamma^{\boldsymbol{\rho}}(\xi)| < \tau$. But this contradicts the τ-regularity of ξ. $\square$

Proof of Proposition 5.5. Fix a τ-regular vector $\xi \in \mathfrak{a}^*$, and let $u \in \widehat{Z}(\xi)$. The vector $L(u)$ satisfies Part(ξ), and we can apply Lemma 5.7; denote the index subset guaranteed by the lemma by $\sigma(u)$.

Let $(B_1(u), \dots, B_r(u))$ be the set of numbers $\{t_i;\ i \in \sigma(u)\}$ arranged in decreasing order, and if $B_j(u) = t_i$, then we will write γ_j for α_i. Then we have

$$|\gamma_j(u)| = e^{-B_j(u)} \text{ with } B_j(u) - B_{j+1}(u) \geq c_1\tau. \tag{5.10}$$

Next assuming that τ is sufficiently large, we need to show that the vectors $\{\gamma_j;\ j = 1, \dots r\}$ are linearly independent. We may use Lemma 5.6. Indeed, assume that there is a linear relation between these vectors. Let γ_j and γ_k be the two vectors with the largest indices that have non-vanishing coefficients in this relation. Then according to (5.9), we would have $|B_j(u) - B_k(u)| < m$ for some constant m that only depends on $\mathfrak{A}$. This clearly cannot happen if τ is sufficiently large.

Next, denote by $F(u)$ the flag in $\mathcal{FL}(\mathfrak{A})$ given by the sequences of subspaces $F_j = \sum_{k=1}^{j}\mathbb{C}\gamma_k$, $j = 1, \dots, r$. Now we estimate $\|L(u) - \widetilde{\mathrm{sol}}(F(u), \xi)\|$. According to Lemma 5.6, for $\alpha_i \in F_j \setminus F_{j-1}$ we have

$$|\log|\alpha_i(u)| - \log|\gamma_j(u)| - \log|m_i|| \leq c_2 e^{-c_1\tau}, \tag{5.11}$$

where m_i is a constant such that $\alpha_i - m_i\gamma_j \in F_{j-1}$, and c_1, c_2 are positive constants depending on $\mathfrak{A}$ only.

Note that this equation implies that F is in $\mathcal{FL}(\mathfrak{A}, \xi)$, provided τ is sufficiently large. Indeed, the equation $\mu(L(u)) = \xi$ implies that

$$\left\| \xi - \left(-\log|\gamma_r(u)|\kappa + \sum_{j=1}^{r-1} (\log|\gamma_{j+1}(u)| - \log|\gamma_j(u)|)\kappa_j^F \right) \right\|$$

is uniformly bounded. This implies that for τ large the vectors κ_j^F have to be linearly independent, as otherwise ξ would not be τ-regular. So F is a proper flag; furthermore, for $j = 1, \dots, r-1$, the number $\log|\gamma_{j+1}(u)| - \log|\gamma_j(u)| = B_j(u) - B_{j+1}(u)$ is positive, bounded below by a quantity that increases linearly with τ. Thus ξ is in the cone $\mathfrak{s}(F, \mathfrak{A})$ and $F \in \mathcal{FL}(\mathfrak{A}, \xi)$.

Let us look more closely at the equation (5.11). The point $\tilde{t} = \sum_{i=1}^n \tilde{t}_i \omega^i$ with coordinate $\tilde{t}_i = -(\log|m_i| + \log|\gamma_j(u)|)$ belongs to the affine space $\widetilde{\mathrm{Eq}}(F)$ defined in (5.8). The point $L(u)$ belongs to a translate $G_u(F)$ of the linear subspace $G(F) \subset \mathfrak{g}^*$, the solution set of $\mathrm{Eq}(F)$. Thus this translate is at distance constant times $e^{-c_1\tau}$ from the solution set $\widetilde{G}(F)$ of $\widetilde{\mathrm{Eq}}(F)$. Since $L(u)$ also satisfies $\mathrm{Part}(\xi)$, using Lemma 5.4, we see that $L(u)$, which is the intersection point of $G_u(F)$ and the solution set of $\mathrm{Part}(\xi)$, is not further from $\widetilde{\mathrm{sol}}(F, \xi)$ than a constant times $e^{-c_1\tau}$. □

Using Proposition 5.5 and its proof, we can describe the structure of $\widehat{Z}(\xi)$ as follows.

Recall the definition and properties of the open sets $U(F, N)$, and the constant N_0 given in Lemma 2.4. Introduce the sets

$$\widehat{Z}^F(\xi) = \widehat{Z}(\xi) \cap U(F, N).$$

Corollary 5.8. *Let $N > N_0$. Then there is a positive τ_0 such that if ξ is τ-regular, with $\tau \geq \tau_0$, then $\widehat{Z}(\xi)$ is the disjoint union of the compact sets $\widehat{Z}^F(\xi) = \widehat{Z}(\xi) \cap U(F, N)$ for $F \in \mathcal{FL}(\mathfrak{A}, \xi)$.*

Note that in (5.10) and (5.11), at the cost of changing the constants c_1 and c_2 we could replace the basis vectors γ_j from $\mathfrak{A}$ by the basis $\boldsymbol{\gamma}^F$ we chose for the flag $F = F(u) \in \mathcal{FL}(\mathfrak{A}, \xi)$. Then these equations, and the observation that the map $L : U(\mathfrak{A}) \to \mathfrak{g}^*$ is proper imply the statement. □

5.3. Smoothness. Next we turn to proving the smoothness of $\widehat{Z}^F(\xi)$. Fix a $\mathbb{Z}$-basis $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of $\Gamma_{\mathfrak{a}}$, and consider the map $p : U(\mathfrak{A}) \to \mathbb{C}^{*r}$ given by

$$p(u) = (p_1(u), p_2(u), \dots, p_r(u)),$$

where, as usual, p_j stands for p_{λ_j}.

As $\widehat{Z}(\xi)$ is the inverse image of a smooth torus under that map p, to prove that it is smooth, it is sufficient to show that the Jacobian matrix of the map $p : U(\mathfrak{A}) \to \mathbb{C}^{*r}$ is non-degenerate for $u \in U(F, N)$. According to Lemma 4.5 (1) and (3), this reduces to the computation of

$$\widetilde{D}_{\mathfrak{A}}(u) = \sum_{\sigma \in \mathrm{BInd}(\mathfrak{A})} \mathrm{vol}_{\mathfrak{a}^*}(\gamma^\sigma)^2 \prod_{i \in \sigma} \frac{1}{\alpha_i(u)}. \tag{5.12}$$

Recall that in Sect. 2 we associated to each flag $F \in \mathcal{FL}(\mathfrak{A})$ a sequence of vectors $(\kappa_1^F, \dots, \kappa_r^F)$, and we fixed a basis γ^F satisfying

$$d\gamma_1^F \wedge d\gamma_2^F \wedge \cdots \wedge d\gamma_r^F = d\mu_\Gamma^{\mathfrak{a}}.$$

We consider the γ_j^Fs as coordinates on $\mathfrak{a}_\mathbb{C}$, and to simplify our notation, we use $u_j = \gamma_j^F(u)$ for $u \in \mathfrak{a}_\mathbb{C}$ and $j = 1, \dots, r$.

Proposition 5.9. *Let $d(F)$ be the integer such that $\kappa_1^F \wedge \cdots \wedge \kappa_{r-1}^F \wedge \kappa = d(F)\ d\mu_\Gamma^{\mathfrak{a}}$. Then for N sufficiently large, we have*

$$\left| \widetilde{D}_{\mathfrak{A}}(u) \prod_{j=1}^r u_j - d(F) \right| \le \frac{\mathrm{const}(\mathfrak{A})}{N}$$

for any $u \in U(F, N)$.

Remark 5.2. Here and below we use the same notation $\mathrm{const}(\mathfrak{A})$ for several constants which depend only on $\mathfrak{A}$.

Proof. Define $\mathrm{BInd}(\mathfrak{A}, F)$ to be the set of those $\sigma \in \mathrm{BInd}(\mathfrak{A})$ for which $\{\alpha_i, i \in \sigma\} \cap F_j$ has j elements. Then the sum formula for $\widetilde{D}_{\mathfrak{A}}(u)$ is divided into two parts $\widetilde{D}_{\mathfrak{A}}(u) = \widetilde{D}_{\mathfrak{A}}^F(u) + R_{\mathfrak{A}}^F(u)$, a dominant and a remainder term, where

$$\widetilde{D}_{\mathfrak{A}}^F(u) = \sum_{\sigma \in \mathrm{BInd}(\mathfrak{A}, F)} \frac{\mathrm{vol}_{\mathfrak{A}}(\gamma^\sigma)^2}{\prod_{i \in \sigma} \alpha_i(u)}$$

and

$$R_{\mathfrak{A}}^F(u) = \sum \frac{\mathrm{vol}_{\mathfrak{A}}(\gamma^\sigma)^2}{\prod_{i \in \sigma} \alpha_i(u)}, \quad \sigma \in \mathrm{BInd}(\mathfrak{A}) \setminus \mathrm{BInd}(\mathfrak{A}, F).$$

Assuming $u \in U(F, N)$, we have the following basic estimates. For $\alpha_i \in F_j \setminus F_{j-1}$, we have

$$\left| \frac{\alpha_i(u)}{u_j} - m_i \right| < \frac{\mathrm{const}(\mathfrak{A})}{N}, \tag{5.13}$$

where $\alpha_i - m_i \gamma_j^F \in F_{j-1}$. This immediately leads to the estimate

$$\left| R_{\mathfrak{A}}^F(u) \prod_{j=1}^r u_j \right| < \frac{\mathrm{const}(\mathfrak{A})}{N}.$$

Now we estimate $\widetilde{D}_{\mathfrak{A}}^F(u)$. First, two simple linear algebraic equalities:

Lemma 5.10. (1) $\prod_{i\in\sigma} m_i = \mathrm{vol}_{\mathfrak{A}}(\gamma^\sigma)$,
(2) $\sum_{\sigma\in\mathrm{BInd}(\mathfrak{A},F)} \mathrm{vol}_{\mathfrak{a}^*}(\gamma^\sigma) = d(F)$.

Proof. The first equality follows from the fact that the matrix of basis-change from $\{\alpha_i;\ i\in\sigma\}$ to the basis γ^F is triangular, with the constants m_i in the diagonal. The second one can be seen by expanding the sums

$$\kappa_j^F = \sum\{\alpha_i|\ \alpha_i \in F_j,\ i = 1,\dots,n\}$$

in the exterior product $\kappa_1^F \wedge\dots\wedge\kappa_r^F$. The non-vanishing terms will exactly correspond to the sum on the left hand side of (2). □

Now we can finish the proof of Proposition 5.9. The sum defining $\widetilde{D}_{\mathfrak{A}}^F(u)$ is indexed by the elements $\sigma\in\mathrm{BInd}(\mathfrak{A},F)$. The term corresponding to σ multiplied by $\prod_{j=1}^r u_j$ may be estimated as follows. Using (5.13) and the first equality in Lemma 5.10 we have

$$\left|\prod_{j=1}^r u_j \prod_{i\in\sigma}\frac{1}{\alpha_i(u)} - \frac{1}{\mathrm{vol}_{\mathfrak{a}^*}(\gamma^\sigma)}\right| < \frac{\mathrm{const}(\mathfrak{A})}{N}.$$

Summing this inequality over $\sigma\in\mathrm{BInd}(\mathfrak{A},F)$ and simplifying the fraction in each term, we obtain that

$$\left|\widetilde{D}_{\mathfrak{A}}^F(u)\prod_{j=1}^r u_j - \sum_{\sigma\in\mathrm{BInd}(\mathfrak{A},F)}\mathrm{vol}_{\mathfrak{a}^*}(\gamma^\sigma)\right| \le \frac{\mathrm{const}(\mathfrak{A})}{N}.$$

Then applying the second equality of Lemma 5.10 completes the proof. □

As we observed earlier, (5.12) is up to a nonzero multiple the Jacobian of the map p. Then Corollary 5.8 together with Proposition 5.9 implies

Corollary 5.11. *For sufficiently large τ and N the compact sets $\widehat{Z}^F(\xi) = \widehat{Z}(\xi)\cap U(F,N)$, $F\in\mathcal{FL}(\mathfrak{A},\xi)$, are smooth manifolds.*

5.4. The homology class. Now we turn to the computation of the homology class of the manifolds $\widehat{Z}^F(\xi)$. Introduce the torus

$$C(\xi) = \left\{(y_1, y_2,\dots,y_r);\ |y_j| = e^{-\langle\xi,\lambda_j\rangle}\right\}\subset\mathbb{C}^{*r}.$$

If we orient $C(\xi)$ by the differential form $d\arg y_1\wedge\dots\wedge d\arg y_r$, then its fundamental class is a generator of $H_r(\mathbb{C}^{*r},\mathbb{Z})$ over $\mathbb{Z}$. Clearly, our set $\widehat{Z}(\xi)$ is the inverse image of $C(\xi)$ by the map $p = (p_1, p_2,\dots,p_r)$:

$$\widehat{Z}(\xi) = \{u\in U(\mathfrak{A});\ |p_1(u)| = e^{-\langle\xi,\lambda_1\rangle},\dots,|p_r(u)| = e^{-\langle\xi,\lambda_r\rangle}\}.$$

We can summarize what we have shown so far as follows. Let τ be sufficiently large, positive and let $N = c_1e^{c_2\tau}$. Then according to Corollary 5.8, for a τ-regular vector ξ the set $\widehat{Z}(\xi)$ breaks up into finitely many compact components $\widehat{Z}^F(\xi) = \widehat{Z}(\xi) \cap U(F, N)$, as F varies in $\mathcal{FL}(\mathfrak{A}, \xi)$. Since ξ is τ-regular, we have $d(F) \neq 0$ for every flag in the family $\mathcal{FL}(\mathfrak{A}, \xi)$. Thus, according to 5.9, the differential of the map p does not vanish on $U(F, N)$ and thus $\widehat{Z}^F(\xi)$ is a smooth compact submanifold of $U(F, N)$,

What remains to prove Theorem 5.1 is that the homology class of the oriented smooth manifold $\widehat{Z}^F(\xi)$ in $H_r((U(F, N), \mathbb{Z})$ is equal to $\nu(F)h(F)$, where $h(F)$ is the fundamental class of the torus $T_F(\epsilon)$ defined in (2.4). We will achieve this using a deformation argument.

Recall that we have fixed an F-basis $(\gamma_1^F, \gamma_2^F, \dots, \gamma_r^F)$ of $\mathfrak{a}^*$. Then for $\alpha_i \in F_j \setminus F_{j-1}$, $i = 1, \dots, n$, we can write

$$\alpha_i = \sum_{k=0}^{j-1} m_{i,k}\gamma_{j-k}^F, \quad \text{with } m_i := m_{i,0} \neq 0. \tag{5.14}$$

Now we define a deformation $\mathfrak{A}_s^F$ of our sequence $\mathfrak{A}$ as follows:

$$\alpha_i^F(s, u) = \sum_{k=0}^{j-1} s^k m_{i,k} u_{j-k} \quad \text{if } \alpha_i \in F_j \setminus F_{j-1},$$

where we again used the simplified notation $u_j = \gamma_j^F$.

In particular, we have $\alpha_i^F(1, \cdot) = \alpha_i$ and $\alpha_i^F(0, \cdot) = m_i\gamma_j^F$. Using the estimate of Lemma 5.6 we see that $\alpha_i^F(s, \cdot)$ does not vanish on $U(F, N)$ for any $s \in [0, 1]$ provided $N \geq N_0$. This means that we obtain a deformation of the map $p = (p_1, \dots, p_r)$ as well. Define $p_j^F(s, u) = \prod_{i=1}^n \alpha_i^F(s, u)^{\langle\alpha_i,\lambda_j\rangle}$. Consider the map $p^F : [0, 1] \times U(F, N) \to \mathbb{C}^{*r}$, $p^F(s, u) = (p_1^F(s, u), \dots, p_r^F(s, u))$, and let $\widehat{Z}_s^F(\xi) = p^F(s, \cdot)^{-1}(C(\xi))$ be the induced deformation of our cycle $\widehat{Z}(\xi)$.

Similarly, we can define the map

$$L^F : [0, 1] \times U(F, N) \to \mathfrak{a}^*, \quad L_s^F(u) = -\sum_{i=1}^n \log\left|\alpha_i^F(s, u)\right| \omega^i.$$

Again, we have $\widehat{Z}_s^F(\xi) = (\mu \circ L_s^F)^{-1}(\xi)$.

Then a direct computation yields the following equalities:

Lemma 5.12. *For $j = 1, \dots, r$, we have $p_j^F(1, u) = p_j(u)$ and*

$$p_j^F(0, u) = \prod_{i=1}^n m_i^{\langle\alpha_i,\lambda_j\rangle} \prod_{l=1}^r u_l^{\langle\kappa_l^F - \kappa_{l-1}^F, \lambda_j\rangle}.$$

Similarly, we have $L_1^F(u) = L(u)$ and

$$L_0^F(u) = -\sum_{j=1}^r \sum_{\alpha_i \in F_j \setminus F_{j-1}} (\log|m_i| + \log|u_j|)\omega^i.$$

Next, we compute the cycles $\widehat{Z}_0^F(\xi)$.

Lemma 5.13. *For a certain sequence of real numbers* $\boldsymbol{\epsilon}$*, we have*

$$\widehat{Z}_0^F(\xi) = T_F(\boldsymbol{\epsilon}) \subset U(F, N), \tag{5.15}$$

where

$$\widehat{Z}_0^F(\xi) = p^F(0, \cdot)^{-1}(C(\xi)) = \left(\mu \circ L_0^F\right)^{-1}(\xi),$$

and the torus $T_F(\boldsymbol{\epsilon})$ *was defined in* (2.4). *In addition, the orientation of the torus* $\widehat{Z}_0^F(\xi)$*, induced by the form* $d \arg p_1^F(0, \cdot) \wedge \cdots \wedge d \arg p_r^F(0, \cdot)$*, will coincide with the orientation of* $T_F(\boldsymbol{\epsilon})$*, induced by the form* $d \arg \gamma_1^F \wedge \cdots \wedge d \arg \gamma_r^F$ *exactly when* $\nu(F) = 1$.

Proof. The fact that $\widehat{Z}_0^F(\xi)$ is a torus immediately follows from the fact that each $p_j^F(0, \cdot)$ is a monomial in the linear forms γ_j^F, $j = 1, \ldots, r$. In fact, it is not hard to compute the sequence $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_r)$: if $\alpha_i \in F_j \setminus F_{j-1}$, then $\epsilon_j = e^{-(t_i - \log|m_i|)}$, where t_i is the ith component of $\widetilde{\mathrm{sol}}(F, \xi)$.

To compare the orientations, observe that

$$\frac{d \arg p_j^F}{d \arg \gamma_l^F} = \langle \kappa_l - \kappa_{l-1}, \lambda_j \rangle, \quad j = 1, \ldots, r.$$

This shows that the two orientations coincide exactly if the basis $\boldsymbol{\kappa}^F$ is oriented the same way as the basis $\boldsymbol{\gamma}^F$. By definition this happens exactly when $\nu(F) = 1$. □

Thus we obtained a deformation of the cycle $\widehat{Z}^F(\xi)$ to a cycle which manifestly represents the homology class $\nu(F)h(F) \in H_r(U(\mathfrak{A}), \mathbb{Z})$. To complete the proof of Theorem 5.1, it remains to show that the homology class of the cycles does not change in this family. This is fairly standard. The background for this material is [3].

If one has a proper smooth map between smooth manifolds $\pi : U \to V$ with $\dim U - \dim V = k$, then there is a natural grade-preserving pull-back map

$$\pi^* : H^\bullet_{\mathrm{comp}}(V, \mathbb{Z}) \longrightarrow H^\bullet_{\mathrm{comp}}(U, \mathbb{Z})$$

on compactly supported cohomology. Via Poincaré duality this induces a natural map

$$\pi^* : H_\bullet(V, \mathbb{Z}) \longrightarrow H_{\bullet+k}(U, \mathbb{Z}).$$

on the homology groups. This has the property that when a compact submanifold $S \subset V$ consists of regular values, then π^* applied to the fundamental class of S is exactly the fundamental class of the manifold $\pi^{-1}(S)$.

These maps are homotopy invariant in the sense that if now π is a proper map from $[0, 1] \times U$ to V, then the maps $\pi^*(0, \cdot)$ and $\pi^*(1, \cdot)$ are equal on the homology groups.

Our map $p : U(\mathfrak{A}) \to \mathbb{C}^{*r}$, and the closely related map $\mu \circ L : U(\mathfrak{A}) \to \mathfrak{a}^*$, however, are *not* proper! Thus we need a slight generalization of the pull-back maps. Since a map is proper if the inverse image of compact sets is compact, we could say that the map π is *proper to* V', for some open subset $V' \subset V$, if the $\pi^{-1}(K)$ is compact for any compact $K \subset V'$.

Lemma 5.14. *Let $N > N_0$. Then for sufficiently large τ, the map $\mu \circ L$ is proper to the set of τ-regular elements of $\mathfrak{a}^*$. Moreover, this map remains proper, when restricted to the set $U(F, N)$, for any $F \in \mathcal{FL}(\mathfrak{A})$.*

The lemma follows from Proposition 5.5 and Corollary 5.8.

Now we can reformulate in these homological terms what we are computing. Consider the connected component of the set of τ-regular elements containing ξ. Then we are trying to show that the pull-back $(\mu \circ L)^*\eta_\xi$ of the generator η_ξ of the zeroth homology of this component is equal to $\sum_{F\in\mathcal{FL}(\mathfrak{A},\xi)} \nu(F)h(F)$. According to Corollary 5.8, to prove this, it is sufficient to show that the pull-back of η_ξ under the map $\mu \circ L$ *restricted* to $U(F, N)$ is $\nu(F)h(F)$. According to Lemma 5.13, the map

$$\mu \circ L^F(0, \cdot) : U(F, N) \to \mathfrak{a}^*$$

has this property, and we need to show the same for the map

$$\mu \circ L^F(1, \cdot) : U(F, N) \to \mathfrak{a}^*.$$

Now this discussion explains that our deformation argument is justified as long as we have the following.

Proposition 5.15. *Let $N > N_0$, and $\xi \in \mathfrak{a}^*$ be a τ-regular vector for τ sufficiently large. Then the restricted map $\mu \circ L^F : [0, 1] \times U(F, N) \to \mathfrak{a}^*$ is proper to the set of τ-regular elements.*

This statement is proved exactly the same way as we proved Corollary 5.8, the analogous statement for the map $\mu \circ L$. It is easy to see that the relevant constants are exactly the ones appearing in the expressions of $\alpha_i^F(s, \cdot)$ via the basis γ^F in (5.14). These constants are clearly uniformly bounded as s varies in $[0, 1]$. This completes the proof of the proposition and that of Theorem 5.1 as well. □

6. The proof of the main results

6.1. The construction of the cycle for the JK-residue. We start with an important observation.

Proposition 6.1. *Let $\mathfrak{A}$ be a projective sequence, and let $\mathfrak{c}$ be a chamber with $\kappa \in \bar{\mathfrak{c}}$. If $\xi \in \mathfrak{c}$ be regular with respect to $\Sigma\mathfrak{A}$, then all flags in $\mathcal{FL}(\mathfrak{A}, \xi)$ are in fact in $\mathcal{FL}^+(\mathfrak{A}, \xi)$.*

Proof. Let $F \in \mathcal{FL}(\mathfrak{A}, \xi)$. Equation Part$(\xi)$ given in (5.6) reads as

$$\xi - B_r\kappa = \sum_{j=1}^{r-1}(B_j - B_{j+1})\kappa_j^F. \tag{6.1}$$

The vector $\sum_{j=1}^{r-1}(B_j - B_{j+1})\kappa_j^F$ is a positive linear combination of those elements of $\mathfrak{A}$ which lie in the subspace F_{r-1}. Thus the right hand side of 6.1 belongs to the closed set $\mathrm{Cone}(\mathfrak{A})_{\mathrm{sing}}$. The vector ξ is in $\mathfrak{c}$, and κ is in $\overline{\mathfrak{c}}$, thus the ray $\xi + \mathbb{R}^{\geq 0}\kappa$ does not touch $\mathrm{Cone}(\mathfrak{A})_{sing}$. Therefore we must have $B_r > 0$. □

Now we can formulate one of the central results of this paper: an explicit construction of a real algebraic cycle which represents the JK-residue.

Theorem 6.2. *Let $\mathfrak{A}$ be a projective sequence and $\mathfrak{c} \subset \mathfrak{a}^*$ be a chamber such that $\kappa \in \overline{\mathfrak{c}}$, and fix an arbitrarily small neighborhood U_0 of the origin in $\mathfrak{a}_{\mathbb{C}}$. If τ is sufficiently large, then for any τ-regular $\xi \in \mathfrak{c}$, the set $\widehat{Z}(\xi)$ is a smooth compact cycle in $U_0 \cap U(\mathfrak{A})$ whose homology class equals the class $h(\mathfrak{c}) \in H_r(U(\mathfrak{A}), \mathbb{Z})$ of the Jeffrey-Kirwan residue.*

Proof. Indeed, we computed the homology class of $\widehat{Z}(\xi)$ in Theorem 5.1, and Theorem 2.6 combined with Proposition 6.1 implies that this class is exactly the homology class realizing the Jeffrey-Kirwan residue.

To prove the other statement of the theorem, let ξ be a τ-regular vector. Then following the argument in the proof of Proposition 6.1, we see that we must have $B_r \geq \tau$ in (6.1). If u is in $\widehat{Z}(\xi)$, then it follows that $L(u)$ is close to the point $\widetilde{\mathrm{sol}}(F, \xi)$. This means that if $\alpha_i \in F_j \setminus F_{j-1}$, then $|\log|\alpha_i(u)| - B_j|$ is less than a constant depending on $\mathfrak{A}$ only. As $B_j \geq B_r \geq \tau$, this means that we have

$$|\alpha_i(u)| \leq \mathrm{const}(\mathfrak{A})\, e^{-\tau}, \quad \text{for } i = 1, \ldots, n.$$

This inequality implies the second statement of the theorem. □

Remark 6.1. Formally, this construction only gives a representative of $h(\mathfrak{c})$ in the case $\kappa \in \overline{\mathfrak{c}}$. Note, however,that for *any* chamber $\mathfrak{c}$ of $\mathfrak{A}$, there exists a sequence $\mathfrak{A}'$ consisting of repetitions of the elements of $\mathfrak{A}$, such that $\kappa' \in \overline{\mathfrak{c}}$, where κ' is the sum of the elements of $\mathfrak{A}'$. Applying the theorem to $\mathfrak{A}'$ will produce a representative for $h(\mathfrak{c})$.

6.2. The proof of the conjecture. Now we are ready to complete the proof of Theorem 4.1. We recall our setup and introduce some new notation. We have a projective, spanning sequence $\mathfrak{A}$ and a chamber $\mathfrak{c}$ containing $\kappa = \sum_{i=1}^{n} \alpha_i$ in its closure. We also picked a $\mathfrak{c}$-positive basis $\boldsymbol{\lambda}$. For $z \in \mathbb{C}^{*n}$, we denote z^{λ_j} by q_j, and introduce the vector $q = (q_1, \ldots q_r) \in \mathbb{C}^{*r}$. For two vectors $\xi_1, \xi_2 \in \mathfrak{a}^*$, write $\xi_1 \overset{\boldsymbol{\lambda}}{<} \xi_2$ if $\langle \xi_1, \lambda_j\rangle < \langle \xi_2, \lambda_j\rangle$ for $j = 1, \ldots, r$.

Fix a small vector $\eta \in \mathfrak{a}^*$ with the property that $\langle \eta, \lambda_j \rangle > 0$ for $j = 1, \ldots, r$, i.e. $0 \overset{\boldsymbol{\lambda}}{<} \eta$. Now we pick $\xi \in \mathfrak{a}^*$ such that every vector ζ satisfying $\xi - \eta \overset{\boldsymbol{\lambda}}{<} \zeta \overset{\boldsymbol{\lambda}}{<} \xi$ is τ-regular with τ sufficiently large. "Sufficiently large" here means large enough to satisfy the conditions of the statements we use in the course of the proof.

For any subset of $S \subset \{1, \ldots, r\}$, define the torus

$$T_S(\xi, \eta) = \left\{ q \in \mathbb{C}^{*r};\ |q_j| = \begin{cases} \exp(-\langle \xi, \lambda_j \rangle) \text{ if } j \notin S, \\ \exp(-\langle \xi - \eta, \lambda_j \rangle) \text{ if } j \in S \end{cases} \right\}$$

with its standard orientation, let $Z_S(\xi, \eta) = p^{-1} T_S(\xi, \eta)$ be the inverse image of this cycle under the map p (see (4.7)), and introduce the ring-like domain

$$R(\xi, \eta) = \{q \in \mathbb{C}^{*r};\ \langle \xi - \eta, \lambda_j \rangle < -\log |q_j| < \langle \xi, \lambda_j \rangle,\ \ j = 1, \ldots, r\},$$

on whose edges these tori are located. Also, denote the associated domain in z-space by $W(\xi, \eta)$:

$$W(\xi, \eta) = \{z \in \mathbb{C}^{*n};\ \xi - \eta \overset{\boldsymbol{\lambda}}{<} \mu \circ L(z) \overset{\boldsymbol{\lambda}}{<} \xi\}. \tag{6.2}$$

Thus $W(\xi, \eta)$ is the pull-back of the domain $R(\xi, \eta)$ under the mapping which associates $q \in \mathbb{C}^{*r}$ to $z \in \mathbb{C}^{*n}$.

We will omit (ξ, η) from the notation if this does not cause confusion. For example, we will use Z_S instead of $Z_S(\xi, \eta)$.

Let the open set U_0 in Theorem 6.2 be the set $\{u \in \mathfrak{a}_{\mathbb{C}};\ |\kappa(u)| < 1\}$, and assume that τ is large enough to satisfy the conditions of Theorem 6.2 with this choice of U_0. Now let

$$U(\boldsymbol{\lambda}, q) = \left\{ u \in \mathfrak{a}_{\mathbb{C}};\ p_j^+(u) \neq q_j p_j^-(u) \right\} \subset \mathfrak{a}_{\mathbb{C}},$$

and recall our meromorphic r-form

$$\Lambda = \frac{P(\alpha_1, \ldots, \alpha_n)\ d\mu_\Gamma^{\mathfrak{a}}}{(1 - \kappa) \prod_{i=1}^n \alpha_i \prod_{j=1}^r (1 - q_j / p_j)}$$

on $\mathfrak{a}_{\mathbb{C}}$, which is regular on $U_0 \cap U(\mathfrak{A}) \cap U(\boldsymbol{\lambda}, q)$, and depends on z.

Now our final argument may be broken up into the following 4 statements.

Proposition 6.3. *Let $z \in W(\xi, \eta)$ and assume that τ is sufficiently large. Then*

(1) $\int_{Z_\emptyset} \Lambda = \langle P \rangle_{\mathfrak{A}, \mathfrak{c}}(z)$

(2) $\int_{Z_S} \Lambda = 0$ *if* $S \neq \emptyset$.

(3) *The cycle* $\sum_S (-1)^{|S|} Z_S$*, where* $|S|$ *denotes the number of elements of* S*, is homologous in* $U_0 \cap U(\mathfrak{A}) \cap U(\boldsymbol{\lambda}, q)$ *to the cycle*

$$Z_\delta(q) = p^{-1}\{y = (y_1, \ldots, y_r);\ |y_j - q_j| = \delta,\ j = 1, \ldots, r\},$$

oriented by the form $d\arg(y_1 - q_1) \wedge \cdots \wedge d\arg(y_r - q_r)$.

(4) $\int_{Z_\delta(q)} \Lambda = \langle P \rangle_{\mathfrak{B}}(z)$.

Proof. We chose τ large enough in order to be able to apply Theorem 6.2 to each of the cycles Z_S, $S \subset \{1, \ldots, n\}$. Thus we know that the homology class of Z_S in $U(\mathfrak{A})$ is $h(\mathfrak{c})$ and that $Z_S \subset U_0$ for every $S \subset \{1, \ldots, n\}$. The inequalities (6.2) defining $W(\xi, \eta)$ imply that for $u \in Z_\emptyset$ we have $|q_j| < |p_j(u)|$ for $j = 1, \ldots, r$, and thus we can apply Proposition 3.4. This proves the first statement of the proposition.

For $m = 1, \ldots, r$, denote by Z_m the cycle $Z_{\{1,\ldots,m\}}$. Since we can permute the elements of the basis $\boldsymbol{\lambda}$, it is sufficient to prove Statement (2) for these cycles. Now reversing the logic of the proof of Proposition 3.4, we can expand the differential form Λ, taking into account that for $u \in Z_m$ and $z \in W(\xi, \eta)$ we have $|p_j(u)| < |q_j|$ for $j = 1, \ldots, m$, and $|p_j(u)| > |q_j|$ for $j = m+1, \ldots, r$. We obtain the convergent expansion

$$\int_{Z_m} \Lambda = \frac{(-1)^m}{\left(2\pi\sqrt{-1}\right)^r} \sum \int_{Z_m} \prod_{j=1}^{m} \frac{p_j^{l_j+1}}{q_j^{l_j+1}} \prod_{j=m+1}^{r} \frac{q_j^{l_j}}{p_j^{l_j}} \cdot \frac{P(\alpha_1, \ldots, \alpha_n)\ d\mu_\Gamma^{\mathfrak{a}}}{(1-\kappa)\prod_{i=1}^n \alpha_i},$$

where the sum runs over $l_j \in \mathbb{Z}^{\geq 0}$, $j = 1, \ldots, r$.

Since Z_m represents the JK-residue, the terms of this series are again of the form $\langle P \rangle_{\lambda, \mathfrak{A}, \mathfrak{c}}$ with $\lambda = \sum_{j=1}^r l_j \lambda_j$, where now $l_j < 0$ for $j = 1, \ldots, m$, and $l_j \geq 0$ for $j = m+1, \ldots, r$. Such an expression vanishes, however, according to Proposition 3.1. This completes the proof of the 2nd statement.

Clearly, (3) will follow from a similar statement formulated for the cycles T_S:

$$\sum_{S \subset \{1,\ldots,n\}} (-1)^{|S|} T_S \text{ is homologous to } \{y \in \mathbb{C}^{*r};\ |y_j - q_j| = \delta,\ j = 1, \ldots, r\}$$

in the open set $\{y \in \mathbb{C}^{*r};\ y_j \neq q_j \text{ for } j = 1, \ldots r\}$. This may be proved by the standard inclusion-exclusion argument and is left to the reader.

Finally, note that the cycle $Z_\delta(q)$ coincides with the cycle $Z_\delta(\boldsymbol{\lambda}, q)$ introduced before Proposition 4.7. Then the 4th statement will follow from Proposition 4.7 as soon as we check the technical conditions that we assumed there. We have done all the groundwork for this; we just need to collect the necessary information here.

First, note that in Corollary 5.8 we show that each point of $O_{\mathfrak{B}}(z)$ is in $U(F, N)$ for some $F \in \mathcal{FL}(\mathfrak{A})$, in Proposition 4.5 we compute the Jacobian of the map p, and in Proposition 5.9 we show that this Jacobian does not vanish on $O_{\mathfrak{B}}(z)$. Thus we can conclude that the set $O_{\mathfrak{B}}(z) =$

$p^{-1}(q)$ is finite. As $D^{\mathfrak{B}}(\alpha_1(u), \ldots, \alpha_r(u))$ coincides with this Jacobian up to a nonzero multiple, we see that it will not vanish on $O_{\mathfrak{B}}(z)$.

Next, it follows from Lemma 5.14, the map p is proper to the domain $R(\xi, \eta)$, and this eliminates the need for intersecting with the small neighborhood $U(z)$ of $O_{\mathfrak{B}}(z)$. Finally, note that we already assumed that $\kappa(u) \neq 1$ for any u such that $p(u) \in R(\xi, \eta)$, thus $1 - \kappa$ will not vanish on $O_{\mathfrak{B}}(z)$. $\square$

Proposition 6.3 proves the equality $\langle P \rangle_{\mathfrak{A},\mathfrak{c}}(z) = \langle P \rangle_{\mathfrak{B}}(z)$ for all $z \in W(\xi, \eta)$, starting from the localized sum definition (4.1) of $\langle P \rangle_{\mathfrak{B}}(z)$. If we use the fact that this localized sum is a toric residue, and thus it is a rational function of z, then we can conclude that the two sides of (4.2) coincide whenever the series on the left hand side converges. In view of of Lemma 3.3, this implies the full statement of Theorem 4.1.

References

1. Batyrev, V.V.: Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties. J. Algebr. Geom. **3**, 493–535 (1994)
2. Batyrev, V.V., Materov, E.N.: Toric residues and mirror symmetry. Dedicated to Yuri I. Manin on the occasion of his 65th birthday. Mosc. Math. J. **2**, 435–475 (2002)
3. Bott, R., Tu, L.: Differential Forms in Algebraic Topology. New York: Springer 1982
4. Brion, M., Vergne, M.: Arrangement of hyperplanes I : Rational functions and Jeffrey-Kirwan residue. Ann. Sci. Éc. Norm. Supér., IV. Sér. **32**, 715–741 (1999)
5. Cattani, E., Cox, D., Dickenstein, A.: Residues in toric varieties. Compos. Math. **108**, 35–76 (1997)
6. Cattani, E., Dickenstein, A., Sturmfels, B.: Residues and resultants. J. Math Sci., Tokyo **5**, 119–148 (1998)
7. Cox, D.: Toric residues. Ark. Mat. **34**, 73–96 (1996)
8. Danilov, V.I.: The geometry of toric varieties. Russ. Math. Surv. **33**, 95–154 (1978)
9. Fulton, W.: Introduction to Toric Varieties. Princeton: Princeton University Press 1993
10. Gelfand, I.M., Kapranov, M.M., Zelevinsky, A.V.: Discriminants, Resultants and Multidimensional Determinants. Boston: Birkhäuser 1994
11. Griffiths, P., Harris, J.: Principles of Algebraic geometry. New York: John Wiley 1978
12. Hausel, T., Sturmfels, B.: Toric hyperKähler varieties. Doc. Math. **7**, 495–534 (2002)
13. Jeffrey, L., Kirwan, F.: Localization for nonabelian group actions. Topology **34**, 291–327 (1995)
14. Morrison, D., Plesser, R.: Summing the instantons: Quantum cohomology and mirror symmetry in toric varieties. Nucl. Phys. B **440**, 279–354 (1995)
15. Orlik, P., Terao, H.: Arrangements of Hyperplanes. In: Grundlehren der Mathematischen Wissenschaften, Vol. 300. Berlin: Springer 1992
16. Szenes, A.: Iterated residues and Bernoulli Polynomials. Int. Math. Res. Not. **18**, 937–956 (1998)
17. Schechtman, V.V., Varchenko, A.N.: Arrangements of hyperplanes and Lie algebra homology. Invent. Math. **106**, 139–194 (1991)
18. Sturmfels, B.: Solving Systems of Polynomial Equations, CBMS Regional Conference Series in Math., no. 97. Providence, RI: AMS 2002
19. Viro, O.: Dequantization of real algebraic geometry on a logarithmic paper, Proc. 3rd European Congress of Mathematics 2000, Vol. I. Prog. Math. **201**, 135–146. Basel: Birkhäuser 2001
20. Witten, E.: Two-dimensional gauge theory revisited. J. Geom. Phys. **9**, 303–368 (1992)