Toric Sylvester forms

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Abstract

In this paper, we investigate the structure of the saturation of ideals generated by sparse homogeneous polynomials over a projective toric variety X with respect to the irrelevant ideal of X. As our main results, we establish a duality property and make it explicit by introducing toric Sylvester forms, under a certain positivity assumption on X. In particular, we prove that toric Sylvester forms yield bases of some graded components of I^{sat}/I, where I denotes an ideal generated by $n+1$ generic forms, n is the dimension of X and I^{sat} the saturation of I with respect to the irrelevant ideal of the Cox ring of X. Then, to illustrate the relevance of toric Sylvester forms we provide three consequences in elimination theory: (1) we introduce a new family of elimination matrices that can be used to solve sparse polynomial systems by means of linear algebra methods, including overdetermined polynomial systems; (2) by incorporating toric Sylvester forms to the classical Koszul complex associated to a polynomial system, we obtain new expressions of the sparse resultant as a determinant of a complex; (3) we give a new formula for computing toric residues of the product of two forms.

Keywords: sparse polynomial systems, toric geometry, sparse resultants, elimination theory.

1 Introduction

The elimination of variables in a system of homogeneous polynomial equations is deeply connected to the saturation of ideals with respect to a certain geometrically irrelevant ideal. Thus, the search and study of universal generators of the saturation of an ideal generated by generic homogeneous polynomials is an important topic in elimination theory. In the classical literature of the previous century, such universal generators were called *inertia forms* by Hurwitz, Mertens, Van der Waerden and many others, including Zariski; see the references in [Jou91; Jou97] and [Zar37]. As examples, Jacobian determinants and resultants associated to a square homogeneous polynomial system are important inertia forms.

To be more specific, consider the ideal $I = (F_0, \dots, F_n)$ where F_i is the generic homogeneous polynomial of degree d_i in the graded polynomial ring $C = A[x_0, \dots, x_n]$, where $\deg(x_i) = 1$ for all $i = 0, \dots, n$ and where A stands for the universal ring of coefficients of the F_i's. The saturation of the ideal I with respect to the ideal $\mathfrak{m} = (x_0, \dots, x_n)$, which we denote by $I^{\text{sat}} = I : \mathfrak{m}^\infty$, is the ideal of inertia forms. In this context, the ideal $\mathfrak{m}$ is the (geometrically) irrelevant ideal of the projective space of dimension n which is associated to C. The elements in I being trivially inertia forms, I^{sat}/I is the natural quotient to study. It turns out that the Jacobian determinant of the F_i's is a generator, as an A-module, of the graded component of I^{sat}/I in degree $\delta = d_0 + \cdots + d_n - (n+1)$ and their resultant is a generator of I^{sat}/I in degree 0. In order to unravel the structure of I^{sat}/I in degrees smaller than δ, Jouanolou introduced and studied the formalism of Sylvester forms [Jou97]. His ideas were based on the fact that for each $\mu = (\mu_0, \dots, \mu_n) \in \mathbb{N}^{n+1}$ such that $|\mu| := \sum_i \mu_i < \min_i d_i$, each polynomial F_i can be decomposed as

$$F_i = \sum_{j=0}^{n} x_j^{\mu_j+1} F_{i,j} \tag{1.1}$$

and one can consider the determinant $\det(F_{i,j})_{0 \le i,j \le n}$. This latter is called a *Sylvester form* of the F_i's and denoted by Sylv_μ. Independently of the choice of decompositions (1.1), the class of Sylv_μ modulo I , which is denoted by

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sylv_μ, gives a nonzero element in $(I^{\text{sat}}/I)_{\delta-|\mu|}$. Moreover, $(I^{\text{sat}}/I)_{\delta-|\mu|}$ is a free A-module which can be generated by the Sylvester forms of degree $\delta-|\mu|$. This result is a consequence of a duality property between Sylvester forms and monomials; namely, for all $\nu < \min_i d_i$ we have an isomorphism of A-modules

$$(I^{\text{sat}}/I)_{\delta-\nu} \simeq \text{Hom}_A(C_\nu, A).$$

More explicitely, this isomorphism corresponds to the equalities

$$x^{\mu'}\,\text{sylv}_\mu = \begin{cases} \text{sylv}_0 & \mu=\mu' \\ 0 & \mu\neq\mu' \end{cases}$$

where sylv_0 is a generator of $(I^{\text{sat}}/I)_\delta$. We note that up to a nonzero multiplicative constant, sylv_0 is equal to the class of the Jacobian determinant of the F_i's; see [Jou97, §3.10].

The definition and main properties of Sylvester forms have been recently extended to the case of $n+1$ generic multi-homogeneous polynomials, i.e. of polynomials defining hypersurfaces over a product of projective spaces of total dimension n; see [BCN22]. In this paper, we develop the theory of Sylvester forms in the general setting of homogeneous polynomials in the coordinate ring of a projective toric variety X_Σ. In addition, to illustrate the importance of these forms in elimination theory, we also provide applications to the construction of elimination matrices for overdetermined polynomial systems, and to the computation of toric resultants and toric residues. As far as we know, these applications are also new results in the context of multi-homogeneous polynomial systems. In what follows we give a brief overview of the main contributions in this paper.

Let k be an algebraically closed field and X_Σ be a n-dimensional projective toric variety over k given by a complete fan Σ in a lattice N. Let R be the homogeneous coordinate ring of X_Σ over k, also known as the *Cox ring* of X_Σ, which is graded using the combinatorics of Σ; see Section 2 or [Cox95] for more details. Assuming that there exists a smooth n-dimensional cone $\sigma\in\Sigma(n)$, we write $x_1,\dots,x_n$ the variables of R associated to σ and we denote by $z_1,\dots,z_r$ the remaining ones. With these notations, a homogeneous polynomial in X_Σ of degree $\alpha\in\text{Cl}(X_\Sigma)$, the class group of X_Σ, is an element in the graded component R_α of R in degree α; it is a k-linear combination of monomials $x^\mu = x_1^{\mu_1}\cdots x_n^{\mu_n} z_1^{\mu_{n+1}}\cdots z_r^{\mu_{n+r}}$ of degree α. Now, the *generic* homogeneous polynomial of degree α is the polynomial $\sum_{x^\mu\in R_\alpha} c_{i,\mu}x^\mu$ where the coefficients $c_{i,\mu}$ are seen as variables. Thus, being given $n+1$ degrees $\alpha_0,\dots,\alpha_n$, the corresponding generic homogeneous polynomial system over X_Σ is given by the $n+1$ homogeneous polynomials

$$F_i = \sum_{x^\mu\in R_{\alpha_i}} c_{i,\mu}x^\mu \in C = A\otimes_k R = A[x_1,\dots,x_n,z_1,\dots,z_r],\ i=0,\dots,n, \tag{1.2}$$

where A is the universal ring of coefficients over k, i.e. $A=k[c_{i,\mu} : x^\mu\in R_{\alpha_i},\ i=0,\dots,n]$. We define the ideal $I=(F_0,\dots,F_n)$ and the ideal $\mathfrak{b} = (\tilde{x}^\sigma : \tilde{x}^\sigma = \prod_{\rho\notin\sigma} x_\rho,\ \sigma\in\Sigma(n))$, which is the *irrelevant ideal* of X_Σ. The saturation of I is the ideal of C defined as $I^{\text{sat}} = (I:\mathfrak{b}^\infty)$.

The first main result of this paper is the following duality property which is a generalization of [BCN22, Theorem A] to the case of a projective toric variety (see Theorem 3.1). We set $\delta=\alpha_0+\cdots+\alpha_n-K_X\in\text{Cl}(X_\Sigma)$, where K_X is the anticanonical class of X_Σ.

Theorem. Let X_Σ be a projective toric variety, let $\sigma\in\Sigma(n)$ be an n-dimensional smooth cone and let $\nu\in\text{Cl}(X_\Sigma)$. Then, with the above notation, there exists a non-empty region $\Gamma\subsetneq\text{Cl}(X_\Sigma)$ such that if $\delta-\nu\notin\Gamma$ then

$$(I^{\text{sat}}/I)_{\delta-\nu} \simeq \text{Hom}_A((C/I)_\nu, A).$$

In the cases where $(C/I)_\nu = C_\nu$, the above duality implies that $(I^{\text{sat}}/I)_{\delta-\nu}$ is a free A-module (see Corollary 3.1) and a natural question is to find explicit bases. To tackle this question, we introduce toric Sylvester forms. We first prove that for suitable ν and $x^\mu\in R_\nu$, each generic homogeneous polynomial F_i can be decomposed into $n+1$ generic homogeneous polynomials $(F_{ij}^\mu)_{0\le j\le n}$, similarly to (1.1). The existence of such decompositions requires a certain property on the smooth cone $\sigma\in\Sigma(n)$; when it holds we will say that X_Σ is σ*-positive* (see Definition 2.1 and Theorem 2.1). Then, from these decompositions we define toric Sylvester forms as the determinants $\text{Sylv}_\mu := \det(F_{ij}^\mu)\in I^{\text{sat}}_{\delta-\nu}$ and show that their classes in $(I^{\text{sat}}/I)_{\delta-\nu}$, denoted by sylv_μ, are independent of the choice of decompositions. Finally, we obtain the following explicit duality property which can be seen as the second main contribution of this paper (see Theorem 4.1).

Theorem. Let X_Σ be a projective toric variety and let $\sigma\in\Sigma(n)$ be a smooth cone such that X_Σ is σ-positive. Then, under suitable conditions on $\nu\in\text{Cl}(X_\Sigma)$, for any pair $x^\mu, x^{\mu'}\in R_\nu$ we have

$$x^\mu\,\text{sylv}_{\mu'} = \begin{cases} \text{sylv}_0 & \mu=\mu' \\ 0 & \text{otherwise,} \end{cases}$$

where sylv_0 is a generator of $(I^{\mathrm{sat}}/I)_\delta$ as an A-module. Therefore, the toric Sylvester forms sylv_μ, for all $x^\mu \in R_\nu$, yield an A-basis of $(I^{\mathrm{sat}}/I)_{\delta-\nu}$.

In the rest of the paper, we provide three applications of toric Sylvester forms in elimination theory. The first application deals with *elimination matrices*. An important question in elimination theory is the study of matrices $\mathbb{M}$ with entries in A such that:

i) their rank drops when the coefficients $c_{i,\mu}$'s are specialized in k and the corresponding polynomial system has solutions in X_Σ,

ii) their corank coincides with the number of solutions, counting multiplicities, when the coefficients $c_{i,\mu}$'s are specialized in k and the corresponding polynomial system has finitely many solutions in X_Σ.

The first property is related to resultant theory (see e.g. [CDS97; GKZ94]) whilst the second is used for solving 0-dimensional polynomial systems (see e.g. [BT22; EM99]). In this paper, we introduce a new family of elimination matrices by adding to a classical Macaulay-block matrix in some degree $\alpha \in \mathrm{Cl}(X_\Sigma)$, a block-matrix built from the toric Sylvester forms of degree α (see Definition 5.1). We call these matrices *hybrid elimination matrices* and prove their main properties in Theorem 5.1. Compared with the more classical Macaulay matrices, this new family yields more compact matrices that can still be used for solving 0-dimensional polynomial systems. In addition, we also prove that the construction of hybrid elimination matrices can be extended to polynomial systems defined by more than $n+1$ polynomials (see Theorem 5.3).

Our second application concerns the *computation of sparse resultants*. A classical result in elimination theory is that the sparse resultant can be computed as the determinant of certain graded components of the Koszul complex built from the considered polynomial system (see [GKZ94]). Generalizing a construction of Cattani, Dickenstein and Sturmfels in [CDS97, §2] using the so-called toric Jacobian, we modify the usual Koszul complex by incorporating the Sylvester forms in its last differential and prove that the determinant of some suitable graded parts of this new complex is equal to the sparse resultant, up to a nonzero multiplicative constant in k (see Theorem 6.1). This result yields new formulas for computing the sparse resultant as a determinant of a complex.

Our third application deals with the *computation of toric residues*. The toric residue of the generic polynomial system (1.2) was defined by Cox in [Cox96]. It is a map that sends any polynomial in $(C/I)_\delta$ to the fraction field $K(A)$ of A. The computation of this residue map by means of determinants has been an active research topic with many contributions, including [Jou97; DK05; CCD97]. In this paper, using toric Sylvester forms we construct matrices whose determinants are used to compute the residue of a product of two forms PQ, where $P \in C_\nu$, $Q \in C_{\delta-\nu}$ and $\nu \in \mathrm{Cl}(X_\Sigma)$. This formula can be seen as an extension of a similar formula proved by Jouanolou in the case $X_\Sigma = \mathbb{P}^n$ [Jou97, Proposition 3.10.27]. It yields more compact matrices in comparison with the formula proved by D'Andrea and Khetan in [DK05, Theorem 5.1] for computing the toric residue of a form of degree δ.

The paper is organised as follows. In Section 2, we present all the tools of toric geometry that are needed in the rest of the paper. In particular, we prove the existence of decompositions of forms in a projective toric variety X_Σ which is σ-positive for a smooth cone $\sigma \in \Sigma(n)$. In Section 3, we show that the claimed duality property holds outside a region $\Gamma \subset \mathrm{Cl}(X_\Sigma)$ which depends on the supports of the local cohomology modules of the corresponding Cox ring. In Section 4, we define Sylvester forms and show that they give an A-basis of $(I^{\mathrm{sat}}/I)_{\delta-\nu}$ for certain degrees $\nu \in \mathrm{Cl}(X_\Sigma)$. In Section 5, we introduce hybrid elimination matrices when X_Σ is assumed to be smooth and σ-positive for a smooth cone $\sigma \in \Sigma(n)$. In Section 6, we prove that the determinant of certain graded parts of a modified Koszul complex in a region $\Gamma_{\mathrm{Res}} \subset \mathrm{Cl}(X_\Sigma)$ is equal to the sparse resultant, up to a nonzero multiplicative constant in k. Finally, in Section 7 we prove a new formula for computing the toric residue of a product of two forms.

2 Preliminaries on toric geometry

In this section, we fix our notation and briefly review some material we will use from toric geometry; we refer to the book by Cox, Little and Schenck [CLS12] for more details. We also prove a decomposition property that we will use in order to introduce toric Sylvester forms later on.

Projective toric varieties. Let k be an algebraically closed field and let M be a lattice of rank n. We denote by $N = \mathrm{Hom}(M, \mathbb{Z})$ the dual of M, by $\mathbb{T}_N = N \otimes k^\times$ the algebraic torus associated to N and we set $M_\mathbb{R} = M \otimes \mathbb{R}$, which are two vector spaces over the real numbers. Let $\mathcal{A} = \{m_1, \ldots, m_s\} \subset M$ be a finite set of lattice points and consider its convex hull $\Delta = \mathrm{conv}(\mathcal{A}) \subset M_\mathbb{R}$. The projective toric variety X_Δ can be defined as the algebraic closure of the image of the map

$$\Phi_\mathcal{A} : \mathbb{T}_N \to \mathbb{P}_k^{s-1} \quad t := (t_1, \ldots, t_n) \to (t^{m_1} : \cdots : t^{m_s}).$$

This variety is called toric because the group action of $\mathbb{T}_N$ on itself extends to X_Δ with good geometric properties.

Example 2.1. If Δ is a product of simplices of the form $\Delta_{n_j} = \{t \in \mathbb{R}^{n_j} : t_k \geq 0, \sum_{k=0}^{n_j} t_k \leq 1\}$ for $j = 1, \ldots, s$, then $X_\Sigma = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$.

When Δ is n-dimensional, another definition of X_Δ, more intrinsic, can be stated from the normal fan $\Sigma \subset N$ of Δ, so that this variety is also denoted by X_Σ (we note that the equivalence between these definitions requires that Δ is very ample [CLS12, Definition 2.1.13, Proposition 3.1.6], but if this is not the case one can always find an integer k such that the polytope $k\Delta$ is very ample [CLS12, Corollary 2.2.18]). The geometric properties of X_Σ are deeply connected with the combinatorial properties of the fan Σ. For instance, X_Σ is a smooth variety if and only if Σ is smooth, which means that the minimal generators of all cones $\sigma \in \Sigma$ are part of a basis of N.

We denote by $\Sigma(r)$ the set of r-dimensional cones of Σ, which are also called rays when $r = 1$. We assume that the generators of the rays $u_\rho \in N$ for $\rho \in \Sigma(1)$ are primitive and span the vector space $N_\mathbb{R}$; by [CLS12, Corollary 3.3.10], this condition is equivalent to the toric variety X_Σ having no torus factors. Moreover, as Δ is a bounded polytope, its normal fan Σ is complete and its cones are strongly convex. Under these assumptions, $\Sigma(1)$ contains at least $n + 1$ rays. In addition, denoting by $\mathrm{Cl}(X_\Sigma)$ the class group of X_Σ, there is a short exact sequence

$$0 \to M \xrightarrow{\mathbf{F}} \mathbb{Z}^{\Sigma(1)} \xrightarrow{\pi} \mathrm{Cl}(X_\Sigma) \to 0, \tag{2.1}$$

where $\mathbf{F}$ is an $(n + r) \times n$ matrix whose rows are the generators of the rays in $\Sigma(1)$ and π is chosen accordingly to be a cokernel matrix; see [CLS12, Theorem 4.1.3].

Finally, we recall that a Cartier divisor D is nef if and only if D is generated by global sections [CLS12, Theorem 6.3.12], and ample if and only if the normal fan of its polytope is Σ [CLS12, Proposition 7.2.3]. As X_Σ is assumed to be a projective toric variety, it follows that there always exists an ample divisor on X_Σ (see [Har77, Chapter 2, Theorem 7.10]).

The Cox ring and a decomposition property. The homogeneous coordinate ring of a projective toric variety X_Σ, also known as the *Cox ring*, is the ring $R = k[x_\rho, \rho \in \Sigma(1)]$ which is $\mathrm{Cl}(X_\Sigma)$-graded by means of the map π defined in (2.1): $R = \oplus_{\alpha \in \mathrm{Cl}(X_\Sigma)} R_\alpha$, with $R_\alpha = H^0(X_\Sigma, \mathcal{O}_\Sigma(D))$ where D is a torus-invariant Weil divisor such that $[D] = \alpha$ and $\mathcal{O}_\Sigma$ is the structure sheaf of X_Σ; see [Cox95] for more details.

In what follows, we will use the following notation for the variables of the Cox ring. Assuming that there exists a maximal smooth cone $\sigma \in \Sigma(n)$, we will denote by $x_1, \ldots, x_n$ the variables associated to the rays $\rho \in \sigma(1)$ and by $z_1, \ldots, z_r$ the remaining variables of R. According to the choice of σ, one can always write a matrix of the map π in (2.1) under the form

$$\pi = \begin{pmatrix} \mathcal{P} & \mathrm{Id}_r \end{pmatrix}, \tag{2.2}$$

where $\mathcal{P}$ is a block matrix $(\mathcal{P}_{i,j})_{1 \leq i \leq r, 1 \leq j \leq n}$ whose rows correspond to the relations between u_ρ for $\rho \notin \sigma$ and the basis given by σ. In order to introduce Sylvester forms later on, we will need the following property which is not standard.

Definition 2.1. For $\sigma \in \Sigma(n)$, the projective toric variety X_Σ is called *σ-positive* if σ is a maximal smooth cone such that a matrix of the map π defined in (2.1) can be written as in (2.2) with the additional condition that $\mathcal{P}_{i,j} \geq 0$ for $i = 1, \ldots, r$ and $j = 1, \ldots, n$.

Intuitively, the above property can be understood as the fact that the vector $-u_\rho$ belongs to σ for all $\rho \notin \sigma(1)$, as each row of π corresponds to the identity $u_{\rho_j} + \sum_{\rho_i \in \sigma(1)} \mathcal{P}_{i,j} u_{\rho_i} = 0$. A first observation is that not all smooth toric varieties is σ-positive for a certain $\sigma \in \Sigma(n)$, as shown in the following example.

Example 2.2. Let Σ be the complete smooth fan in $N_\mathbb{R} = \mathbb{R}^2$ with the following rays:

$$\rho_1 = (1, 0)\ \rho_2 = (0, 1)\ \rho_3 = (-1, 1)\ \rho_4 = (-1, 0)\ \rho_5 = (-1, -1)\ \rho_6 = (0, -1).$$

It is straightforward to check that for every $\sigma \in \Sigma(2)$, there is $\rho \notin \sigma(1)$ such that $-u_\rho \notin \sigma$.

On the other hand, most of the projective toric varieties that are of interest for our applications are σ-positive for some smooth maximal cone σ. For instance, this property is preserved under product of toric varieties. To be more precise, recall that the product of two toric varieties is defined by the product fan (see [CLS12, Theorem 2.4.7]). Any cone of this fan is of the form $\sigma_1 \times \sigma_2$, where the elements are considered as pairs (u, v) for $u \in \sigma_1$ and $v \in \sigma_2$. It is easy to check that $\dim \sigma_1 \times \sigma_2 = \dim \sigma_1 + \dim \sigma_2$.

Lemma 2.1. If X_1, resp. X_2, is a toric variety which is σ_1-positive, resp. σ_2-positive, for some maximal cone σ_1 in a fan Σ_1, resp. σ_2 in a fan Σ_2, then the product $X_1 \times X_2$ is $\sigma_1 \times \sigma_2$-positive.

Proof. Any ray ρ of the product fan is generated by an element of the form $(u_{\rho_1}, 0)$, resp. $(0, u_{\rho_2})$, for ρ_1 a ray of σ_1 and ρ_2 a ray σ_2. By assumption, $-u_{\rho_1}$ can be written as a positive combination of elements in σ_1, therefore of $\sigma_1 \times \sigma_2$. ☐

Example 2.3. The projective space $\mathbb{P}^n$ is σ-positive as the map π can be written as $\pi = (1 \cdots 1)$. Therefore, any product of projective spaces is σ'-positive by Lemma 2.1. Another classical family of smooth toric varieties are Hirzebruch surfaces $\mathcal{H}_r \subset \mathbb{R}^2$: for each $r \in \mathbb{Z}_{>0}$, it is the variety corresponding to the fan Σ_r with rays

$$\rho_1 = (1,0)\ \rho_2 = (0,1)\ \rho_3 = (-1,-r)\ \rho_4 = (0,-1).$$

Hirzebruch surfaces are smooth and are σ-positive with respect to the smooth maximal cone $\sigma = (\rho_1, \rho_2)$ as π can be written as

$$\pi = \begin{pmatrix} 1 & r & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

We are now ready to prove the existence of certain decompositions of homogeneous polynomials that we will use in Section 4 in order to define toric Sylvester forms. Let J be an ideal of R generate by homogeneous polynomials $f_0, \dots, f_n$ of degree $\alpha_0, \dots, \alpha_n$, respectively.

Theorem 2.1. Let X_Σ be a projective toric variety of dimension n such that X_Σ is σ-positive with respect to a smooth cone $\sigma \in \Sigma(n)$. Let $\nu \in \mathrm{Cl}(X_\Sigma)$ be a nef class and let Δ_ν be the corresponding polytope, written as in (2.3), satisfying $0 \le \nu_j < \min_i a_{i,j}$ for $\rho_j \notin \sigma(1)$. Then, the two following properties hold:

- $R_\nu = (R/J)_\nu$.
- For every $x^\mu \in R_\nu$ and $f_i \in R_{\alpha_i}$ and $i = 0, \dots, n$, there exists a decomposition of the form

$$f_i = z_1^{\mu_{n+1}+1} \cdots z_r^{\mu_{n+r}+1} f_{i,0}^\mu + x_1^{\mu_1+1} f_{i,1}^\mu + \cdots + x_n^{\mu_n+1} f_{i,n}^\mu$$

  where the $f_{i,j}^\mu$, $i,j = 0, \dots, n$, are homogeneous polynomials in R.

Proof. The graded quotient map $R_\nu \to (R/J)_\nu$ is surjective. Using the degree constraint, its kernel must be zero, giving the first property. On the other hand, we have to prove that for $x^\mu = x_1^{\mu_1} \dots x_n^{\mu_n} z_1^{\mu_{n+1}} \dots z_r^{\mu_{n+r}} \in R_\nu$ (we recall that $\mu = \mathbf{F}m + \nu$ for $m \in \mathcal{A}_i$), every monomial $x^{\mathbf{F}m + a_i} \in R_i$ that is not divisible by $z_1^{\mu_{n+1}+1} \cdots z_r^{\mu_{n+r}+1}$ is divisible by some of the monomials $x_1^{\mu_1+1}, \dots, x_n^{\mu_n+1}$. Using the toric homogenization, the fact that the first does not hold implies that

$$\langle u_{n+j}, m \rangle + a_{i,j} \le \mu_{n+j} \quad j = 1, \dots, r.$$

Considering any of these j and using that $\nu_j = \mu_{n+j} + \sum_{k=1}^n \mathcal{P}_{k,j}\mu_k$, via the map π defined in Definition 2.1, we get:

$$\langle u_{n+j}, m \rangle + \min a_{i,j} \le \mu_{n+j} \implies \langle u_{n+j}, m \rangle + \nu_j < \mu_{n+j} \implies$$
$$\langle u_{n+j}, m \rangle + \sum_{k=1}^n \mathcal{P}_{k,j}\mu_k < 0 \implies \sum_{k=1}^n \mathcal{P}_{k,j}(\mu_k - \langle u_k, m \rangle) < 0.$$

As X_Σ is σ-positive, there must exists $k \in \{1, \dots, n\}$ such that $\mu_k - \langle u_k, m \rangle < 0$. This implies that $x_k^{\mu_k+1}$ divides $x^{\mathbf{F}m+a_i}$, as the coefficient of x_k in this monomial is $\mathbf{F}u_k + a_{i,k} = \langle u_k, m \rangle$. ☐

Corollary 2.1. Assume that the projective toric variety X_Σ is σ-positive for some $\sigma \in \Sigma(n)$. If Δ_i is n-dimensional for all $i = 0, \dots, n$, then Theorem 2.1 holds for $\nu = 0$.

Proof. If Δ_i is n-dimensional, then $a_{i,j} > 0$ for $j > n$ thanks to the positivity property. Therefore, $0 < \min_i a_{i,j}$ for $\rho_j \notin \sigma(1)$, which proves the claim. ☐

Generic sparse homogeneous polynomial systems. Let $\Delta_0, \dots, \Delta_n$ be rational polytopes in $M_\mathbb{R}$, let Σ be the normal fan of the Minkowski sum $\Delta = \sum_{i=0}^n \Delta_i$ and let X_Σ be the corresponding projective toric variety (as X_Σ is defined by the normal fan of Δ, X_Σ is complete and Δ corresponds to an ample divisor [CLS12, Proposition 6.1.4], which implies that X_Σ is projective).

Suppose that X_Σ admits a maximal smooth cone $\sigma \in \Sigma(n)$ whose corresponding variables in the Cox ring R of X_Σ are $x_1, \dots, x_n$, as in (2.2). The polytopes Δ_i can be seen as elements $a_i = (a_{i,j}) \in \mathbb{Z}^{\Sigma(1)}$ using the following facet presentations

$$\Delta_i = \{m \in M_\mathbb{R} : \langle m, u_j \rangle \ge -a_{i,j},\ \rho_j \in \Sigma(1)\},\ i = 0, \dots, n, \tag{2.3}$$

which are chosen to be minimal so that the Δ_i's correspond to basepoint free divisors. These presentations relate each of the polytopes Δ_i to Weil divisors that can be written as $\sum_j a_{i,j} D_j$ where D_j is the torus invariant divisor associated with the ray ρ_j. Using (2.1), we see that two polytopes that map to the same class in $\mathrm{Cl}(X_\Sigma)$ are translations of each other. For each class $\alpha_i \in \mathrm{Cl}(X_\Sigma)$, we choose this presentation so that the vertex associated to the cone σ is $0 \in M$. In particular, this implies that $a_{i,j} = 0$ for $\rho_j \in \sigma(1)$.

Now, let $\alpha_0, \ldots, \alpha_n$ be nef classes in $\mathrm{Cl}(X_\Sigma)$ associated to $\Delta_0, \ldots, \Delta_n$, and $R_{\alpha_0}, \ldots, R_{\alpha_n}$ be the corresponding graded components in the Cox ring, respectively. These graded components are finite k-vector spaces and have a monomial basis given by $x^\mu := x_1^{\mu_1} \cdots x_n^{\mu_n} z_1^{\mu_{n+1}} \cdots z_r^{\mu_r} \in R$. Let $A = k[c_{i,\mu} : x^\mu \in R_{\alpha_i}, i = 0, \ldots, n]$ and $C = A[x_1, \ldots, x_n, z_1, \ldots, z_r]$. The *generic homogeneous sparse polynomial system of degree* $\alpha_0, \ldots, \alpha_n$ is the system defined by the polynomials

$$F_i = \sum_{x^\mu \in R_{\alpha_i}} c_{i,\mu} x^\mu \in C = A[x_1, \ldots, x_n, z_1, \ldots, z_r], \quad i = 0, \ldots, n. \tag{2.4}$$

The ring C can be interpreted as the Cox ring of the toric variety $X_\Sigma \times_k \mathrm{Spec}(A)$ over generic coefficients and its graded components are given by $C_\alpha = R_\alpha \otimes_k A$.

If the system is dehomogenized by setting $z_1 = \cdots = z_r = 1$, the Newton polytope of F_i is Δ_i for $i = 0, \ldots, n$. Conversely, the polynomials $F_0, \ldots, F_n$ can be defined as the homogenization of the system of polynomials $\tilde{F}_0 = \cdots = \tilde{F}_n = 0$ with supports in the subsets $\mathcal{A}_i = \Delta_i \cap M$ for $i = 0, \ldots, n$. More precisely, the homogeneization of the polynomial

$$\tilde{F}_i = \sum_{m \in \mathcal{A}_i} c_{i,m} x^m \in \tilde{C} = A[x_1, \ldots, x_n]$$

is the polynomial

$$F_i = \sum_{m \in \mathcal{A}_i} c_{i,m} x^{\mathbf{F}m + a_i} \in C = A[x_1, \ldots, x_n, z_1, \ldots, z_r]$$

where $\mathbf{F}$ and a_i are defined in (2.1) and (2.3), respectively. We note that by homogenizing the monomials associated to the lattice points in $\mathcal{A}_i$, we can choose a monomial basis of R_{α_i} using $\mu = \mathbf{F}m + a_i$. We refer to [BT22, Section 2.2] for more details about homogenization and dehomogenization of sparse polynomial systems.

Finally, we note that that if X_Σ is assumed to be σ-positive, then Theorem 2.1 can be easily extended to our setting and yield a decomposition of the generic homogeneous sparse polynomials F_i, $i = 0, \ldots, n$, over $X_\Sigma \times_k A$, where $F_{i,j}^\mu$ are homogeneous polynomials in C.

Torsion and local cohomology. From the fan Σ of a toric variety X_Σ, the irrelevant ideal $\mathfrak{b}$ of its homogeneous coordinate ring $C = A[x_\rho, \rho \in \Sigma(1)]$ is defined as

$$\mathfrak{b} = (x^{\overline{\sigma}} \text{ such that } \sigma \in \Sigma(n)), \text{ where } x^{\overline{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho.$$

The $\mathfrak{b}$-torsion of a graded C-module S is classically defined as

$$\Gamma_{\mathfrak{b}}(S) = \{a \in S : \mathfrak{b}^k a = 0, k \in \mathbb{N}\}$$

and the local cohomology modules $H^i_{\mathfrak{b}}(S)$ are then the derived functors of $S \to \Gamma_{\mathfrak{b}}(S)$. When the module S is a quotient ring $B = C/I$ for $I = \langle F_0, \ldots, F_n \rangle$ the ideal defined by (2.4), the 0-th local cohomology is $H^0_{\mathfrak{b}}(B) = I^{\mathrm{sat}}/I$ where I^{sat} denotes the saturation of the ideal I with respect to the irrelevant ideal of C, i.e. $I^{\mathrm{sat}} := (I : \mathfrak{b}^\infty) = \{p \in C : \exists k \in \mathbb{Z} \quad \mathfrak{b}^k p \subset I\}$.

Local cohomology modules are strongly related to sheaf cohomology modules. More precisely, let S be a finitely generated $\mathrm{Cl}(X_\Sigma)$-graded R-module with associated coherent sheaf $\mathcal{S}$ in X_Σ and $\alpha \in \mathrm{Cl}(X_\Sigma)$. If $p \geq 2$, then

$$H^p_{\mathfrak{b}}(S)_\alpha \simeq H^{p-1}(X_\Sigma, \mathcal{S}(\alpha)). \tag{2.5}$$

Furthermore, the following exact sequence holds (see [CLS12, Theorem 9.5.7] for proofs):

$$0 \to H^0_{\mathfrak{b}}(S)_\alpha \to S_\alpha \to H^0(X_\Sigma, \mathcal{S}(\alpha)) \to H^1_{\mathfrak{b}}(S)_\alpha \to 0.$$

If $S = R$, then $R_\alpha = H^0(X_\Sigma, \mathcal{O}_\Sigma(\alpha))$ and therefore

$$H^0_{\mathfrak{b}}(R) = H^1_{\mathfrak{b}}(R) = 0, \tag{2.6}$$

which implies that $H^i_{\mathfrak{b}}(C) = 0$ for $i = 0, 1$.

Notation 2.1. For the sake of simplicity in the notation, for any Cartier divisor D and any integer $p \geq 0$, we will write $H^p(X_\Sigma, \alpha)$ in place of $H^p(X_\Sigma, \mathcal{O}_\Sigma(D))$, where $\alpha = [D] \in \mathrm{Cl}(X_\Sigma)$.

The following theorems by Demazure and Batyrev-Borisov are the main tools that we will rely on in order to analyze the vanishing of sheaf cohomology modules of toric varieties.

Theorem 2.2. [Dem70, Corollary 1] Let X_Σ be a toric variety and $\alpha \in \mathrm{Cl}(X_\Sigma)$ be a nef class, then $H^p(X_\Sigma, \alpha) \simeq 0$ for all $p > 0$.

Theorem 2.3. [BB11, Theorem 2.5] Let X_Σ be a toric variety and $\alpha \in \mathrm{Cl}(X_\Sigma)$ be a nef class, then

$$H^p(X_\Sigma, -\alpha) \simeq \begin{cases} 0 & \text{if } p \neq \dim \Delta_\alpha \\ \oplus_{m \in \mathrm{Relint}(\Delta_\alpha) \cap M} k\chi^{-m} & \text{if } p = \dim \Delta_\alpha \end{cases}$$

Another important result we will use is the toric version of Serre duality (see [CLS12, Theorem 9.2.10] for a proof): for any $\alpha \in \mathrm{Cl}(X_\Sigma)$ and any integer $p \geq 0$,

$$H^p(X_\Sigma, \alpha) \cong H^{n-p}(X_\Sigma, -K_X - \alpha)^\vee,$$

where K_X the anticanonical class in $\mathrm{Cl}(X_\Sigma)$.

Hilbert functions and the Grothendieck-Serre formula. Let X_Σ be a smooth projective toric variety and let R be its Cox ring. The Hilbert function of graded R-module S is defined as

$$\begin{aligned} \mathrm{HF}(S, -) : \mathrm{Cl}(X_\Sigma) \quad &\to \quad \mathbb{Z}_{\geq 0} \\ \alpha \quad &\mapsto \quad \mathrm{HF}(S, \alpha) := \dim_k(S_\alpha). \end{aligned}$$

For $\alpha \gg 0$ (component-wise), this function becomes a (multivariate) polynomial which is called the Hilbert polynomial and is denoted by $\mathrm{HP}(S, \alpha)$ (see [MS03, Lemma 2.8]).

Remark 2.1. If $S = R/J$ with J an ideal of R defining a 0-dimensional subscheme in X_Σ, then the Hilbert polynomial of S is a constant which is equal to the number of points counted with multiplicity.

An important relation between the Hilbert function, the Hilbert polynomial and local cohomology modules is given by the Grothendieck-Serre formula (see [MS03, Proposition 2.14]): for any $\alpha \in \mathrm{Cl}(X_\Sigma)$,

$$\mathrm{HF}(S, \alpha) = \mathrm{HP}(S, \alpha) + \sum_{i=0}^{n} (-1)^i \dim_k H^i_{\mathfrak{b}}(S)_\alpha. \tag{2.7}$$

3 A duality theorem

Let X_Σ be a projective toric variety of dimension n such that it admits a maximal smooth cone $\sigma \in \Sigma(n)$. In this section, we consider the ideal generated by $n+1$ generic homogeneous sparse polynomials (see Section 2) and analyze some graded components of its saturation via a duality property. For that purpose, we take again the notation (2.4): $F_0, \ldots, F_n$ are the generic homogeneous polynomials of degree $\alpha_0, \ldots, \alpha_n$, respectively; they are of the form

$$F_i = \sum_{x^\mu \in C_i} c_{i,\mu} x^\mu \in C = A[x_1, \ldots, x_n, z_1, \ldots, z_r]. \tag{3.1}$$

As a preliminary result, we first show that $F_0, \ldots, F_n$ form a regular sequence outside $V(\mathfrak{b}) \subset \mathrm{Spec}(C)$.

Lemma 3.1. The homogeneous generic polynomials $F_0, \ldots, F_n$ define a regular sequence in the localization ring $C_\tau := C_{x^{\overline{\tau}}}$ for any $\tau \in \Sigma(n)$.

Proof. We claim that F_0 is a nonzero divisor in C. This follows as a corollary of Dedekind-Mertens Lemma [BJ14, Corollary 2.8], which says that F is a nonzero divisor in $A[x_1, \ldots, x_n]$ if its content ideal is a nonzero divisor in A. The content ideal is generated by the coefficients $c_{0,\mu}$ for $x^\mu \in C_0$ and they are all nonzero divisors. Therefore, F_0 is a nonzero divisor also in C_τ for all $\tau \in \Sigma(n)$.

Now, as Σ always refines the normal fan of Δ_i, we can always find a vertex $a_\tau \in \mathcal{A}_i$ corresponding to each maximal cone $\tau \in \Sigma(n)$. Let $c_{i,\tau}$ be the coefficient associated to this vertex. Then, similarly to [BCN22, Lemma 3.2], for any $t \in \{1, \ldots, n-1\}$ there is a isomorphism of algebras

$$B^t_\tau = \left(A[x_1, \ldots, x_n, z_1, \ldots, z_r]/\langle F_0, \ldots, F_t \rangle\right)_\tau \xrightarrow{\sim} (A^t_\tau[x_1, \ldots, x_n, z_1, \ldots, z_r])_\tau$$

where $A^t_\tau = k[c_{i,\mu} \quad c_{i,\mu} \neq c_{i,\tau} \quad 0 \leq i \leq t]$ and which maps $c_{i,\tau} x^{a_\tau}$ to $F_i - c_{i,\tau} x^{a_\tau}$ and which leaves invariant the other variables and coefficients. Applying again Dedekind-Mertens Lemma as above, we deduce that F_{t+1} is a nonzero divisor in $(A^t_\tau[x_1, \ldots, x_n, z_1, \ldots, z_r])_\tau$, and therefore in B^t_τ. $\square$

Next, we consider the two canonical spectral sequences associated with the Čech-Koszul double complex $\mathcal{C}^\bullet_{\mathfrak{b}}(K_\bullet(F))$, where $K_\bullet(F)$ denotes the Koszul complex of the sequence of homogeneous polynomials $F_0,\dots,F_n$ in C. The terms of this Koszul complex are graded free C-modules and we denote their homology modules by H_p for simplicity in the notation. If we start taking homologies horizontally, the second page is:

$$\begin{array}{cccccc}
H^0_{\mathfrak{b}}(H_{n+1}) & H^0_{\mathfrak{b}}(H_n) & H^0_{\mathfrak{b}}(H_{n-1}) & \cdots & H^0_{\mathfrak{b}}(H_0) = I^{\mathrm{sat}}/I \\
0 & 0 & 0 & \cdots & H^1_{\mathfrak{b}}(H_0) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & H^n_{\mathfrak{b}}(H_0) \\
0 & 0 & 0 & \cdots & H^{n+1}_{\mathfrak{b}}(H_0)
\end{array}.$$

The vanishing of the local cohomology modules $H^i_{\mathfrak{b}}(H_j)$ for $i>0$ and $j>0$ follows from the fact that the F_i's form a regular sequence after localization by a generator of $\mathfrak{b}$ by Lemma 3.1. In addition, we deduce that H_p are geometrically supported on $V(\mathfrak{b})$ for all $p>0$ by a classical property of Koszul complexes, and hence that $H^0_{\mathfrak{b}}(H_p)=H_p$ for all $p>0$.

On the other hand, if we start taking homologies vertically, we obtain the following first page:

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & \cdots & 0 \\
0 & & 0 & & 0 & \cdots & 0 \\
\vdots & & \vdots & & \vdots & & \vdots \\
H^n_{\mathfrak{b}}(C(-\sum_j \alpha_j)) & \to & H^n_{\mathfrak{b}}(\oplus_k C(-\sum_{j\neq k}\alpha_j)) & \to & H^n_{\mathfrak{b}}(\oplus_{k,k'} C(-\sum_{j\neq k,k'}\alpha_j)) & \cdots & H^n_{\mathfrak{b}}(C) \\
H^{n+1}_{\mathfrak{b}}(C(-\sum_j \alpha_j)) & \to & H^{n+1}_{\mathfrak{b}}(\oplus_k C(-\sum_{j\neq k}\alpha_j)) & \to & H^{n+1}_{\mathfrak{b}}(\oplus_{k,k'} C(-\sum_{j\neq k,k'}\alpha_j)) & \cdots & H^{n+1}_{\mathfrak{b}}(C)
\end{array}$$

using that $K_j(F)=\oplus_{|J|=j}C(-\sum_{k\in J}\alpha_k)$ for $J\subset\{0,\dots,n\}$. We note that the vanishing of the two first rows follows from (2.6), and also that the vanishing of $H^p_{\mathfrak{b}}(C)$ for all $p>n+1$ is a consequence of Grothendieck's vanishing Theorem [Gro57, Theorem 3.6.5].

Notation 3.1. The support $\operatorname{Supp} S$ of a graded module S is the subset of $\nu\in\mathrm{Cl}(X_\Sigma)$ such that $S_\nu\neq 0$. We denote by Γ_1 the support of the modules on the main diagonal, expect on the last row, and by Γ_0 the support of the modules in the diagonal under Γ_1, except on the last row again, i.e

$$\Gamma_i=\operatorname{Supp}(\oplus_{p=0}^n H^p_{\mathfrak{b}}(K_{p+i-1}(F)))\quad i=0,1.$$

In addition, we define Γ_{Res} to be the support of all the cohomology modules that are appearing above the diagonal in the first page of the second spectral sequence, i.e. $\Gamma_{\mathrm{Res}}=\cup_{i<j}\operatorname{Supp} H^i_{\mathfrak{b}}(K_j(F))$. Moreover, from now on, we denote by δ the divisor class $\alpha_0+\cdots+\alpha_n-K_X$ where K_X denotes the anticanonical divisor of X_Σ.

The comparison of the two above spectral sequences leads to the following duality.

Theorem 3.1. Let X_Σ be a projective toric variety which admits a maximal smooth cone $\sigma\in\Sigma(n)$ and let $\nu\in\mathrm{Cl}(X_\Sigma)$. If $\delta-\nu\notin\Gamma_0\cup\Gamma_1$ then

$$(I^{\mathrm{sat}}/I)_{\delta-\nu}\simeq \operatorname{Hom}_A((C/I)_\nu,A).$$

Proof. From the comparison of the two spectral sequences associated to the double complex $\mathcal{C}^\bullet_{\mathfrak{b}}(K_\bullet(F))$, for all $\delta-\nu\notin\Gamma_0\cup\Gamma_1$ we get an isomorphism

$$(I^{\mathrm{sat}}/I)_{\delta-\nu}\simeq \operatorname{Ker}\left(H^{n+1}_{\mathfrak{b}}(C(-\sum_j\alpha_j))\to H^{n+1}_{\mathfrak{b}}(\oplus_k C(-\sum_{j\neq k}\alpha_j))\right)_{\delta-\nu}.$$

Moreover, using toric Serre duality and the relation between sheaf and local cohomology modules, we obtain

$$H^{n+1}_{\mathfrak{b}}(C(-\sum_j \alpha_j))_{\delta-\nu} \simeq H^n(X_\Sigma, -\nu - K_X) \simeq H^0(X_\Sigma, \nu)^\vee \simeq \mathrm{Hom}_A(C_\nu, A).$$

By the same argument, we also have $H^{n+1}_{\mathfrak{b}}(\oplus_k C(-\sum_{j\neq k}\alpha_i))_{\delta-\nu} \simeq \mathrm{Hom}_A(I_\nu, A)$. Using the first isomorphism, we get the duality property. ☐

Theorem 3.1 holds if $\delta - \nu \in \mathrm{Supp}\, H^{n+1}_{\mathfrak{b}}(\oplus_j C(-\sum_{i\neq j}\alpha_i)))$, which is a priori not contained in Γ_0 or Γ_1, but if it does not belong to this support, then we get the following important consequence.

Corollary 3.1. Let X_Σ be a projective toric variety which admits a maximal smooth cone $\sigma \in \Sigma(n)$. Let $\nu \in \mathrm{Cl}(X_\Sigma)$ be a nef class and Δ_ν be the corresponding polytope, written as in (2.3), satisfying $0 \le \nu_j < \min_i a_{i,j}$ for $\rho_j \notin \sigma(1)$. Then,

$$(I^{\mathrm{sat}}/I)_{\delta-\nu} \simeq \mathrm{Hom}_A(C_\nu, A).$$

In particular, $(I^{\mathrm{sat}}/I)_{\delta-\nu}$ is a free A-module whose rank is equal to the rank of C_ν.

Proof. We note that these conditions imply that $C_\nu = (C/I)_\nu$, as in Theorem 2.1, but without requiring the property of being σ-positive in this case. ☐

We also note that the case $\nu = 0$, which corresponds to the isomorphism $(I^{\mathrm{sat}}/I)_\delta \simeq A$, was already known; see [BC93; CCD97].

4 Toric Sylvester forms

Take again the notation of Section 3. As a consequence of Corollary 3.1, some graded components of I^{sat}/I are free A-modules and hence a natural question is to provide explicit A-bases for them. This is precisely the goal of this section. We will first describe the graded component $(I^{\mathrm{sat}}/I)_\delta$, which essentially follows from [CCD97], and then introduce Sylvester forms to deal with the other cases. In what follows, we assume that the projective toric variety X_Σ is σ-positive with respect to the maximal smooth cone $\sigma \in \Sigma(n)$.

Following [CCD97], we find a nonzero element of $(I^{\mathrm{sat}}/I)_\delta \simeq A$ as follows. Using Corollary 2.1, if $\Delta_0, \dots, \Delta_n$ are n-dimensional, one can decompose each polynomial as

$$F_i = x_1 F_{i,1} + \cdots + x_n F_{i,n} + z_1 \cdots z_r F_{i,n+1}, \tag{4.1}$$

and consider the determinant

$$\mathrm{Sylv}_0 = \det\left(F_{i,j}\right)_{0\le i,j\le n}.$$

This homogeneous polynomial is called the *toric jacobian*; we will denote its class modulo I by sylv_0.

Proposition 4.1. The element Sylv_0 belongs to $(I^{\mathrm{sat}})_\delta$. Moreover, sylv_0 is independent on the choices of decompositions (4.1) and the choice of σ (as long as X_Σ is σ-positive). In addition, sylv_0 is a generator of $(I^{\mathrm{sat}}/I)_\delta$, which is a free A-module of rank 1.

Proof. The fact that $\mathrm{Sylv}_0 \in I^{\mathrm{sat}}$ follows from $x_i \,\mathrm{Sylv}_0 \in I$ for $i = 1, \dots, n$ and $z_1 \cdots z_r \,\mathrm{Sylv}_0 \in I$. The A-module $(I^{\mathrm{sat}}/I)_\delta$ is free of rank 1 by Theorem 3.1 and the fact that $C_0 \simeq A$. The fact that sylv_0 is nonzero is a consequence of [CCD97, Theorem 0.2] and the independence of the choice of the decomposition (4.1) is a consequence of the classical Wiebe's lemma; see [Jou95, Proposition 3.8.1.6].

The property $\mathrm{Sylv}_0 \notin I$ and the independence from σ is proved in [CCD97, Theorem 0.2]. In this paper the hypothesis that the α_i 's are $\mathbb{Q}$-ample, for $i = 0, \dots, n$, is used in order to derive the decomposition (4.1). In our context, we already derived such a decomposition in Theorem 2.1 so, as claimed in [CCD97, Remark 2.12, iv], the same property holds in this case.

In order to prove that sylv_0 has degree δ, we find the degree of each entry (i, j) of the matrix defined by the $F_{i,j}$. In (4.1), we divided a set of monomials of degree α_i, by a monomial of degree

$$\begin{cases} \pi(e_j) & \text{if the monomial is } x_j \text{ for } j = 1, \dots, n, \\ \pi(\sum_{k=n+1} e_j) & \text{if the monomial is } z_1 \cdots z_r, \end{cases}$$

where $\{e_j\}_{j=1}^{n+r}$ is the canonical basis of $\mathbb{Z}^{\Sigma(1)}$. On the other hand, the anticanonical class K_X coincides with the degree of the monomial $x_1 \cdots x_n z_1 \cdots z_r$, which is equal to $\pi(\sum_{j=1}^{n+r} e_j)$. Therefore, the degree of each of the summands constituting the determinant is equal to:

$$\sum_{i=0}^{n} \left(\alpha_i - \pi(e_{\tau(i)})\right) = \left(\sum_{i=0}^{n} \alpha_i\right) - K_X = \delta,$$

where $\tau \in \mathfrak{S}_n$ is the element of the symmetric group corresponding to such summand. ☐

We note that applying Theorem (2.3), the Sylvester form sylv_0 is in correspondence with the unique lattice point in the interior of the polytope Δ_Σ associated to the anticanonical divisor K_X:

$$(I^{\mathrm{sat}}/I)_\delta \simeq H^{n+1}_{\mathfrak{b}}(C(-\sum \alpha_i))_\delta \simeq H^n(X_\Sigma, -K_X) \simeq \oplus_{m\in \mathrm{Relint}(\Delta_\Sigma)} A\chi^{-m}.$$

So far we proved that the toric Jacobian sylv_0 yields an A-basis of $(I^{\mathrm{sat}}/I)_\delta \simeq A$. The next step is to construct an A-basis of $(I^{\mathrm{sat}}/I)_{\delta-\nu}$ when it is a free A-module.

Definition 4.1. Let X_Σ be a projective toric variety which is σ-positive for some $\sigma \in \Sigma(n)$. Let $\nu \in \mathrm{Cl}(X_\Sigma)$ be a nef class and Δ_ν be the corresponding polytope, written as in (2.3), satisfying $0 \leq \nu_j < \min_i a_{i,j}$ for $\rho_j \notin \sigma(1)$. According to Theorem 2.1, for any $x^\mu \in R_\nu$ and for any $i \in \{0,\dots,n\}$ the polynomial F_i can be decomposed as

$$F_i = z_1^{\mu_{n+1}+1}\cdots z_r^{\mu_{n+r}+1}F^\mu_{i,0} + x_1^{\mu_1+1}F^\mu_{i,1} + \cdots + x_n^{\mu_n+1}F^\mu_{i,n} \tag{4.2}$$

and we define the *toric Sylvester form* Sylv_μ as the determinant

$$\mathrm{Sylv}_\mu = \det(F^\mu_{i,j})_{0\leq i,j\leq n}.$$

The class of Sylv_μ modulo I is denoted by sylv_μ.

Theorem 4.1. Let X_Σ be a projective toric variety which is σ-positive for some $\sigma \in \Sigma(n)$. Then, for any $\nu \in \mathrm{Cl}(X_\Sigma)$ satisfying the hypotheses of Theorem 2.1 and any pair $x^\mu, x^{\mu'} \in R_\nu$:

$$x^{\mu'}\,\mathrm{sylv}_\mu = \begin{cases} \mathrm{sylv}_0 & \mu = \mu' \\ 0 & \text{otherwise.}\end{cases}$$

The element Sylv_μ belongs to $(I^{\mathrm{sat}})_{\delta-\nu}$. Its class $\mathrm{sylv}_\mu \in (I^{\mathrm{sat}}/I)_{\delta-\nu}$ is a nonzero element which is independent of the choices of decompositions and of σ (as long as X_Σ has the σ-positive property). Therefore, $\{\mathrm{sylv}_\mu\}_{x^\mu\in C_\nu}$ gives a basis of $(I^{\mathrm{sat}}/I)_{\delta-\nu}$.

Proof. First, the fact that Sylv_μ has degree $\delta-\nu$ follows by using the same reasonning as the one at the end of Proposition 4.1. Now, applying Cramer's rule from the decompositions (4.2) we get

$$x_j^{\mu_j+1}\,\mathrm{Sylv}_\mu = \det\begin{pmatrix} \cdots & x_j^{\mu_j+1}F_{0,j} & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & x_j^{\mu_j+1}F_{0,j} & \cdots \end{pmatrix} = \det\begin{pmatrix} \cdots & F_0 & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & F_n & \cdots \end{pmatrix} \in I,$$

and the same holds for the monomial $z_1^{\mu_{n+1}+1}\cdots z_r^{\mu_{n+r}+1}$. This proves that $\mathrm{Sylv}_\mu \in I^{\mathrm{sat}}_{\delta-\nu}$. Now, suppose that for $x^\mu \neq x^{\mu'} \in R_\nu$ there exists $j \in \{1,\dots,n\}$ such that $\mu'_j > \mu_j$ and $x_j^{\mu_j+1}$ divides $x^{\mu'}$. Then,

$$x^{\mu'}\,\mathrm{Sylv}_\mu = \frac{x^{\mu'}}{x_j^{\mu_j+1}}x_j^{\mu_j+1}\,\mathrm{Sylv}_\mu \in I \implies x^{\mu'}\,\mathrm{sylv}_\mu = 0 \in (I^{\mathrm{sat}}/I)_{\delta-\nu}.$$

If this does not hold, then $\mu'_j \leq \mu_j$ for all $j \in \{1,\dots,n\}$. Using the σ-positive property, this implies that $\sum_{j=1}^n \mathcal{P}_{j,k}\mu'_j \leq \sum_{j=1}^n \mathcal{P}_{j,k}\mu_j$ for $k = 1,\dots,r$, but if it was an equality, then:

$$\nu_k = \mu'_{n+k} + \sum_{j=1} \mathcal{P}_{j,k}\mu'_j = \mu_{n+k} + \sum_{j=1}\mathcal{P}_{j,k}\mu_j \quad k = 1,\dots,r$$

since x^μ and $x^{\mu'}$ have the same degree ν. This would imply that $x^\mu = x^{\mu'}$, which yields a contradiction. Otherwise, $\mu'_{n+k} > \mu_{n+k}$ for all $k = 1,\dots,r$, implying:

$$x^{\mu'}\,\mathrm{Sylv}_\mu = \frac{x^{\mu'}}{z_1^{\mu_{n+1}+1}\cdots z_r^{\mu_{n+r}+1}}z_1^{\mu_{n+1}+1}\cdots z_r^{\mu_{n+r}+1}\,\mathrm{Sylv}_\mu \in I \implies x^{\mu'}\,\mathrm{sylv}_\mu = 0 \in (I^{\mathrm{sat}}/I)_{\delta-\nu}.$$

On the other hand, we have

$$x^\mu\,\mathrm{Sylv}_\mu = x_1^{\mu_j}\cdots x_n^{\mu_n}z_1^{\mu_{n+1}}z_r^{\mu_{n+r}}\det(F_{i,j}) = \det(x_j^{\mu_j}F_{i,j})$$

but at the same time, the decomposition

$$F_i = x_1x_1^{\mu_1}F_{i,1} + x_2x_2^{\mu_2}F_{i,2} + \cdots + z_1\cdots z_r z_1^{\mu_{n+1}}\cdots z_r^{\mu_{n+r}}F_{i,n+1}$$

gives the Sylvester form sylv_0, implying the equality and that $\mathrm{Sylv}_\mu \notin I$. The fact that they form a basis follows from the duality in Theorem 3.1 and from the observation that sylv_0 is a basis of $(I^{\mathrm{sat}}/I)_\delta$ ☐

Remark 4.1. We note that over the field k, the duality between Sylvester forms and monomials in Theorem 4.1 could also be deduced from the global transformation law in [CCD97, Theorem 0.2]. The approach we developed above allows us to work over the ring A, which is the universal ring of coefficients over k.

5 Application to toric elimination matrices

An important motivation for studying the structure of the saturation of an ideal generated by generic sparse polynomials is for applications in elimination theory, in particular for solving sparse polynomial systems. In this section we introduce a family of matrices whose construction involves toric Sylvester forms. It yields new compact elimination matrices that can be used for solving 0-dimensional sparse polynomial systems via linear algebra methods; we refer the reader to [EM99; BT22; Tel20] for a thorough exposition of such solving methods that we will not discuss in this paper.

In what follows, we will consider a *smooth* projective toric variety X_Σ which is σ-positive for some maximal cone σ, and a generic sparse polynomial system defined by homogeneous polynomials $F_0, \ldots, F_n$ as defined in (2.4). We need to assume that X_Σ is smooth because we will use the Grothendieck-Serre formula (see Section 2). This setting covers many cases that are of interest for applications. We note that the smoothness assumption is not very restrictive as X_Σ can be replaced by one of its desingularization variety (see e.g. [CLS12, Chapters 10, 11]). However, it is not straightforward that the desingularization of a toric variety satisfying the positivity property will itself satisfy this property.

Notation 5.1. The elimination matrices we will consider are universal with respect to the coefficients of the F_i's, so we introduce the following notation to study rigorously their properties under specialization of these coefficients. Recall that I denotes the ideal in C generated by $F_0, \ldots, F_n$.

Any specialization (i.e. ring morphism) $\theta : A \to k$ induces a surjective map $C \to R$ where $R = k[x_\rho : \rho \in \Sigma(1)]$ (this map leaves invariant the variables x_ρ). For all $i = 0, \ldots, n$ we define $f_i = \theta(F_i) \in R$ and we denote by $I(f)$ the homogeneous ideal $(f_0, \ldots, f_n)$ of R and set $B(f) = R/I(f)$. Moreover, we also set $B^{\mathrm{sat}} = C/I^{\mathrm{sat}}$, $B(f)^{\mathrm{sat}} = R/I(f)^{\mathrm{sat}}$ and $B^{\mathrm{sat}}(f) = C/I^{\mathrm{sat}}(f)$ (observe that $I(f)^{\mathrm{sat}}$ and $I^{\mathrm{sat}}(f)$ are in general not the same ideals). Finally, for any matrix $\mathbb{M}$ with coefficients in A, we denote by $\mathbb{M}(f)$ its specialization by $\theta : A \to k$.

Finally, we note that we will consider $\mathrm{Pic}(X_\Sigma)$ instead of $\mathrm{Cl}(X_\Sigma)$ as all Weil divisors are Cartier in a smooth variety (see [CLS12, Proposition 4.2.6]).

5.1 Hybrid elimination matrices

We begin by describing precisely what we mean by an elimination matrix $\mathbb{M}$ associated to the polynomials $F_0, \ldots, F_n$. It is a matrix whose columns are filled with coefficients of some homogeneous forms that are of the same degree and that all belong to the saturated ideal $I^{\mathrm{sat}} \subset C$. Thus, its entries are polynomials in A. Moreover, it is required that for any specialization map $\theta : A \to k$ the following two properties hold :

i) The corank of $\mathbb{M}(f)$ is equal to zero if and only if $f_0 = \cdots = f_n = 0$ has no solution in X_Σ.

ii) If the number of solutions of $f_0 = \cdots = f_n = 0$ is finite in X_Σ and equals κ, then the corank of $\mathbb{M}(f)$ is κ.

We note that the first property yields a certificate of existence of a common root of the f_i's, which is equivalent to the vanishing of their sparse resultant; we will come back to resultants in the next section. The second property is mainly required for solving 0-dimensional polynomial systems by means of linear algebra techniques based on eigen-computations because in this approach, the common roots of the f_i's are extracted from the cokernel of $\mathbb{M}(f)$.

A very classical family of elimination matrices is obtained by filling columns with all the multiples of the F_i's of a certain degree. These matrices are usually called Macaulay-type matrices and are widely used for solving 0-dimensional polynomial systems (see [BT22]). To be more precise, these matrices, that we will denote by $\mathbb{M}_\alpha$, are presentation matrices of the A-module B_α, i.e. are matrices of the maps

$$\begin{array}{rcl} \left(\oplus_{i=0}^n C(-\alpha_i)\right)_\alpha & \to & C_\alpha \\ (G_0, \ldots, G_n) & \mapsto & \displaystyle\sum_{i=0}^n G_i F_i. \end{array} \tag{5.1}$$

Of course, some conditions on $\alpha \in \mathrm{Pic}(X_\Sigma)$ are required in order to guarantee that $\mathbb{M}_\alpha$ is an elimination matrix; we refer to [EM99] and to [Tel20, Chapter 5] for more details. Applying results we proved in the previous sections, we extend the family of Macaulay-type matrices by using toric Sylvester forms. We recall that Sylvester forms belong to I^{sat} by Theorem 4.1.

Definition 5.1. Let α be such that $\left(I^{\mathrm{sat}}/I\right)_\alpha \simeq \oplus_\mu A$ is a free A-module; e.g. $\alpha \notin \Gamma_0 \cup \Gamma_1$ and $I_{\delta-\alpha} = 0$ as in Corollary 3.1. Consider the map

$$\begin{array}{rcl} (\oplus_{i=0}^n C(-\alpha_i))_\alpha \oplus \left(I^{\mathrm{sat}}/I\right)_\alpha & \to & C_\alpha \\ (G_0, \ldots, G_n) \oplus (l_\mu)_{x^\mu \in C_{\delta-\alpha}} & \mapsto & \displaystyle\sum_{i=0}^n G_i F_i + \sum_{x^\mu \in C_{\delta-\alpha}} l_\mu \,\mathrm{Sylv}_\mu \end{array} \tag{5.2}$$

where we recall that $l_\mu \in A$ for all μ. Its matrix is called a *hybrid elimination matrix* and will be denoted by $\mathbb{H}_\alpha$.

The matrices $\mathbb{H}_\alpha$ are called *hybrid* because they are composed of two blocks, one from the classical Macaulay-type matrices and another one built from toric Sylvester forms. In particular, $\mathbb{M}_\alpha = \mathbb{H}_\alpha$ if $(I^{\text{sat}}/I)_\alpha = 0$, so that the family of matrices $\mathbb{H}_\alpha$ can be seen as an extension of the family of Macaulay-type matrices $\mathbb{M}_\alpha$; from now on we will use the notation $\mathbb{H}_\alpha$ instead of $\mathbb{M}_\alpha$. Our next step is to prove that these matrices are elimination matrices.

5.2 Main properties

In this section, we first prove that the matrices $\mathbb{H}_\alpha$ introduced in Definition 5.1 are elimination matrices. Then, we give an illustrative example and also provide another criterion to construct the matrices $\mathbb{H}_\alpha$ without relying on the computation of the supports Γ_0 and Γ_1.

First, suppose given a specialization map (see Notation 5.1) and a degree α. From the results of Section 3 and Section 4, and also Definition 5.1, we deduce that the image of the matrix $\mathbb{H}_\alpha(f)$ is $I^{\text{sat}}(f)_\alpha$, so that its corank is $\text{HF}(B^{\text{sat}}(f), \alpha)$. Therefore, a natural question is to compare this Hilbert function of $B^{\text{sat}}(f)$ with the one of $B(f)^{\text{sat}}$ in degrees for which hybrid matrices $\mathbb{H}$ are defined; see Definition 5.1. We recall that we take again the notation of Section 3 and we assume that the toric variety X_Σ is smooth and σ-positive for a maximal cone $\sigma \in \Sigma(n)$.

Lemma 5.1. Let $\alpha \notin \Gamma_0 \cup \Gamma_1 \subset \text{Pic}(X_\Sigma)$ and suppose given specialized polynomials $f_0, \ldots, f_n$ defining a 0-dimensional subscheme in X_Σ, possibly empty, of κ points, counted with multiplicity. Then,

$$\text{HF}(B(f)^{\text{sat}}, \alpha) = \text{HF}(B^{\text{sat}}(f), \alpha) = \kappa.$$

Proof. This proof goes along the same lines as [BCN22, Lemma 2.7]. First, one observes that $I(f) \subset I^{\text{sat}}(f) \subset I(f)^{\text{sat}}$ so that $B(f)^{\text{sat}}$, $B^{\text{sat}}(f)$ and $B(f)$ have the same Hilbert polynomial, which is the constant κ by our assumption.

Now, $H^i_{\mathfrak{b}}(B(f)^{\text{sat}}) = 0$ for $i = 0$ and for all $i > 1$ since $V(I(f))$ is finite. Applying Grothendieck-Serre formula, it follows that $\text{HF}(B(f)^{\text{sat}}, \alpha) = \kappa$ for all α such that $H^1_{\mathfrak{b}}(B(f)^{\text{sat}})_\alpha = 0$. Analyzing the two spectral sequences associated to the Čech-Koszul complex of $f_0, \ldots, f_n$, we get that the above vanishing holds for all $\alpha \notin \Gamma_0 \cup \Gamma_1$.

Similarly, Grothendieck-Serre formula and the finiteness of $V(I(f))$ imply that $\text{HF}(B^{\text{sat}}(f), \alpha) = \kappa$ for all α such that $H^0_{\mathfrak{b}}(B(f)^{\text{sat}})_\alpha = H^1_{\mathfrak{b}}(B(f)^{\text{sat}})_\alpha = 0$. By [Cha13, Proposition 6.3], the vanishing of these modules can be derived from the similar vanishing conditions $H^0_{\mathfrak{b}}(B^{\text{sat}})_\alpha = H^1_{\mathfrak{b}}(B^{\text{sat}})_\alpha = 0$. These latter conditions hold for all $\alpha \notin \Gamma_0 \cup \Gamma_1$, which concludes the proof. □

Remark 5.1. As a consequence of the above lemma, the canonical map from I^{sat}_α to $I(f)^{\text{sat}}_\alpha$, which is induced by a specialization ρ, is surjective, i.e. generators of $I(f)^{\text{sat}}_\alpha$ can be computed by means of universal formulas.

Theorem 5.1. Assume that the toric variety X_Σ is smooth and σ-positive for a maximal cone $\sigma \in \Sigma(n)$. Then, for any $\alpha \notin \Gamma_0 \cup \Gamma_1 \subset \text{Pic}(X_\Sigma)$ the matrix $\mathbb{H}_\alpha$ is an elimination matrix, i.e. :

i) $\text{corank}(\mathbb{H}_\alpha(f)) = 0$ if and only if $V(I(f))$ is empty in X_Σ,

ii) If $V(I(f))$ is a finite subscheme of degree κ in X_Σ, then $\text{corank}(\mathbb{H}_\alpha(f)) = \kappa$.

Proof. The proof of i) follows from Lemma 5.1. For ii): if $V(I(f))$ is empty, equivalently $B(f)^{\text{sat}} = 0$ (this equivalence follows by the Grothendieck-Serre formula which requires the smoothness of X_Σ), then $\text{HF}(B^{\text{sat}}(f), \alpha) = 0$ by Lemma 5.1. If $V(I(f)) \neq \emptyset$, then the f_i's have a common solution, say the point $p \in X_\Sigma$ (over k) with defining ideal I_p (radical and maximal in R). Therefore, since $I^{\text{sat}}(f) \subset I(f)^{\text{sat}} \subset I_p$ and $\text{HF}(R/I_p, \beta) = 1$ for all $\beta \in \text{Pic}(X_\Sigma)$ by the maximality of I_p, we deduce that $\text{HF}(R/I^{\text{sat}}(f), \alpha) \neq 0$ for any α. □

Example 5.1. Let $M = \mathbb{Z}^2$ and X_Σ be the Hirzebruch surface $\mathcal{H}_1$ described in Example 2.3. Consider the following polytope presentations:

$$\Delta_i = \{m \in \mathbb{Z}^2 \,:\, \langle m, (1,0)\rangle \geq 0,\ \langle m, (0,1)\rangle \geq 0,\ \langle m, (-1,-1)\rangle \geq -2,\ \langle m, (0,-1)\rangle \geq -1\},\ i = 0, 1, 2.$$

$\mathcal{H}_1$ has the σ-positive for $\sigma = \langle (1,0), (0,1)\rangle$. The class in $\text{Pic}(\mathcal{H}_1) = \mathbb{Z}^2$ corresponding to these polytopes is $\alpha_i = (2,1)$ and we write the corresponding generic sparse homogeneous polynomials as:

$$F_0 = a_0 z_1^2 z_2 + a_1 x_1 z_1 z_2 + a_2 x_1^2 z_2 + a_3 x_2 z_1 + a_4 x_1 x_2 \quad \text{resp. } F_1, F_2 \text{ with coefficients } b_i, c_i \quad i = 0, \ldots, 4.$$

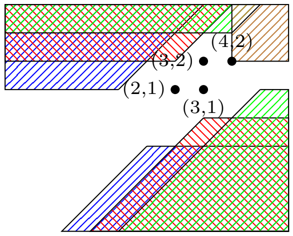

Figure 1: This is the picture of the regions $\Gamma_0, \Gamma_1, \Gamma_{\mathrm{Res}}, \Gamma \subset \mathrm{Pic}(X_\Sigma) = \mathbb{Z}^2$ (being the last one defined in Section 6). The blue region corresponds to Γ_0, the red region corresponds to Γ_1, the green region corresponds to Γ_{Res} and the brown region corresponds to Γ. We marked in orange those α with $(I^{\mathrm{sat}}/I)_\alpha \neq 0$. We derived the local cohomology of $\mathcal{H}_1$ from [Alt+20]; see also [EMS00; Bot11].

Figure 1 describes the supports $\Gamma_0, \Gamma_1, \Gamma_{\mathrm{Res}}$ (also Γ, which will be defined in Section 6). We deduce that elimination matrices $\mathbb{H}_\alpha$ are obtained for $\alpha \in \{(4,2),(3,2),(3,1),(2,1)\}$. In the cases $\alpha = (4,2)$ and $\alpha = (3,2)$, we get two Macaulay-type matrices. The two other cases give the following matrices:

- Case $\alpha = (3,1)$. This matrix corresponds to $\alpha = \delta$ and in this case, we are introducing a Sylvester form. This form is Sylv_0 and can be computed, as before, by a determinant that we write as:

$$\det\begin{pmatrix} a_1z_1z_2 + a_2x_1z_2 + a_4x_2 & a_3z_1 & a_0z_1 \\ b_1z_1z_2 + b_2x_1z_2 + b_4x_2 & b_3z_1 & b_0z_1 \\ c_1z_1z_2 + c_2x_1z_2 + c_4x_2 & c_3z_1 & c_0z_1 \end{pmatrix} = [130]z_1^3z_2 + [230]x_1z_1^2z_2 + [430]x_2z_1^2,$$

where $[ijk] = \det\begin{pmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{pmatrix}$. Therefore, the elimination matrix $\mathbb{H}_\alpha$ is of the form:

$$\mathbb{H}_{(3,1)} = \begin{pmatrix} a_0 & 0 & b_0 & 0 & c_0 & 0 & [130] \\ a_1 & a_0 & b_1 & b_0 & c_1 & c_0 & [230] \\ a_2 & a_1 & b_2 & b_1 & c_2 & c_1 & 0 \\ 0 & a_2 & 0 & b_2 & 0 & c_2 & 0 \\ a_3 & 0 & b_3 & 0 & c_3 & 0 & [430] \\ a_4 & a_3 & b_4 & b_3 & c_4 & c_3 & 0 \\ 0 & a_4 & 0 & b_4 & 0 & c_4 & 0 \end{pmatrix}.$$

This type of matrices for $\alpha = \delta$ were already known from [CDS97] as the Δ_i's are all equal and ample in $\mathcal{H}_1$. Nevertheless, we note that the block of Sylvester forms is different with our construction, in particular it is more sparse.

- Case $\alpha = (2,1)$. We obtain the following matrix $\mathbb{H}_\alpha$ which is built from two different Sylvester forms:

$$\mathbb{H}_{(2,1)} = \begin{pmatrix} a_0 & b_0 & c_0 & [013] & [023] \\ a_1 & b_1 & c_1 & [023]+[014] & [024]+[123] \\ a_2 & b_2 & c_2 & [024] & [124] \\ a_3 & b_3 & c_3 & 0 & 0 \\ a_4 & b_4 & c_4 & 0 & 0 \end{pmatrix}$$

where the Sylvester forms correspond to the monomial basis $\{z_1, x_1\}$ in C_ν for $\nu = (1,0)$. As far as we know, this kind of matrices are new.

Example 5.2. Consider again Example 5.1 but suppose now that $\alpha_2 = (1,1)$. This implies that the corresponding generic sparse homogeneous polynomial is:

$$F_2 = c_0z_1z_2 + c_1x_1z_2 + c_3x_2. \tag{5.3}$$

In this case, the Newton polytopes Δ_i's are not scaled copies of a fixed ample class and α_2 is not even ample in $\mathcal{H}_1$. Now, $\delta = (2,1)$ and the corresponding Sylvester is

$$\det\begin{pmatrix} a_1z_1z_2 + a_2x_1z_2 + a_4x_2 & a_3z_1 & a_0z_1 \\ b_1z_1z_2 + b_2x_1z_2 + b_4x_2 & b_3z_1 & b_0z_1 \\ c_1z_2 & c_3 & c_0 \end{pmatrix} = [130]z_1^2z_2 + [230]x_1z_1z_2 + [430]x_2z_1,$$

where $[ijk] := \det\begin{pmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{pmatrix}$, with the convention that $c_i = 0$ if this coefficient does not appear in F_2. Then, the corresponding elimination matrix is

$$\mathbb{H}_{(2,1)} = \begin{pmatrix} a_0 & b_0 & c_0 & 0 & [013] \\ a_1 & b_1 & c_1 & c_0 & [023]+[014] \\ a_2 & b_2 & 0 & c_1 & 0 \\ a_3 & b_3 & c_3 & 0 & [024] \\ a_4 & b_4 & 0 & c_3 & 0 \end{pmatrix}.$$

This example illustrates that Theorem 5.1 extends the results of [CCD97] in the case $\alpha = \delta$, with the only restriction that the Δ_i's must be n-dimensional.

As illustrated in Example 5.1, the construction of elimination matrices $\mathbb{H}_\alpha$ requires the computation of the support of the local cohomology modules $H^i_{\mathfrak{b}}(R)$. This task can be delicate, although several results are known; see for instance [Alt+20], especially when Σ splits or the rank of $\mathrm{Pic}(X_\Sigma)$ is 2 or 3, or [EMS00; Bot11]. In order to avoid such computations, our next result yields some combinatorial sufficient conditions to get hybrid elimination matrices.

We recall that we use the same notation as in Section 3. In particular, we write $\alpha_i \in \mathrm{Pic}(X_\Sigma)$ for the classes associated to the homogeneous polynomial system, K_X for the anticanonical divisor, $\delta = \alpha_0 + \cdots + \alpha_n - K_X$, $\nu, \alpha \in \mathrm{Pic}(X_\Sigma)$ as elements in the class group.

Theorem 5.2. Assume that the toric variety X_Σ is smooth and σ-positive for some maximal cone $\sigma \in \Sigma(n)$. Moreover, assume that all the polytopes Δ_i are n-dimensional and that $\alpha \in \mathrm{Pic}(X_\Sigma)$ satisfies one of the two following properties:

i) $\alpha = \delta + \nu$ with $\nu \in \mathrm{Pic}(X_\Sigma)$ a nef class,

ii) $\alpha = \delta - \nu$ for $\nu \in \mathrm{Pic}(X_\Sigma)$ a nef class satisfying the hypotheses of Theorem 2.1, and $\alpha_i - \nu$ is nef for all $i = 0, \dots, n$.

Then, $\mathbb{H}_\alpha$ is an elimination matrix. In addition, it is purely of Macaulay-type if and only if α satisfies only i).

Proof. First, we recall that the $K_j(F)$'s denote the modules involved in the Koszul complex associated to $F_0, \dots, F_n$. We will also denote by J subsets of $\{0, \dots, n\}$.

We begin with the case i). If $\alpha = \delta + \nu$, we have

$$H^i_{\mathfrak{b}}(K_j(F))_{\delta+\nu} \simeq H^i_{\mathfrak{b}}(\oplus_{|J|=j} C(-\sum_{i\in J}\alpha_i))_{\delta+\nu} \simeq \oplus_{|J|=j} H^i_{\mathfrak{b}}(C)_{\delta+\nu-\sum_{i\in J}\alpha_i},$$

where we use that the local cohomology functors are exact. Using (2.5), for $i \geq 2$ and $J \subset \{0, \dots, n\}$, we get:

$$H^i_{\mathfrak{b}}(C)_{\delta+\nu-\sum_{i\in J}\alpha_i} \simeq H^{i-1}(X_\Sigma, \sum_{i\notin J}\alpha_i - K_X + \nu) \simeq H^{n-i+1}(X_\Sigma, -\sum_{i\notin J}\alpha_i - \nu),$$

which is zero by Theorem 2.3, unless $\nu = 0$ and $J = \{0, \dots, n\}$. The vanishing of $H^i_{\mathfrak{b}}(C)$ for $i = 0, 1$ follows from (2.6). From here, we check that $H^0_{\mathfrak{b}}(B)_\alpha \simeq (I^{\mathrm{sat}}/I)_\alpha = 0$ and $H^1_{\mathfrak{b}}(B)_\alpha = 0$, unless $\nu = 0$, in which case $(I^{\mathrm{sat}}/I)_\delta = \langle \mathrm{sylv}_0 \rangle$ and we have the hybrid elimination matrix $\mathbb{H}_\delta$.

Now, we turn to the case ii). We have:

$$H^i_{\mathfrak{b}}(K_j(F))_{\delta-\nu} \simeq H^i_{\mathfrak{b}}(\oplus_{|J|=j} C(-\sum_{i\in J}\alpha_i))_{\delta-\nu} \simeq \oplus_{|J|=j} H^i_{\mathfrak{b}}(C)_{\delta-\nu-\sum_{i\in J}\alpha_i}.$$

For $i > 1$, using Serre duality we get:

$$H^i_{\mathfrak{b}}(C)_{\delta-\nu-\sum_{j\in J}\alpha_j} \simeq H^{i-1}(X_\Sigma, \sum_{j\notin J}\alpha_j - K_X - \nu) \simeq H^{n-i+1}(X_\Sigma, \nu - \sum_{j\notin J}\alpha_j).$$

As we supposed that $\alpha_i - \nu$ is nef, if $J \neq \{0, \dots, n\}$, $\sum_{j\notin J}\alpha_j - \nu$ is also nef and we can apply Theorem 2.3. Moreover, if $J = \{0, \dots, n\}$, $H^{n-i+1}(X_\Sigma, \nu)$ vanishes by Theorem 2.2, unless $i = n+1$. In such cases, we have $H^{n+1}_{\mathfrak{b}}(K_{n+1})_{\delta-\nu} \simeq C_\nu$. The claimed result follows using the fact that $\delta - \nu \notin \Gamma_0 \cup \Gamma_1$ and $(I^{\mathrm{sat}}/I)_{\delta-\nu}$ has a basis of toric Sylvester forms. ☐

Corollary 5.1. Assume that the toric variety X_Σ is smooth and σ-positive for some maximal cone $\sigma \in \Sigma(n)$. If all the Δ_i's are n-dimensional then $\mathbb{H}_\delta$ is an elimination matrix.

Proof. If the Δ_i's are full-dimensional then, similarly to Corollary 2.1, we have that $0 = \nu_j < \min \alpha_{ij}$. Moreover, $\nu = 0$ is a nef divisor. Therefore, this must be the only case in which the hypothesis i) and ii) of Theorem 5.2 are satisfied, yielding the elimination matrix $\mathbb{H}_\delta$. □

Example 5.3. Taking again Example 5.1, we see that many elimination matrices are obtained from Theorem 5.2. Thus, the matrix of Macaulay-type with $\alpha = (4, 2)$ corresponds to the case i). The matrices with $\alpha = (3, 1), (2, 1)$ correspond to case ii) as $\nu = (0, 0), (1, 0)$ are nef divisors and so are $\alpha_i - \nu = (2, 1), (1, 1)$ for $i = 0, 1, 2$. However, the matrix with $\alpha = (3, 2)$ does not belong to either of the two cases as $\nu = (0, 1)$ is not a nef divisor.

5.3 Overdetermined sparse polynomial systems

In this section we extend the construction of hybrid elimination matrices to the case of homogeneous polynomial systems that are defined by $r + 1$ equations with $r \geq n$. Such systems, so-called overdetermined, appear often in various applications.

Notation 5.2. We assume that the projective toric variety X_Σ is smooth and σ-positive for some maximal cone σ. In what follows, $F_0, \ldots, F_r$ are generic homogeneous sparse polynomials corresponding to the nef classes $\alpha_0, \ldots, \alpha_r$, I denotes the ideal they generate and $B = C/I$ the corresponding quotient ring. For each subset $T \subset \{0, \ldots, r\}$ of cardinality $n + 1$, we set $I_T = (F_i : i \in T)$, $B_T = C/I_T$ and $\delta_T = \sum_{i\in T} \alpha_i - K_X$. We denote by $\mathrm{Sylv}_{\mu,T}$ the Sylvester forms that can be formed from $\{F_i\}_{i\in T}$; see Section 4. We also denote by $K_\bullet(F)$ the Koszul complex of $F_0, \ldots, F_r$ and by $K_{T,\bullet}(F)$ the Koszul complex built from the generators of I_T.

The following result is a generalization of [BCP23, Chapter 3, Proposition 3.23] which deals with the particular case $X_\Sigma = \mathbb{P}^n$.

Theorem 5.3. Using the previous notation, suppose that there is a subset $S \subset \{0, \ldots, r\}$ of cardinality $n+1$ and a nef class $\nu \in \mathrm{Pic}(X_\Sigma)$ satisfying the hypotheses of Theorem 2.1 such that

$$\forall i \in S \quad j \notin S \quad \alpha_i - \alpha_j \text{ nef} \text{ and } \forall i \in S \quad \alpha_i - \nu \text{ is nef.}$$

Then, the set of Sylvester forms

$$\{\mathrm{sylv}_{\mu,T} : T \subset \{0, \ldots, r\} \text{ such that } |T| = n + 1 \text{ and } x^\mu \in C_{\delta_T - \delta_S + \nu}\}$$

yields a generating set of the A-module $(I^{\mathrm{sat}}/I)_{\delta_S-\nu}$.

Proof. First, we use Serre duality and Theorem 2.3 in order to compute the local cohomology modules $H^i_\mathfrak{b}(K_j(F))_{\delta_S-\nu}$, for $i, j = 0, \ldots, n+1$, similarly to what we did in Theorem 5.2. Namely, for $i \geq 2$ we get

$$H^i_\mathfrak{b}(K_j(F))_{\delta_S-\nu} \simeq \oplus_{|T|=j} H^i_\mathfrak{b}(C(-\sum_{k\in T} \alpha_k))_{\delta_S-\nu} \simeq H^{n+1-i}(X_\Sigma, \sum_{k\in T} \alpha_k - \sum_{k'\in S} \alpha_{k'} + \nu).$$

The elements in $S \cap T$ cancel each other, and the rest of elements $k' \in S$ can be either (i) paired up with α_k for $k \in T$ satisfying that $\alpha_k - \alpha_{k'}$ is nef, (ii) paired up with ν satisfying that $\alpha_{k'} - \nu$ is nef or, (iii) they are nef themselves. Therefore, the previous cohomology module is of the form $H^{n+1-i}(X_\Sigma, -\alpha)$ with α a sum of nef divisors, and applying Theorem 2.3 we deduce:

$$H^i_\mathfrak{b}(K_j(F))_{\delta_S-\nu} \simeq \begin{cases} \oplus_{|T|=n+1} C^\vee_{\sum_{j\in T}\alpha_j - \sum_{i\in S}\alpha_i + \nu} & \text{if } i, j = n+1 \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, from the comparison of the two spectral sequences that are considered in Theorem 3.1, we obtain the following transgression map, which is an isomorphism of graded modules:

$$\tau : H_{n+1}(K_\bullet(F), H^n_\mathfrak{b}(C))_{\delta_S-\nu} \xrightarrow{\sim} H^0_\mathfrak{b}(B)_{\delta_S-\nu}.$$

For any $T \subset \{0, \ldots, r\}$, let τ_T be the corresponding transgression map for $K_{T,\bullet}(F)$ and B_T. For each of these Koszul complexes, we have a canonical morphism of complexes $K_{T,\bullet}(F) \to K_\bullet(F)$ that all together induce the morphism of complexes:

$$L_\bullet(F) = \bigoplus_{|T|=n+1} K_{T,\bullet}(F) \to K_\bullet(F).$$

It follows that there is a commutative diagram:

$$\begin{array}{ccc} \oplus_{|T|=n+1} H_{n+1}(K_{T,\bullet}(F), H^{n+1}_\mathfrak{b}(C))_{\delta_S-\nu} & \longrightarrow & H_{n+1}(K_\bullet(F), H^{n+1}_\mathfrak{b}(C))_{\delta_S-\nu} \\ {\scriptstyle \oplus_T \tau_T}\big\downarrow & & \big\downarrow{\scriptstyle \tau} \\ \oplus_{|T|=n+1} H^0_\mathfrak{b}(B_T)_{\delta_S-\nu} & \longrightarrow & H^0_\mathfrak{b}(B)_{\delta_S-\nu} \end{array} \tag{5.4}$$

As the two vertical arrows are isomorphisms, in order to show that the bottom arrow is surjective, it is enough to show that the top arrow is surjective. For that purpose, observe that $L_{n+1} = K_{n+1}$ by construction and also

$$\oplus_{|T|=n+1} H_{n+1}(K_{T,\bullet}(F), H^{n+1}_{\mathfrak{b}}(C))_{\delta_S-\nu} = \ker(H^{n+1}_{\mathfrak{b}}(L_{n+1}(F)) \to H^{n+1}_{\mathfrak{b}}(L_n(F)))_{\delta_S-\nu}.$$

However, by the same argument as before $H^{n+1}_{\mathfrak{b}}(L_n(F)))_{\delta_S-\nu} = 0$, so

$$\oplus_{|T|=n+1} H_{n+1}(K_{T,\bullet}(F), H^{n+1}_{\mathfrak{b}}(C))_{\delta_S-\nu} \simeq \oplus_{|T|=n+1} C^{\vee}_{\sum_{j\in T}\alpha_j - \sum_{i\in S}\alpha_i+\nu}.$$

which is generated by the Sylvester forms at each of these degrees. On the other hand,

$$\begin{aligned} &H_{n+1}(K_\bullet(F), H^{n+1}_{\mathfrak{b}}(C))_{\delta_S-\nu} \simeq \\ &\qquad \ker(H^{n+1}_{\mathfrak{b}}(K_{n+1})_{\delta_S-\nu} \to H^{n+1}_{\mathfrak{b}}(K_n)_{\delta_S-\nu}) / \operatorname{Im}(H^{n+1}_{\mathfrak{b}}(K_{n+2})_{\delta_S-\nu} \to H^{n+1}_{\mathfrak{b}}(K_{n+1})_{\delta_S-\nu}).\end{aligned}$$

As above, $H^{n+1}_{\mathfrak{b}}(K_n)_{\delta_S-\nu} = 0$ and:

$$H_{n+1}(K_\bullet(F), H^{n+1}_{\mathfrak{b}}(C))_{\delta_S-\nu} \simeq H^{n+1}_{\mathfrak{b}}(K_{n+1})_{\delta_S-\nu} / \operatorname{Im}(H^{n+1}_{\mathfrak{b}}(K_{n+2})_{\delta_S-\nu} \to H^{n+1}_{\mathfrak{b}}(K_{n+1})_{\delta_S-\nu}).$$

This implies that the top map in the diagram (5.4) is surjective, as we wanted to prove. It follows that the basis of Sylvester forms of $\oplus_{|T|=n+1} H^0_{\mathfrak{b}}(B_T)_{\delta_S-\nu}$ is a set of generators of $H^0_{\mathfrak{b}}(B)_{\delta_S-\nu} = (I^{\mathrm{sat}}/I)_{\delta_S-\nu}$. □

We are now ready to extend the construction of hybrid elimination matrices to overdetermined homogeneous polynomial systems.

Definition 5.2. Under the assumptions of Theorem 5.3, we denote by $\mathbb{H}_\alpha$ the matrix of the following map:

$$\begin{array}{ccc} (\oplus_{i=0}^n C(-\alpha_i))_\alpha \bigoplus_{\substack{T\subset\{0,\ldots,r\}\\ |T|=n+1}} \left(I_T^{\mathrm{sat}}/I_T\right)_\alpha & \to & C_\alpha \\ (G_0,\ldots,G_n)\oplus(\ldots,l_{\mu,T},\ldots) & \mapsto & \displaystyle\sum_{i=0}^n G_iF_i + \sum_{\substack{T\subset\{0,\ldots,r\}\\ |T|=n+1}} \sum_{x^\mu\in C_{\delta_T-\alpha}} l_{\mu,T}\operatorname{Sylv}_{\mu,T} \end{array} \tag{5.5}$$

where $\alpha = \delta_S - \nu$ and where we recall that $l_{\mu,T} \in A$ for all μ and T.

Theorem 5.4. Under the assumptions of Theorem 5.3, $\mathbb{H}_\alpha$ is an elimination matrix, where $\alpha = \delta_S - \nu$.

Proof. The proof goes along the same lines as the proof of Theorem 5.1 for the case $r = n$. □

Example 5.4. Taking again the notation of Example 2.3, we add another polynomial with degree $\alpha_3 = (2,1)$ in $\mathcal{H}_1$ and write it in homogeneous coordinates as

$$F_3 = d_0z_1^2z_2 + d_1x_1z_1z_2 + d_2x_1^2z_2 + d_3x_2z_1 + d_4x_1x_2.$$

Following Theorem 5.4, the matrix $\mathbb{H}_{\delta_S}$ is

$$\begin{pmatrix} a_0 & 0 & b_0 & 0 & c_0 & 0 & d_0 & 0 & [130]_{abc} & [130]_{abd} & [130]_{acd} & [130]_{bcd} \\ a_1 & a_0 & b_1 & b_0 & c_1 & c_0 & d_1 & d_0 & [230]_{abc} & [230]_{abd} & [230]_{acd} & [230]_{bcd} \\ a_2 & a_1 & b_2 & b_1 & c_2 & c_1 & d_2 & d_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & b_2 & 0 & c_2 & 0 & d_2 & 0 & 0 & 0 & 0 \\ a_3 & 0 & b_3 & 0 & c_3 & 0 & d_3 & 0 & [430]_{abc} & [430]_{abd} & [430]_{acd} & [430]_{bcd} \\ a_4 & a_3 & b_4 & b_3 & c_4 & c_3 & d_4 & d_3 & 0 & 0 & 0 & 0 \\ 0 & a_4 & 0 & b_4 & 0 & c_4 & 0 & d_4 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $[ijk]_{abc} = \begin{pmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{pmatrix}$, and $[ijk]_{abd}, [ijk]_{acd}, [ijk]_{bcd}$ defined accordingly. It is an elimination matrix for the overdetermined polynomial system defined by F_0, F_1, F_2 and F_3.

We conclude this section with a comment on the computational impact of the hybrid elimination matrices obtained in Theorem 5.4. Indeed, these matrices are designed for solving (overdetermined) 0-dimensional polynomial systems by means of eigenvalues and eigenvectors computations, over a projective space, a multi-projective space or more generally a smooth projective toric variety which is σ-positive for some maximal cone σ. In comparison with the more classical Macaulay-type matrices, hybrid elimination matrices yield in general more compact matrices, in particular matrices with a smaller number of rows, which is a key ingredient with respect to computational

Type of system	degree α		number of rows	
	[BT21]	hybrid matrices	[BT21]	hybrid matrices
Polynomials of deg. 2 in $\mathbb{P}^3$	4	3	35	20
Polynomials of deg. 10 in $\mathbb{P}^3$	28	27	4495	4060
Polynomials of deg. $(2,1)$ in $\mathbb{P}^2 \times \mathbb{P}^1$	(6,3)	(4,2)	112	45
Polynomials in $\mathcal{H}_1 \times \mathbb{P}^1$	$3(\Delta \times [0,1])$	$2(\Delta \times [0,1])$	88	36

Table 1: The first column describes the type of system of 6 random polynomials which is considered. The second column provides the degree α for which the Macaulay-type matrices in [BT21] and the hybrid elimination matrices are constructed. The third column gives the corresponding number of rows of these two matrices.

complexity [BT21]. Indeed, this number of rows is controlled by the vanishing of the local cohomology modules at certain degrees, including the control of the saturation index of the homogeneous ideal $I(f)$ generated by general polynomials $f_0, \ldots, f_r$ of degree $\alpha_0, \ldots, \alpha_r$. In the case of hybrid elimination matrices, the situation is similar with the difference that now one considers the homogeneous ideal generated by $f_0, \ldots, f_r$ and their toric Sylvester forms, whose saturation index is necessarily smaller than the one of $I(f)$.

To be more concrete, we considered some specific polynomial systems for which we report in Table 1, the number of rows of hybrid elimination matrices and of the Macaulay-type matrices built in [BT21] by means of a "degree-by-degree" strategy (which is more efficient than using the classical Macaulay-type matrices construction). We considered random systems of 6 polynomials in 3-dimensional varieties in four different settings of Newton polytopes and degrees (in the case of $\mathcal{H}_1 \times \mathbb{P}^1$, the Newton polytope is $\Delta \times [0,1]$, where Δ corresponds to the same degrees as in Example 2.3).

Finally, we notice that the number of columns of hybrid elimination matrices may increase fast when the number of equations is large compared to the dimension of the ground projective toric variety. Further work is needed to analyze if some toric Sylvester forms can be avoided or combined to gain in efficiency. A more practical approach for future improvements would be to add Sylvester forms step by step until the expected corank is achieved, similarly to the "degree by degree" approach developed in [BT21].

6 Sylvester forms and sparse resultants

Resultants are central tools in elimination theory and there is a huge literature on various methods to compute them. A classical result is that the sparse resultant can be computed as the determinant of certain graded components of the Koszul complex built from the considered polynomial system; see for instance [GKZ94; DD00; WZ92; Ben+21]. In this section, we show that Sylvester forms can be incorporated in the usual Koszul complex and obtain this way new expressions for the sparse resultant as the determinant of a complex. This extends results in [CDS97, §2] by providing more compact formulas.

In what follows, we assume that X_Σ is a smooth projective toric variety which is σ-positive for some maximal cone $\sigma \in \Sigma(n)$. We take again the notation of Section 3 and we consider the generic homogeneous sparse polynomials $F_0, \ldots, F_n$ defined by (2.4). We recall that their supports $\mathcal{A}_i$, $i = 0, \ldots, n$, can be seen as the lattice points in $\Delta_i \subset M_\mathbb{R}$ (see Section 2).

To begin with, we first recall briefly the definition of the sparse resultant. The space of coefficients of the F_i's has a natural structure of multi-projective space, as the equations $F_i = 0$ are not modified after multiplication by a nonzero scalar. We denote it as $\prod_{i=0}^n \mathbb{P}^{\mathcal{A}_i}$, where $\mathbb{P}^{\mathcal{A}_i}$ stands for the projective space associated to the coefficients of the polynomial F_i. Let

$$Z(F) = \{x \times (\ldots, c_{i,\mu}, \ldots) \in X_\Sigma \times \prod_{i=0}^n \mathbb{P}^{\mathcal{A}_i} \quad F_0 = \cdots = F_n = 0\}$$

be the incidence variety of $F_0, \ldots, F_n$ and consider the canonical projection onto the second factor

$$\pi : X_\Sigma \times \prod_{i=0}^n \mathbb{P}^{\mathcal{A}_i} \to \prod_{i=0}^n \mathbb{P}^{\mathcal{A}_i}.$$

Whenever the image of $Z(F)$ via π is an irreducible hypersurface, the *sparse resultant*, denoted $\mathrm{Res}_\mathcal{A}$, is defined as an equation of the direct image $\pi_*(Z(\mathbf{F}))$. Thus, by universality $\mathrm{Res}_\mathcal{A}$ is a primitive and irreducible polynomial in the coefficients of $F_0, \ldots, F_n$, which is defined up to sign. We note that if $\pi_*(Z(F))$ has codimension at least 2, then $\mathrm{Res}_\mathcal{A}$ is set to 1.

To compute $\mathrm{Res}_{\mathcal{A}}$, a classical method is to consider the Koszul complex $K_\bullet(F)$ of the sequence of polynomials $F_0, \dots, F_n$, which is of the form

$$K_\bullet(F) : K_{n+1} = C(-\sum \alpha_i) \to \dots \to K_2 = \oplus_{k,k'} C(-\alpha_k - \alpha_{k'}) \to K_1 = \oplus_k C(-\alpha_k) \to C.$$

It is a graded complex of free graded A-modules. In [GKZ94] it is proved that the determinant of some of its graded components is equal to $\mathrm{Res}_{\mathcal{A}}$, up to multiplication by a nonzero scalar in k. More precisely, for all $\alpha \notin \Gamma_{\mathrm{Res}} \subset \mathrm{Pic}(X_\Sigma)$ the strand $K_\bullet(F)_\alpha$ is an acyclic complex of free A-modules and $H_0(K_\bullet(F)_\alpha) = B_\alpha$. Moreover, if in addition $(I^{\mathrm{sat}}/I)_\alpha = 0$ then $\det(K_\bullet(F)_\alpha)$ equals the sparse resultant $\mathrm{Res}_{\mathcal{A}}$ up to a nonzero scalar (see [GKZ94, Chapter 3, Theorem 4.2] for proofs). Observe that the map on the far right of the complex $K_\bullet(F)_\alpha$ is nothing but the Macaulay-type map (5.1), whose matrix is an elimination matrix of the form $\mathbb{M}_\alpha$.

In order to incorporate Sylvester forms in the above construction we proceed as follows: we consider a graded strand $K_\bullet(F)_\alpha$ of the Koszul complex, such that $(I^{\mathrm{sat}}/I)_\alpha$ is a nonzero free A-module, and we define a new complex, denoted $K_\bullet^{\mathrm{sat}}(F)_\alpha$, by adding Sylvester forms to the map on the far right: i.e.

$$K_\bullet^{\mathrm{sat}}(F)_\alpha = C(-\sum \alpha_i)_\alpha \xrightarrow{d_n} \dots \to \oplus_{k,k'} C(-\alpha_k - \alpha_{k'})_\alpha \xrightarrow{d_1} \oplus_k C(-\alpha_k)_\alpha \oplus (I^{\mathrm{sat}}/I)_\alpha \to C_\alpha.$$

It is a graded complex of free A-modules. By definition, $H_0(K_\bullet^{\mathrm{sat}}(F)_\alpha) = (B^{\mathrm{sat}})_\alpha$ and the map on the far right is precisely the map we used to define hybrid elimination matrices $\mathbb{H}_\alpha$. Observe also that if $(I^{\mathrm{sat}}/I)_\alpha = 0$ then we recover the strand of usual Koszul complex $K_\bullet(F)_\alpha$.

The following result generalizes [GKZ94, Chapter 3, Theorem 4.2], as well as [CDS97, Theorem 2.2] where a formula for the sparse resultant as the determinant of a complex incorporating the toric Jacobian (i.e. $\alpha = \delta$) is proved (under the assumption that the polytopes Δ_i are scaled copies of a given polytope).

Theorem 6.1. Assume that X_Σ is a smooth projective toric variety which is σ-positive for a maximal cone σ. Let $\alpha \notin \Gamma_{\mathrm{Res}}$, then $K_\bullet^{\mathrm{sat}}(F)_\alpha$ is an acyclic complex of free A-modules. Moreover, if $\alpha = \delta - \nu$ as in Theorem 5.2 ii), then $\det(K_\bullet^{\mathrm{sat}}(F)_\alpha)$ is equal to $\mathrm{Res}_{\mathcal{A}}$ up to a nonzero multiplicative scalar in k.

Proof. The acyclicicity of $K_\bullet^{\mathrm{sat}}(F)_\alpha$ follows from the acyclicity of the Koszul complex $K_\bullet(F)_\alpha$ beacuse the image of d_1 does not map to $(I^{\mathrm{sat}}/I)_\alpha)$ and $(I^{\mathrm{sat}}/I)_\alpha$ is a free A-module. Now, the acyclicity of $K_\bullet^{\mathrm{sat}}(F)_\alpha$, together with the fact that $H_0(K_\bullet^{\mathrm{sat}}(F)_\alpha) = (B^{\mathrm{sat}})_\alpha$, imply that $\det(K_\bullet^{\mathrm{sat}}(F)_\alpha)$ and $\mathrm{Res}_{\mathcal{A}}$ are two polynomials in A that vanish under the same specializations in k. As a consequence of the projective Nullstellenstaz, we only have to compare their degrees in order to prove that they are the same polynomial, up to multiplication by a nonzero constant in k.

As proved in [GKZ94, Section 3, Theorem 14], the determinant of a complex of vector spaces

$$V_\bullet : V_{n+1} \to \dots \to V_1 \to V_0$$

is given by the formula

$$\det(V_\bullet) = \bigotimes_i \bigwedge^{\mathrm{rk}(F_i)} V_i^{(-1)^i}.$$

This result implies that the degree of determinants of complexes can be calculated as alternate sums. In our setting, we know that $\det(K_\bullet^{\mathrm{sat}}(F)_\alpha) = \mathrm{Res}_{\mathcal{A}}$ (up to multiplication by a nonzero constant) if $(I^{\mathrm{sat}}/I)_\alpha = 0$, and also that $\mathrm{HF}(C, \alpha) = \mathrm{HP}(C, \alpha)$ for $\alpha \gg 0$ (component-wise). Therefore, for $\alpha \gg 0$,

$$\deg(\mathrm{Res}_{\mathcal{A}}) = \deg\det(K_\bullet^{\mathrm{sat}}(F)_\alpha) = \sum_{J \subset \{0,\dots,n\}} (-1)^{|J|} \mathrm{HF}(C, \alpha - \sum_{j \in J} \alpha_i) = \sum_{J \subset \{0,\dots,n\}} (-1)^{|J|} \mathrm{HP}(C, \alpha - \sum_{j \in J} \alpha_i).$$

This alternate sum yields a polynomial whose degree coincides with the degree of the resultant. Therefore, for $\alpha = \delta - \nu$ as in the statement, we have $(I^{\mathrm{sat}}/I)_{\delta-\nu} = \mathrm{Hom}_A(C_\nu, A) \neq 0$ and we can check that the difference of degrees between the previous alternate sum and the degree of the resultant is compensated by $(I^{\mathrm{sat}}/I)_{\delta-\nu}$ as follows:

$$\deg\det(K_\bullet(F)_{\delta-\nu}) - \deg(\mathrm{Res}_{\mathcal{A}}) = \sum_{J \subset \{0,\dots,n\}} (-1)^{|J|} \mathrm{HF}(C, \delta - \nu - \sum_{j \in J} \alpha_j) - \mathrm{HP}(C, \delta - \nu - \sum_{j \in J} \alpha_j).$$

Using Grothendieck-Serre formula (2.7), we deduce that this coincides with the quantity

$$\sum_{J \subset \{0,\dots,n\}} (-1)^{|J|} \sum_{i=0}^{n+1} (-1)^i \dim_k H_{\mathfrak{b}}^i(C)_{\delta - \nu - \sum_{j \in J} \alpha_j}.$$

Now, applying Theorem 5.2 ii), all the elements in the above sum vanish except the term $H_{\mathfrak{b}}^{n+1}(C)_{-K_X - \nu}$, which is counted with the sign $(-1)^{2(n+1)} = 1$. In particular, using Serre duality we get $H_{\mathfrak{b}}^{n+1}(C)_{-K_X-\nu} \cong C_\nu$ which is dual to $(I^{\mathrm{sat}}/I)_{\delta-\nu}$, so the difference of degrees is compensated in the complex $K_\bullet^{\mathrm{sat}}(F)_{\delta-\nu}$, which concludes the proof. □

We note that if the polytopes Δ_i are n-dimensional for $i = 0, \dots, n$, then Theorem 6.1 applied with $\nu = 0$ recovers the results in [CDS97], with the slight improvement that the α_i's are not necessarily of the form $k_i\beta$ for $k_i > 0$ and β an ample class.

Remark 6.1. The relation between $(I^{\text{sat}}/I)_{\delta-\nu}$ and $\text{HP}(C, \nu)$ in the previous result can be seen as an instance of multivariate Ehrhart reciprocity; see [Bec02, Theorem 2]. This result shows that if $\text{HP}(C, \nu)$ is the multivariate Hilbert polynomial in X_Σ corresponding to the number of lattice points in Δ_ν and if $\text{HP}^\circ(C, \nu)$ is another Hilbert polynomial associated to the number of lattice points in the interior of Δ_ν, then $\text{HP}(C, \nu) = (-1)^n \,\text{HP}^\circ(C, \nu)$.

From the above result, we can also identify cases where the matrices $\mathbb{H}_\alpha$ are square matrices, and therefore their determinant (in the usual sense of the determinant of a matrix) is equal to the sparse resultant, up to a nonzero multiplicative constant.

Corollary 6.1. Let $\Gamma = \text{Supp} \oplus_{k,k'} C(-\alpha_k - \alpha_{k'})$. For $\alpha \notin \Gamma$, we have $\det(\mathbb{H}_\alpha) = \text{Res}_{\mathcal{A}}$, up to a nonzero multiplicative constant.

Proof. If $\alpha \notin \Gamma$, then the complex $K_\bullet(F)_\alpha$ has only two terms and therefore $\det(K_\bullet(F)_\alpha) = \det(\mathbb{H}_\alpha)$. □

Remark 6.2. Computing the determinant of a complex can be done using some techniques such as Cayley determinants; see [GKZ94, Appendix A], but it is not very practical. However, Theorem 6.1 yields new expressions of the sparse resultant as a ratio of two determinants if $\alpha \notin \text{Supp} \oplus_{k,l,m} C(-\alpha_k - \alpha_l - \alpha_m)$; see [CDS97, Corollary 2.4] for a combinatorial characterization of such case.

We close this section with a comment related to the well-known Canny-Emiris formula. For Macaulay-type formulas of the form $\mathbb{M}_\alpha$, the Canny-Emiris formula gives a possible way to choose a nonzero minor; see [CE93] for the formula and [DJS22] for a proof and the non-vanishing of the minor using tropical deformations. It is an open problem to see whether the conditions on the proof of the Canny-Emiris formula [DJS22] coincide with the Cayley determinant for such choice of a minor. In the case of hybrid elimination matrices $\mathbb{H}_\alpha$, the Canny-Emiris formula has only been explored in for $n = 2$ and $\alpha = \delta$ (see [DE01]).

Example 6.1. Let's consider the four matrices provided in Example 5.1, which correspond to the cases $\alpha \in \{(4, 2), (3, 2), (3, 1), (2, 1)\}$. The last three are square matrices while the first one is not. We have drawn the region Γ in brown in Figure 1, in order to indicate the elements that provide a square matrix, as well as Γ_{Res}, in green, for the acyclicity of the complex. For the Macaulay-type matrices, we can combinatorially describe a maximal minor of $\mathbb{M}_{(4,2)}$ using the Canny-Emiris formula; see [CE93; DJS22]. The matrix $\mathbb{H}_{(3,2)}$ is square,

$$\mathbb{H}_{(3,2)} = \begin{pmatrix} a_0 & 0 & 0 & b_0 & 0 & 0 & c_0 & 0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & 0 & c_1 & c_0 & 0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\ 0 & a_2 & a_1 & 0 & b_2 & b_1 & 0 & c_2 & c_1 \\ 0 & 0 & a_2 & 0 & 0 & b_2 & 0 & 0 & c_2 \\ a_3 & 0 & 0 & b_3 & 0 & 0 & c_3 & 0 & 0 \\ a_4 & a_3 & 0 & b_4 & b_3 & 0 & c_4 & c_3 & 0 \\ 0 & a_4 & a_3 & 0 & b_4 & b_3 & 0 & c_4 & c_3 \\ 0 & 0 & a_4 & 0 & 0 & b_4 & 0 & 0 & c_4 \end{pmatrix},$$

and it might be obtained using a greedy approach to the same formula (see [CP93; CE22]), but as far as we know, there was no certificate of its existence as a resultant formula until now. The hybrid matrices for $\alpha = (3, 1), (2, 1)$ are square, but if they weren't, a procedure for choosing a minor is known for $n = 2$ and $\alpha = \delta = (3, 1)$; see [DE01].

7 Toric residue of the product of two forms

Another topic for which Sylvester forms are of interest is the computation of toric residues. These objects were initially introduced by Cox as a way to relate the residue of a family of $n + 1$ forms to the integral of a certain form in a toric variety X_Σ (see [Cox96]). Being given $F_0, \dots, F_n$ generic homogeneous polynomials as in (2.4), and denoting by $K(A)$ the quotient field of the universal ring of coefficients A, Cox proved the existence of a residue map

$$\text{Residue}_F : B_\delta \to K(A)$$

(recall that $I = (F_0, \dots, F_n)$ and $B = C/I$) which has the following property: for any specialization $\theta : A \to k$ (see Notation 5.1) such that the specialized system $f_0 = \cdots = f_n = 0$ has no solution in X_Σ, the residue map $\text{Residue}_f : (R/I(f))_\delta \to k$ is an isomorphism. Cox defined residue maps through trace maps of Čech cohomology, but it can be characterized through the fact that, if there is no solution in X_Σ, $\rho(\text{sylv}_0)$ is sent to $\pm 1 \in k$, so generically $\mathrm{Residue}_F(\mathrm{sylv}_0) = \pm 1$, as we used in Proposition 4.1. Many authors contributed formulas based on elimination matrices and resultants to compute residues [KS05; DK05; CCD97; CDS97] and also used them in other applications such as polynomial interpolation [Sop07] or mirror symmetry [BM02]. In particular, in [DK05] an explicit formula for computing the toric residue of a form of degree δ as a quotient of two determinants "à la Macaulay" is proved.

If a form G of degree δ can be written as a product $G = PQ$, a natural question is to ask whether one can take advantage of this factorization in the computation of the residue of $G = PQ$ with respect to the polynomial system $F_0, \dots, F_n$. In the case $X_\Sigma = \mathbb{P}^n$, Jouanolou proved that this is indeed possible by exploiting the duality between the degrees $\delta - \nu$ and ν of P and Q, respectively (see [Jou97, Proposition 3.10.27]). In what follows, we generalize Jouanolou's formula to a general smooth projective toric variety X_Σ which is σ-positive for a maximal cone σ. For that purpose, we use toric Sylvester forms and the elimination matrices $\mathbb{H}_{\delta-\nu}$ we introduced in Section 5.1. The new formulas we obtain can be seen as an extension of the rational formula "à la Macaulay" proved in [DK05, Corollary 3.4].

Remark 7.1. We note that in [Jou97] the residue is defined as a map onto A, and not in $K(A)$, by multiplying with $\mathrm{Res}_{\mathcal{A}}$ in the image; see also [CDS97, Theorem 1.4] for a proof that the product of the residue and the resultant lies in A.

Let $\mathbb{H}_{\delta-\nu}$ be an elimination matrix that satisfies the assumptions of Theorem 5.2 ii), and let $\mathcal{H}_{\delta-\nu}$ be a maximal minor of $\mathbb{H}_{\delta-\nu}$ which contains the entire block built with Sylvester forms. Now, being given two generic forms $P \in C_\nu$ and $Q \in C_{\delta-\nu}$, we consider the matrix

$$\Theta_{\delta-\nu} = \begin{pmatrix} & \mathcal{H}_{\delta-\nu} & & \mathbf{q} \\ 0 & & \mathbf{p}^T & 0 \end{pmatrix} \tag{7.1}$$

where $\mathbf{p}$, respectively $\mathbf{q}$, stands for the vector of coefficients of P, respectively Q. Recall that by construction of the matrix $\mathbb{H}_{\delta-\nu}$, the matrix $\mathcal{H}_{\delta-\nu}$ is built as the join of a Macaulay-type block-matrix and another column-block matrix built from Sylvester forms. Thus, the row $\mathbf{p}^T$ is aligned with the second column-block, built from Sylvester forms, of $\mathcal{H}_{\delta-\nu}$; see Example 5.4 for an illustration.

We first prove that the residue of the product of two monomials can be computed as a quotient of determinants. In what follows, we denote by $\mathcal{H}_{\alpha,\beta}$ the submatrix of $\mathcal{H}_{\delta-\nu}$ that is obtained by deleting the column corresponding to the monomial $x^\alpha \in C_\nu$ and the row corresponding to the monomial $x^\beta \in C_{\delta-\nu}$.

Lemma 7.1. Assume that X_Σ is a smooth projective toric variety which is σ-positive for a maximal cone σ. Let $F_0, \dots, F_n$ be a system of homogeneous polynomials in C as in (2.4), then for any pair of monomials $x^\alpha \in C_\nu$ and $x^\beta \in C_{\delta-\nu}$,

$$\mathrm{Residue}_F(x^{\alpha+\beta}) = (-1)^{\alpha+\beta} \frac{\det(\mathcal{H}_{\alpha,\beta})}{\det(\mathcal{H}_{\delta-\nu})}.$$

Proof. Let H^β be the matrix obtained by multiplying the row of $\det(\mathcal{H}_{\delta-\nu})$ corresponding to x^β by the monomial x^β itself. Then, by expanding the determinant along this row, one gets:

$$x^\alpha x^\beta \det(\mathcal{H}_{\delta-\nu}) = x^\alpha \det(H^\beta) = x^\alpha (\sum G_i F_i + \sum_{\alpha' \in C_\nu} c_{\alpha',\beta} \mathrm{Sylv}_{\alpha'}) = \sum x^\alpha G_i F_i + c_{\alpha,\beta} \mathrm{Sylv}_0 .$$

Taking residues at both sides, we deduce that

$$\mathrm{Residue}_F(x^{\alpha+\beta}) \det(\mathcal{H}_{\delta-\nu}) = (-1)^{\alpha+\beta} c_{\alpha,\beta}.$$

Finally, from the expansion of the determinant $\det(H^\beta)$, one sees immediately that $c_{\alpha,\beta} = \det(\mathcal{H}_{\alpha,\beta})$. ☐

We are now ready to prove the claimed formula for the residue of the product of two forms.

Theorem 7.1. Assume that X_Σ is a smooth projective toric variety which is σ-positive for a maximal cone σ. Let $F_0, \dots, F_n$ be a system of homogeneous polynomials in C as in (2.4), then for any pair of forms $P \in C_\nu$ and $Q \in C_{\delta-\nu}$,

$$\mathrm{Residue}_F(PQ) = \frac{\det(\Theta_{\delta-\nu})}{\det(\mathcal{H}_{\delta-\nu})}.$$

Proof. Write $P = \sum_{x^\alpha \in C_\nu} p_\alpha x^\alpha$ and $Q = \sum_{x^\beta \in C_{\delta-\nu}} q_\beta x^\beta$. Then, by linearity of residues, we have:

$$\mathrm{Residue}_F(PQ) = \sum_{x^\alpha \in C_\nu \ x^\beta \in C_{\delta-\nu}} p_\alpha q_\beta \mathrm{Residue}_F(x^{\alpha+\beta}) = \frac{\sum_{\alpha,\beta} (-1)^{\alpha+\beta} p_\alpha q_\beta \det(\mathcal{H}_{\alpha,\beta})}{\det(\mathcal{H}_{\delta-\nu})}.$$

The numerator is precisely the expansion of the determinant $\det(\Theta_{\delta-\nu})$ of the matrix defined in (7.1), with respect to the last row and column. ☐

Example 7.1. In Example 5.1, the elimination matrix $\mathbb{H}_\alpha$ for $\alpha = (2,1)$ is square, therefore we take

$$\mathcal{H}_{(2,1)} = \mathbb{H}_{(2,1)} = \begin{pmatrix} a_0 & b_0 & c_0 & [013] & [023] \\ a_1 & b_1 & c_1 & [023]+[014] & [024]+[123] \\ a_2 & b_2 & c_2 & [024] & [124] \\ a_3 & b_3 & c_3 & 0 & 0 \\ a_4 & b_4 & c_4 & 0 & 0 \end{pmatrix}$$

Let $P = p_0 z_1 + p_1 x_1$ and $Q = q_0 z_1^2 z_2 + q_1 z_1 z_2 x_1 + q_2 z_2 x_1^2 + q_3 z_1 x_2 + q_4 x_1 x_2$ be homogeneous forms in $C_{(1,0)}$ and $C_{(2,1)}$, respectively, then

$$\Theta_{(2,1)} = \begin{pmatrix} a_0 & b_0 & c_0 & [013] & [023] & q_0 \\ a_1 & b_1 & c_1 & [023]+[014] & [024]+[123] & q_1 \\ a_2 & b_2 & c_2 & [024] & [124] & q_2 \\ a_3 & b_3 & c_3 & 0 & 0 & q_3 \\ a_4 & b_4 & c_4 & 0 & 0 & q_4 \\ 0 & 0 & 0 & p_0 & p_1 & 0 \end{pmatrix}.$$

and aplying Theorem 7.1 we deduce that $\operatorname{Residue}_F(PQ) = \frac{\det(\Theta_{(2,1)})}{\det(\mathcal{H}_{(2,1)})}$. For the sake of comparison, let us examine the formula we obtain by developing the product of P and Q. In this case, we apply Theorem 7.1 with $\delta = (3,1)$ and $\nu = 0$, so we have to consider the matrix $\Theta_{(3,1)}$ which is of the form:

$$\Theta_{(3,1)} = \begin{pmatrix} a_0 & 0 & b_0 & 0 & c_0 & 0 & [130] & p_0q_0 \\ a_1 & a_0 & b_1 & b_0 & c_1 & c_0 & [230] & p_0q_1 + p_1q_0 \\ a_2 & a_1 & b_2 & b_1 & c_2 & c_1 & 0 & p_0q_2 + p_1q_1 \\ 0 & a_2 & 0 & b_2 & 0 & c_2 & 0 & p_1q_2 \\ a_3 & 0 & b_3 & 0 & c_3 & 0 & [430] & p_0q_3 \\ a_4 & a_3 & b_4 & b_3 & c_4 & c_3 & 0 & p_0q_4 + p_1q_3 \\ 0 & a_4 & 0 & b_4 & 0 & c_4 & 0 & p_1q_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

since the product PQ is equal to

$$p_0q_0z_1^3z_2 + (p_0q_1 + p_1q_0)z_1^2z_2x_1 + (p_0q_2 + p_1q_1)z_1z_2x_1^2 + p_0q_3z_1^2x_2 + (p_0q_4 + p_1q_3)z_1x_1x_2 + p_1q_2z_2x_1^3 + p_1q_4x_1^2x_2.$$

The expansion of the determinant of $\Theta_{(3,1)}$ with respect to the last row leads to the same formula as in [DK05, Corollary 3.4].

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