Math. Ann. 295, 1–24 (1993)

Mathematische Annalen

Toric varieties, lattice points and Dedekind sums

James E. Pommersheim

Department of Mathematics, University of Chicago, 5734 S. University Ave., Chicago, IL 60637, USA

Received December 4, 1991; in revised form March 13, 1992

Mathematics Subject Classification (1991): 14M25, 11F20, 52B20

Introduction

In this paper, we prove a formula for the Todd class of a toric variety, which we use to obtain results about lattice polyhedra and Dedekind sums. These applications include a formula for the number of lattice points in an arbitrary lattice tetrahedron, and a generalization of Rademacher's three-term reciprocity formula for Dedekind sums. This paper is written in three parts, with separate introductions so that the parts may be read independently. Readers who are interested primarily in the applications to lattice polyhedra or Dedekind sums are encouraged to skip to Part II or III.

It is well known that the Chern classes of a nonsingular toric variety are expressed nicely as the sum of the classes of certain special subvarieties. For simplicial but possibly singular toric varieties, we use this same sum to define the *mock Chern class*, and then define the *mock Todd class* via the Todd polynomials. We prove that in the dimension of the singular locus, the difference between the actual Todd class and the mock Todd class has a local expression. The codimension two part of this difference is expressed explicitly in terms of Dedekind sums. In this way, we obtain an expression for the codimension two part of the Todd class of an arbitrary toric variety given in terms of Dedekind sums.

This leads to several number-theoretic applications. First, we give a formula for the number of lattice points in an arbitrary lattice tetrahedron in terms of six Dedekind sums, one for each edge of the tetrahedron. This formula generalizes the lattice point formula of Mordell. We also use the Todd class result to prove facts about Dedekind sums. We derive a formula expressing the sum of two arbitrary Dedekind sums in terms of a third, as well as an n-term reciprocity law which generalizes the three-term law of Rademacher.

Part I: The Todd class of simplicial toric varieties

1 Introduction

In this first part, we investigate the Todd class of simplicial toric varieties. Toric varieties form a very special class of rational varieties. They arise from combinatorial objects called fans, which are collection of cones in a lattice. Toric varieties are of interest both in their own right as algebraic varieties, and in their application to the theory of convex polytopes. For example, Danilov established the direct connection between the Todd class of toric varieties and the problem of counting the number of lattice points in a convex lattice polytope [Dan, p. 134]. Thus the problem of finding explicit expressions for the Todd class of a toric variety is of interest not only to algebraic geometers.

For nonsingular toric varieties, we may obtain an expression for the Todd class in the following manner: Let Σ be a fan and let X_Σ denote the toric variety associated to Σ. To each cone of Σ there corresponds a special subvariety of X_Σ. In the case that X_Σ happens to be nonsingular, the total Chern class of X_Σ is simply the sum of the classes of these subvarieties. The Todd class may then be computed from the Chern classes using the Todd polynomials [Dan, p. 132]. Motivated by this result, we define the *mock Chern class* of a simplicial toric variety X_Σ as the sum of the classes of the special subvarieties (those corresponding to the cones of Σ). Since the Chow groups of a simplicial toric variety have a natural ring structure [Dan, p. 131], we may define the *mock Todd class* to be the Todd polynomials in the mock Chern classes.

We shall investigate the difference between the Todd class and the mock Todd class of a simplicial toric variety. It is not hard to show that this difference lies on the singular locus, and so vanishes in codimensions smaller than d, the codimension of the singular locus. Our main theorem is that in codimension d, the difference between the Todd class and the mock Todd class is the sum of the special cycles of codimension d with coefficients computable in terms of the local combinatorics of the fan.

To make this precise, we let Σ be a simplicial fan and X_Σ be the associated toric variety. We then define the mock Chern classes by:

$$C_i X_\Sigma = \sum \frac{1}{\operatorname{mult} \tau} F_\tau$$

where the sum ranges over all i-dimensional cones τ in Σ, and F_τ is the class in $(A^i X_\Sigma)_\mathbb{Q}$ of the subvariety corresponding to τ. The mock Todd class is then defined by:

$$\mathrm{TD}\, X_\Sigma = \sum_{i \geqq 0} \mathrm{TD}^i X_\Sigma ,$$

where $\mathrm{TD}^i X_\Sigma$ is the i^{th} Todd polynomial [Hir] in the classes $C_1, \ldots, C_i$. Every algebraic variety has a naturally defined Todd class [Ful], and we denote the codimension i part of this class by $\mathrm{Td}^i X$.

We then have:

Theorem 1. *Let N be a lattice and let $\mathscr{S}$ be the set of all d-dimensional simplicial* *cones in N with non-singular $(d-1)$-dimensional faces. Then there is a unique function*

$$t: \mathscr{S} \to \mathbb{Q}$$

with the property that if Σ is a complete simplicial fan in N all of whose $(d-1)$-dimensional cones are non-singular (so that $\operatorname{codim}(\operatorname{Sing} X_\Sigma) \geqq d$*), then*

$$\mathrm{Td}^d X_\Sigma - \mathrm{TD}^d X_\Sigma = \sum t(\tau) F_\tau$$

with the sum taken over all d-dimensional cones $\tau \in \Sigma$.

In the case $d = 2$, we show that the function t is expressed in terms of a Dedekind sum. Precisely, if τ is a two-dimensional cone generated by e_1 and $pe_1 + qe_2$ where $\{e_1, e_2\}$ forms a lattice basis in the plane containing τ, then

$$t(\tau) = s(p, q) - \frac{1}{4q} + \frac{1}{4}$$

where $s(p, q)$ is the classical Dedekind sum. In this way, we obtain a formula for the Todd class of a toric variety in codimension two in terms of Dedekind sums. Given a fan Σ, it is in practice quite easy to compute $\mathrm{TD}^2 X_\Sigma$, as the ring $A^* X_\Sigma$ is given rather explicitly. Thus, the previous theorem gives a computable expression for $\mathrm{Td}^2 X_\Sigma$ in terms of Dedekind sums.

Worth noting is the similarity between the present work and that of Hirzebruch and Zagier [HiZa], in which Dedekind sums appear as "signature defects." They examine the difference between the signature of a singular quotient variety M/G, and the expression

$$\frac{1}{|G|} \operatorname{Sign} M ,$$

which gives the true signature in the case that M/G is nonsingular. Applying these ideas to certain algebraic surfaces yields number-theoretic results, just as in the present case, where "Todd class defects" yield number-theoretic results similar to those explored in Parts II and III of this paper.

Also worth noting is the difference in approach between this work and that of Morelli [Mor], who found formulas for the Todd class of singular toric varieties after suitably extending the coefficient field. Morelli's formulas are additive on the cones of a fixed lattice, whereas the formulas of this paper are invariant under lattice automorphisms.

2 General facts about toric varieties

In this section, we state without proof the facts about toric varieties that we will need in the remainder of Part I. We also establish notation used in future sections. The reader may find proofs of these results Oda's book [Oda] or in the survey article [Dan].

2.1 Basic facts and notations

Toric varieties arise from combinatorial objects called fans. Here are some facts about fans and the relation between the combinatorics of a fan and the geometry of the associated toric variety.

Let N be a lattice of dimension n. A *half-space* in N is a set of the form $\lambda^{-1}(\mathbb{Z}_{\geqq 0})$, where $\lambda \in \operatorname{Hom}(N, \mathbb{Z})$. A *cone* in N is a finite intersection of half-spaces. Equivalently, we may think of a cone as the convex hull in $N \otimes \mathbb{R}$ of a finite set of rays starting at the origin. If $n_1, \ldots, n_k \in N$, we use $\langle n_1, \ldots, n_k \rangle$ to denote the convex hull of the rays $\overline{On_1}, \ldots, \overline{On_k}$, which is called the cone generated by $\{n_1, \ldots, n_k\}$. Throughout, we shall assume that our cones contain no linear subspace of positive dimension.

A *fan* in N is a finite collection Σ of cones such that

(1) For all $\sigma, \tau \in \Sigma$, $\sigma \cap \tau$ is a common face of σ and τ.

(2) If $\sigma \in \Sigma$, then all faces of σ are also in Σ.

We use $\Sigma^{(i)}$ to denote the set of i-dimensional cones of Σ. One-dimensional cones are called *rays* or *edges*. If ρ is a ray, we shall also use ρ to denote the unique primitive element of N lying on ρ.

To each fan Σ in N, there is an associated toric variety X_Σ. To each $\sigma \in \Sigma$, there corresponds a subvariety $V(\sigma)$ of X_Σ such that

(1) $\dim \sigma = \operatorname{codim} V(\sigma)$

(2) $\tau \subset \sigma \Leftrightarrow V(\tau) \supset V(\sigma)$.

The construction of X_Σ, as well as the subvarieties $V(\sigma)$ may be found in [Oda, Sects. 1.2 and 1.3].

The following properties show the relation between the combinatorics of Σ and the geometry of X_Σ:

(1) A fan is called *complete* if its cones cover the lattice N. It is then true that X_Σ is complete if and only if Σ is complete (cf. [Oda, Sect. 1.4]).

(2) A cone σ is called *non-singular* if it is generated by a subset of a basis for the lattice N. A fan is said to be non-singular if all of its cones are non-singular. Then it is true that X_Σ is non-singular if and only if Σ is non-singular (cf. [Oda, Sect. 1.4]). In fact, the singular locus of X_Σ is $\bigcup V(\tau)$, the union being taken over all singular cones $\tau \in \Sigma$.

(3) A cone σ of dimension k is called *simplicial* if σ is generated by k elements of N. In particular, any non-singular cone or any 2-dimensional cone is simplicial. A fan is called simplicial if all of its cones are simplicial. If σ is a simplicial cone generated by primitive elements $n_1, \ldots, n_k \in N$, then we define $\operatorname{mult} \sigma$ to be $\#(P/(n_1, \ldots, n_k))$, where P is the k-plane in N containing σ.

2.2 Resolution of singularities

A *subdivision* of a cone τ is a fan Γ such that the union of the cones of Γ is τ. Γ is called an *interior subdivision* if every ray of Γ (except the rays of τ) lies in the interior of τ. If Σ is a fan, $\tau \in \Sigma$, and Γ is a subdivision of τ, then we obtain a new fan Σ' from Σ by replacing τ with the cones of Γ, and replacing cones of Σ which intersect τ with suitably subdivided cones. In this case, we obtain a proper, birational map of toric varieties:

$$\pi: X_{\Sigma'} \to X_\Sigma$$

(cf. [Oda, Sect. 1.5]). If Γ is an interior subdivision of τ, then the map π is an isomorphism except above $V(\tau)$.

Given any fan Σ we may obtain a non-singular fan Σ' through a sequence of such interior subdivisions of singular cones. In this way, we obtain a resolution of singularities for an arbitrary toric variety. If $\dim \Sigma = 2$, this resolution is described explicitly in terms of continued fractions (cf. [Oda, Sect. 1.6]).

2.3 The Chow ring

Because a simplicial toric variety is locally the quotent of a smooth variety by a finite group, the rational Chow groups of a simplicial toric variety have a natural ring structure described explicitly below [Dan, p. 127]. Throughout, A^*X is used to abbreviate $(A^*X)_{\mathbb{Q}}$.

Let Σ be a simplicial fan in a lattice N, and let $M = \operatorname{Hom}(N, \mathbb{Z})$ be the dual lattice with $\langle\,,\rangle\colon M \times N \to \mathbb{Z}$ the natural pairing. For $\sigma \in \Sigma$, we denote by F_σ the class in $A^{(\dim \sigma)}X_\Sigma$ of the subvariety $V(\sigma)$. Then the ring A^*X_Σ is generated by $\{F_\rho \mid \rho \in \Sigma^{(1)}\}$, the classes of the special divisors, with the relations:

(1) For each $m \in M$,

$$\sum_{\rho \in \Sigma^{(1)}} \langle m, \rho\rangle F_\rho = 0\,.$$

(2) If $\rho_1, \ldots, \rho_k$ are distinct, then

$$F_{\rho_1} \cdot \ldots \cdot F_{\rho_k} = \begin{cases} \dfrac{1}{\operatorname{mult}\langle \rho_1, \ldots, \rho_k\rangle} F_{\langle \rho_1, \ldots, \rho_k\rangle} & \text{if } \langle \rho_1, \ldots, \rho_k\rangle \in \Sigma \\ 0 & \text{otherwise}\,. \end{cases}$$

2.4 Push-forward of invariant cycles

Let Σ be a simplicial fan and let Σ' be a subdivision of Σ, with $\pi\colon X_{\Sigma'} \to X_\Sigma$. If $\sigma' \in \Sigma'$, we will use $E_{\sigma'}$ to denote the class in $A^*X_{\Sigma'}$ of the subvariety $V(\sigma')$, and for $\sigma \in \Sigma$, F_σ will denote the class in A^*X_Σ corresponding to $V(\sigma)$.

In this case, it is easy to describe the push-forward map

$$\pi_*\colon A^*X_{\Sigma'} \to A^*X_\Sigma\,.$$

Let $\sigma' \in \Sigma'$, and let σ be the smallest cone of Σ such that $\sigma' \subset \sigma$. Then

$$\pi_*(E_{\sigma'}) = \begin{cases} F_\sigma & \text{if } \dim \sigma' = \dim \sigma \\ 0 & \text{otherwise}\,. \end{cases}$$

This fact is well-known and is proven by unraveling the construction of the map π given in [Oda, Sect. 1.5].

2.5 Chern classes of a non-singular toric variety

If Σ is a non-singular fan, the Chern classes of X_Σ are given by the sum of the special subvarieties:

$$c_i X_\Sigma = \sum_{\tau \in \Sigma^{(i)}} F_\tau\,.$$

As X_Σ is non-singular, the Todd class Td X_Σ may then be computed from the Chern classes using the Todd polynomials, in the usual way. See [Dan, p. 114].

3 Push-forward of products

This section contains a theorem about pushing forward a product of cycles under a proper, birational map of toric varieties. This result will be the key step in proving the theorems about Todd classes found in the following sections.

We start with a simplicial fan Σ, and let $\tau \in \Sigma^{(d)}$. Let Σ' be obtained from Σ by an interior subdivision of τ. As mentioned in Sect. 2.2, this gives a map $\pi: X_{\Sigma'} \to X_\Sigma$ of toric varieties which is an isomorphism except above $V(\tau)$.

Let $W_1', \ldots, W_r' \in A^* X_{\Sigma'}$ with $W_i' \in A^{e_i} X_{\Sigma'}$, and let $W_i = \pi_* W_i'$. We now consider the difference

$$\delta = \pi_* \prod W_i' - \prod W_i \in A^e X_\Sigma ,$$

where $e = \sum e_i$. Since π is an isomorphism above $X_\Sigma \setminus V(\tau)$, this difference vanishes when restricted to $A^e(X_\Sigma \setminus V(\tau))$, and hence lies in the image of $A^{e-d} V(\tau) \to A^e X_\Sigma$.

If $e < d$, we see that δ vanishes.

Our theorem concerns the case $e = d$. In this case, we see immediately that δ is some rational multiple of F_τ. In examining exactly what rational number occurs here, it suffices to consider the case in which the W_i' are all divisors ($i = 1, \ldots, d$), since the Chow rings above are generated by divisors. We now show that the rational number in question does not depend on all of Σ, but only on the cone τ and its subdivision.

Theorem 2. *Given a d-dimensional simplicial cone τ and a simplicial interior subdivision Γ of τ, there exists a unique function*

$$f_{\tau,\Gamma}: N^d \to \mathbb{Q}$$

with the properties:

(1) *$f_{\tau,\Gamma}(\beta_1, \ldots, \beta_d) = 0$ unless all β_i are primitive elements of rays of Γ.*

(2) *$f_{\tau,\Gamma}$ is symmetric in its d variables.*

(3) *If $\beta_1, \ldots, \beta_d$ are distinct rays of Γ, then*

$$f_{\tau,\Gamma}(\beta_1, \ldots, \beta_d) = \begin{cases} \dfrac{1}{\operatorname{mult}\langle \beta_1, \ldots, \beta_d \rangle} & \text{if } \langle \beta_1, \ldots, \beta_d \rangle \in \Gamma \\ \dfrac{-1}{\operatorname{mult}\langle \beta_1, \ldots, \beta_d \rangle} & \text{if } \langle \beta_1, \ldots, \beta_d \rangle = \tau \\ 0 & \text{otherwise} . \end{cases}$$

(4) *If $m \in M = \operatorname{Hom}(N, \mathbb{Z})$ and $\beta_2, \ldots, \beta_d \in N$, then*

$$\sum_{\beta \in \Gamma^{(1)}} \langle m, \beta \rangle f_{\tau,\Gamma}(\beta, \beta_2, \ldots, \beta_d) = 0 .$$

This function has an additional property:

(5) *Let Σ be a complete simplicial fan in N such that $\tau \in \Sigma$, and Σ' be the fan obtained from Σ by performing the subdivision Γ of τ, with $\pi: X_{\Sigma'} \to X_\Sigma$ the natural*

map. Then for any edges $\beta_1, \ldots, \beta_d$ of Σ', we have

$$\pi_* \prod_{i=1}^{d} E_{\beta_i} - \prod_{i=1}^{d} \pi_* E_{\beta_i} = f_{\tau,\Gamma}(\beta_1, \ldots, \beta_d) F_\tau .$$

Proof. That (1)–(4) determine $f_{\tau,\Gamma}$ uniquely is a straightforward induction—(3) determines the value for distinct β_i, and (4) allows us to reduce the number of coincidences among the β_i until they are all distinct.

To prove the existence of $f_{\tau,\Gamma}$ as well as (5), let $\pi: X_{\Sigma'} \to X_\Sigma$ be as in the statement of the theorem, and let $\beta_1, \ldots, \beta_d$ be edges of Σ'. Then, as noted at the beginning of this section, it follows from general principles of intersection theory that

$$\pi_* \prod_{i=1}^{d} E_{\beta_i} - \prod_{i=1}^{d} \pi_* E_{\beta_i} \in A^d X_\Sigma$$

is a rational multiple of F_τ. Thus, we may write

$$\pi_* \prod_{i=1}^{d} E_{\beta_i} - \prod_{i=1}^{d} \pi_* E_{\beta_i} = g(\beta_1, \ldots, \beta_d) F_\tau .$$

It follows from the completeness of X_Σ and facts of Sect. 2.3 that $F_\tau \neq 0$, and hence the above equation defines g uniquely. We will show that g satisfies (1)–(4), and hence $g = f_{\tau,\Gamma}$, establishing the existence of $f_{\tau,\Gamma}$ as well as property (5). (Note that *a priori*, g depends on all of Σ', while $f_{\tau,\Gamma}$ depends only on τ and its subdivision.)

g satisfies (1). We must show that if some β_j is not a ray of Γ, then $\pi_* \prod_{i=1}^{d} E_{\beta_i} - \prod_{i=1}^{d} \pi_* E_{\beta_i} = 0$. This follows easily from intersection theory, as follows: Let $D_i' = V(\beta_i) \subset X_{\Sigma'}$. If $\beta_i \in \Sigma^{(1)}$, let $D_i = \pi(D_i') = V(\beta_i) \subset X_\Sigma$, and if $\beta_i \notin \Sigma^{(1)}$, let $D_i = 0$. We are then interested in

$$\delta = \pi_* \prod [D_i'] - \prod [D_i] \in A^d X_\Sigma .$$

The condition $\beta_j \nsubseteq \tau$ ensures that $D_j = V(\beta_j) \nsupseteq V(\tau)$. Thus, if we consider δ living in $A^{d-1} D_j$, and apply the exact sequence

$$A^*(D_j \cap V(\tau)) \xrightarrow{\alpha} A^*(D_j) \to A^*(D_j \backslash V(\tau)) \to 0 ,$$

we see on the one hand that δ lies in the image of α, and on the other hand, δ is a class of dimension $n - d$. However, $\dim(D_j \cap V(\tau)) < n - d$ as $V(\tau) \nsubseteq D_j$. We conclude that $\delta = 0$.

g satisfies (2). This is evident from the commutativity of intersection products.

g satisfies (3). Let $\beta_1, \ldots, \beta_d$ be distinct rays of Σ.

(A) If $\langle \beta_1, \ldots, \beta_d \rangle \in \Gamma$, then $\prod_{i=1}^{d} E_{\beta_i} = \dfrac{1}{\operatorname{mult}\langle \beta_1, \ldots, \beta_d \rangle} E_{\langle \beta_1, \ldots, \beta_d \rangle}$, so

$$\pi_* \prod_{i=1}^{d} E_{\beta_i} = \frac{1}{\operatorname{mult}\langle \beta_1, \ldots, \beta_d \rangle} F_\tau ,$$

while downstairs, $\langle \beta_1, \ldots, \beta_d \rangle \notin \Sigma$, so

$$\prod_{i=1}^{d} \pi_* E_{\beta_i} = 0 .$$

This gives $g(\beta_1, \ldots, \beta_d) = \dfrac{1}{\operatorname{mult}\langle \beta_1, \ldots, \beta_d \rangle}$.

(B) If $\langle \beta_1, \ldots, \beta_d \rangle = \tau$, then since $\tau \notin \Sigma'$,

$$\prod_{i=1}^{d} E_{\beta_i} = 0 ,$$

while downstairs,

$$\prod_{i=1}^{d} \pi_* E_{\beta_i} = \prod_{i=1}^{d} F_{\beta_i} = \frac{1}{\operatorname{mult}\tau} F_\tau ,$$

so $g(\beta_1, \ldots, \beta_d) = \dfrac{-1}{\operatorname{mult}\tau}$.

(C) Otherwise, $\langle \beta_1, \ldots, \beta_d \rangle$ is neither a cone of Σ' nor of Σ, so both $\prod_{i=1}^{d} E_{\beta_i}$ and $\prod_{i=1}^{d} \pi_* E_{\beta_i}$ vanish, and hence $g(\beta_1, \ldots, \beta_d) = 0$.

g satisfies (4). It will be convenient to set $F_\beta = 0 \in A^1 X_\Sigma$ when $\beta \in \Sigma'^{(1)} \backslash \Sigma^{(1)}$. With this notation, it is then true that for any $\beta \in \Sigma'^{(1)}$, $\pi_* E_\beta = F_\beta$. Now let $\beta_2, \ldots, \beta_d \in \Gamma^{(1)}$ and let $m \in M$. We then have

$$(*) \qquad \sum_{\beta \in \Sigma'^{(1)}} \langle m, \beta \rangle E_\beta = 0 \quad \text{in } A^* X_{\Sigma'}$$

and

$$\sum_{\beta \in \Sigma^{(1)}} \langle m, \beta \rangle F_\beta = 0 \quad \text{in } A^* X_\Sigma ,$$

which implies

$$(**) \qquad \sum_{\beta \in \Sigma'^{(1)}} \langle m, \beta \rangle F_\beta = 0 .$$

$(*)$ and $(**)$ now yield:

$$\sum_{\beta \in \Sigma'^{(1)}} \langle m, \beta \rangle E_\beta \cdot E_{\beta_2} \cdot \ldots \cdot E_{\beta_d} = 0 \quad \text{and} \quad \sum_{\beta \in \Sigma'^{(1)}} \langle m, \beta \rangle F_\beta \cdot F_{\beta_2} \cdot \ldots \cdot F_{\beta_d} = 0 .$$

Together, these yield

$$\sum_{\beta \in \Sigma'^{(1)}} \langle m, \beta \rangle (\pi_*(E_\beta \cdot E_{\beta_2} \cdot \ldots \cdot E_{\beta_d}) - F_\beta \cdot F_{\beta_2} \cdot \ldots \cdot F_{\beta_d}) = 0 .$$

Hence,

$$\sum_{\beta \in \Sigma'^{(1)}} \langle m, \beta \rangle (\pi_*(E_\beta \cdot E_{\beta_2} \cdot \ldots \cdot E_{\beta_d}) - \pi_* E_\beta \cdot \pi_* E_{\beta_2} \cdot \ldots \cdot \pi_* E_{\beta_d}) = 0 ,$$

and finally

$$\left(\sum_{\beta \in \Sigma'^{(1)}} \langle m, \beta \rangle g(\beta, \beta_2, \ldots, \beta_d) \right) F_\tau = 0 .$$

Since $F_\tau \neq 0$ in $A^d X_\Sigma$, and using that g satisfies (1), we get

$$\sum_{\beta \in \Gamma^{(1)}} \langle m, \beta \rangle g(\beta, \beta_2, \ldots, \beta_d) = 0 ,$$

which completes the proof.

4 Todd class formula

In this section, we prove the main theorem of Part I, which asserts that in the codimension of the singular locus, the difference between the Todd class and the mock Todd class of a simplicial toric variety may be computed locally. We first recall some facts and definitions.

If Σ is a complete non-singular fan, the Chern classes of X_Σ are given by

$$cX_\Sigma = \sum_{\tau \in \Sigma} F_\tau = \prod_{\rho \in \Sigma^{(1)}} (1 + F_\rho) ,$$

the second equality following easily from the description of the Chow ring given in Sect. 2.3.

For a simplicial toric variety X_Σ, we define the *mock Chern class* of X_Σ by

$$CX_\Sigma = \sum_{\tau \in \Sigma} \frac{1}{\operatorname{mult} \tau} F_\tau = \prod_{\rho \in \Sigma^{(1)}} (1 + F_\rho) .$$

Again the second inequality follows from Sect. 2.3. We let $C_i X_\Sigma$ be the codimension i part of CX_Σ, i.e.

$$C_i X_\Sigma = \sum_{\tau \in \Sigma^{(i)}} \frac{1}{\operatorname{mult} \tau} F_\tau .$$

We then define the *mock Todd class* of X_Σ by

$$\operatorname{TD} X_\Sigma = \sum_{i \geqq 0} \operatorname{TD}^i X_\Sigma ,$$

where $\operatorname{TD}^i X_\Sigma$ is the i^{th} Todd polynomial in the class $C_1, \ldots, C_i$.

We now examine the difference

$$\operatorname{Td}^i X_\Sigma - \operatorname{TD}^i X_\Sigma .$$

For $i < d = \operatorname{codim}(\operatorname{Sing} X_\Sigma)$, the difference vanishes. This is because we may find a non-singular subdivision Σ' of Σ such that the map $\pi\colon X_{\Sigma'} \to X_\Sigma$ is an isomorphism except over a set of codimension d. So if $i < d$, then as we have seen at the beginning of the previous section, a product in codimension i pushes forward, and we have $\pi_* \operatorname{TD}^i X_{\Sigma'} = \operatorname{TD}^i X_\Sigma$. But since $X_{\Sigma'}$ is non-singular, $\operatorname{TD}^i X_{\Sigma'} = \operatorname{Td}^i X_{\Sigma'}$, and this gives $\pi_* \operatorname{Td}^i X_{\Sigma'} = \operatorname{TD}^i X_\Sigma$. Finally, we use the fact that the Todd class pushes forward under proper birational morphisms [Ful, p. 353] to obtain $\operatorname{Td}^i X_\Sigma = \operatorname{TD}^i X_\Sigma$.

Before we examine the above difference in the case $i = d$, we will prove a lemma about pushing forward an arbitrary polynomial of degree d in the mock Chern classes.

Lemma 1. *Let P be a polynomial of graded degree d in the variables $C_1, \ldots, C_d$ (C_i having degree i). If Σ is a simplicial fan, $\tau \in \Sigma^{(d)}$, and Γ is a simplicial interior subdivision of τ, and Σ' is the fan obtained from Σ by subdividing τ, with $\pi\colon X_{\Sigma'} \to X_\Sigma$ the induced map of varieties, then*

$$\pi_* P(C_1 X_{\Sigma'}, \ldots, C_d X_{\Sigma'}) - P(C_1 X_\Sigma, \ldots, C_d X_\Sigma) = rF_\tau ,$$

where $r \in \mathbb{Q}$ depends only on τ and Γ (but not on all of Σ).

Proof. By linearity, it suffices to consider the case in which P is a monomial:

$$P = C_1^{n_1} \ldots C_d^{n_d} \quad \text{where} \sum_{i=1}^{d} in_i = d\,.$$

In this case, the above difference is equal to:

$$\pi_* \left[\left(\sum_{\rho_1 \in \Sigma'^{(1)}} E_{\rho_1} \right)^{n_1} \left(\sum_{\rho_1, \rho_2 \in \Sigma'^{(1)}} E_{\rho_1} E_{\rho_2} \right)^{n_2} \cdots \left(\sum_{\rho_1, \ldots, \rho_d \in \Sigma'^{(1)}} E_{\rho_1} \ldots E_{\rho_d} \right)^{n_d} \right]$$
$$- \left(\sum_{\rho_1 \in \Sigma'^{(1)}} F_{\rho_1} \right)^{n_1} \left(\sum_{\rho_1, \rho_2 \in \Sigma'^{(1)}} F_{\rho_1} F_{\rho_2} \right)^{n_2} \cdots \left(\sum_{\rho_1, \ldots, \rho_d \in \Sigma'^{(1)}} F_{\rho_1} \ldots F_{\rho_d} \right)^{n_d}$$

where each sum is taken over all distinct subsets $\{\rho_1, \ldots, \rho_i\}$ of $\Sigma'^{(1)}$ of size i. (Again, we set $F_\rho = 0$ if $\rho \in \Sigma'^{(1)} \backslash \Sigma^{(1)}$.) We may rewrite this difference as

$$\sum \left[\pi_* \left(\prod_{i=1}^{d} \prod_{j=1}^{n_i} \prod_{\rho \in A_i^j} E_\rho \right) - \prod_{i=1}^{d} \prod_{j=1}^{n_i} \prod_{\rho \in A_i^j} F_\rho \right]$$

where the sum ranges over all sequences $A_1^1, \ldots, A_1^{n_1}, A_2^1, \ldots, A_2^{n_2}, \ldots, A_d^1, \ldots, A_d^{n_d}$ of subsets of $\Sigma'^{(1)}$ such that $\#(A_i^j) = i$.

Finally, by properties (1) and (5) of the theorem of the preceeding section, we see that the above quantity equals

$$\left[\sum f_{\tau, \Gamma}(A_1^1, \ldots, A_1^{n_1}, A_2^1, \ldots, A_2^{n_2}, \ldots, A_d^1, \ldots, A_d^{n_d}) \right] F_\tau\,,$$

where once again, $A_1^1, \ldots, A_1^{n_1}, A_2^1, \ldots, A_2^{n_2}, \ldots, A_d^1, \ldots, A_d^{n_d}$ range over all sequences of subsets of $\Sigma'^{(1)}$ such that $\#(A_i^j) = i$.

Note that the bracketed coefficient above gives a recipe for calculating r in terms of τ and Γ.

Theorem 1 now follows from:

Lemma 2. *If Σ is a simplicial fan, $\tau \in \Sigma^{(d)}$ is a cone whose $(d-1)$-dimensional faces are non-singular, and Γ is a non-singular interior subdivision of τ, then letting Σ' be the fan obtained by subdividing τ, and letting $\pi\colon X_{\Sigma'} \to X_\Sigma$, we have*

$$\pi_* \mathrm{TD}^d X_{\Sigma'} - \mathrm{TD}^d X_\Sigma = r_\tau F_\tau\,,$$

where r_τ depends only on τ (not on all of Σ nor on the subdivision Γ).

Proof. Since the d^{th} mock Todd class is a polynomial of graded degree d in the mock Chern classes, the previous lemma asserts that r_τ exists and depends only on τ and Γ, so all we must show is that r_τ is independent of Γ.

Let Γ_1 and Γ_2 be two non-singular interior subdivisions of τ. To show that r_{τ, Γ_1} and r_{τ, Γ_2} agree, we let Σ be a complete simplicial fan such that $\tau \in \Sigma^{(d)}$, and such that every other cone in $\Sigma^{(d)}$ is non-singular. (We may construct such a Σ by choosing an arbitrary complete simplicial fan containing τ and desingularizing all cones in $\Sigma^{(d)}$ except τ.) Let Σ_1' and Σ_2' be the fans obtained from Σ by replacing τ with Γ_1 and Γ_2, respectively, and let $\pi_1\colon X_{\Sigma_1'} \to X_\Sigma$ and $\pi_2\colon X_{\Sigma_2'} \to X_\Sigma$ be the induced maps of toric varieties.

Note that since $X_{\Sigma_1'}$ and $X_{\Sigma_2'}$ are non-singular in codimension d, we have

$$\pi_{1*}\mathrm{TD}^d X_{\Sigma_1'} = \pi_{1*}\mathrm{Td}^d X_{\Sigma_1'} = \mathrm{Td}^d X_{\Sigma},$$

$$\text{and} \quad \pi_{2*}\mathrm{TD}^d X_{\Sigma_2'} = \pi_{2*}\mathrm{Td}^d X_{\Sigma_2'} = \mathrm{Td}^d X_{\Sigma}.$$

(Once again, we are using the fact that the Todd class pushes forward under proper birational morphisms.)

But r_{τ, Γ_1} and r_{τ, Γ_2} are defined by

$$\pi_{1*}\mathrm{TD}^d X_{\Sigma_1'} - \mathrm{TD}^d X_{\Sigma} = r_{\tau, \Gamma_1} F_{\tau}$$

$$\text{and} \quad \pi_{2*}\mathrm{TD}^d X_{\Sigma_2'} - \mathrm{TD}^d X_{\Sigma} = r_{\tau, \Gamma_2} F_{\tau},$$

so we see that $r_{\tau, \Gamma_1} F_{\tau} = r_{\tau, \Gamma_2} F_{\tau}$. Since $F_{\tau} \neq 0$ in $A^d X_{\Sigma}$, we conclude that $r_{\tau, \Gamma_1} = r_{\tau, \Gamma_2}$, as desired.

Proof of Theorem 1. The lemma guarantees that there is a function $t: \mathscr{S} \to \mathbb{Q}$ such that each time we desingularize a cone τ of Σ, there is a contribution of $t(\tau) F_{\tau}$ to the difference $\pi_* \mathrm{TD}^d X_{\Sigma'} - \mathrm{TD}^d X_{\Sigma}$. The theorem then easily follows by induction, desingularizing one d-dimensional cone at a time.

Remark. Throughout this section, the hypothesis that Σ or Γ be simplicial may be weakened to "simplicial in dimension d," by which we mean that all d-dimensional cones are simplicial. For if Σ is simplicial in dimension d, then we may find a simplicial subdivision Σ_1' such that $\Sigma_1'^{(d)} = \Sigma^{(d)}$, and so the map $\pi: X_{\Sigma_1'} \to X_{\Sigma}$ is an isomorphism except above a subvariety of codimension strictly greater than d. It is then easy to check that all of the assertions of this section make sense and remain true when applied to the fan Σ. In the next section, in which we consider the case $d = 2$, we may drop the simplicial hypothesis altogether, since any fan is simplicial in dimension two.

5 The codimension two formula

We now show that in codimension two, the function t of the previous section is given in terms of a Dedekind sum. This leads to a formula expressing the Todd class of a toric variety in codimension two in terms of Dedekind sums.

The Dedekind sum $s(p, q)$ for relatively prime integers p and q is defined by

$$s(p, q) = \sum_{i=1}^{q} \left(\left(\frac{i}{q}\right)\right)\left(\left(\frac{pi}{q}\right)\right),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & x \notin \mathbb{Z} \\ 0 & x \in \mathbb{Z}. \end{cases}$$

These sums occur in many contexts and may be characterized in many ways, including a cotangent formula and a reciprocity law. The book [RaGr] contains a nice collection of properties of these sums.

Dedekind sums may also be defined by means of continued fractions. If $0 \leqq p < q$ with $(p, q) = 1$, and

$$\frac{q}{q-p} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_k}}}, \qquad b_i \in \mathbb{N}$$

is the negative-regular continued fraction expansion of $\frac{q}{q-p}$, and $pp' \equiv 1 (\bmod q)$ with $0 \leqq p' < q$, then

$$s(p, q) = \frac{1}{12}\left(\sum_{i=1}^{k} (3 - b_i) + \frac{1}{q}(p + p') - 2 \right).$$

This follows from Theorem 1 of [My], which explores the connection between Dedekind sums and continued fractions.

We are now ready to find the value of $t(\tau)$ for a two-dimensional cone τ. Given a two-dimensional cone τ in a lattice N, τ may be written as $\langle e_1, pe_1 + qe_2 \rangle$, where $\{e_1, e_2\}$ forms a basis of N in the plane containing τ, $q \geqq 1$, $(p, q) = 1$, and $0 \leqq p < q$. With this representation, q is uniquely determined, but p is determined only up to multiplicative inverses modulo q. In this situation, we say that τ has *type* (p, q). We may say equally well, however, that τ has type (p', q), where $pp' \equiv 1$ modulo q. These facts are easily verified. See [Oda, p. 24].

We then have

Theorem 3. *Let Σ be a complete simplicial fan. For each $\tau \in \Sigma^{(2)}$, let (p_τ, q_τ) be the type of τ. Then*

$$\mathrm{Td}^2 X_\Sigma = \mathrm{TD}^2 X_\Sigma + \sum_{\tau \in \Sigma^{(2)}} \left[s(p_\tau, q_\tau) + \frac{1}{4} - \frac{1}{4q_\tau} \right] F_\tau .$$

In other words, for a two-dimensional cone τ of type (p, q), we have

$$t(\tau) = s(p, q) + \frac{1}{4} - \frac{1}{4q}.$$

Proof. Let τ be a cone of type (p, q) in a two-dimensional lattice. In order to compute $t(\tau)$, we consider any complete fan Σ such that $\tau \in \Sigma$ and all other cones of Σ are non-singular. It is now a routine lattice computation to find the value of $t(\tau)$ using the explicit description of the function t (in terms of a subdivision of τ) which was given in Sect. 4. We will also need facts about the explicit resolution of singularities of toric surfaces (cf. [Oda, Sect. 1.6]). We now summarize these facts.

Let τ be a cone of type (p, q). For simplicity, assume that $\tau = \langle (1, 0), (p, q) \rangle$ in $\mathbb{Z}^2$, with $(p, q) = 1$ and $0 \leqq p < q$. Then there is a unique minimal non-singular subdivision Γ of τ which may be described explicitly using continued fractions.

Here is what we will need about Γ:

(1) Consider the convex hull H of the set $\tau \cap N \setminus \{(0, 0)\}$, and let

$$\rho_0 = (1, 0), \rho_1, \ldots, \rho_k, \rho_{k+1} = (p, q)$$

be, in this order, the primitive elements of N on the compact faces of the boundary of H. Then Γ, the unique minimal non-singular subdivision of τ is given by

$$\Gamma^{(1)} = \{\rho_0, \ldots, \rho_{k+1}\}, \quad \text{and}$$
$$\Gamma^{(2)} = \{\langle \rho_0, \rho_1\rangle, \langle \rho_1, \rho_2\rangle, \ldots, \langle \rho_k, \rho_{k+1}\rangle\}.$$

It follows from $0 \leqq p < q$ that $\rho_1 = (1, 1)$ and $\rho_k = (p + q', q - p')$, where $pp' + qq' = 1$ and $0 \leqq p' < q$.

(2) If

$$\frac{q}{q-p} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_k}}}, \qquad b_i \in \mathbb{N}$$

is the negative-regular continued fraction expansion of $\frac{q}{q-p}$, then for $0 < i \leqq k$, we have

$$\rho_{i+1} = b_i \rho_i - \rho_{i-1}.$$

To finish the proof of Theorem 3, we must compute $t(\tau) = \pi_* \mathrm{TD}^2 X_{\Sigma'} - \mathrm{TD}^2 X_\Sigma$, where Σ is a two-dimensional fan containing τ, and Σ' is obtained from Σ by subdividing τ in the manner described above. By definition,

$$\pi_* \mathrm{TD}^2 X_{\Sigma'} - \mathrm{TD}^2 X_\Sigma = \tfrac{1}{12}[(\pi_* C_1^2 X_{\Sigma'} - C_1^2 X_\Sigma) + (\pi_* C_2 X_{\Sigma'} - C_2 X_\Sigma)].$$

By the computation of Lemma 1, this becomes

$$\frac{1}{12}\left(\sum_{\alpha_1, \alpha_2 \in \Gamma^{(1)}} f_{\tau,\Gamma}(\alpha_1, \alpha_2) + \sum_{\{\gamma_1, \gamma_2\} \subset \Gamma^{(1)}} f_{\tau,\Gamma}(\gamma_1, \gamma_2)\right),$$

where the second sum ranges over all two-element subsets of $\Gamma^{(1)}$.

From this point, we will write f for $f_{\tau,\Gamma}$.

By property (3) of Theorem 2, we have

$$f(\rho_i, \rho_j) = 0 \quad \text{if } |i-j| > 1 \quad \text{and} \quad \{i, j\} \neq \{0, k+1\},$$

so the first sum above reduces to

$$\frac{1}{12}\left(\sum_{i=0}^{k+1} f(\rho_i, \rho_i) + 2\sum_{i=0}^{k} f(\rho_i, \rho_{i+1}) + 2f(\rho_0, \rho_{k+1})\right),$$

and the second becomes

$$\frac{1}{12}\left(\sum_{i=0}^{k} f(\rho_i, \rho_{i+1}) + f(\rho_0, \rho_{k+1})\right).$$

Using $f(\rho_i, \rho_{i+1}) = 1$ and $f(\rho_0, \rho_{k+1}) = -\frac{1}{q}$ (Property (3) again), the total equals

$$\frac{1}{12}\left(3(k+1) - \frac{3}{q} + \sum_{i=0}^{k+1} f(\rho_i, \rho_i)\right),$$

and we must compute only $f(\rho_i, \rho_i)$ for $i = 0, 1, \ldots, k+1$.

$i = 0$: Using $\rho_0 = (1, 0)$, $\rho_1 = (1, 1)$ and $\rho_{k+1} = (p, q)$, and taking $m = (1, 0)$, Property (4) gives

$$\sum_{i=0}^{k+1} \langle m, \rho_j \rangle f(\rho_j, \rho_0) = 0 \ ,$$

which becomes

$$\langle m, \rho_0 \rangle f(\rho_0, \rho_0) + \langle m, \rho_1 \rangle f(\rho_1, \rho_0) + \langle m, \rho_{k+1} \rangle f(\rho_{k+1}, \rho_0) = 0 \ ,$$

and so $f(\rho_0, \rho_0) = \dfrac{p}{q} - 1$.

$i = k + 1$: This is similar. This time we use $\rho_0 = (1, 0)$, $\rho_k = (p + q', q - p')$ and $\rho_{k+1} = (p, q)$, and take $m = (1, 0)$. Property (4) then gives

$$\sum_{i=0}^{k+1} \langle m, \rho_j \rangle f(\rho_j, \rho_{k+1}) = 0 \ ,$$

which becomes

$$\langle m, \rho_0 \rangle f(\rho_0, \rho_{k+1}) + \langle m, \rho_k \rangle f(\rho_k, \rho_{k+1}) + \langle m, \rho_{k+1} \rangle f(\rho_{k+1}, \rho_{k+1}) = 0 \ ,$$

and so $f(\rho_{k+1}, \rho_{k+1}) = \dfrac{p'}{q} - 1$.

$1 \leqq i \leqq k + 1$: In this case, $\{\rho_{i-1}, \rho_i\}$ is a basis of $\mathbb{Z}^2$ and $\rho_{i+1} = b_i \rho_i - \rho_{i-1}$. Take $m \in (\mathbb{Z}^2)^*$ such that $\langle m, \rho_{i-1} \rangle = 0$ and $\langle m, \rho_1 \rangle = 1$. Then $\langle m, \rho_{i+1} \rangle = b_1$, and the equation

$$\sum_{i=0}^{k+1} \langle m, \rho_j \rangle f(\rho_j, \rho_i) = 0 \ ,$$

tells us that

$$\langle m, \rho_{i-1} \rangle f(\rho_{i-1}, \rho_i) + \langle m, \rho_i \rangle f(\rho_i, \rho_i) + \langle m, \rho_{i+1} \rangle f(\rho_{i+1}, \rho_i) = 0 \ ,$$

and so $f(\rho_i, \rho_i) = -b_i$.

Adding it all up, we get

$$\frac{1}{12}\left(3(k+1) - \frac{3}{q} + \left(\frac{p}{q} - 1 \right) + \left(\frac{p'}{q} - 1 \right) - \sum_{i=1}^{k} b_i \right)$$

$$= \frac{1}{12}\left[\left(\sum_{i=1}^{k} (3 - b_i) + \frac{1}{q}(p + p') - 2 \right) + \left(3 - \frac{3}{q} \right) \right]$$

$$= s(p, q) + \frac{1}{4} - \frac{1}{4q} \ ,$$

as was to be shown.

Part II: Counting lattice points in a tetrahedron

6 Introduction

For a long time, mathematicians have been interested in the problem of counting the number of lattice points in integral convex polytopes, i.e., convex polytopes with vertices at lattice points. For such a polytope Δ of dimension n in a lattice M, it is convenient to introduce the function $l_\Delta(k)$ to denote the number of lattice points in Δ dilated by a factor of the integer k:

$$l_\Delta(k) = \#(k\Delta \cap M) \qquad k \in \mathbb{Z}^+ .$$

l_Δ turns out to be a polynomial function in k of degree n with rational coefficients:

$$l_\Delta(k) = a_n k^n + a_{n-1} k^{n-1} + \cdots + a_0$$

and is called the *Ehrhart polynomial* of Δ [Ehr]. One hopes to find expressions for the coefficients a_i in terms of the geometry of Δ. Ehrhart showed that:

(1) a_n equals the volume of Δ.

(2) a_{n-1} equals half the sum of the volumes of the $(n-1)$-dimensional faces of Δ. (Here and throughout, the volume of a k-dimensional face of Δ is measured with respect to the k-dimensional lattice in the k-plane containing Δ.)

(3) $a_0 = 1$.

For $n = 2$, this gives a complete answer, namely Pick's Theorem (cf. [Ham]), which says that if Δ is a lattice polygon, then

$$l_\Delta(k) = \mathrm{vol}(\Delta)k^2 + \tfrac{1}{2}S(\Delta)k + 1 ,$$

where $S(\Delta)$ denotes the sum of the lattice lengths of the edges of Δ.

For a three-dimensional integral convex polytope Δ, (1)–(3) give

$$l_\Delta(k) = \mathrm{vol}(\Delta)k^3 + \tfrac{1}{2}S(\Delta)k^2 + a_1 k + 1 ,$$

where $S(\Delta)$ denotes the sum of the lattice volumes of the two-dimensional faces of Δ. Our problem is thus reduced to determining a_1. By analogy with Pick's Theorem, one would hope to express a_1 in terms of the volumes of the one-dimensional faces (edges) of Δ. However this is not possible [Ree]. To see this, consider the tetrahedron Δ_r in $\mathbb{Z}^3$ with vertices at (0, 0, 0), (1, 0, 0), (0, 1, 0), and (1, 1, r), where $r \in \mathbb{Z}^+$. It is easily seen that $a_1 = 1 - \frac{r}{6}$, but the lattice volumes of the one-dimensional faces (and the two-dimensional faces) of Δ_r are independent of r. Thus, even in the case of a tetrahedron, a formula for the coefficient a_1 must involve ingredients other than just the volumes of the faces of Δ.

In this paper, we present a formula for a_1 in the case of a general lattice tetrahedron Δ given in terms of:

(1) The lattice volumes of the one and two-dimensional faces of Δ, and

(2) certain functions of the dihedral angles formed at each edge, computed in terms of Dedekind sums.

We now define these functions.

Let P and Q be distinct non-parallel planes in a three-dimensional lattice M. P and Q then form a dihedral angle (i.e., a cone with one-dimensional cospan), to which we attach two numbers, the *multiplicity*, $m(P, Q)$, and the *Dedekind measure*, $d(P, Q)$, as follows:

Let L be the line where P and Q intersect. Choose distinct elements $v_0, v_1 \in L$, and choose points $v_2 \in P \setminus L$, $v_3 \in Q \setminus L$. Let ρ be the unique primitive element of M on the ray from the origin through $v_1 - v_0$. Take M' to be the lattice $M/\mathbb{Z}\rho$, and let v_2' and v_3' be the images in M' of v_2 and v_3. Let v_2'' and v_3'' denote the primitive elements on the rays in M' from the origin through v_2' and v_3'. Then it is easy to see that a basis $\{e_1, e_2\}$ of M' may be chosen so that $\{v_2'', v_3''\} = \{e_1, pe_1 + qe_2\}$, where $0 \leqq p < q$. (In this case we say that the cone in M' generated by v_2'' and v_3'' has type (p, q).) We then define

$$m(P, Q) = q \,,$$

$$\text{and} \quad d(P, Q) = -s(p, q) + \tfrac{1}{4} \,.$$

Here $s(p, q)$ denotes the classical Dedekind sum. (For the definition of $s(p, q)$, see the beginning of Sect. 5.)

It is easy to check that $m(P, Q)$ and $d(P, Q)$ are well-defined (i.e., independent of the choices of v_0, v_1, v_2, v_3) and are invariant under translations and lattice automorphisms.

$m(P, Q)$ and $d(P, Q)$ may be easily computed by picking v_0, v_1, v_2, v_3 as above and then choosing a coordinate system so that

$$\begin{aligned} v_0 &= (0, 0, 0)\,, & & \\ v_1 &= (r, 0, 0)\,, & &\text{so that } \rho = (1, 0, 0)\,, \\ v_2 &= (t, u, 0)\,, & &\text{and} \\ v_3 &= (x, y, z)\,, & & \end{aligned}$$

where $u, z > 0$. It then follows easily that $p = \dfrac{y}{\gcd(y, z)}$, and $q = \dfrac{z}{\gcd(y, z)}$, and we may compute $m(P, Q)$ and $d(P, Q)$ from these.

Now let v_0, v_1, v_2, v_3 be the vertices of a tetrahedron Δ in a three-dimensional lattice M. For $0 \leqq i < j \leqq 3$, let Γ_{ij} and Γ_{ij}' be the two two-dimensional faces of Δ containing the edge $v_i v_j$, and let P_{ij} and P_{ij}' be the planes containing these faces.

We then have the following formula for a_1, given in terms of the volumes of the faces of Δ, and the multiplicity and Dedekind measures of the six dihedral angles formed by Δ:

Theorem 4. *With the above notation,*

$$a_1 = \sum_{0 \leqq i < j \leqq 3} \left[\frac{1}{36 m(P_{ij}, P_{ij}')} \left(\frac{\operatorname{vol}(\Gamma_{ij})}{\operatorname{vol}(\Gamma_{ij}')} + \frac{\operatorname{vol}(\Gamma_{ij}')}{\operatorname{vol}(\Gamma_{ij})} \right) + d(P_{ij}, P_{ij}') \right] \operatorname{vol}(v_i v_j) \,.$$

The connection between counting lattice points in a tetrahedron and Dedekind sums was known to Mordell, who considered the tetrahedron $\Delta(a, b, c)$ with vertices at $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$. Mordell gave a formula for the number of lattice points in $\Delta(a, b, c)$, expressed in terms of three Dedekind sums, in the case that a, b, c are pairwise relatively prime [Mor]. In the Sect. 9, we use Theorem 4 to give a formula for the number of lattice points in $\Delta(a, b, c)$ for arbitrary positive integers a, b, and c. This is

Theorem 5. *If* $a, b, c > 0$ *with* $\gcd(a, b, c) = 1$, *and* Δ *is the tetrahedron in* $\mathbb{Z}^3$ *with vertices* $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, *and* $(0, 0, c)$, *then*

$$\begin{aligned}\#(k\Delta \cap \mathbb{Z}^3) = {} & \frac{abc}{6}k^3 + \left(\frac{ab + ac + bc + d}{4}\right)k^2 \\ & + \left[\frac{1}{12}\left(\frac{ac}{b} + \frac{bc}{a} + \frac{ab}{c} + \frac{d^2}{abc}\right) + \frac{a + b + c + A + B + C}{4}\right. \\ & \left. - As\left(\frac{bc}{d}, \frac{aA}{d}\right) - Bs\left(\frac{ac}{d}, \frac{bB}{d}\right) - Cs\left(\frac{ab}{d}, \frac{cC}{d}\right)\right]k \\ & + 1\,,\end{aligned}$$

where $A = \gcd(b, c)$, $B = \gcd(a, c)$, $C = \gcd(a, b)$, *and* $d = ABC$.

The derivation of the formula of Theorem 4 relies on the well-established connection between convex polytopes and toric varieties. To each integral convex polytope Δ, there is an associated toric variety X_Δ (see Sect. 7). Theorems of algebraic geometry applied to the variety X_Δ often yield results about the polytope Δ. Danilov [Dan, p. 134] showed that expressions for the Todd class of X_Δ give rise to formulas for the Ehrhart polynomial of Δ. Theorems 1 and 2 will be derived from Theorem 3, which expresses the codimension two part of the Todd class of a toric variety in terms of Dedekind sums. This may be used to obtain into an expression for the a_{n-2} coefficient of the Ehrhart polynomial of an arbitrary integral convex polytope.

7 Polytopes and toric varieties

In this section, we present background information about the connection between polytopes and toric varieties. In Sect. 7.1, we describe how an integral convex polytope Δ gives rise to a toric variety X_Δ. In Sect. 7.2, we state the relation, due to Danilov, between the Todd class of X_Δ and the Ehrhart polynomial of Δ. In Sect. 7.3, we state a formula for the Todd class of a toric variety in codimension two (a restatement of Theorem 3), which will be essential for the proof of Theorem 4.

7.1 The toric variety associated to a polytope

Let Δ be an integral convex polytope in an n-dimensional lattice M. Then there is an associated n-dimensional toric variety X_Δ. To each face Γ of Δ, there is a special subvariety $V(\Gamma)$, whose complex dimension is the same as the real dimension of Γ. The construction of X_Δ and its special subvarieties may be found in [Oda] or [Dan]. X_Δ is obtained as the toric variety associated to a certain fan Σ_Δ in the dual lattice $N = \mathrm{Hom}(M, \mathbb{Z})$. We briefly describe the construction of Σ_Δ.

Δ determines a polytope in the vector space $M_{\mathbb{R}} = M \otimes \mathbb{R}$. If Γ is a face of Δ, define a cone σ_Γ in M by

$$\sigma_\Gamma = \bigcup_{m \in \Gamma} \bigcup_{r \geqq 0} r(\Delta - m)\,.$$

We then define

$$\Sigma_\Delta = \{\check{\sigma}_\Gamma \mid \Gamma \text{ is a face of } \Delta\} ,$$

where the *dual* $\check{\sigma}$ of a cone σ in M is defined by

$$\check{\sigma} = \{n \in N_{\mathbb{R}} \mid \langle n, m \rangle \geqq 0 \text{ for all } m \in \sigma\} ,$$

and $\langle\,,\rangle\colon N \times M \to \mathbb{Z}$ is the natural pairing. Σ_Δ is a fan in N which determines a toric variety as in [Oda, Sect. 1.2].

7.2 The Todd class and lattice points

Every algebraic variety has a naturally defined Todd class [Ful]. Danilov [Dan, p. 134] showed how to determine the Ehrhart polynomial of an integral convex polytope Δ from the Todd class of the associated toric variety X_Δ. This relation, a consequence of the Riemann-Roch Theorem, allows us to prove Theorem 4 using toric varieties.

Let $[V(\Gamma)]$ denote the class in $(A_* X_\Delta)_{\mathbb{Q}}$ of the special subvariety $V(\Gamma)$. If the Todd class of X_Δ has an expression of the form

$$\mathrm{Td}\, X_\Delta = \sum r_\Gamma [V(\Gamma)]$$

with $r_\Gamma \in \mathbb{Q}$, then the coefficient of a_k in the Ehrhart polynomial of Δ is given by

$$a_k = \sum_{\dim \Gamma = k} r_\Gamma \operatorname{vol}(\Gamma) .$$

7.3 The Todd class in codimension two

We restate here the formula of Theorem 3 in the context of a toric variety X_Δ associated to a lattice tetrahedron Δ. This formula allows us to compute the codimension two part of the Todd class of X_Δ.

Let Δ be a three-dimensional integral convex polytope. For each edge E of Δ, we will use $d(E)$ and $m(E)$ to denote $d(P, P')$ and $m(P, P')$, where P and P' are the planes containing the two two-dimensional faces of Δ which meet at E.

The Todd class formula may now be stated as follows:

$$\mathrm{Td}^2 X_\Delta = \mathrm{TD}^2 X_\Delta + \sum \left(d(E) - \frac{1}{4m(E)} \right) [V(E)] ,$$

where

$$\mathrm{TD}^2 X_\Delta = \frac{1}{12}\left[\left(\sum [V(F)] \right)^2 + \sum \frac{1}{m(E)} [V(E)] \right] ,$$

with the sums taken over all two-dimensional faces F and all edges E of Δ.

Remark. It is easy to see that this is a restatement of Theorem 3. The quantities $d(E)$ and $m(E)$ are defined in terms of the cone σ_E. If this cone modulo its cospan has type (p, q), then the dual $\check{\sigma}_E$ has type $(-p, q)$. The expression $t(\check{\sigma}_E) = s(-p, q) + -\frac{1}{4q} + \frac{1}{4}$ of the above-mentioned theorem thus becomes $d(E) - \frac{1}{4m(E)}$, as stated above.

8 Proof of the lattice point formula

In this section, we use the toric variety results stated in the previous section to derive the formula of Theorem 4. Because of the results of Sect. 7.2, to prove Theorem 4 it is enough to prove the following result about the Todd class of the corresponding toric variety X_Δ.

Let E_{ij} denote the edge $v_i v_j$ of Δ.

Theorem 6. *With the notation of Theorem* 4,

$$\mathrm{Td}^2 X_\Delta = \sum_{0 \leqq i < j \leqq 3} \left[\frac{1}{36m(P_{ij}, P'_{ij})} \left(\frac{\mathrm{vol}(\Gamma_{ij})}{\mathrm{vol}(\Gamma'_{ij})} + \frac{\mathrm{vol}(\Gamma'_{ij})}{\mathrm{vol}(\Gamma_{ij})} \right) + d(P_{ij}, P'_{ij}) \right] [V(E_{ij})] \, .$$

Proof. By the Todd class formula of Sect. 7.3, we have

$$\mathrm{Td}^2 X_\Delta = \frac{1}{12} \left[\left(\sum [V(F)] \right)^2 + \sum \frac{1}{m(E)} [V(E)] \right] + \sum \left(d(E) - \frac{1}{4m(E)} \right) [V(E)] \, ,$$

where F ranges over the two-dimensional faces and E over the edges of Δ. The cycles corresponding to faces multiply as follows [Dan, p. 127]:

$$(*) \qquad [V(F)][V(F')] = \frac{1}{m(F \cap F')} [V(F \cap F')] \, ,$$

where $F \neq F'$ are two-dimensional faces of Δ. Thus, we get

$$(**) \qquad \mathrm{Td}^2 X_\Delta = \frac{1}{12} \sum [V(F)]^2 + \sum d(E) [V(E)] \, .$$

We will compute the first sum above using

Lemma. *Let $F \neq F'$ be two-dimensional faces of Δ. Then*

$$\mathrm{vol}(F')[V(F)] = \mathrm{vol}(F)[V(F')] \, .$$

Proof. Let $F = v_i v_j v_k$ and $F' = v_i v_j v_l$. Let ρ and ρ' be the primitive elements of N dual to F and F', respectively. First note that

$$\frac{\mathrm{vol}(F)}{\mathrm{vol}(F')} = \frac{\langle \rho', v_k - v_l \rangle}{\langle \rho, v_l - v_k \rangle} \, .$$

This was pointed out to me by Burt Totaro. It is perhaps easiest to see this by choosing coordinates so that

$$v_i = (0, 0, 0) \, ,$$
$$v_j = (x, 0, 0) \, ,$$
$$v_k = (y, z, 0) \, , \qquad \text{and}$$
$$v_l = (w, v, u) \, .$$

Then $\mathrm{vol}(v_i v_j v_k) = \frac{xz}{2}$, $\mathrm{vol}(v_i v_j v_l) = \frac{x}{2}\gcd(v, u)$, and $\langle \rho', v_k - v_l \rangle = \frac{zu}{\gcd(v, u)}$, $\langle \rho, v_l - v_k \rangle = u$, and the above equality follows.

The lemma now follows easily from

$$\langle \rho', v_k - v_l \rangle [V(F')] + \langle \rho, v_k - v_l \rangle [V(F)] = 0 ,$$

which is a basic relation in the Chow ring of a toric variety. See [Dan, p. 127] or Sect. 2.3.

It is now easy to complete the proof of Theorem 6. For by the lemma,

$$[V(F)]^2 = \frac{\mathrm{vol}(F)}{\mathrm{vol}(F')}[V(F)][V(F')] .$$

By ($*$), this becomes

$$[V(F)]^2 = \frac{\mathrm{vol}(F)}{\mathrm{vol}(F')}\frac{1}{m(E)}[V(E)] ,$$

where $E = F \cap F'$. It then easily follows that

$$\sum [V(F)]^2 = \frac{1}{3} \sum_{0 \leqq i < j \leqq 3} \left(\frac{\mathrm{vol}(\Gamma_{ij})}{\mathrm{vol}(\Gamma'_{ij})} + \frac{\mathrm{vol}(\Gamma'_{ij})}{\mathrm{vol}(\Gamma_{ij})} \right) [V(E_{ij})] .$$

Putting this into ($**$) yields the equation of Theorem 6.

9 A specific tetrahedron

In this section, we prove Theorem 5, which gives the Ehrhart polynomial of the tetrahedron $\Delta(a, b, c)$ with vertices at

$$v_0 = (0, 0, 0) ,$$
$$v_1 = (a, 0, 0) ,$$
$$v_2 = (0, b, 0) , \qquad \text{and}$$
$$v_3 = (0, 0, c) ,$$

Without loss of generality, we assume $\gcd(a, b, c) = 1$. We set $A = \gcd(b, c)$, $B = \gcd(a, c)$, $C = \gcd(a, b)$, and $d = ABC$.

We first compute the coefficients a_3, a_2, and a_0 using Ehrhart's results. (See (1)–(3) of Sect. 5.) (These results are easy consequences of facts about Todd classes of toric varieties.)

$$a_3 = \mathrm{vol}(\Delta) = \frac{abc}{6}$$

$$a_2 = \frac{1}{2}S(\Delta) = \frac{ab + ac + bc + d}{4}, \qquad \text{and}$$

$$a_0 = 1 .$$

It remains to compute a_1, for which we use Theorem 4. One easily verifies that

$$\operatorname{vol}(v_0 v_1) = a, \quad \operatorname{vol}(v_0 v_2) = b, \quad \operatorname{vol}(v_0 v_3) = c\,,$$
$$\operatorname{vol}(v_1 v_2) = C, \quad \operatorname{vol}(v_1 v_3) = B, \quad \operatorname{vol}(v_2 v_3) = A\,,$$
$$\operatorname{vol}(v_0 v_1 v_2) = \frac{ab}{2}, \quad \operatorname{vol}(v_0 v_1 v_3) = \frac{ac}{2}\,,$$
$$\operatorname{vol}(v_0 v_2 v_3) = \frac{bc}{2}, \quad \operatorname{vol}(v_1 v_2 v_3) = \frac{d}{2}\,.$$

We also have $m(P_{0i}, P'_{0i}) = 1$, and $d(P_{0i}, P'_{0i}) = \frac{1}{4}$ for $i = 1, 2, 3$, while

$$m(P_{12}, P'_{12}) = \frac{cC}{d}, \qquad d(P_{12}, P'_{12}) = -\, s\left(\frac{ab}{d}, \frac{cC}{d}\right) + \frac{1}{4},$$
$$m(P_{13}, P'_{13}) = \frac{bB}{d}, \qquad d(P_{13}, P'_{13}) = -\, s\left(\frac{ac}{d}, \frac{bB}{d}\right) + \frac{1}{4},$$
$$m(P_{23}, P'_{23}) = \frac{aA}{d}, \qquad d(P_{23}, P'_{23}) = -\, s\left(\frac{bc}{d}, \frac{aA}{d}\right) + \frac{1}{4}.$$

The desired formula now follows easily from Theorem 4.

Part III: Dedekind sum relations

10 Introduction

In this part, we use toric varieties to prove a law expressing the sum of two arbitrary Dedekind sums in terms of a third. This is seen to be a generalization of Rademacher's three-term law for Dedekind sums [Ra]. A consequence of our law is an n-term reciprocity law for Dedekind sums. The proofs of these results are based on the formula of Theorem 3, which relates the Todd class of toric varieties to Dedekind sums.

The Dedekind sum $s(p, q)$ for relatively prime integers p and q is defined by

$$s(p, q) = \sum_{i=1}^{q} \left(\left(\frac{i}{q}\right)\right)\left(\left(\frac{pi}{q}\right)\right),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & x \notin \mathbb{Z} \\ 0 & x \in \mathbb{Z}\,. \end{cases}$$

These sums first appeared in Dedekind's work on the eta-function, and since then have arisen in a variety of contexts, including the lattice point formula of Mordell [Mor], and the work of Hirzebruch and Zagier [HiZa], which connects them with signatures of quotient spaces. Dedekind sums may be characterized in many ways, including the reciprocity law

$$s(p, q) + s(q, p) = -\frac{1}{4} + \frac{1}{12}\left(\frac{p}{q} + \frac{q}{p} + \frac{1}{pq}\right),$$

which is due to Dedekind [Ded]. [RaGr] contains a number of elementary proofs of this law. Rademacher [Ra] found a three-term reciprocity law:

$$s(bc', a) + s(ca', b) + s(ab', c) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right),$$

where a, b, c are pairwise coprime and $aa' \equiv 1 \pmod b$, $bb' \equiv 1 \pmod c$, and $cc' \equiv 1 \pmod a$. It is easy to see that this is a generalization of Dedekind's two-term law.

We prove the following theorem which gives the sum of two arbitrary Dedekind sums in terms of a third.

Theorem 7. *Let $p, q, u, v \in \mathbb{N}$ with $(p, q) = (u, v) = 1$. Then*

$$s(p, q) + s(u, v) = s(pu' - qv', pv + qu) - \frac{1}{4} + \frac{1}{12}\left(\frac{q}{vt} + \frac{v}{tq} + \frac{t}{qv}\right),$$

where $t = pv + qu$ and u', v' are any integers which satisfy $uu' + vv' = 1$.

The special case of this formula with q and v relatively prime is equivalent to Rademacher's three-term law. (Given a, b, c pairwise coprime, set $q = a$, $v = b$, and find $p, q \in \mathbb{Z}$ such that $pa + qb = c$. The equation of the corollary then gives Rademacher's formula.)

In order to state the n-term law, let M be a two-dimensional lattice and let $\det \in \Lambda^2 M$ denote one of the two possible choices of a determinant on M (so that $\det(e_1, e_2) = \pm 1$ whenever $\{e_1, e_2\}$ is a basis of M).

Given $m_1, m_2 \in M$, we say that the pair $\langle m_1, m_2 \rangle$ has *type* (p, q) if there exists a basis $\{e_1, e_2\}$ of M such that

$$m_1 = e_1 \quad \text{and}$$

$$m_2 = pe_1 + qe_2 .$$

In this case, q is determined up to sign, and p is determined modulo q.

We then have

Theorem 8. *Let $m_1, m_2, \ldots, m_n$ be distinct primitive elements of M such that $\det(m_i, m_{i+1}) > 0$ for $i = 1, \ldots, n$. (For notational purposes set $m_{n+1} = m_1$, and $m_{n+2} = m_2$.) If necessary, reorder the m_i so that for any $i, j \in \{1, \ldots, n\}$ we have $\det(m_i, m_j) \leqq 0$ or $\det(m_j, m_{i+1}) \leqq 0$. (This is to insure that the m_i "go around" M exactly once.) Suppose that $\langle m_i, m_{i+1} \rangle$ has type (p_i, q_i). Then*

$$\sum_{i=1}^{n} s(p_i, q_i) = 1 - \frac{n}{4} + \frac{1}{12}\sum_{i=1}^{n} \frac{\det(m_i, m_{i+2})}{\det(m_i, m_{i+1})\det(m_{i+1}, m_{i+2})}.$$

It is not hard to prove this theorem by induction using Theorem 7.

Hirzebruch and Zagier [HiZa] were the first to use geometric techniques to prove results about Dedekind sums. By considering "signature defects" of certain four-dimensional quotient spaces, they were able to derive Rademacher's three-term law and other facts about Dedekind sums.

11 Proof of the generalized three-term law

In Sect. 11.1, we present background information about toric varieties necessary for the proofs in Sect. 11.2. As noted, Theorem 8 is a straightforward consequence of Theorem 7. However, a direct proof of Theorem 8 is more natural and no more difficult than a proof of Theorem 7. Thus we choose to prove Theorem 8 first, and derive Theorem 7 as an easy corollary.

11.1 Toric variety facts

Each collection $\Gamma = \{m_1, m_2, \ldots, m_n\}$ of elements of M as in Theorem 8 form the set of rays of a unique complete fan in M, and hence determines a two-dimensional toric variety X_Γ. To each m_i there is an associated divisor $V(m_1) \subset X_\Gamma$ (cf. [Oda, Sects. 1.2 and 1.3]). The two main facts we'll need about X_Γ are:

(1) The codimension two part of the Todd class of X_Γ is given by

$$\mathrm{Td}^2 X_\Gamma = 1$$

under the identification $(A^2 X_\Gamma)_{\mathbb{Q}} \simeq \mathbb{Q}$. This is because toric varieties are rational. See [Oda, Sect. 1.2].

(2) Theorem 3 in this context becomes:

$$\mathrm{Td}^2 X_\Gamma = \mathrm{TD}^2 X_\Gamma + \sum_{i=1}^{n} \left(s(p_i, q_i) - \frac{1}{4q_i} + \frac{1}{4} \right),$$

where

$$\mathrm{TD}^2 X_\Gamma = \frac{1}{12}\left[\left(\sum_{i=1}^{n} [V(m_i)] \right)^2 + \sum_{i=1}^{n} \frac{1}{q_i} \right].$$

Here $[V(m_i)]$ denotes the class of the divisor $V(m_i)$ in $(A^1 X_\Gamma)_{\mathbb{Q}}$.

We will also need to know how to multiply the special cycles. For this, we use the description of the Chow ring of a toric variety given in [Dan, p. 127] (or see Sect. 2.3). We have:

$$(*) \qquad [V(m_i)][V(m_{i+1})] = \frac{1}{\det(m_i, m_{i+1})}, \quad \text{and}$$

$$(**) \qquad [V(m_i)]^2 = -\frac{\det(m_i, m_{m+2})}{\det(m_i, m_{i+1})\det(m_{i+1}, m_{i+2})}$$

while all other products $[V(m_i)][V(m_j)]$ vanish.

11.2 Proof of the n-term law

It is now quite easy to prove Theorems 7 and 8. By (1) and (2), we get

$$1 = \frac{1}{12}\left[\left(\sum_{i=1}^{n} [V(m_i)] \right)^2 + \sum_{i=1}^{n} \frac{1}{q_i} \right] + \sum_{i=1}^{n} \left(s(p_i, q_i) - \frac{1}{4q_i} + \frac{1}{4} \right).$$

Thus it follows that

$$(3)\qquad \sum_{i=1}^{n} s(p_1, q_i) = 1 - \frac{n}{4} - \frac{1}{12}\left(\sum_{i=1}^{n} [V(m_i)]\right)^2 + \frac{1}{6}\sum_{i=1}^{n}\frac{1}{q}.$$

By (*),

$$\left(\sum_{i=1}^{n} [V(m_i)]\right)^2 = 2\sum_{i=1}^{n}\frac{1}{q_i} + \sum_{i=1}^{n} [V(m_i)]^2$$

$$= 2\sum_{i=1}^{n}\frac{1}{q_i} - \sum_{i=1}^{n}\frac{\det(m_i, m_{m+2})}{\det(m_i, m_{i+1})\det(m_{i+1}, m_{i+2})} \quad \text{by } (**)\,.$$

Putting this into (3), we get the equation of Theorem 8.

To prove Theorem 7, set $n = 3$ and take $m_1 = (p, q)$, $m_2 = (-1, 0)$, and $m_3 = (u, -v)$ in the lattice $\mathbb{Z}^2$. One then computes that

$$\langle m_1, m_2\rangle \text{ has type } (-p, q)\,,$$

$$\langle m_2, m_3\rangle \text{ has type } (-u, v)\,, \quad \text{and}$$

$$\langle m_3, m_1\rangle \text{ has type } (pu' - qv', pv + qu)\,.$$

Theorem 7 now follows.

Acknowledgements. I would like to express great thanks to William Fulton for his advice and encouragement in the preparation of this paper. I would also like to thank Burt Totaro for suggesting an improvement in the statement of Theorem 4, and Stavros Garoufalidis, who introduced me to Dedekind sums.

References

[Dan] Danilov, V.I.: The geometry of toric varieties. Russ. Math. Surv. **33:2**, 97–154 (1978)

[Ehr] Ehrhart, E.: Sur un problème de géométrie diophantine linéaire. J. Reine Angew. Math. **227**, 1–29 (1967)

[Ful] Fulton, W.: Intersection theory. Berlin Heidelberg New York: Springer 1984

[Ham] Hammer, J.: Unsolved problems concerning lattice points. London San Francisco Melbourne: Pitman 1977

[Hir] Hirzebruch, F.: Topological methods in algebraic geometry. Berlin Heidelberg New York: Springer 1966

[HiZa] Hirzebruch, F., Zagier, D.: The Atiyah-Singer index theorem and elementary number theory. Berkeley: Publish or Perish 1974

[Mo] Mordell, L.J.: Lattice points in a tetrahedron and generalized Dedekind sums. J. Indian Math. **15**, 41–46 (1951)

[Mor] Morelli, R.: Pick's theorem and the Todd class of a toric variety (to appear)

[My] Myerson, G.: On semi-regular continued fractions. Arch. Math. **48**, 420–425 (1987)

[Oda] Oda, T.: Convex bodies and algebraic geometry. Berlin Heidelberg New York: Springer 1987

[Ra] Rademacher, H.: Generalization of the reciprocity formula for Dedekind sums. Duke Math. J. **21**, 391–397 (1954)

[RaGr] Rademacher, H., Grosswald, E.: Dedekind sums. (Carus Math. Monogr. no. 16) Washington: Mathematical Association of America 1972

[Ree] Reeve, J.E.: On the volume of lattice polyhedra. Proc. Lond. Math. Soc., III. Ser. **7**, 378–395 (1957)