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Toric varieties, monoid schemes and cdh descent

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Abstract. We give conditions for the Mayer–Vietoris property to hold for the algebraic K-theory of blow-up squares of toric varieties and schemes, using the theory of monoid schemes. These conditions are used to relate algebraic K-theory to topological cyclic homology in characteristic p. To achieve our goals, we develop many notions for monoid schemes based on classical algebraic geometry, such as separated and proper maps and resolution of singularities.

0. Introduction

The goal of this paper is to prove Haesemeyer's theorem [19], [7, Theorem 3.12] for toric schemes in any characteristic. It is proven below as Corollary 14.4.

Theorem 0.1. *Assume k is a commutative regular noetherian ring containing an infinite field and let $\mathcal{G}$ be a presheaf of spectra defined on the category of schemes of finite type over k. If $\mathcal{G}$ satisfies the Mayer–Vietoris property for Zariski covers, finite abstract blow-up squares, and blow-ups along regularly embedded closed subschemes, then $\mathcal{G}$ satisfies the Mayer–Vietoris property for all abstract blow-up squares of toric k-schemes obtained from subdividing a fan.*

The application we have in mind is to understand the relationship between the algebraic K-theory $K_*(X) = \pi_*\mathcal{K}(X)$ and topological cyclic homology $TC_*(X) = \{\pi_* TC^\nu(X, p)\}$ of a toric scheme over a regular ring of characteristic p (and in particular of toric varieties over a field of characteristic p). Thus we consider the presheaf of homotopy fibers $\{\mathcal{F}^\nu(X)\}$ of the map of pro-spectra from $\mathcal{K}(X)$ to $\{TC^\nu(X, p)\}$. Work of Geisser–Hesselholt ([11] and [12, Theorem B]) shows that this homotopy fiber (regarded as a pro-presheaf of spectra) satisfies the hypotheses of Theorem 0.1 and hence a slight modification of the proof of our theorem implies that it satisfies the Mayer–Vietoris property for all abstract blow-up squares of toric schemes. We will give a rigorous proof of this in Corollary 14.8 below.

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One major tool in our proof will be a theorem of Bierstone–Milman [1] which says that the singularities of a toric variety (or scheme) can be resolved by a sequence of blow-ups $X_C \to X$ along a center C that is a smooth, equivariant closed subscheme of X along which X is normally flat. If one only had to consider toric schemes, this would allow one to use Haesemeyer's original argument to prove Theorem 0.1, since toric schemes over a regular ring are normal and Cohen–Macaulay. However, examples show that the blow-up of a toric scheme along a smooth center (even a point) can be non-normal. Thus, even starting with a toric scheme, the tower of blow-ups constructed by Bierstone–Milman will often involve non-normal schemes with a torus action. The proof of our theorem requires us to work with a larger class of schemes, one containing all the schemes in this tower. Beyond this, we need a class of schemes which is closed under passage to (possibly non-reduced) equivariant closed subschemes, pullbacks and blow-ups.

It turns out that all these operations may be lifted to the category of *monoid schemes* of finite type, and that the realizations of monoid schemes over a commutative regular ring k containing a field form a class of schemes with the above-mentioned properties. The k-realization of an affine monoid scheme is a scheme of the form $\operatorname{Spec} k[A]$, with A an abelian monoid; the k-realization of a monoid scheme (Definition 5.3) is a scheme over k which is covered by affine open subschemes of this form, with homomorphisms of the underlying monoids inducing the gluing maps between these open subschemes.

To achieve our goals, it is easier to work directly with the category of monoid schemes, and Sections 1–3 of this paper are devoted to a introduction to monoid schemes. Toric monoid schemes are introduced in Section 4 and the relation to toric varieties is carefully described. In Sections 5 and 6, we prove that the k-realization functor preserves limits and show that many monoid scheme-theoretic properties translate well into algebraic geometry. Projective monoid schemes, blow-ups and proper maps are introduced in Sections 7 and 8. After introducing the technical notion of pctf monoid schemes in Section 9, birational maps and resolution of singularities are given in Sections 10 and 11.

The last part of this paper (Sections 12–14) is devoted to the notion of cohomological descent (Definition 12.11), the proof of our Main Theorem 0.1 and its application to algebraic K-theory and topological cyclic homology.

As far as the authors are aware, this paper presents the first attempt at a systematic study of geometric properties of monoid schemes within the category of monoid schemes, and the relationship of these with the geometric properties of their realizations. The idea of a monoid scheme itself goes back at least to Kato [25], and general definitions were given by Deitmar in [8] and (under the name $\mathcal{M}_0$-schemes) by Connes, Consani and Marcolli in [4]. Deitmar studies notions of flatness and étaleness for monoid schemes, and introduces discrete valuation monoids. New in this paper is our systematic investigation of separatedness, properness, general valuation monoids and the valuative criteria, projectivity and blowing up, and the introduction of a class of monoid schemes (the above mentioned pctf monoid schemes) with better formal properties than only those given by fans, yet avoiding the worst pathologies of non-cancellative monoids.

1. Monoids

Since we know of no suitable reference for the facts we need concerning monoids and their prime spectra, we begin with a short exposé of this basic material.

Unless otherwise stated, a *monoid* in this paper is a pointed abelian monoid, i.e., an abelian monoid object in the symmetric monoidal category of pointed sets with smash product as monoidal product. More explicitly, a monoid is a pointed set A with basepoint 0, equipped with a pairing $\mu : A \wedge A \to A$ (written $\mu(a,b) = ab$) that is associative and commutative and has an identity element 1. The basepoint is unique because it is characterized by the property that $0a = 0$ for all $a \in A$. For example, if R is a commutative ring, then forgetting addition gives a monoid $(R, \times)$ of this type. Sometimes $+$ notation is used for μ, for example in applications to toric varieties; in these cases we write 0 for the identity element, and ∞ for the basepoint.

We can convert any unpointed abelian monoid B into a pointed abelian monoid B_* by adjoining a basepoint. Neither the zero monoid $\{0\}$ nor the monoid $\{0, 1, t\}$ with $t^2 = 0$ are of this form.

A morphism of monoids is a map of pointed sets preserving the multiplicative identity and multiplication. The initial monoid is $S^0 = \{0, 1\}$ with $1 \cdot 1 = 1$, and the initial map $\iota_A : S^0 \to A$ is such that the identity on A equals the composition

$$A \xrightarrow{\cong} S^0 \wedge A \xrightarrow{\iota_A \wedge \mathrm{id}} A \wedge A \xrightarrow{\mu} A.$$

Localization. A *multiplicatively closed* subset $S \subset A$ is a subset containing 1 and closed under multiplication. Given a multiplicatively closed subset S of A, the *localization* $S^{-1}A$ consists of equivalence classes of fractions of the form $\frac{a}{s}$ with $a \in A$ and $s \in S$. As usual, $\frac{a}{s} = \frac{a'}{s'}$ if and only if $as's'' = a'ss''$ for some $s'' \in S$, and the operation in $S^{-1}A$ is given by multiplication of fractions. There is a canonical monoid homomorphism $A \to S^{-1}A$ sending a to $\frac{a}{1}$, and $a, b \in A$ are mapped to the same element of $S^{-1}A$ if and only if $as = bs$ for some $s \in S$.

An *ideal* I in a monoid A is a pointed subset such that $AI \subseteq I$. If $I \subset A$ is an ideal, A/I is the monoid obtained by collapsing I to 0 – i.e., it is canonically isomorphic to $(A \setminus I) \cup \{0\}$ with the unique multiplication rule that makes the canonical surjection $A \twoheadrightarrow A/I$ into a morphism of monoids. More generally, any surjective homomorphism of monoids $A \to B$ is the quotient by a *congruence*, i.e., an equivalence relation compatible with the monoid operation.

Every nonzero monoid A has a unique maximal ideal (written $\mathfrak{m}_A$), namely the complement of the submonoid of units

$$U(A) := \{a \in A : ab = 1 \text{ for some } b\}.$$

We say that a monoid morphism $g : A \to B$ is *local* if $g(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$ or, equivalently, if $g^{-1}(U(B)) \subseteq U(A)$.

A *prime ideal* is a proper ideal $\mathfrak{p}$ ($\mathfrak{p} \neq A$) whose complement $S = A \backslash \mathfrak{p}$ is closed under multiplication; in this case we write $A_\mathfrak{p}$ for the localization $S^{-1}A$. The *dimension* of A is the supremum of the lengths of all chains of prime ideals, and the *height* of $\mathfrak{p}$ is the dimension of $A_\mathfrak{p}$. Since the intersection of an arbitrary chain of primes is prime, every prime ideal contains a minimal prime ideal (by Zorn's lemma).

Lemma 1.1. *For every multiplicatively closed subset S of A with $0 \notin S$, there is a prime ideal $\mathfrak{p}$ of A such that $S^{-1}A = A_\mathfrak{p}$.*

Proof. Since $S^{-1}A$ is a nonzero monoid, it has a maximal (proper) ideal $\mathfrak{m}$; the inverse image of $\mathfrak{m}$ in A is a prime ideal $\mathfrak{p}$. Let T denote $A \setminus \mathfrak{p}$; then $S \subset T$ and any $t \in T$ is a unit in $S^{-1}A$. Hence there are homomorphisms $S^{-1}A \to T^{-1}A = A_\mathfrak{p}$ and $T^{-1}A \to S^{-1}A$ covering the identity of A. Hence both composites $S^{-1}A \to S^{-1}A$ and $A_\mathfrak{p} \to A_\mathfrak{p}$ are identity maps, by the universal property of localization. □

We let $\mathrm{MSpec}(A)$ denote the set of prime ideals of A; it is a topological space when equipped with its Zariski topology, in which closed subsets are those of the form

$$V(I) = \{\mathfrak{p} : I \subset \mathfrak{p}\}$$

for an ideal I of A. The principal open subsets

$$D(s) = \{\mathfrak{p} \in \mathrm{MSpec}(A) : s \notin \mathfrak{p}\} = \mathrm{MSpec}(A[1/s])$$

form a basis for the Zariski topology. The space $\mathrm{MSpec}(A)$ is quasi-compact, since any open $D(s)$ containing the unique maximal ideal $\mathfrak{m}_A$ must have $D(s) = \mathrm{MSpec}(A)$.

There is a sheaf of monoids $\mathcal{A}$ on $\mathrm{MSpec}(A)$ whose stalk at $\mathfrak{p}$ is $A_\mathfrak{p}$; if U is open, then $\mathcal{A}(U)$ is the subset of $\prod_{\mathfrak{p}\in U} A_\mathfrak{p}$ consisting of elements which locally come from some $S^{-1}A$. Explicitly,

$$\mathcal{A}(U) = \left\{ a \in \prod_{\mathfrak{p}\in U} A_\mathfrak{p} : (\forall \mathfrak{p} \in U)(\exists s \notin \mathfrak{p}, x \in A)(\forall \mathfrak{q} \in U) s \notin \mathfrak{q} \Rightarrow a_\mathfrak{q} = \frac{a}{s} \right\}.$$

In particular, $A = \mathcal{A}_{\mathfrak{m}_A}$, and $\mathcal{A}(D(s)) = A[1/s]$. More generally any ideal I of A determines a sheaf $\mathcal{I}$ on $\mathrm{MSpec}\, A$ by

$$\mathcal{I}(U) = \{a \in \mathcal{A}(U) : (\forall \mathfrak{p} \in U)\ \ a_\mathfrak{p} \in A_\mathfrak{p} \cdot I\}$$

Example 1.2. The free (abelian) pointed monoid on the set $\{t_1, \dots, t_n\}$ is the multiplicative monoid F_n consisting of all monomials in the polynomial ring $\mathbb{Z}[t_1, \dots, t_n]$ (together with 0). Each of the 2^n subsets of $\{t_1, \dots, t_n\}$ generates a prime ideal $\mathfrak{p}$, and every prime ideal of F_n has this form. We write $\mathbb{A}^n$ for $\mathrm{MSpec}(F_n)$.

If $A \to B$ is a morphism of monoids, then the inverse image of a prime ideal is a prime ideal, and we have a continuous map $\mathrm{MSpec}(B) \to \mathrm{MSpec}(A)$. If I is an ideal of A, then $\mathrm{MSpec}(A/I) \to \mathrm{MSpec}(A)$ is a closed injection onto $V(I)$. If S is multiplicatively closed in A, then either $S^{-1}A = 0$ (in which case $\mathrm{MSpec}\, S^{-1}A = \emptyset$) or $S^{-1}A = A_\mathfrak{p}$ for some $\mathfrak{p}$ (Lemma 1.1); in either case $\iota : \mathrm{MSpec}(S^{-1}A) \to \mathrm{MSpec}(A)$ is an injection onto the set of primes that are disjoint from S. The restriction $\iota^{-1}(\mathcal{A})$ to this subset is the sheaf of monoids on $\mathrm{MSpec}(A_\mathfrak{p})$.

Recall that a point x_1 of a topological space X is called *generalization* of a point x_0 (and x_0 is called a *specialization* of x_1) if x_0 is in the closure of x_1. For example, if $\mathfrak{p}, \mathfrak{q} \in \mathrm{MSpec}\, A$, then $\mathfrak{p}$ generalizes $\mathfrak{q}$ if and only if $\mathfrak{p} \subset \mathfrak{q}$.

Lemma 1.3. *Let $\mathfrak{p}$ be a prime ideal in a monoid A. Then* $\mathrm{MSpec}(A_\mathfrak{p}) \to \mathrm{MSpec}(A)$ *is an injection, closed under generalization, and the following are equivalent:*

(i) $\mathrm{MSpec}(A_\mathfrak{p})$ *is open in* $\mathrm{MSpec}(A)$.

(ii) $\mathrm{MSpec}(A_\mathfrak{p}) = D(s)$ *for some $s \in A$.*

(iii) *There is an $s \in A$ such that $A_\mathfrak{p} = A[1/s]$.*

Proof. The first assertion was observed above. Since $D(s) = \mathrm{MSpec}(A[1/s])$, (iii) is equivalent to (ii), a special case of (i). Conversely, suppose that $U = \mathrm{MSpec}(A_{\mathfrak{p}})$ is the complement of $V(I)$ for some ideal I of A. Then $U = \bigcup_{s \in I} D(s)$. In particular, there is an s in I such that $\mathfrak{p} \in D(s)$. But then $U \subseteq D(s)$ and hence $U = D(s)$. □

Example 1.4. Let A be the free pointed abelian monoid generated by the infinite set $\{t_1, t_2, \dots\}$. If $\mathfrak{p}$ is the prime ideal generated by some finite subset of the elements t_i, then $\mathrm{MSpec}(A_{\mathfrak{p}})$ cannot be open in $\mathrm{MSpec}(A)$. Indeed, if it were open, then by Lemma 1.3 it would have the form $D(s)$ for some element $s \in A$. But any s involves only a finite number of variables, so the prime ideal $t_j A$ belongs to $D(s)$ for infinitely many $t_j \notin \mathfrak{p}$. In particular, $D(s)$ cannot be contained in MSpec $A_{\mathfrak{p}}$.

Lemma 1.5. *If A is finitely generated as a monoid, then* $\mathrm{MSpec}(A)$ *is a finite partially ordered set. If S is a multiplicative subset of A, then $S^{-1}A$ is also finitely generated, and* $\mathrm{MSpec}(S^{-1}A)$ *is open in* $\mathrm{MSpec}(A)$.

Proof. Suppose A is generated by $x_1, \dots, x_m$. Then for any prime ideal $\mathfrak{p}$, the multiplicative subset $S = A \setminus \mathfrak{p}$ is generated by $\{x_i : x_i \notin \mathfrak{p}\}$. Indeed, if $s \in S$, then $s = \prod_i x_i^{e_i}$ with $e_i = 0$ whenever $x_i \in \mathfrak{p}$. Thus A has at most 2^m prime ideals.

By Lemma 1.1, we may assume $S = A \setminus \mathfrak{p}$ for some prime $\mathfrak{p}$. If s is the product of the generators of S, then $A_{\mathfrak{p}} = A[1/s]$. By Lemma 1.3, $\mathrm{MSpec}(A_{\mathfrak{p}})$ is open. □

We say A is *cancellative* if for $a, b, c \in A$ the conditions $ab = ac$ and $a \neq 0$ together imply that $b = c$. In this case, the unpointed monoid $A \setminus \{0\}$ injects into its group completion and $\{0\}$ is the unique minimal prime ideal of A. We define the *pointed group completion* of A to be the pointed monoid A^+ obtained by adjoining a basepoint to the usual group completion of the unpointed monoid $A \setminus \{0\}$. Note that A is a pointed submonoid of A^+, and that A^+ is the localization $A_{\{0\}}$ of A at the minimal prime ideal.

We say A is *torsionfree* if whenever $a^n = b^n$ for $a, b \in A$ and some $n \geq 1$, we have $a = b$. The monoid $\{0, \pm 1\}$ is cancellative but not torsionfree. If A is cancellative and $A^+ \setminus \{0\}$ is a torsionfree abelian group, then A is torsionfree.

An element is *nilpotent* if $a^n = 0$ for some n, and the *nilradical* of A is the set $\mathrm{nil}(A)$ of nilpotent elements. It is easy to prove (using Zorn's lemma as in ring theory) that $\mathrm{nil}(A)$ is the intersection of the minimal prime ideals of A. We say that A is *reduced* if $\mathrm{nil}(A) = 0$, and set $A_{\mathrm{red}} = A/\mathrm{nil}(A)$.

Any closed subset Z of $X = \mathrm{MSpec}(A)$ defines a largest ideal I such that $Z = V(I)$, and A/I is a reduced monoid. Indeed, if $Z = V(I_0)$, then $A/I = (A/I_0)_{\mathrm{red}}$; I is the intersection of the prime ideals containing I_0. Anticipating Lemma 2.9, we write $\bar{Z}^{\mathrm{eq}}$ for $\mathrm{MSpec}(A/I)$ and call it the *equivariant closure* of Z in X. For example, $\bar{X}^{\mathrm{eq}}$ is $\mathrm{MSpec}(A_{\mathrm{red}})$. Another important special case is when $Z = \{\mathfrak{p}_1, \dots, \mathfrak{p}_l\}$ is a set of prime ideals of A; in this case $\bar{Z}^{\mathrm{eq}} = \mathrm{MSpec}(A/\cap \mathfrak{p}_i)$.

Definition 1.6. The *normalization* of a cancellative monoid A is defined to be the submonoid

$$A_{\mathrm{nor}} = \{\alpha \in A^+ : \alpha^n \in A \text{ for some } n \geq 1\}$$

of A^+. We say that A is *normal* if it is cancellative and $A = A_{\mathrm{nor}}$. The normalization of $S^{-1}A$ is $S^{-1}A_{\mathrm{nor}}$. If A is torsionfree, then so is A_{nor}.

Remark 1.6.1. If A is cancellative, then $\mathrm{MSpec}(A_{\mathrm{nor}}) \to \mathrm{MSpec}(A)$ is a topological homeomorphism. Indeed, if $\mathfrak{p}$ is a prime ideal of A, then $\mathfrak{p}_{\mathrm{nor}} := \{b \in A_{\mathrm{nor}} : (\exists n) b^n \in \mathfrak{p}\}$ is a prime ideal of A_{nor} and $\mathfrak{p} = \mathfrak{p}_{\mathrm{nor}} \cap A$. It is easily seen that every prime ideal of A_{nor} has the form $\mathfrak{p}_{\mathrm{nor}}$ for some $\mathfrak{p}$.

Remark 1.6.2. If A is normal and $\mathfrak{p}$ is a prime ideal, then $A/\mathfrak{p}$ is also normal. Indeed, if $x, y \in A$ and $s \in A \setminus \mathfrak{p}$ are such that x^n and $s^n y$ are mapped to the same element of $A/\mathfrak{p}$, then either $x^n = s^n y$ in A or $x, y \in \mathfrak{p}$. Since A is assumed normal, it follows that either $x \in \mathfrak{p}$ or there is a $z \in A$ such that $x = sz$ in A.

More generally, let $f : A \to B$ be a morphism of monoids. We say that f is *integral* if for every $b \in B$ there is an integer $n \geq 1$ such that b^n lies in the image of A, and we say that f is *finite* if there exist $b_1, \dots, b_n \in B$ ($n \geq 1$) such that $B = \bigcup_i Ab_i$. The normalization $A \to A_{\mathrm{nor}}$ is integral but not always finite.

Lemma 1.7. *Let $A \xrightarrow{f} B$ be a monoid morphism with B finitely generated over A.*

(i) *If f is integral, then f is finite.*

(ii) *If f is finite and B is cancellative, then f is integral.*

Proof. Choose a surjection $A[t_1, \dots, t_n] \twoheadrightarrow B$, with the t_i mapping onto generators b_i of B over A. If f is integral, then there is an m such that b_i^m is in the image of A for all i; thus every element of B can be written as a product $f(a)c_j$, where $a \in A$ and c_j is a monomial on the b_i with exponents $\leq m$. This proves (i).

Next assume that f is finite and that B is cancellative. Let $b_1, \dots, b_n \in B$ be such that $B = \bigcup_i Ab_i$. For each i, we choose an index $\pi(i)$ and $a_i \in A$ such that $b_i^2 = a_i b_{\pi(i)}$; then π is a map from the finite set $\{1, \dots, n\}$ to itself. For each fixed i, the iterates $\pi^r(i)$ cannot all be distinct, so there exist $s \geq 1$ and $r \geq 1$ such that $j = \pi^r(i)$ satisfies $\pi^s(j) = j$. Hence there is an $a \in A$ and $m \geq 1$ such that $b_j^m = ab_j$. Because B is cancellative, this implies that $b_j^{m-1} = f(a)$. Thus b_j and hence b_i is integral over A, as required. □

Remark 1.7.1. The hypothesis that B be cancellative in part (ii) of Lemma 1.7 is necessary. For example, the monoid B generated by x, y subject to $y^2 = xy$ contains the free monoid A generated by x; the extension $A \subset B$ is finite but not integral.

For a pointed set X and commutative ring k, $k[X]$ denotes the free k-module on X, modulo the summand indexed by the base point of X. If A is a pointed monoid, $k[A]$ is a ring in the usual way, with multiplication given by the product rule for A. If B is an unpointed monoid, $k[B_*]$ coincides with the usual monoid ring for B with k coefficients. If I is an ideal of the monoid A, then $k[I]$ is an ideal of the ring $k[A]$, and $k[A/I] = k[A]/k[I]$. If I is prime, $k[I]$ need not be a prime ideal.

The category of pointed monoids has all small colimits. For example, the coproduct of A_1 and A_2 is the smash product $A_1 \wedge A_2$; the maps from A_1 and A_2 to $A_1 \wedge A_2$ send a_1 to $a_1 \wedge 1$ and a_2 to $1 \wedge a_2$. The functor $A \mapsto k[A]$ preserves colimits since it has a right adjoint, sending an algebra R to $(R, \times)$, the underlying multiplicative monoid of R; in particular, the natural map

$$k[A_1] \otimes_k k[A_2] \to k[A_1 \wedge A_2]$$

is an isomorphism. More generally, the pushout $A_1 \wedge_C A_2$ of a diagram

(1.8)
$$\begin{array}{ccc} C & \xrightarrow{f} & A_2 \\ {\scriptstyle g}\downarrow & & \downarrow \\ A_1 & \dashrightarrow & A_1 \wedge_C A_2 \end{array}$$

is the quotient of $A_1 \wedge A_2$ by the congruence generated by $(a_1 f(c), a_2) \sim (a_1, g(c)a_2)$. Note that $k[A_1 \wedge_C A_2] \cong k[A_1] \otimes_{k[C]} k[A_2]$.

Lemma 1.9. *Every prime ideal $\mathfrak{p}$ of $A_1 \wedge A_2$ has the form $\mathfrak{p}_1 \wedge A_2 \cup A_1 \wedge \mathfrak{p}_2$ for unique prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$. Explicitly, $\mathfrak{p}_i$ is the inverse image of $\mathfrak{p}$ under the canonical inclusion $A_i \to A_1 \wedge A_2$.*

Proof. Given a prime ideal $\mathfrak{p}$ of $A_1 \wedge A_2$, set

$$\mathfrak{p}_1 = \mathfrak{p} \cap A_1, \quad \mathfrak{p}_2 = \mathfrak{p} \cap A_2 \quad \text{and} \quad \mathfrak{q} = \mathfrak{p}_1 \wedge A_2 \cup A_1 \wedge \mathfrak{p}_2.$$

Then $\mathfrak{q}$ is prime because its complement is $(A_1 \backslash \mathfrak{p}_1) \times (A_2 \backslash \mathfrak{p}_2)$, which is multiplicatively closed. Clearly $\mathfrak{q} \subseteq \mathfrak{p}$; to see that $\mathfrak{q} = \mathfrak{p}$, consider an element $a_1 \wedge a_2$ of $\mathfrak{p}$. As $\mathfrak{p}$ is prime, either $a_1 \wedge 1$ or $1 \wedge a_2$ is in $\mathfrak{p}$. In the first case, $a_1 \in \mathfrak{p}_1$ so $a_1 \wedge a_2$ is in $\mathfrak{p}_1 \wedge A_2 \subseteq \mathfrak{q}$; in the second case, $a_2 \in \mathfrak{p}_2$ so $a_1 \wedge a_2$ is in $A_1 \wedge \mathfrak{p}_2 \subseteq \mathfrak{q}$. □

Example 1.10. If T is the free monoid on one element t, then $A \wedge T$ is the analogue of a polynomial ring over A, and $k[A \wedge T] = k[A][t]$. For any prime ideal $\mathfrak{p}$ of A there are exactly two primes of $A \wedge T$ over $\mathfrak{p}$: the extended prime $\mathfrak{p} \wedge T$ and the prime generated by $\mathfrak{p}$ and t (i.e., $\mathfrak{p} \wedge T \cup A \wedge \{t^n : n \geq 1\}$). The map $\operatorname{MSpec}(A \wedge T) \to \operatorname{MSpec}(A)$ induced by the canonical inclusion $A \to A \wedge T$ is both open and closed, because the image of $D(at^n)$ is $D(a)$ and the image of $V(I)$ is $V(I \cap A)$.

Proposition 1.11. *Given a pushout diagram* (1.8)*, every prime ideal of $A_1 \wedge_C A_2$ has the form $\mathfrak{p}_1 \wedge A_2 \cup A_1 \wedge \mathfrak{p}_2$ for unique prime ideals $\mathfrak{p}_1$ in A_1, $\mathfrak{p}_2$ in A_2.*

Moreover, the ideal $\mathfrak{p}_1 \wedge A_2 \cup A_1 \wedge \mathfrak{p}_2$ of $A_1 \wedge_C A_2$ is prime if and only if $\mathfrak{p}_1$ and $\mathfrak{p}_2$ have a common inverse image in C.

Proof. If $\mathfrak{p}$ is a prime in $A_1 \wedge_C A_2$, its inverse image in $A_1 \wedge A_2$ is prime; by Lemma 1.9 it has the form $\mathfrak{p}_1 \wedge A_2 \cup A_1 \wedge \mathfrak{p}_2$, where $\mathfrak{p}_i \subset A_i$ are the inverse images of $\mathfrak{p}$. Since $A_1 \wedge_C A_2$ is a quotient, this proves the first assertion; because (1.8) commutes, $\mathfrak{p}_1$ and $\mathfrak{p}_2$ have a common inverse image in C.

Conversely, suppose that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ have a common inverse image $\mathfrak{q}$ in C, and set

$$S_1 = A_1 \backslash \mathfrak{p}_1, \quad S_2 = A_2 \backslash \mathfrak{p}_2 \quad \text{and} \quad I = \mathfrak{p}_1 \wedge A_2 \cup A_1 \wedge \mathfrak{p}_2 \subset A_1 \wedge_C A_2.$$

To see that the ideal I is prime, it suffices to show that the image of $S_1 \times S_2$ in $A_1 \wedge_C A_2$ is disjoint from I. Since $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime, $a_1 f(c) \in S_1$ if and only if $a_1 \in S_1$ and $c \notin \mathfrak{q}$, while $g(c)a_2 \in S_2$ if and only if $a_2 \in S_2$ and $c \notin \mathfrak{q}$. It follows that $(a_1 f(c), a_2)$ is in $S_1 \times S_2$ if and only if $(a_1, g(c)a_2)$ is. Thus $S_1 \times S_2$ is closed under the equivalence relation defining $A_1 \wedge_C A_2$, and its image in $A_1 \wedge_C A_2$ is disjoint from I. □

2. Monoid schemes

We will need to consider *monoid schemes*, sometimes known as "schemes over the field with one element". These are the objects which result by gluing together spectra of pointed monoids along open subsets, and will be related to classical schemes in Section 5. The theory of monoid schemes was developed by Kato [25], Deitmar [8], Connes–Consani–Marcolli [4–6], etc. The survey [26] by López Peña and Lorscheid gives a nice overview of this notion and related ideas (but see Remark 4.4.1 below).

A *monoid space* is a pair $(X, \mathcal{A}_X)$ consisting of a topological space X and a sheaf $\mathcal{A}_X$ of pointed abelian monoids on X. A *morphism of monoid spaces* from $(X, \mathcal{A}_X)$ to $(Y, \mathcal{A}_Y)$ is given by a continuous map $f : X \to Y$ together with a morphism of sheaves

$$f_\# : f^{-1}\mathcal{A}_Y \to \mathcal{A}_X$$

on X (or, equivalently, a morphism $f^\# : \mathcal{A}_Y \to f_*\mathcal{A}_X$ of sheaves on Y) that is *local* in the sense that the maps on stalks $\mathcal{A}_{Y,f(x)} \to \mathcal{A}_{X,x}$ are local morphisms of monoids, for all $x \in X$. By abuse of notation, we will often simply write X for the monoid space $(X, \mathcal{A}_X)$.

The association $A \mapsto \mathrm{MSpec}(A)$ extends to a fully faithful contravariant functor from monoids to monoid spaces, which we will call MSpec by abuse of notation. An *affine monoid scheme* is a monoid space isomorphic to $\mathrm{MSpec}(A)$ for some monoid A. A *monoid scheme* is a monoid space $(X, \mathcal{A})$ such that every point has an open neighborhood U such that $(U, \mathcal{A}|_U)$ is isomorphic to an affine monoid scheme. If $(U, \mathcal{A}|_U) \cong \mathrm{MSpec}\,A$, we shall often abuse notation and write $U = \mathrm{MSpec}\,A$. A morphism of monoid schemes is just a morphism of the underlying monoid spaces. The *dimension* of a monoid scheme is the largest dimension of its affine open neighborhoods.

Lemma 2.1. *Let $(X, \mathcal{A})$ be a monoid scheme. For any open $U \subseteq X$, the monoid space $(U, \mathcal{A}|_U)$ is a monoid scheme.*

The scheme $(U, \mathcal{A}|_U)$ is called the open subscheme *of X associated to U.*

Proof. If $x \in U$ and $V = \mathrm{MSpec}(A)$ is an affine open neighborhood of x in X, $U \cap V$ is also open. Since $U \cap V$ is the union of basic open subschemes $D(s)$ of V, x has a neighborhood of the form $D(s)$, and $D(s) = \mathrm{MSpec}(A[1/s])$ is affine. □

We say that a monoid scheme is cancellative (resp., reduced, normal, ...) if its stalks are cancellative monoids (resp., reduced, normal, ... monoids), or equivalently, if its monoids of sections are cancellative (resp., ...).

Example 2.2. The projective line $\mathbb{P}^1$ is obtained by gluing $\mathrm{MSpec}(\{t^n, n \geq 0\}_*)$ and $\mathrm{MSpec}(\{t^n, n \leq 0\}_*)$ along $\mathrm{MSpec}(\{t^n, n \in \mathbb{Z}\}_*)$. This monoid scheme is connected, torsion-free and normal.

Partial order, maximal and minimal points. Recall that the points of any topological space may be partially ordered by the relation that $x \leq y$ if and only if y is in the closure of $\{x\}$. In this way we can speak of maximal and minimal points. The maximal points are the closed points; minimal points are also called *generic* points. For the topological space $\mathrm{MSpec}(A)$ of a monoid A, we have $\mathfrak{p} \leq \mathfrak{q}$ if and only if $\mathfrak{p} \subseteq \mathfrak{q}$. Minimal points exist in any monoid scheme because, as noted before Lemma 1.1, every prime ideal contains a minimal prime ideal.

Lemma 2.3. *Each cancellative monoid scheme X decomposes as the disjoint union of (closed and open) monoid subschemes X_η, each the closure of a unique minimal point η of X. In particular, if X is connected, then it has a unique minimal point.*

Proof. For each minimal point $\eta \in X$, let X_η denote the closure of η in X. Given $x \in X$, choose an affine neighborhood $U_x = \mathrm{MSpec}(A)$ of x. If y is the point of X corresponding to $\mathfrak{m}_A$, then $A = \mathcal{A}_y$. Since A is cancellative, U_x has a unique minimal point η, so $U_x \subseteq X_\eta$. It follows that $X_\eta = \bigcup U_x$ is open (and closed) in X, and that X is the disjoint union of the subschemes X_η. □

Lemma 2.4. *Let X be a monoid scheme and $U \subseteq X$ an open subscheme. Then the following are equivalent.*

(i) *U is an affine monoid scheme.*

(ii) *U has a unique maximal point.*

If $X = \mathrm{MSpec}(A)$, every affine open subscheme is $\mathrm{MSpec}(A_\mathfrak{p})$ for some $\mathfrak{p}$.

Proof. Since monoids have unique maximal ideals, (i) implies (ii). Conversely, suppose that U has a unique maximal point x. Note that $U = \{y : y \le x\}$ by definition of the order relation. If $\mathrm{MSpec}(A)$ is an affine open neighborhood of x, then $U \subseteq \mathrm{MSpec}(A)$, so we may assume that $X = \mathrm{MSpec}(A)$. In this case $U = \mathrm{MSpec}(A_x)$ by Lemma 1.3. □

Definition 2.5. Let $f : Y \to X$ be a map of monoid schemes. We say that f is a *closed immersion* if it induces a homeomorphism of Y onto its image (equipped with the subspace topology), and for every affine open subscheme $U = \mathrm{MSpec}(A)$ of X:

(i) the open subscheme $V = U \times_X Y$ of Y is affine (possibly empty),

(ii) the map $\mathcal{A}_X(U) \to \mathcal{A}_Y(V)$ is surjective.

A *closed subscheme* of a monoid scheme X is an isomorphism class of closed immersions into X. Each closed subscheme is represented by a monoid scheme $(Z, \mathcal{A}_Z)$ such that f is a subspace inclusion $Z \subset X$.

A closed immersion $f : Y \to X$ is called *equivariant* if in addition each such map $\mathcal{A}_X(U) \to \mathcal{A}_Y(V)$ is the quotient by an ideal.

The terminology "equivariant closed immersion" comes from the theory of toric varieties: the equivariant closed subschemes of a toric variety are precisely those closed subschemes that are equivariant for the action of the underlying torus. We will see in Section 4 that a toric variety has an associated toric monoid scheme, and that the equivariant closed subschemes of the monoid scheme determine equivariant closed subschemes of the toric variety.

Example 2.6. Given a closed subset Z of a monoid scheme X, there is an equivariant reduced closed subscheme Z_{red} associated to Z, defined by patching; if $X = \mathrm{MSpec}(A)$ and $Z = V(I)$, then $Z_{\mathrm{red}} = \mathrm{MSpec}(A/I)_{\mathrm{red}}$.

Lemma 2.7. *Any surjection of monoids $A \xrightarrow{\pi} B$ determines a closed immersion*

$$\mathrm{MSpec}(B) \subseteq \mathrm{MSpec}(A).$$

If $B = A/I$, then it is an equivariant closed subscheme.

Proof. Set $Y = \mathrm{MSpec}(B)$ and $X = \mathrm{MSpec}(A)$. The map $\pi^* : Y \to X$ of underlying spaces is injective, since if $\mathfrak{q}_1 \neq \mathfrak{q}_2$, then $\pi^{-1}(\mathfrak{q}_1) \neq \pi^{-1}(\mathfrak{q}_2)$. If $a \in A$, the image of the basic open $D(\pi(a)) \subseteq Y$ is $D(a) \cap \pi^*(Y)$. Thus Y is homeomorphic to $\pi^*(Y)$.

Let $U \subseteq \mathrm{MSpec}(A)$ be an affine open subscheme. By Lemma 2.4 there is a prime $\mathfrak{p}$ of A such that $U = \mathrm{MSpec}(A_{\mathfrak{p}})$; by Lemma 1.3, $U = D(s)$ for some s. Hence

$$U \cap Y = D(\pi(s)) = \mathrm{MSpec}(B[1/s]),$$

which is affine or empty. Since $A[1/s] \to B[1/s]$ is onto, $Y \to X$ is a closed immersion. □

Remark 2.7.1. A closed subscheme $Y \subset X$ need not determine a closed subset of the underlying topological space. For example, the diagonal embedding $\mathbb{A}^1 \to \mathbb{A}^2$ is a closed immersion by Lemma 2.7, but it is not topologically closed, because it takes the generic point of $\mathbb{A}^1$ to the generic point of $\mathbb{A}^2$ and the maximal point to the maximal point; the intermediate points are not in the image.

Definition 2.8. If $(X, \mathcal{A})$ is a monoid scheme, a sheaf of ideals $\mathcal{I}$ is said to be *quasi-coherent* if its restriction to any affine open subscheme U of X is the sheaf associated to the ideal $\mathcal{I}(U)$ of the monoid $\mathcal{A}(U)$. Given any closed immersion $i : Y \to X$, the inverse image $\mathcal{I}$ of 0 under $\mathcal{A}_X \to i_*\mathcal{A}_Y$ is quasi-coherent. Lemma 2.7 shows that conversely any quasi-coherent sheaf $\mathcal{I}$ defines an equivariant closed immersion.

Lemma 2.9. *For any monoid scheme X and any subset Z of the underlying poset, there is an equivariant closed subscheme $\bar{Z}^{\mathrm{eq}}$ of X that contains Z and is contained in every other equivariant closed subscheme of X containing Z. We call $\bar{Z}^{\mathrm{eq}}$ the* equivariant closure *of Z in X.*

If U is an open subscheme of X, then $\bar{Z}^{\mathrm{eq}} \cap U$ is $\overline{Z \cap U}^{\mathrm{eq}}$.

Proof. We saw in Section 1 that if Z is any subset of $\mathrm{MSpec}(A)$, there is an equivariant closed subscheme $\bar{Z}^{\mathrm{eq}} = \mathrm{MSpec}(A/I)$ which contains Z (and its closure), and which is minimal with this property. Indeed, if the closure of Z is $V(I_0)$, then $A/I = (A/I_0)_{\mathrm{red}}$. Since

$$S^{-1}(A/I) = (S^{-1}A/S^{-1}I_0)_{\mathrm{red}},$$

this construction patches to give a general construction. □

Remark 2.9.1. If every point in Z has height at least i in X, then every point in $\bar{Z}^{\mathrm{eq}}$ has height at least i in X. This follows from the local description of $\bar{Z}^{\mathrm{eq}}$.

Finite type. We say that a monoid scheme has *finite type* if it admits a finite open cover by affine monoid schemes associated to finitely generated monoids. These monoid schemes are the analogues of noetherian schemes, just as finitely generated monoids are the analogues of commutative noetherian rings: if A is a finitely generated monoid, then every ideal is finitely generated, and A has the ascending chain condition on ideals. (The usual proof of the Hilbert Basis Theorem works.)

By Lemma 1.5, if $(X, \mathcal{A})$ is a monoid scheme of finite type, then X is a finite poset, with the poset topology. The sheaf of monoids $\mathcal{A}$ of a monoid scheme X determines a (contravariant) functor A from the poset X to monoids, called the *stalk functor* of $(X, \mathcal{A})$, sending x to $\mathcal{A}_x$.

It is useful to introduce the notion of a monoid poset as a context for thinking about a stalk functor A.

A *monoid poset* is a pair (Y, B) consisting of a poset Y and a contravariant functor B from Y to monoids. There is a category of monoid posets; a morphism $f : (X, A) \to (Y, B)$ of monoid posets is a poset map $g : X \to Y$ and a natural transformation $B \circ f \Rightarrow A$ such that $B(f(x)) \to A(x)$ is a local monoid morphism for every $x \in X$. With this terminology, the stalk functor induces a functor F from the category of monoid schemes to the category of monoid posets, sending $(X, \mathcal{A})$ to (X, A).

If (X, A) is a monoid poset and $x \in X$, then we can restrict A to the downward-closed subset $W(x) = \{y \in X : y \le x\}$. There is a morphism of monoid posets

$$\iota_x : (W(x), A|_{W(x)}) \to F(\mathrm{MSpec}\, A(x)) \tag{2.10}$$

whose poset map sends a point y of $W(x)$ to the inverse image $\mathfrak{p}_y \in \mathrm{MSpec}\, A(x)$ of the maximal ideal of $A(y)$ under $A(x) \to A(y)$; the maps $A(x)_{\mathfrak{p}_y} \to A(y)$ determine the natural transformation $A \circ \iota_x \Rightarrow A|_{W(x)}$. If the morphism (2.10) is an isomorphism for all $x \in X$, we will say that the monoid poset (X, A) is *scheme-like* and (by abuse of notation) we will call A a *stalk functor*.

We say that a monoid poset (X, A) is of *finite type* if X is a finite poset and each $A(x)$ is a finitely generated monoid. If X is a monoid scheme of finite type, then $F(X)$ is a monoid poset of finite type. The following proposition shows that the stalk functor is always enough to determine a monoid scheme of finite type.

Proposition 2.11. *The functor $F(X, \mathcal{A}) = (X, A)$ induces an equivalence between the full subcategory of monoid schemes of finite type and the full subcategory of scheme-like monoid posets (X, A) of finite type.*

Proof. If (X, A) is a monoid poset, we may equip X with the poset topology, and define the sheaf $\mathcal{A}$ on X by the formula

$$\mathcal{A}(U) = \varprojlim_{x \in U} A(x).$$

Thus $G(X, A) = (X, \mathcal{A})$ is a monoid space. It is clear from the formula for $\mathcal{A}(U)$ that a morphism $(Y, B) \to (X, A)$ of monoid posets induces a morphism $G(Y, B) \to G(X, A)$ of monoid spaces. Thus G is a functor. Because each $W(x)$ has x as its maximal point, $\mathcal{A}(W(x)) = A(x)$. Thus $F(G(X, A)$ is isomorphic to (X, A).

If (X, A) is scheme-like of finite type, then $G(X, A)$ is a monoid scheme of finite type. Conversely, if X is a monoid scheme of finite type and U is an affine open in X, we know by Lemma 2.4 that there is a unique $x \in X$ such that $U = \mathrm{MSpec}(A(x))$ and hence $A(x) = \mathcal{A}(U)$. Given an open U in X, any point y in U lies in an affine open $V \subset U$, and $V = \mathrm{MSpec}(A(x))$ for some $x \in U$ with $y \le x$ by Lemma 2.4. It follows that $GF(X) \cong X$. □

A monoid scheme $(X, \mathcal{A})$ of finite type will often be specified by its monoid poset, viz., (X, A). To avoid confusion, we shall use roman letters for stalk functors and script letters for sheaves.

Remark 2.12. The proof of Proposition 2.11 shows that any scheme-like monoid poset (X, A) can be recovered from the monoid space $G(X, A)$ because $FG(X, A) \cong (X, A)$. If

$(X, \mathcal{A})$ is an arbitrary monoid scheme with stalk functor A, then the topology of X may be coarser than the poset topology. However the argument of the proof of the proposition shows that we can recover $\mathcal{A}$ from A and the topological space underlying X, using the formula $\mathcal{A}(U) = \varprojlim_{x \in U} A(x)$.

3. Basechange and separated morphisms

It is useful to simplify constructions using base-change. For this, we need pullback squares in the category of monoid schemes.

There is a canonical morphism $\nu : X \to \mathrm{MSpec}(\mathcal{A}(X))$ which is universal for maps from X to affine monoid schemes. It sends a point x to the preimage ν_x of the maximal ideal of $\mathcal{A}_x$. The sheaf homomorphism $\nu^\#$ is that induced by the canonical maps

$$\mathcal{A}(X)[1/s] \to \mathcal{A}(\nu^{-1}(D(s))).$$

The universal property shows that the (contravariant) functor

$$X \mapsto \mathcal{A}_X(X)$$

from monoid schemes to monoids is left adjoint to the functor MSpec, i.e., that affine monoid schemes are a reflective subcategory of all monoid schemes. It follows that MSpec converts pushouts of diagrams of monoids to pullbacks of diagrams in the category of all monoid schemes. In particular, for any pushout diagram of monoids (1.8), the induced diagram is cartesian:

$$\begin{array}{ccc} \mathrm{MSpec}(A_1 \wedge_C A_2) & \longrightarrow & \mathrm{MSpec}\, A_2 \\ \downarrow & & \downarrow \\ \mathrm{MSpec}\, A_1 & \longrightarrow & \mathrm{MSpec}\, C. \end{array}$$

Proposition 3.1. *The pullback $X \times_S Y$ of a diagram of monoid schemes*

$$\begin{array}{ccc} X \times_S Y & \dashrightarrow & X \\ \vdots & & \downarrow \\ Y & \longrightarrow & S \end{array}$$

exists in the category of all monoid schemes. Its underlying topological space is the pullback $X \times_S Y$ in the category of topological spaces.

Proof. Existence of the pullback $X \times_S Y$ is derived from the existence of pullbacks of affine monoid schemes, just as for usual schemes ([20, Theorem 3.3]).

To prove the assertion about underlying topological spaces, it suffices to consider the affine case. Using the notation of (1.8), write P for the pullback of $\mathrm{MSpec}(A_1)$ and $\mathrm{MSpec}(A_2)$ over $\mathrm{MSpec}(C)$ in Top. The canonical map $f : \mathrm{MSpec}(A_1 \wedge_C A_2) \to P$ is a continuous bijection by Proposition 1.11. To show that f is a homeomorphism, it suffices to show that it takes any basic open set $D(s)$ to an open set of P. Write $s = s_1 \wedge s_2$; then we have $s \notin \mathfrak{p}$ if and only if $s_1 \wedge 1, 1 \wedge s_2 \notin \mathfrak{p}$. We saw in Proposition 1.11 that if $\mathfrak{p}$ maps to $(\mathfrak{p}_1, \mathfrak{p}_2)$, then $\mathfrak{p} = \mathfrak{p}_1 \wedge A_2 \cup A_1 \wedge \mathfrak{p}_2$, and that $s_1 \wedge 1 \notin \mathfrak{p}$ (resp., $1 \wedge s_2 \notin \mathfrak{p}$) is equivalent to $s_1 \notin \mathfrak{p}_1$ (resp., $s_2 \notin \mathfrak{p}_2$). This shows that f takes $D(s)$ to the open set $(D(s_1) \times D(s_2)) \cap P$, as required. □

Example 3.1.1. The product $X \times Y$ is just the pullback when S is the terminal monoid scheme $\mathrm{MSpec}(S^0)$.

Remark 3.1.2. Let X and Y be monoid schemes of finite type, over a common S. Then the pullback $X \times_S Y$ has finite type. Indeed, it has a finite cover by affine opens of the form $\mathrm{MSpec}(A_1 \wedge_C A_2)$, and in each case $A_1 \wedge_C A_2$ is finitely generated because A_1 and A_2 are.

Example 3.2. Proposition 3.1 shows that given two closed subschemes Z_1, Z_2 of X, the pullback $Z_1 \times_X Z_2$ is a subscheme whose underlying topological space is the intersection of the two subspaces of X. More generally, given any family of closed immersions $Z_i \hookrightarrow X$, we can form the inverse limit $\lim Z_i \hookrightarrow X$ by patching the inverse limits on each affine open $\mathrm{MSpec}(A)$, because the colimit of a family of surjections $A \twoheadrightarrow B_i$ exists and is a surjection.

Separated morphisms. An important hypothesis in many theorems about monoid schemes, often overlooked in the literature, is that they be separated.

Definition 3.3. A morphism $f : X \to S$ of monoid schemes is *separated* if the diagonal map $\Delta : X \to X \times_S X$ is a closed immersion. We say that X is *separated* if it is separated over $\mathrm{MSpec}(S^0)$ where we recall $S^0 = \{0, 1\}$.

Being separated is local on the base: if S has an open cover $\{U\}$, then f is separated if and only if each $f^{-1}(U) \to U$ is separated.

Lemma 3.4. *If $A \to B$ is a morphism of monoids, then* $\mathrm{MSpec}(B) \to \mathrm{MSpec}(A)$ *is a separated morphism of monoid schemes.*

In particular, closed immersions are separated.

Proof. By Proposition 3.1, the diagonal map Δ corresponds to the multiplication map $B \wedge_A B \to B$, which is surjective. By Lemma 2.7, Δ is a closed immersion. □

Remark 3.4.1. Example 1.10 shows that $X \times \mathbb{A}^1 \to X$ is separated and universally closed for every monoid scheme X. This shows that "separated and universally closed" does not provide a good notion of proper morphism of monoid schemes; we will discuss an appropriate definition in Section 8.

Example 3.5. Here is an example of a monoid scheme which is non-separated. Let A and B each be the free abelian monoid with two generators, F_2 (see Example 1.2). Let U be the open subset of each of $\mathrm{MSpec}(A)$ and $\mathrm{MSpec}(B)$ given by removing the unique closed point (associated to the maximal ideal in each monoid); explicitly

$$U = \{\langle t_1 \rangle, \langle t_2 \rangle, \{0\}\}.$$

Then we may glue $\mathrm{MSpec}(A)$ and $\mathrm{MSpec}(B)$ along U to form a monoid scheme X of finite type. As a poset, X has five elements, two of which are maximal – the two copies of $\langle t_1, t_2 \rangle$ – and the rest are in U.

The k-realization of X (defined in Definition 5.3 below) is the non-separated scheme given by the affine plane with the origin doubled.

Lemma 3.6. *A map $f : (X, \mathcal{A}) \to (S, \mathcal{B})$ of monoid schemes is separated if and only if for every x_1, x_2 in X such that $f(x_1) = f(x_2)$ and such that* $\mathrm{MSpec}(\mathcal{A}_{x_1})$ *and* $\mathrm{MSpec}(\mathcal{A}_{x_2})$ *are open, either there is no lower bound for $\{x_1, x_2\}$ in the poset X or else there is a unique maximal lower bound $x_0 = x_1 \cap x_2$, and $\mathcal{A}_{x_1} \wedge_{\mathcal{B}_{f(x_1)}} \mathcal{A}_{x_2} \to \mathcal{A}_{x_0}$ is onto.*

Proof. By Proposition 3.1 and Lemma 2.4, an affine open subset of $X \times_S X$ has the form $U = (U_1 \times U_2) \cap (X \times_S X)$, where the maximal point (x_1, x_2) of U determines the affine open subsets $U_i = \mathrm{MSpec}(\mathcal{A}_{x_i})$ of X. Since $\Delta^{-1}(U) = U_1 \cap U_2$, Proposition 3.1 implies that $X \to \Delta(X)$ is a homeomorphism and that the poset underlying $U_1 \cap U_2$ is the subset $\{z \in X : z \le x_1, z \le x_2\}$ of lower bounds for $\{x_1, x_2\}$. If $U_1 \cap U_2 = \emptyset$, $\{x_1, x_2\}$ has no lower bound.

By Lemma 2.4, $U_1 \cap U_2$ is nonempty affine if and only if it has a unique maximal element. Thus Δ is a closed immersion if and only if, in the above situation, whenever $U_1 \cap U_2$ is nonempty it is affine (and hence has a unique maximal lower bound x_0), and

$$\mathcal{A}_{x_1} \wedge_C \mathcal{A}_{x_2} \to \mathcal{A}_{x_0}$$

is onto, where $s = f(x_1) = f(x_2)$ and $C = \mathcal{B}_s$. □

Corollary 3.7. *If X is a monoid scheme of finite type with stalk functor A, then X is separated if and only if whenever two points x_1, x_2 of X have a lower bound they have a greatest lower bound $x_1 \cap x_2$, and $A(x_1) \wedge A(x_2) \to A(x_1 \cap x_2)$ is onto.*

Proof. Combine Lemma 3.6 and Proposition 2.11. □

Corollary 3.8. *The intersection of two affine open subschemes of a separated monoid scheme is affine.*

Proof. Suppose X is a separated monoid scheme, with U_1, U_2 affine and open in X. Let x_1, x_2 be the unique closed points of U_1, U_2. If x_1 and x_2 do not have a common lower bound in X, then $U_1 \cap U_2 = \emptyset$. Otherwise, by Lemma 3.6, they have a greatest lower bound, which is the unique maximal point of $U_0 = U_1 \cap U_2$. By Lemma 2.4, U_0 is affine. □

4. Toric monoid schemes

As observed by Kato [25] and Deitmar [8], the fan associated to a toric variety produces a monoid scheme. In this section we clarify this correspondence, using the following definition.

Definition 4.1. A *toric monoid scheme* is a separated, connected, torsionfree, normal monoid scheme of finite type.

Recall that a *fan* consists of a free abelian group N of finite rank (written additively) together with a finite collection Δ of strongly convex rational polyhedral cones σ in $N_{\mathbb{R}}$ (hereafter referred to as just *cones*), satisfying the following conditions:

(1) every face of a member of Δ is also a member of Δ,

(2) the intersection of any two members of Δ is a face of each.

Here a *strongly convex rational polyhedral cone* is a cone with apex at the origin, generated by finitely many elements of N, that contains no lines through the origin.

Note that Δ is a finite poset under containment; we now construct a monoid poset (Δ, A) and use Proposition 2.11 to define the associated monoid scheme.

Construction 4.2. Given a fan (N, Δ), set $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ and $M_{\mathbb{R}} = M \otimes \mathbb{R}$. We define a contravariant functor A from Δ to monoids (written additively) by

$$A(\sigma) = (\sigma^{\vee} \cap M)_{*}, \quad \sigma^{\vee} = \{m \in M_{\mathbb{R}} : m(\sigma) \geq 0\}.$$

Each such monoid is torsionfree, normal and finitely generated (Gordon's lemma). If τ is a face of σ, then there is an $m \in A(\sigma)$ such that $A(\tau) = A(\sigma)[-m]$. Hence by Lemma 1.1 there is a prime ideal $P_{\sigma}(\tau)$ of $A(\sigma)$ such that $A(\tau) = A(\sigma)_{P(\tau)}$. By Proposition 2.11 and Corollary 3.7, A is the stalk functor of a toric monoid scheme $X(N, \Delta)$, which by abuse of notation we write as

$$X(\Delta) = (\Delta, A).$$

Thus any fan Δ determines a toric monoid scheme in the sense of Definition 4.1.

A morphism of fans, from (N, Δ) to (N', Δ'), is given by a group homomorphism $\phi : N \to N'$ such that the image of each cone in Δ under the induced map $N_{\mathbb{R}} \to N'_{\mathbb{R}}$ is contained in a cone in Δ'. Such a map of fans induces a poset map $\Delta \to \Delta'$, sending σ to the smallest cone σ' in Δ' that contains $\phi(\sigma)$, and precomposition with ϕ yields a natural transformation $((\sigma')^{\vee} \cap M')_{*} \to (\sigma^{\vee} \cap M)_{*}$ of stalk functors. According to Proposition 2.11, this data determines a morphism of monoid schemes:

$$X(\phi) : X(\Delta) \to X(\Delta').$$

If $\phi_1 \neq \phi_2$, then $X(\phi_1) \neq X(\phi_2)$, as $\phi_1^{*} \neq \phi_2^{*}$ on some $A(\sigma)$. Thus we have a faithful functor X from fans to toric monoid schemes.

Example 4.3. For the cone σ in the plane spanned by $(0, 1)$ and $(1, -2)$, $A(\sigma) = \sigma^{\vee} \cap M$ is the submonoid of $\mathbb{Z}^2$ spanned by $\{(1, 0), (1, 1), (1, 2)\}$. If Δ is the fan spanned by σ and its faces, then $(\Delta, A) = \mathrm{MSpec}\, A(\sigma)$.

If (X, A) is a toric monoid scheme and $x \in X$, we will write M_x for the group completion of the unpointed monoid $A(x) \setminus \{0\}$. Each M_x is a torsionfree abelian group of finite rank. The groups M_x are all isomorphic, because X has a unique minimal point η by Lemma 2.3, and $M_x \to M_{\eta} = A(\eta) \setminus \{0\}$ is an isomorphism for all x.

Theorem 4.4. *The faithful functor $\Delta \mapsto X(\Delta)$ from fans to toric monoid schemes defined by Construction* 4.2 *has the following properties:*

(1) *Every toric monoid scheme (X, A) is isomorphic to $X(N, \Delta)$, where:*
   - (a) *The lattice N is the $\mathbb{Z}$-linear dual of $M = M_{\eta}$, where η is the unique minimal point of X.*
   - (b) *The poset Δ of cones in $N_{\mathbb{R}}$ is isomorphic to the poset underlying X. For each $x \in X$, the cone σ_x in $N_{\mathbb{R}}$ is the dual cone of the convex hull of $A(x) \setminus \{0\}$ in $M_{\mathbb{R}}$.*

(2) *For fans (N, Δ) and (N', Δ'), a morphism $f : X(\Delta) \to X(\Delta')$ of monoid schemes is given by a (necessarily unique) morphism of fans if and only if f maps the generic (i.e., minimal) point η of $X(\Delta)$ to the generic point η' of $X(\Delta')$. In this case, the map of fans $(N, \Delta) \to (N', \Delta')$ is given by the $\mathbb{Z}$-linear dual of the group homomorphism*

$$f_{\eta}^{\#} : M' = (A'(\eta') \setminus \{0\}) \to (A(\eta) \setminus \{0\}) = M.$$

Proof. Throughout this proof, for a cancellative monoid A, we write A^o for the unpointed monoid $A \setminus \{0\}$, written additively, and we identify each $A^o(x)$ with a submonoid of M. Let (X, A) be a toric monoid scheme. We first show that (N, Δ) as defined in the statement is a fan. For $x \in X$, let $\sigma_x^\vee \subset M_\mathbb{R}$ denote the convex hull of $A^o(x)$ in $M_\mathbb{R}$. Note that this defines a cone $\sigma_x = (\sigma_x^\vee)^\vee$ in N via the identification $N = N^{**}$. The cone $\sigma_x^\vee$ is a rational polyhedral cone because it is spanned by a finite set $\{a_i\}$ of generators of $A^o(x)$; the cone σ_x is thus also a rational polyhedral cone, and it is strongly convex since $A^o(x)^+ = M$.

To see that $A(x) = (\sigma_x^\vee \cap M)_*$, let

$$b = \sum_i q_i a_i$$

be an element of $\sigma_x^\vee \cap M$, written as a positive $\mathbb{Q}$-linear combination of the a_i. Clearing denominators, nb is a positive $\mathbb{Z}$-linear combination of the a_i for some positive integer n and hence is in $A^o(x)$. Because $A(x)$ is normal, b is in $A^o(x)$, as required.

If τ is a face of σ_x, it is defined by the vanishing of some $m \in \sigma_x^\vee$. Clearing denominators and using again that $A(x)$ is normal, we may assume $m \in A^o(x)$. By definition, τ is the set of linear functionals on $M_\mathbb{R}$ that are non-negative on $A^o(x)[-m]$. By Lemma 1.1, $A(x)[-m]$ coincides with $A(y)$ for some $y \le x$, and thus the face τ is the element σ_y of Δ.

If $x, y \in X$, we claim that the intersection $\sigma_x \cap \sigma_y$ is a cone in of Δ. Since X is separated, x and y have a unique greatest common lower bound, written $x \cap y$, and the map $A(x)^o \times A(y)^o \to A^o(x \cap y)$ is surjective, by the additive version of Corollary 3.7; moreover because X is cancellative, it is an isomorphism. A linear functional on $M_\mathbb{R}$ is non-negative on $A^o(x) \times A^o(y)$ if and only if it is non-negative on $A^o(x)$ and $A^o(y)$, and thus we have the required identity:

$$\sigma_x \cap \sigma_y = \sigma_{x \cap y}.$$

Moreover, $\sigma_{x\cap y}$ is a face of both σ_x and σ_y, because by Lemmas 1.3 and 1.5 there are m_1, m_2 such that

$$A^o(\sigma_{x\cap y}) = A^o(\sigma_x)[-m_1] = A^o(\sigma_y)[-m_2].$$

This proves that Δ is a fan.

By Construction 4.2, the fan (N, Δ) determines a monoid scheme (Δ, B). The bijection $\sigma : X \to \Delta$ $(x \mapsto \sigma_x)$ is order preserving, because if $x < y$ in X, then $A^o(y) \subset A^o(x) \subseteq M$. By construction, we have a natural isomorphism $A(x) = (\sigma_x^\vee \cap M)_* = B(\sigma_x)$. This proves that σ determines an isomorphism of monoid schemes, completing the proof of property (1).

Construction 4.2 shows that the condition in property (2) is necessary, since a morphism of fans sends the zero cone to the zero cone. Conversely, if $f(\eta) = \eta'$, then $f_\eta^\#$ induces a monoid map $A'(\eta') = M'_* \to M_* = A(\eta)$; since any such map sends units to units, it induces a group homomorphism $M' \to M$. Let $\phi : N \to N'$ be the $\mathbb{Z}$-linear dual of this map. Since for each $x \in X$, the map $f_x^\#$ is the restriction of $f_\eta^\#$, it follows that $f = X(\phi)$, as desired. □

Remark 4.4.1. There are differing assertions in the literature related to Theorem 4.4. Using a different definition of 'toric variety' it is claimed in [8, Theorem 4.1] that any connected cancellative monoid scheme of finite type yields a toric variety, but not every such "toric variety" is associated to a fan. For example, MSpec of the cusp monoid $C = \{t^2, t^3, \dots\}_*$ yields the cusp. In [26, Section 2.1], the flawed [8, Theorem 4.1] is used to claim that the functor of Theorem 4.4 is an equivalence, under the weaker hypothesis that A has no torsion; the cusp monoid is also a counterexample to the assertion in loc. cit.

We conclude this section with a description of separated normal monoid schemes. If X is connected and cancellative, with minimal prime η, then M_η is a finitely generated abelian group. Therefore there is a non-canonical isomorphism $M_\eta \cong M \times T$, where M is a free abelian group and T is a finite torsion group.

Proposition 4.5. *Any separated, connected, normal monoid scheme of finite type decomposes as a cartesian product of monoid schemes*

$$X \cong (X, A) \times \mathrm{MSpec}(T_*),$$

where (X, A) *is a toric monoid scheme and* T *is a finite abelian group.*

Proof. If $\mathrm{MSpec}(A)$ is an affine open of X, then A is a submonoid of $A_\eta = (M \times T)_*$; since A is normal, T_* is a submonoid of A. Every element of $A_\eta \backslash \{0\}$ can be written uniquely as a product mt with $m \in M$ and $t \in T$; since $t \in A$, if $mt \in A$, then $m \in A \cap M$. Thus if we set $B = A \cap M_*$, there is a decomposition $A \cong B \wedge T_*$. In other words,

$$\mathrm{MSpec}(A) \cong \mathrm{MSpec}(B) \times \mathrm{MSpec}(T_*).$$

Since every localization of A has the form $A_{\mathfrak{p}} = B' \wedge T_*$, the affine open subsets of $\mathrm{MSpec}(A)$ are all of the form $\mathrm{MSpec}(B') \times \mathrm{MSpec}(T_*)$. Gluing these together gives the decomposition of X. □

Note that the factorization in Proposition 4.5 is not unique; it depends upon the choice of isomorphism $A_\eta \cong (M \times T)_*$.

Corollary 4.6. *If* $f : X \to X'$ *is a morphism between separated and connected normal monoid schemes of finite type, inducing an isomorphism* $f^* : \mathcal{A}'_{\eta'} \to \mathcal{A}_\eta$ *of group completions, then* f *is isomorphic to the product of a morphism* $X(\Delta) \to X(\Delta')$ *of toric monoid schemes and an isomorphism* $\mathrm{MSpec}(T_*) \to \mathrm{MSpec}(T'_*)$.

Proof. By assumption, f maps the generic point η of X to the generic point η' of X'. Choosing a decomposition $\mathcal{A}_\eta \cong (M \times T)_*$, we have an implicitly defined decomposition

$$\mathcal{A}'_{\eta'} \cong (M \times T)_*.$$

Then for each $x \in X$ the decompositions $\mathcal{A}_x \cong B_x \wedge T_*$, $\mathcal{A}'_{f(x)} \cong B'_{f(x)} \wedge T_*$ of Proposition 4.5 satisfy

$$f^*(B'_{f(x)}) \subseteq B_x \subseteq M_*.$$

Therefore the map $\mathcal{A}'_{f(x)} \to \mathcal{A}_x$ factors as a product of $f^*(B'_{f(x)}) \subseteq B_x$ and $T_* \cong T_*$, for each x. The result follows. □

Remark 4.6.1. Not every morphism $(X, A) \times \mathrm{MSpec}(T_*) \to (X', A') \times \mathrm{MSpec}(T'_*)$ between connected normal monoid schemes of finite type will factor as a cartesian product of maps $(X, A) \to (X', A')$ and $\mathrm{MSpec}(T_*) \to \mathrm{MSpec}(T'_*)$. For example, this fails for the canonical $\mathrm{MSpec}((\mathbb{Z}/n)_*) \to \mathrm{MSpec}(\mathbb{Z}_*)$. However, such a map determines both a toric map $(X, A) \to (X', A')$ and a map $\mathrm{MSpec}(T_*) \to \mathrm{MSpec}(T'_*)$.

5. Realizations of monoid schemes

In this section we fix a commutative ring k. If A is a monoid, the ring $k[A]$ gives rise to a scheme $\mathrm{Spec}(k[A])$, which is called the *k-realization* of $\mathrm{MSpec}(A)$. The affine spaces $\mathbb{A}^n_k = \mathrm{Spec}(k[t_1, \dots, t_n])$ of Example 1.2 are useful examples. The k-realization is a faithful functor from monoids to affine k-schemes; a monoid morphism $A \to B$ naturally gives rise to a morphism $\mathrm{Spec}(k[B]) \to \mathrm{Spec}(k[A])$.

If X is an affine monoid scheme, we write X_k for its realization:

$$\mathrm{MSpec}(A)_k = \mathrm{Spec}(k[A]).$$

We saw in (1.8) that the k-realization functor commutes with pullback for affine monoid schemes, because it has a left adjoint (defined on the category of affine k-schemes) sending $\mathrm{Spec}(R)$ to $\mathrm{MSpec}(R, \times)$, where $(R, \times)$ is the multiplicative monoid whose underlying pointed set is R. Thus if $X = \mathrm{MSpec}(A)$ is an affine monoid, the adjunction

$$\mathrm{Hom}(\mathrm{Spec}(R), X_k) \cong \mathrm{Hom}_{\mathrm{MSch}}(\mathrm{MSpec}(R, \times), X)$$

means that X_k represents the functor sending $\mathrm{Spec}\ R$ to $\mathrm{Hom}_{\mathrm{MSch}}(\mathrm{MSpec}(R, \times), X)$.

Definition 5.1. Let X be a monoid scheme and k be a ring. Define a contravariant functor F_X from the category of affine k-schemes to sets to be the Zariski sheafification of the presheaf

$$\mathrm{Spec}\ R \mapsto \mathrm{Hom}_{\mathrm{MSch}}(\mathrm{MSpec}(R, \times), X).$$

If X is affine, the presheaf is already a sheaf since it is represented by X_k.

Recall from [9, Theorem VI-14] that a contravariant functor F from affine k-schemes to sets is represented by a unique k-scheme X if and only if F is a Zariski sheaf and F admits a covering by open subfunctors F_α, each of which is represented by an affine scheme U_α. If so, the representing scheme X is obtained by gluing the U_α together. Here, a subfunctor $F_\alpha \subseteq F$ is *open* if for every k-algebra R and every morphism $\mathrm{Hom}(-, \mathrm{Spec}\ R) \to F$, i.e., for every element of $F(\mathrm{Spec}\ R)$, the pullback functor $F_\alpha \times_F \mathrm{Hom}(-, \mathrm{Spec}\ R)$ is represented by an open subscheme of $\mathrm{Spec}\ R$. A collection of subfunctors $\{F_\alpha\}$ of F *covers* F if for every k-algebra L which is a field, we have $F(\mathrm{Spec}\ L) = \bigcup_\alpha F_\alpha(\mathrm{Spec}\ L)$.

Theorem 5.2. *The functor F_X is represented by a scheme X_k.*

Proof. Suppose that $U = \mathrm{MSpec}(A)$ is any affine monoid subscheme of X. As sheafification preserves monomorphisms such as $\mathrm{Hom}(-, U) \subseteq \mathrm{Hom}(-, X)$, F_U is a subfunctor of F_X. If Λ is a local k-algebra and $L = \mathrm{Spec}(\Lambda)$, then

$$F_X(L) = \mathrm{Hom}_{\mathrm{MSch}}(\mathrm{MSpec}(\Lambda, \times), X). \tag{5.2a}$$

Since $\mathrm{MSpec}(\Lambda, \times)$ has a unique point, each map $\mathrm{MSpec}(\Lambda, \times) \to X$ factors through an affine open submonoid U. Therefore F_X is covered by the collection of subfunctors F_U, as U ranges over all affine open monoid subschemes of X. We will show that the F_U are open subfunctors of F_X; we have seen that each F_U is represented by the affine scheme U_k. By [9, Theorem VI-14], this will prove that F_X is representable by the k-scheme which is obtained by gluing the affine schemes U_k.

Fix an affine open monoid subscheme $U = \mathrm{MSpec}(A)$. To prove that F_U is open, fix a k-algebra R and consider a morphism $\mathrm{Hom}(-, \mathrm{Spec}\, R) \to F_X$ and its corresponding element $\phi \in F_X(\mathrm{Spec}\, R)$. We have to show that the pullback $G = F_U \times_{F_X} \mathrm{Hom}(-, \mathrm{Spec}\, R)$ is represented by an open subscheme V of $\mathrm{Spec}(R)$. Since F_X is a sheaf, $\mathrm{Spec}(R)$ has an affine open covering $\{\mathrm{Spec}\, R[1/s] \mid s \in \mathcal{S}\}$ such that the restriction of ϕ to $F_X(\mathrm{Spec}\, R[1/s])$ is represented by a morphism $\phi_s : \mathrm{MSpec}(R[1/s], \times) \to X$ of monoid schemes. By Observation 5.2.1 below, there are continuous maps

$$\mathrm{Spec}(R[1/s]) \hookrightarrow \mathrm{MSpec}(R[1/s], \times) \xrightarrow{\phi_s} X.$$

Let V'_s denote the inverse image of U under ϕ_s and let V_s denote the open subspace $V'_s \cap \mathrm{Spec}(R[1/s])$; we regard V_s as an open subscheme of $\mathrm{Spec}(R[1/s])$ and hence of $\mathrm{Spec}(R)$. We claim that G is represented by the open subscheme $V = \cup V_s$ of $\mathrm{Spec}\, R$. To prove our claim, it suffices to consider a local k-scheme $L = \mathrm{Spec}(\Lambda)$ and prove that

$$G(L) = \mathrm{Hom}(L, V)$$

as subsets of $\mathrm{Hom}(L, \mathrm{Spec}\, R)$. Since L is local, we have $F_U(L) = \mathrm{Hom}(A, (\Lambda, \times))$, and (5.2a) holds for X. Thus $G(L)$ is the set of all $f : L \to \mathrm{Spec}\, R$ such that

$$\mathrm{MSpec}(\Lambda, \times) \xrightarrow{f^\times} \mathrm{MSpec}(R, \times) \xrightarrow{\phi} X$$

maps the closed point $\mathfrak{m}$ of L into U. If the image of f lies in V, $\mathfrak{m}$ lands in some V_s and hence $f^\times$ maps the closed point $(\mathfrak{m}, \times)$ of $\mathrm{MSpec}(\Lambda, \times)$ into V'_s. It follows that $\phi f^\times(\mathfrak{m}, \times) \in U$, i.e., $f \in G(L)$. Thus $\mathrm{Hom}(L, V) \subseteq G(L)$.

Conversely, if $f : L \to \mathrm{Spec}(R)$ is in $G(L)$, then f factors through some

$$f_s : L \to \mathrm{Spec}(R[1/s])$$

and $\phi_s f_s^\times$ maps the closed point $(\mathfrak{m}, \times)$ of $\mathrm{MSpec}(\Lambda, \times)$ to a point in the subset U of X, so $f_s(\mathfrak{m}) \in V_s$. But since L is local, this implies that $f_s(L) \subseteq V_s$. The desired equality

$$G(L) = \mathrm{Hom}(L, V)$$

follows. □

Observation 5.2.1. Let R be any commutative ring, and $(R, \times)$ be its underlying multiplicative monoid. If $\mathfrak{p}$ is a prime ideal of the ring R, then $(\mathfrak{p}, \times)$ is a prime ideal of the monoid $(R, \times)$. The resulting inclusion $\mathrm{Spec}(R) \hookrightarrow \mathrm{MSpec}(R, \times)$ is continuous because if $s \in R$, the open subspace $D(s)$ of $\mathrm{MSpec}(R)$ intersects $\mathrm{Spec}(R)$ in the open subspace $\{\mathfrak{p} \subset R : s \notin \mathfrak{p}\}$. If R is local, the maximal ideal $\mathfrak{m}$ of R maps to the maximal prime $(\mathfrak{m}, \times)$ of $\mathrm{MSpec}(R, \times)$.

Definition 5.3. Given a commutative ring k and a scheme $(X, \mathcal{A})$, we define its *k-realization* X_k to be the scheme representing F_X.

Remark 5.3.1. Observe that $X_k = X_{\mathbb{Z}} \times_{\mathrm{Spec}\, \mathbb{Z}} \mathrm{Spec}\, k$ for any monoid scheme X and commutative ring k. Those preferring the notion of a field with one element ($\mathbb{F}_1$) might prefer writing X_k as $X \times_{\mathrm{Spec}\, \mathbb{F}_1} \mathrm{Spec}\, k$ or just $X \times_{\mathbb{F}_1} k$.

Corollary 5.4. *The k-realization functor $X \mapsto X_k$ preserves arbitrary limits (when they exist). In particular, it preserves pullbacks.*

Proof. Suppose that $\{X_i : i \in I\}$ is a diagram of monoid schemes and that its limit X exists in the category of monoid schemes. It suffices to prove the canonical map

$$F_X \to F = \varprojlim F_{X_i}$$

is an isomorphism of sheaves on the category of affine k-schemes. Recall that the limit of a diagram of sheaves exists and coincides with the limit as presheaves. That is, we have

$$F(\mathrm{Spec}\, R) = \varprojlim F_{X_i}(\mathrm{Spec}\, R).$$

When R is local, we have $F_X(\mathrm{Spec}\, R) = \mathrm{Hom}(\mathrm{MSpec}(R, \times), X)$ and also

$$F(\mathrm{Spec}\, R) = \varprojlim \mathrm{Hom}(\mathrm{MSpec}(R, \times), X_i) \cong \mathrm{Hom}(\mathrm{MSpec}(R, \times), X),$$

where the second isomorphism holds since $X = \varprojlim_i X_i$. Since the sheaf map $F_X \to F$ is an isomorphism on all local rings, it is an isomorphism of sheaves. □

In Proposition 5.7 below we shall give an explicit construction of X_k for separated X. We need some preliminaries.

Lemma 5.5. *If S is multiplicatively closed in A, $S^{-1}k[A] \cong k[S^{-1}A]$.*

Proof. The monoid map $A \to S^{-1}A$ is initial among monoid maps $A \to B$ that take S to units. Similarly, the map $k[A] \to S^{-1}k[A]$ is initial among k-algebra homomorphisms $k[A] \to C$ that take S to units. Being a left adjoint, the functor $k[-]$ preserves initial objects. □

Remark 5.6. Let A be a monoid. Any affine open monoid subscheme of MSpec(A) has the form MSpec($A_{\mathfrak{p}}$) for some prime ideal $\mathfrak{p}$ of A, by Lemma 2.4, and $A_{\mathfrak{p}} = A[1/s]$ by Lemma 1.3. Hence $\mathrm{Spec}(k[A_{\mathfrak{p}}]) \to \mathrm{Spec}(k[A])$ is an open immersion, by Lemma 5.5.

For the next proposition, let us say that a point x in a monoid scheme X is *nice* if the canonical map $U = \mathrm{MSpec}(\mathcal{A}_x) \to X$ is an open immersion. Every closed point is nice by Lemma 2.4, but the points of Example 1.4 are not nice. If X is of finite type, then every point is nice by Lemma 1.5. The nice points $x \in X$ are a cofinal subset of the poset underlying X by Lemmas 1.3 and 2.4, because the closed points in any open subscheme are nice. If $x < y$ are two nice points, then $\mathrm{Spec}(k[\mathcal{A}_x]) \to \mathrm{Spec}(k[\mathcal{A}_y])$ is an open immersion by Lemma 5.5. The criterion for separatedness in Lemma 3.6 uses nice points.

Proposition 5.7. *Let k be a commutative ring and $(X, \mathcal{A})$ a separated monoid scheme. Then the k-realization of X is*

$$X_k = \varinjlim_{x \in X} \mathrm{Spec}(k[\mathcal{A}_x]).$$

Proof. Put $U_x = \mathrm{MSpec}(\mathcal{A}_x)$. Because nice points are cofinal in the poset underlying X, the limit can be taken over the nice points. If x is nice, then $U_x \subset X$ is an open immersion; set $V_x = (U_x)_k$. If y is also nice, then $U_x \cap U_y$ is an affine open, because the intersection of two affine open subschemes of a separated monoid scheme is affine open by Corollary 3.8. By Corollary 5.4 we have $(U_x \cap U_y)_k = V_x \times_{X_k} V_y$. Let $V_{x,y}$ be the image of the projection $\pi_x : V_x \times_{X_k} V_y \to V_x$. Then $V_{x,y}$ is open in V_x and we have an isomorphism $\psi_{x,y} = \pi_y(\pi_x)^{-1} : V_{x,y} \to V_{y,x}$. Hence the family of schemes V_x indexed by the nice points of X together with the open subschemes $V_{x,y} \subset V_x$ and the isomorphisms $\psi_{x,y}$ satisfy the hypothesis of [17, Chapitre 0, (4.1.7)] (or [20, Example II.2.12]). Therefore the limit of the proposition exists, and is the scheme obtained by gluing the realizations of the open affine subschemes of X. Since this is also the definition of X_k, the proposition follows. □

The k-realization functor from monoid schemes to k-schemes is faithful, because it is so locally: $\mathrm{MSpec}(A)_k = \mathrm{Spec}(k[A])$. (This is clear if X is separated, and follows from Theorem 5.2 if it is not separated.) It is not full because k-schemes such as $\mathbb{A}^1_k$ have many more endomorphisms than their monoidal counterparts.

The realization functor loses information, because distinct monoid schemes can have isomorphic realizations. This is a well-known phenomenon even for toric varieties, where the additional data of a (faithful) torus action is needed to recover the fan.

Example 5.8. For a fan Δ and any field k, the variety $X(\Delta)_k$ is the usual toric k-variety associated to Δ. This is clear from Construction 4.2.

Example 5.9. Let T be a finite abelian group. The k-realization of $\mathrm{MSpec}(T_*)$ is the cogroup scheme $\mathrm{Spec}(k[T])$. If $|T|$ is a unit (or nonzerodivisor) in k, then $k[T]$ is reduced, but this fails if k is a field of characteristic $p > 0$ and T has p-torsion.

Lemma 5.10. *Let k be an integral domain and let A be a cancellative monoid. Set $X = \mathrm{MSpec}(A)$ and $U = \mathrm{MSpec}(A^+)$.*

(1) *If A^+ is torsionfree, then $k[A]$ is a domain (i.e., X_k is integral).*

(2) *Suppose that k is a normal domain containing a field; if $\mathrm{char}(k) = p > 0$, assume also that A^+ has no p-torsion. Then $k[A^+]$ is normal and its subalgebra $k[A]$ is reduced. That is, U_k is normal and X_k is reduced.*

(3) *Suppose that $\mathrm{char}(k) = p > 0$ and A^+ has p-torsion. Then $k[A]$ is not reduced, and we have $k[A]_{\mathrm{red}} = k[B]$, where the monoid B is the quotient of A by the congruence relation that $a_1 \sim a_2$ if and only if $a_1^{p^e} = a_2^{p^e}$ for some $e \geq 0$.*

Proof. Since A is the union of its finitely generated submonoids A_i and since we have $k[A] = \bigcup k[A_i]$, we may assume that A is finitely generated. As noted before Proposition 4.5, we can write $A^+ = (M \times T)_*$ where M is a free abelian group and T is a finite torsion group. Since A is a submonoid of A^+, it follows that $k[A]$ is a subalgebra of $k[A^+]$. If T is trivial, $k[A]$ is a subring of $k[M]$, which is manifestly a domain. If $k \supset \mathbb{Q}$ or if $\mathrm{char}(k) = p$ and $p \nmid |T|$, then $k \to k[T]$ is a finite étale extension and $k[A]$ is a subring of $k[A^+] = k[T][M]$, which is manifestly normal if k is normal. Hence $k[A^+]$ and its subalgebra $k[A]$ are reduced in this case.

Finally, suppose that $\mathrm{char}(k) = p$ and that the p-torsion subgroup T_p of T is non-trivial. Since $k[T_p]_{\mathrm{red}} = k$ and $k[A^+/T_p]$ is reduced by (2), we have

$$k[A^+]_{\mathrm{red}} = k[A^+/T_p].$$

If B is the image of $A \to A^+/T_p$, then $k[A]_{\rm red}$ is the image $k[B]$ of $k[A] \to k[A^+/T_p]$. Two elements $a_1, a_2 \in A$ go to the same element of A^+/T_p if and only if their quotient is p-torsion, i.e., if and only if they are congruent under the relation $\sim$ of the lemma. It follows that $B = A/{\sim}$; this concludes the proof. □

Remark 5.10.1. If (X, A) is a cancellative monoid scheme of finite type, and k is of characteristic $p > 0$, Lemma 5.10 (3) implies that $(X_k)_{\rm red}$ is the k-realization of (X, B), where $B = A/{\sim}$ is the quotient stalk functor of A defined as in Lemma 5.10 (3).

Proposition 5.11. *If $(Y, \mathcal{B}) \xrightarrow{f} (X, \mathcal{A})$ is a closed immersion of monoid schemes, then $f_k : Y_k \to X_k$ is a closed immersion of schemes for all rings k.*

Proof. If $V \subseteq X$ is an affine open subscheme, then by Lemma 2.4 there exists an $x \in X$ such that $V = \mathrm{MSpec}(\mathcal{A}_x)$. We shall abuse notation and write $V \cap Y$ for $V \times_X Y$. If $V \cap Y = \emptyset$, then $V_k \cap Y_k = (V \cap Y)_k = \emptyset$. Otherwise $V \cap Y = \mathrm{MSpec}(\mathcal{B}_y)$ for some y, and $\mathcal{A}_x \to \mathcal{B}_y$ is onto, by Definition 2.5. Since k-realization preserves pullbacks by Corollary 5.4, we have

$$f_k^{-1}(V_k) = f^{-1}(V)_k = \mathrm{Spec}(k[\mathcal{B}_y])$$

and the restriction

$$f_k^{-1}(V_k) \to V_k = \mathrm{Spec}\, k[\mathcal{A}_x]$$

of f is induced by the surjection $k[\mathcal{A}_x] \to k[\mathcal{B}_y]$. This proves that the restriction $Y_k \cap V_k \to V_k$ of f_k is a closed immersion. Since V is an arbitrary affine open subscheme of X, this proves that $Y_k \to X_k$ is a closed immersion. □

A partial converse of this proposition is true.

Lemma 5.12. *Suppose $i : Y \to X$ is a morphism of monoid schemes such that the underlying map of topological spaces induces a homeomorphism onto its image. For any ring k, if $i_k : Y_k \to X_k$ is a closed immersion, then i is a closed immersion of monoid schemes.*

Proof. It suffices to prove that if $X = \mathrm{MSpec}(A)$ is affine, then Y is also affine and the associated map of monoids is surjective. Let $\mathcal{B}$ be the sheaf of monoids for the scheme Y and set $B = \Gamma(Y, \mathcal{B})$. The map $Y \to X$ factors as

$$Y \to \mathrm{MSpec}\, B \to \mathrm{MSpec}\, A.$$

Upon taking k-realizations we have $Y_k = \mathrm{Spec}(R)$ and the map induced by $Y_k \to X_k$ is a surjection: $k[A] \twoheadrightarrow R$. Since this surjection factors through the map $k[A] \to k[B]$, which is induced by a map of monoids $A \to B$, we see that $k[B] \twoheadrightarrow R$ is surjection as well. Let

$$Y = \bigcup_j W_j$$

be a covering by open affine subschemes, with $W_j = \mathrm{MSpec}\, B_j$. Then the map $B \to \prod_j B_j$ is injective and hence so is the map

$$k[B] \to \prod_j k[B_j].$$

Since the latter map factors as

$$k[B] \to R \to \prod_J k[B_j],$$

it follows that $k[B] \xrightarrow{\cong} R$ is an isomorphism. That is, the k-realization of $Y \to \mathrm{MSpec}(B)$ is an isomorphism. Moreover, since $k[A] \twoheadrightarrow k[B]$ is onto, so is the map $A \twoheadrightarrow B$, and hence $\mathrm{MSpec}(B) \to X$ is a closed immersion. In particular, the map of underlying topological spaces is a homeomorphism onto its image. It follows from this (and our assumption) that the map of topological spaces underlying $Y \to \mathrm{MSpec}(B)$ is a homeomorphism onto its image.

We may thus assume that the k-realization $Y_k \to X_k = \mathrm{Spec}(k[A])$ is an isomorphism. We next claim that $Y \to X$ is a surjection on points, and hence (by our assumption that Y is homeomorphic to its image) a homeomorphism on underlying topological spaces. To see this, fix a point $\mathfrak{p} \in X$ and consider the monoid map $i_{\mathfrak{p}} : A \to S^0 = \{0, 1\}$ sending $\mathfrak{p}$ to 0 and $A \setminus \mathfrak{p}$ to 1. Let Y' denote the pullback of $Y \to X$ along the map

$$\mathrm{MSpec}\, S^0 \xrightarrow{i_{\mathfrak{p}}} X.$$

By Corollary 5.4, the map $Y'_k \to (\mathrm{MSpec}\, S^0)_k = \mathrm{Spec}\, k$ is an isomorphism, so in particular Y' is nonempty. By Proposition 3.1, it follows that $Y \to X$ is onto.

Since X has a unique maximal point, so does Y. By Lemma 2.4, Y is affine. Since

$$Y_k \cong \mathrm{Spec}(k[A]),$$

we conclude that $Y \cong X$. □

Proposition 5.13. *For any ring k and morphism of monoid schemes $f : Y \to X$, the map f is a separated morphism of monoid schemes if and only if its k-realization $f_k : Y_k \to X_k$ is a separated morphism of schemes.*

Proof. One direction is immediate from Corollary 5.4 and Proposition 5.11.

Assume f_k is separated. Since the underlying topological space of $Y \times_X Y$ is given by the pullback in the category of topological spaces, it follows that

$$Y \xrightarrow{\Delta} Y \times_X Y$$

is a homeomorphism onto its image. (Observe that $Y \to \Delta(Y)$ and $\Delta(Y) \xrightarrow{\pi_1} Y$ are continuous, and both compositions are the identity, where $\Delta(Y) \subset Y \times_X Y$ is given the subspace topology.) Since Δ_k is a closed immersion, Lemma 5.12 applies to finish the proof. □

6. Normal and smooth monoid schemes

Throughout this section, k denotes an integrally closed domain containing a field. The normalization A_{nor} of a cancellative monoid A is defined in Definition 1.6; since we have $(A_{\mathfrak{p}})_{\mathrm{nor}} = (A_{\mathrm{nor}})_{\mathfrak{p}_{\mathrm{nor}}}$, it makes sense to talk about the normalization of any cancellative monoid scheme.

The k-realization of X cannot be normal unless X_k is reduced. Lemma 5.10 shows that $k[A]$ is reduced unless $p > 0$ and A^+ has p-torsion, in which case $k[A]_{\mathrm{red}}$ is $k[B]$, where B is a particular quotient of A, described there.

Proposition 6.1. *Let $X = (X, A)$ be a cancellative monoid scheme of finite type such that its k-realization X_k is a reduced scheme. Then:*

(1) *The normalization of X_k is the k-realization of (X, A_{nor}).*

(2) *If X is normal, connected and separated, there is a decomposition*

$$X_k = X'_k \times_k \operatorname{Spec} k[T]$$

where X'_k is a toric k-variety and $k[T]$ is finite étale over k.

As in Remark 4.6.1, the decomposition in Proposition 6.1 (2) is not natural in X.

Proof. Part (2) is immediate from Proposition 4.5 and Corollary 5.4.

Since the normalization of a reduced scheme is the scheme constructed by patching together the normalizations of an affine cover, we may assume that X is affine, i.e., assume $X = \operatorname{MSpec}(A)$. Since $k[A_{\text{nor}}]$ is integral over $k[A]$, we may assume that $A = A_{\text{nor}}$. In this situation, where A is a normal monoid of finite type, Proposition 4.5 states that $A \cong A' \wedge T_*$ where A' is torsionfree and T is a finite abelian group. Since X_k is reduced, we know from Lemma 5.10 (3) and Example 5.9 that T has no p-torsion and $k[T]$ is finite étale over k. Since $k[A] = k[T][A']$, we are reduced to the case in which A is normal and torsionfree, i.e., $X = \operatorname{MSpec}(A)$ is an affine toric monoid scheme. By Theorem 4.4, X is associated to a fan Δ; by Example 5.8, X_k is the toric variety associated to Δ, and in particular X_k is normal. □

Remark 6.1.1. It is possible to give an elementary proof of this result using that if A is a torsionfree normal monoid, then $k[A]$ is integrally closed; see [14, Corollary 12.6].

Finite morphisms. We will need to know that the normalization of a monoid scheme is a finite morphism, at least when X is of finite type.

We say that a morphism of monoid schemes $f : Y \to X$ is *affine* if X can be covered by affine open subschemes $U_i = \operatorname{MSpec}(A_i)$ such that $f^{-1}(U_i)$ is affine. Equivalently, f is affine if $f^{-1}(U)$ is affine for every affine open subscheme $U \subset X$.

Definition 6.2. Let $f : Y \to X$ be a morphism of monoid schemes. We say that f is *finite* if it is affine and $\mathcal{A}_X(U) \to \mathcal{A}_Y(f^{-1}(U))$ is finite for every affine subscheme $U \subset X$. We say that f is *integral* if it is affine and $\mathcal{A}_X(U) \to \mathcal{A}_Y(f^{-1}(U))$ is integral for every affine subscheme $U \subset X$.

If X is cancellative, its normalization $X_{\text{nor}} \to X$ is an integral morphism. To see this, we may assume $X = \operatorname{MSpec}(A)$ is affine so that $X_{\text{nor}} \to X$ is given by $A \hookrightarrow A_{\text{nor}}$, where the normalization A_{nor} is integral by Definition 1.6. We now show that if X is also of finite type, then $X_{\text{nor}} \to X$ is finite.

Proposition 6.3. *If X is a cancellative monoid scheme of finite type, the normalization $X_{\text{nor}} \to X$ is a finite morphism.*

Proof. It suffices to show that if A is a cancellative monoid of finite type, then $A \to A_{\text{nor}}$ is finite. Since A_{nor} is integral over A, it suffices by Lemma 1.7 (i) to show that A_{nor} is of finite type. Because the group completion A^+ is finitely generated, it has the form $(M \times T)_*$ where

T is a finite abelian group and M is free abelian. Since $A[T] = \bigcup At$ is finite over A, we may replace A by $A[T]$ to assume that $T \subset A$. As in the proof of Proposition 4.5, this implies that $A = B \wedge T_*$ where $B = A \cap M_*$ is a finitely generated submonoid of M. If β is the rational convex polyhedral cone of $M_{\mathbb{R}}$ spanned by the generators of B, B_{nor} is $(\beta \cap M)_*$. By Gordon's lemma [10], B_{nor} is finitely generated. A fortiori, $A_{\text{nor}} = B_{\text{nor}} \wedge T_*$ is finitely generated. □

Smoothness. We start with the following definition.

Definition 6.4. Let p be a prime. A separated monoid scheme of finite type is *p-smooth* if each stalk (equivalently, each maximal stalk) is the smash product $S \wedge T_*$, where $S = G_* \wedge F$ is the smash product of a free abelian group with a point adjoined and a free abelian monoid, and T is a finite abelian group having no p-torsion. A separated monoid scheme is *0-smooth* if each stalk has the form $S \wedge T_*$ with T an arbitrary finite abelian group.

We will say that X is *smooth* if it is p-smooth for all p, i.e., if each stalk is the product of a free group of finite rank and a free monoid of finite rank.

A cone in a fan (N, Δ) is said to be *nonsingular* if it is spanned by part of a $\mathbb{Z}$-basis for the lattice N, in which case each monoid $\sigma^\vee \cap M$ is the product of a free abelian group and a free abelian monoid. A fan is said to be nonsingular if all its cones are nonsingular.

Proposition 6.5. *Let $X = (X, A)$ be a separated cancellative monoid scheme of finite type. Its k-realization X_k is smooth over a field k of characteristic $p \geq 0$ if and only if X is p-smooth. If X is connected and p-smooth then, under the decomposition*

$$X = (X, A') \times \text{MSpec}(T)$$

of Proposition 6.1*, the fan underlying (X, A') is nonsingular.*

Proof. Recall from [10, Section 2.1] that the toric variety associated to a fan is smooth if and only if each of its cones is nonsingular. Therefore the proposition is an immediate corollary of Proposition 6.1 and Lemma 5.10. □

Example 6.5.1. The hypothesis in Proposition 6.5 that X be cancellative is necessary. For example, consider the monoid $A = \langle t, e \mid e = e^2 = te \rangle$, which has $k[A] \cong k[x] \times k$. Thus $X = \text{MSpec}(A)$ is not p-smooth but X_k is smooth for every k.

7. MProj and blow-ups

An $\mathbb{N}$-grading of a monoid A is a pointed set decomposition

$$A = \bigvee_{i=0}^{\infty} A_i$$

such that $A_i \cdot A_j \subseteq A_{i+j}$; $\mathbb{Z}$-gradings are defined similarly. For each nonzero a in A, let $|a|$ denote the unique i such that $a \in A_i$. For every multiplicative set S, the localization $S^{-1}A$ is $\mathbb{Z}$-graded by $|a/s| = |a| - |s|$. For example, if $s \in A_i$ is nonzero, we have

$$A[\tfrac{1}{s}]_0 = \left\{ \frac{a}{s^n} : |a| = |s^n| = ni,\ n \geq 0 \right\} \cup \{0\}.$$

Let $A_{\geq 1}$ denote the ideal

$$\bigvee_{i\geq 1} A_i = \{a : |a| > 0\} \cup \{0\},$$

so that $A/A_{\geq 1} \cong A_0$; the image of the corresponding map $\mathrm{MSpec}(A_0) \to \mathrm{MSpec}(A)$ consists of the prime ideals of A containing $A_{\geq 1}$.

Definition 7.1. If A is an $\mathbb{N}$-graded monoid, we define $\mathrm{MProj}(A) = (X, \mathcal{B})$ to be the following monoid scheme. The underlying topological space is $X = \mathrm{MSpec}(A) \setminus \mathrm{MSpec}(A_0)$ – i.e., the open subspace of those prime ideals of A that do not contain $A_{\geq 1}$. The stalks of $\mathcal{B}$ on X are defined by sending $\mathfrak{p} \in \mathrm{MSpec}(A) \setminus \mathrm{MSpec}(A_0)$ to $\mathcal{B}_{\mathfrak{p}} = (A_{\mathfrak{p}})_0$, the degree 0 part of $A_{\mathfrak{p}}$. If $\mathrm{MSpec}(A_{\mathfrak{p}}) \subset X$ is open, that is, if $A_{\mathfrak{p}} = A[1/s]$ for some $s \in A_{\geq 1}$, then the map $\mathrm{MSpec}(A_{\mathfrak{p}}) \to \mathrm{MSpec}(A_{\mathfrak{p}})_0$ is a homeomorphism. Indeed, this follows from the fact that a prime ideal $\mathfrak{q}$ of $A[1/s]$ contains an element a/s^n if and only if $\mathfrak{q} \cap (A[1/s])_0$ contains $a^{n|s|}/s^{n|a|}$. Thus $\mathrm{MProj}(A)$ is covered by the affine open subschemes $D_+(s) = \mathrm{MSpec}(A\big[\frac{1}{s}\big]_0)$ where $s \in A_{\geq 1}$, and moreover, every affine open subscheme is of this form. Hence $\mathrm{MProj}(A)$ is a monoid scheme of finite type whenever A is a finitely generated monoid. The maps $A_0 \to (A_{\mathfrak{p}})_0$ induce a structure morphism $\mathrm{MProj}(A) \to \mathrm{MSpec}(A_0)$.

Remark 7.1.1. The k-realization of A is the graded ring $k[A]$, and $k[A\big[\frac{1}{s}\big]_0]$ is the degree 0 part of the ring $k[A]\big[\frac{1}{s}\big]$, so the k-realization of $\mathrm{MProj}(A)$ is $\mathrm{Proj}(k[A])$.

Observation 7.1.2. The construction is natural in A for maps $A \to A'$ of graded monoids such that $A' = A \cdot A'_0$. For such maps there is a canonical morphism

$$\mathrm{MProj}(A') \to \mathrm{MProj}(A)$$

induced by the restriction of

$$\mathrm{MSpec}(A') \to \mathrm{MSpec}(A).$$

If $s \in A_{\geq 1}$, the affine open $\mathrm{MSpec}(A'\big[\frac{1}{s}\big]_0)$ maps to the affine open $\mathrm{MSpec}(A\big[\frac{1}{s}\big]_0)$.

If $S \subset A_0$ is multiplicatively closed, $S^{-1}A$ is graded and

$$\mathrm{MProj}(S^{-1}A) = \mathrm{MProj}(A) \times_{\mathrm{MSpec}(A_0)} \mathrm{MSpec}(S^{-1}A_0).$$

It follows that this construction may be sheafified: for any monoid scheme $(X, \mathcal{A}_0)$ and any sheaf $\mathcal{A}$ of graded monoids on X with $(\mathcal{A}_x)_0 = (\mathcal{A}_0)_x$ for all $x \in X$, there is a monoid scheme $\mathrm{MProj}(\mathcal{A})$ over X whose stalk at each x is $\mathrm{MProj}(\mathcal{A}_x)$. Moreover, if $f : (X', \mathcal{A}'_0) \to (X, \mathcal{A}_0)$ is a morphism of monoid schemes, equipped with sheaves $\mathcal{A}'$ and $\mathcal{A}$ of graded monoids as above, any graded extension $f^{-1}\mathcal{A} \to \mathcal{A}'$ of $f^{-1}\mathcal{A}_0 \to \mathcal{A}'_0$ such that $\mathcal{A}' = f^{-1}\mathcal{A} \cdot \mathcal{A}'_0$ induces a canonical morphism $\mathrm{MProj}(\mathcal{A}') \to \mathrm{MProj}(\mathcal{A})$ over f.

Lemma 7.2. *If $f : A \to B$ is a surjective homomorphism of graded monoids, then the induced map* $\mathrm{MProj}(B) \to \mathrm{MProj}(A)$ *is a closed immersion.*

Proof. As noted above, any affine open subscheme $U \subset \mathrm{MProj}(A)$ is of the form

$$U = \mathrm{MSpec}(A\big[\tfrac{1}{s}\big]_0)$$

for some $s \in A_{\geq 1}$. But $U \cap \mathrm{MProj}(B) = \mathrm{MSpec}(B\big[\frac{1}{f(s)}\big]_0)$ is affine, so we are in the case of Lemma 2.7. □

Projective monoid schemes. For a monoid A and indeterminates $T_0, \dots, T_n$, let $A[T_0, \dots, T_n]$ denote the monoid freely generated by A and the T_i. It is a graded monoid, where each element of A has degree 0 and each T_i has degree 1, and we define $\mathbb{P}^n_A$ to be $\mathrm{MProj}(A[T_0, \dots, T_n])$. More generally, for any monoid scheme $X = (X, \mathcal{A})$, define $\mathbb{P}^n_X$ to be $\mathrm{MProj}(\mathcal{B})$ where $\mathcal{B}$ is the sheaf of graded monoids on X defined by sending an open subset U to $\mathcal{A}(U)[T_0, \dots, T_n]$. In other words, $\mathbb{P}^n_X$ is defined by patching together the monoid schemes of the form $\mathbb{P}^n_A$ as $\mathrm{MSpec}(A)$ ranges over affine open subschemes of X. If X has finite type, so does $\mathbb{P}^n_X$.

A morphism of monoid schemes $Y \to X$ is *projective* if, locally on X, it factors as a closed immersion $Y \to \mathbb{P}^n_X$ for some n followed by the projection $\mathbb{P}^n_X \to X$.

Lemma 7.3. *Projective morphisms are separated.*

Although this follows from Proposition 5.13, we give an elementary proof here.

Proof. Since closed immersions are separated by Lemma 3.4, it suffices to show that the morphisms $\mathbb{P}^n_X \to X$ are separated. We may assume that $X = \mathrm{MSpec}(A)$, so that

$$\mathbb{P}^n_X = \mathrm{MProj}(A[T_0, \dots, T_n]).$$

By Definition 7.1, points of $\mathbb{P}^n_X$ correspond to prime ideals in $A[T_0, \dots, T_n]$ not containing $\{T_0, \dots, T_n\}$. By Lemma 1.9 and Example 1.2, every such prime ideal has the form

$$P_{S,\mathfrak{p}} = A \wedge \langle S \rangle \cup \mathfrak{p}[T_0, \dots, T_n]$$

where $\mathfrak{p}$ is a prime ideal of A and $\langle S \rangle$ is the prime ideal generated by a proper subset S of $\{T_0, \dots, T_n\}$; moreover $\mathfrak{p}$ and S are unique and the projection to $\mathrm{MSpec}(A)$ sends the point $P_{S,\mathfrak{p}}$ to $\mathfrak{p}$. According to Lemma 3.6, it suffices to observe that for every $P_{S,\mathfrak{p}}$ and $P_{S',\mathfrak{p}}$ the prime $P_{S\cap S',\mathfrak{p}}$ is a unique lower bound. (The surjectivity condition of Lemma 3.6 is easy, and left to the reader.) □

Example 7.3.1. If B is a finitely generated graded monoid, then

$$\mathrm{MProj}(B) \to \mathrm{MSpec}(B_0)$$

is projective and hence separated by Lemma 7.3. Indeed, this is a particular case of Lemma 7.2, since B is a quotient of some $B_0[T_0, \dots, T_n]$.

Blow-ups. Given a monoid A and an ideal I, we shall consider the graded monoid $A \vee I \vee I^2 \vee \cdots$, where I^n has degree n. It is useful to introduce a variable t, and rewrite this as

$$A[It] = \bigvee_{n\geq 0} I^n t^n \subseteq A \wedge F_1.$$

If S is multiplicatively closed in A, then $S^{-1}(A[It]) \cong (S^{-1}A)[S^{-1}It]$. It follows that if $\mathcal{I}$ is a quasi-coherent sheaf of ideals in a monoid scheme $(X, \mathcal{A})$, then there is a monoid scheme $\mathrm{MProj}(\mathcal{A}[\mathcal{I}t])$ over $(X, \mathcal{A})$ obtained by patching the $\mathrm{MProj}(A[It])$ in the evident manner.

Definition 7.4. If $X = (X, \mathcal{A})$ is a monoid scheme and $Z \subseteq X$ is an equivariant closed subscheme, given by a quasi-coherent sheaf of ideals $\mathcal{I}$, we define the *blow-up* of X along Z to be the monoid scheme $X_Z = \mathrm{MProj}(\mathcal{A}[\mathcal{I}t])$.

Remark 7.4.1. If $X = \mathrm{MSpec}(A)$ is affine and $Z = \mathrm{MSpec}(A/I)$, then

$$X_Z = \mathrm{MProj}(A[It]),$$

together with the structure morphism $\mathrm{MProj}(A[It]) \to \mathrm{MSpec}(A)$. Since

$$\mathrm{MProj}(A[t]) \cong \mathrm{MSpec}(A),$$

it follows that for $U = X \setminus Z$ we have $X_Z \times_X U \cong U$.

The blow-up construction is natural in the pair (A, I) in the following sense. If $A \to B$ is a morphism of monoids, I is an ideal of A and $J = IB$, there is a canonical graded morphism $A[It] \to B[Jt]$ satisfying the hypotheses of Observation 7.1.2. Hence there is a morphism $\mathrm{MProj}(B[Jt]) \to \mathrm{MProj}(A[It])$ of the blow-ups over $\mathrm{MSpec}(B) \to \mathrm{MSpec}(A)$. More generally, if $f : X' \to X$ is a morphism of monoid schemes, $\mathcal{I}$ is a quasi-coherent sheaf of ideals on X and $\mathcal{J} = f^{-1}\mathcal{I} \cdot \mathcal{A}'$, then the morphism $f^{-1}\mathcal{A}[\mathcal{I}t] \to \mathcal{A}'[\mathcal{J}t]$ induces a canonical morphism $\mathrm{MProj}(\mathcal{A}'[\mathcal{J}t]) \to \mathrm{MProj}(\mathcal{A}[\mathcal{I}t])$ over f, described in Observation 7.1.2.

Remark 7.4.2. The blow-up of X along a quasi-coherent sheaf of ideals $\mathcal{I}$ is projective provided $\mathcal{I}$ is given locally on X by finitely generated ideals, by Example 7.3.1. For example, if X has finite type, then the blowup of X along any quasi-coherent sheaf of ideals is projective.

Example 7.5. Suppose N is a free abelian group with basis $\{v_1, \ldots, v_n\}$, and suppose $\{x_1, \ldots, x_n\}$ is the dual basis of M. Let σ be the cone in $N_{\mathbb{R}}$ generated by $\{v_1, \ldots, v_d\}$, the corresponding affine monoid scheme is $X(\sigma) = \mathrm{MSpec}(A)$, where A is generated by $x_1, \ldots, x_n$ and $x_{d+1}^{-1}, \ldots, x_n^{-1}$ subject to $x_i x_i^{-1} = 1$ for $d < i \leq n$. The blow-up of $X(\sigma)$ along the ideal generated by $x_1, \ldots, x_d$ is the toric monoid scheme $X(\Delta)$, where Δ is the subdivision of the fan $\{\sigma\}$ given by insertion of the ray spanned by $v_0 = v_1 + \cdots + v_d$. To see this, it suffices to copy the corresponding argument for toric varieties given in [10, p. 41].

Example 7.5.1. If Z is an equivariant closed subscheme of X, defined by a quasi-coherent sheaf of ideals $\mathcal{I}$, and $f : X' \to X$ is a morphism, then by naturality of the blow-up construction, discussed above, there is a canonical morphism over f, from the blow-up $X'_{Z'}$ of X' along the pullback $Z' = Z \times_X X'$ to the blow-up X_Z.

Lemma 7.6. *Let $f : X' \to X$ be a finite morphism of monoid schemes* (6.2). *Let Z be an equivariant closed subscheme of X, X_Z the blow-up along Z, and $X'_{Z'}$ the blow-up of X' along the pullback $Z' = Z \times_X X'$. Then $\tilde{f} : X'_{Z'} \to X_Z$ is a finite morphism.*

Proof. We may assume that X, and hence X', is affine. Then f is induced by a map $A \to A'$, Z is defined by an ideal $I \subset A$ and Z' is defined by $J = I\,A'$. Moreover because f is assumed finite, there are elements $c_1, \ldots, c_r \in B$ such that $B = \bigcup_i Ac_i$. If $a_0, \ldots, a_n$ generate I, and $b_0, \ldots, b_n$ are their images in B, then $\tilde{f}$ restricts to maps $D_+(b_i) \to D_+(a_i)$ induced by the monoid maps $A_i = A[a_0/a_i, \ldots, a_n/a_i] \to B_i = B[b_0/b_i, \ldots, b_n/b_i]$. By inspection, $B_i = \bigcup_{j=1}^n A_i c_j$. □

Proposition 7.7. *Let Z be an equivariant closed subscheme of a monoid scheme X of finite type. Then for any commutative ring k the blow-up of X_k along Z_k is canonically isomorphic to the k-realization of the blow-up of X along Z.*

Proof. It suffices to consider the case

$$X = \mathrm{MSpec}(A), \quad Z = \mathrm{MSpec}(A/I).$$

In this case $S = k[A[It]]$ is the usual Rees ring $k[A][Jt]$, $J = k[I]$. Since the blowing-up of $X_k = \mathrm{Spec}(k[A])$ along $Z_k = \mathrm{Spec}(k[A/I])$ is $\mathrm{Proj}(S)$, we have the desired identification

$$\mathrm{Proj}(S) = \mathrm{Proj}(k[A[It]]) = \mathrm{MProj}(A[It])_k. \qquad \square$$

We conclude this section by observing that blow-ups of monoid schemes satisfy a universal property analogous to that for blow-ups of usual schemes. To state it, we need some notation. We define a *principal invertible* ideal of A to be an ideal I such that there is an $x \in I$ such that the map

$$A \xrightarrow{x} I \quad (a \mapsto ax)$$

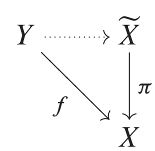
is a bijection. If I is a principal invertible ideal of A, then the canonical map

$$\mathrm{MProj}(A[It]) \to \mathrm{MSpec}(A)$$

is an isomorphism.

A quasi-coherent sheaf of ideals of a monoid scheme X is said to be *invertible* if X can be covered by affine open subschemes U such that $\mathcal{I}(U)$ is a principal invertible ideal of $\mathcal{A}_X(U)$. If $(X, \mathcal{A})$ is a monoid scheme and $\mathcal{I} \subset \mathcal{A}$ is a quasi-coherent sheaf of ideals (see Definition 2.8), we say that a morphism $f : Y \to X$ *inverts* $\mathcal{I}$ if $f^{-1}\mathcal{I} \cdot \mathcal{B}$ is an invertible sheaf on Y.

Proposition 7.8. *Let X be a monoid scheme of finite type, Z be an equivariant closed subscheme defined by a quasi-coherent sheaf of ideals $\mathcal{I}$, and $\pi : \widetilde{X} \to X$ be the blow-up of X along Z. Then π inverts $\mathcal{I}$ and is universal with this property in the sense that if Y is of finite type and $f : Y \to X$ inverts $\mathcal{I}$, then the dotted arrow in the diagram below exists and is unique.*



Proof. We may assume that $X = \mathrm{MSpec}(A)$ for some finitely generated monoid A, that $\mathcal{I}$ corresponds to an ideal I of A, and that $\widetilde{X} = \mathrm{MProj}(A[It])$. The map π inverts I because the restriction of $\pi^{-1}\mathcal{I}$ to $D_+(s)$ is generated by s for each $s \in I$. Let $\mathcal{B}$ be the structure sheaf of Y, and write $\mathcal{J}$ for the sheaf of ideals $f^{-1}\mathcal{I} \cdot \mathcal{B}$. By Example 7.5.1, there is a unique morphism from the blow-up $\widetilde{Y} = \mathrm{MProj}(\mathcal{B}[\mathcal{J}t])$ to $\widetilde{X}$ over f. By assumption, $\mathcal{J}$ is an invertible sheaf. Hence $\widetilde{Y} \to Y$ is an isomorphism, because locally $\mathcal{J}$ is a principal invertible ideal J of B and $\mathrm{MProj}(B[Jt]) \cong \mathrm{MSpec}(B)$. $\square$

8. Proper morphisms

A monoid V is called a *valuation monoid* if V is cancellative and for every nonzero element $\alpha \in V^+$, at least one of α or $\frac{1}{\alpha}$ belongs to V. For example, if R is a valuation ring, then the underlying multiplicative monoid $(R, \times)$ is a valuation monoid. Also, the free pointed monoid on one generator is a valuation monoid. Given a valuation monoid V, the monoid $V \wedge M_*$ is also a valuation monoid for any abelian group M. For example, the monoid $\langle y_1^{\pm 1}, \dots, y_n^{\pm 1}, x \rangle$ is a valuation monoid.

Given a valuation monoid V, the units $U(V)$ are a subgroup of $V^+ \setminus 0$ and the quotient group $(V^+ \setminus 0)/U(V)$ is a totally ordered abelian group with the total ordering defined by $x \geq y$ if and only if $\frac{x}{y}$ belongs to the image of $V \setminus 0$. To conform to usual custom, we convert the group law for $(V^+ \setminus 0)/U(V)$ into $+$. We also adjoin a base point, written ∞, to obtain the totally ordered pointed (additive) monoid

$$\Gamma := \big((V^+ \setminus 0)/U(V)\big)_*.$$

We extend the total ordering to Γ by declaring that $\gamma \leq \infty$ for all $\gamma \in \Gamma$. We call Γ the *value monoid* of the valuation monoid V. The canonical surjection

$$\text{ord} : V^+ \twoheadrightarrow \Gamma \tag{8.1}$$

is called the *valuation map* of V. The monoid V is then identified with the set of $x \in V^+$ such that $\text{ord}(x) \geq 0$ (where, recall 0 is the identity of Γ), and the maximal ideal $\mathfrak{m}$ of V is $\{x : \text{ord}(x) > 0\}$ (since $\text{ord}(x) = 0$ just in case x is a unit of V).

Note that the surjection in (8.1) satisfies $\text{ord}(x) \leq \infty$, $\text{ord}(xy) = \text{ord}(x) + \text{ord}(y)$ and $\text{ord}(x) = \infty$ if and only if $x = 0$. Conversely, given an abelian group M and a surjective morphism $\text{ord} : M_* \to \Gamma$ onto a totally ordered monoid $(\Gamma, +, 0, \infty)$ that satisfies these conditions, the set $C = \{a \in M : \text{ord}(a) \geq 0\}$ is a valuation monoid whose pointed group completion is M_* and whose associated valuation map is ord.

Lemma 8.2. *A valuation monoid V has no finite extensions contained in V^+.*

Proof. Suppose that $V \subseteq B \subseteq V^+$ with B finite over V. By Lemma 1.7 (ii), B is integral over V. For every nonzero $b \in B$ there is an $n \geq 1$ so that $b^n \in V$ and hence $n\ \text{ord}(b) \geq 0$, which implies that $\text{ord}(b) \geq 0$ and thus $b \in V$. □

Example 8.3. A *discrete valuation monoid* is a valuation monoid whose value monoid is isomorphic to $\mathbb{Z} \cup \{\infty\}$ with its canonical ordering. In this case, a lifting of the generator $1 \in \mathbb{Z}$ to an element π in the discrete valuation monoid V is a generator of the maximal ideal of V and every nonzero element of V^+ may written uniquely as $u\pi^n$ for $n \in \mathbb{Z}$ and $u \in U(V)$. Let us call such an element a *uniformizing parameter*.

Observe that if R is a discrete valuation ring, then $(R, \times)$ is a discrete valuation monoid and the notion of a uniformizing parameter has its usual meaning.

If V is a discrete valuation monoid, its valuation map induces a surjection

$$\pi : V^+ \setminus 0 \twoheadrightarrow \mathbb{Z};$$

write $M = \ker \pi$. A choice of uniformizing parameter t is equivalent to a section of π and identifies

$$V^+ \setminus 0 = M \times \langle t^{\pm 1} \rangle.$$

Under this identification, π is the evident projection. Thus, every discrete valuation monoid V is isomorphic to $U(V)_* \wedge \langle t \rangle$, where $\langle t \rangle$ is the free abelian monoid on one generator and $U(V)$ is the group of units of V. Any element of the form $u \wedge t$ with $u \in U(M)$ is a uniformizing parameter.

Remark 8.3.1. It is well known that a valuation ring is noetherian if and only if it is a discrete valuation ring; see [34, Section VI.10, Theorem 16] for a proof. The same argument shows that a valuation monoid is finitely generated if and only if it is a discrete valuation monoid with a finitely generated group of units.

Definition 8.4. A map $f : Y \to X$ of monoid schemes satisfies the *valuative criterion for properness* if for every valuation monoid V and commutative square

$$\text{(8.4a)} \qquad \begin{array}{ccc} \mathrm{MSpec}(V^+) & \longrightarrow & Y \\ \big\downarrow & \nearrow & \big\downarrow f \\ \mathrm{MSpec}(V) & \longrightarrow & X \end{array}$$

there is a unique map $\mathrm{MSpec}(V) \to Y$ causing both triangles to commute.

We say f satisfies the *valuative criterion of separatedness* if each such square has at most one completion.

A map $Y \to X$ of monoid schemes of finite type is said to be *proper* if it satisfies the valuative criterion for properness.

Remark 8.4.1. We are not certain what the correct definition of "proper" is for monoid schemes not of finite type. (Recall from Remark 3.4.1 that "separated and universally closed" is clearly not the correct definition.)

Given any morphism $f : \mathrm{MSpec}(V) \to X$, any affine open $U \subset X$ containing $f(\mathfrak{m})$ (where $\mathfrak{m}$ is the unique closed point of $\mathrm{MSpec}(V)$) will contain the image of $\mathrm{MSpec}(V)$. Hence the valuative criterion of properness and separatedness are local on the base: if $Y|_U \to U$ satisfies one of these criteria for every U in a covering of X, then so does $Y \to X$.

It is immediate from Definition 8.4 that the class of maps satisfying the valuative criterion of properness (resp., separatedness) is closed under composition and pullback.

Proposition 8.5. *A finite morphism between monoid schemes satisfies the valuative criterion of properness.*

Proof. Suppose $Y \to X$ is finite and consider a commutative square (8.4a) with V a valuation monoid. We may assume $Y \to X$ is a map of affine schemes, say given by a map of monoids $A \to B$. Then the square (8.4a) is associated to the square

$$\begin{array}{ccc} V^+ & \longleftarrow & B \\ \big\uparrow & & \big\uparrow \\ V & \longleftarrow & A \end{array}$$

of monoids. The image of B in V^+ is finite over V, but V is closed under finite extensions in V^+, by Lemma 8.2. It follows that the map $B \to V^+$ actually lands in V, which gives the diagonal map we seek. □

Corollary 8.6. *Closed immersions satisfy the valuative criterion of properness.*

Construction 8.7. To prove Theorem 8.9 below, we need a technical construction: Let V be a valuation monoid with group completion V^+ and value monoid $(\Gamma, +)$. Recall that totally ordered groups are necessarily torsionfree, and hence, for any field k, the ring $k[\Gamma]$ is an integral domain by Lemma 5.10.

For an element

$$\alpha = \sum_{\gamma} a_\gamma \gamma$$

of $k[\Gamma]$ (where for this ring we have rewritten Γ using $\cdot$ instead of $+$ notation), define

$$\mathrm{ord}(\alpha) = \min\{\gamma \in \Gamma : a_\gamma \neq 0\}.$$

(For $\alpha = 0$, set $\mathrm{ord}(0) = \infty$.) It is easily verified that $\mathrm{ord} : (k[\Gamma], \times) \to \Gamma$ is a monoid map such that $\mathrm{ord}(\alpha) = \infty$ if and only if $\alpha = 0$.

It follows that we get an induced map of pointed group completions

$$\mathrm{ord} : (k(\Gamma), \times) \to \Gamma$$

where $k(\Gamma)$ denotes the field of fractions of $k[\Gamma]$. Moreover, the composition

$$V^+ \longrightarrow (k(\Gamma), \times) \xrightarrow{\mathrm{ord}} \Gamma$$

coincides with the original valuation map $\mathrm{ord} : V^+ \to \Gamma$.

Finally, the pair $(k(\Gamma), \mathrm{ord})$ is a valuation in the usual ring-theoretic sense. To prove this, it remains to show

$$\mathrm{ord}(\alpha + \beta) \geq \min\{\mathrm{ord}(\alpha), \mathrm{ord}(\beta)\} \quad \text{for all } \alpha, \beta \in k(\Gamma).$$

One easily reduces to the case when $\alpha, \beta \in k[\Gamma]$, where it is obvious from the definition of ord.

Proposition 8.8. *Given a valuation monoid V, with pointed group completion V^+ and value monoid Γ, let* ord *be the valuation map on the field $k(\Gamma)$ given in Construction* 8.7*, and let $R \subset k(\Gamma)$ denote the associated valuation ring. Then the square of affine monoid schemes*

$$\begin{array}{ccc} \mathrm{MSpec}(k(\Gamma), \times) & \longrightarrow & \mathrm{MSpec}(V^+) \\ \downarrow & & \downarrow \\ \mathrm{MSpec}(R, \times) & \longrightarrow & \mathrm{MSpec}(V) \end{array}$$

is a pushout square in the category of monoid schemes.

Proof. For any monoid scheme T, suppose two morphisms $f : \mathrm{MSpec}(V^+) \to T$ and $g : \mathrm{MSpec}(R, \times) \to T$ are given causing the evident square to commute. Let $t \in T$ be the image of the unique closed point of $\mathrm{MSpec}(R, \times)$ under g, and let $U \subset T$ be any affine open subscheme of T containing t. Then g factors through U. Since $\mathrm{MSpec}(k(\Gamma), \times) \to \mathrm{MSpec}(V^+)$ is a bijection on underlying sets (each is a one-point set), the unique point of $\mathrm{MSpec}(V^+)$ also lands in U and hence f too factors through U. We may thus assume $T = U$ is affine. That is, it suffices to prove

$$\begin{array}{ccc} V & \longrightarrow & (R, \times) \\ \downarrow & & \downarrow \\ V^+ & \longrightarrow & (k(\Gamma), \times) \end{array}$$

is a pullback square in the category of pointed monoids. But this is evident since

$$\begin{array}{ccc} V^+ & \longrightarrow & (k(\Gamma), \times) \\ {\scriptstyle \text{ord}}\downarrow & & \downarrow{\scriptstyle \text{ord}} \\ \Gamma & \xrightarrow{=} & \Gamma \end{array}$$

commutes, $V = \{\alpha \in V^+ : \text{ord}(\alpha) \geq 0\}$ and $R = \{\beta \in k(\Gamma) : \text{ord}(\beta) \geq 0\}$. □

Recall that a map of (classical) k-schemes $Y_k \to X_k$, where k is a field, is said to satisfy the *valuative criterion of properness* (resp., *separatedness*) if every solid arrow square

$$\begin{array}{ccc} \text{Spec}(F) & \longrightarrow & Y_k \\ \downarrow & \nearrow & \downarrow \\ \text{Spec}(R) & \longrightarrow & X_k \end{array}$$

has a unique (resp., at most one) completion making both triangles commute whenever R is a valuation ring (which is necessarily a k-algebra) and F is its field of factions.

Theorem 8.9. *Let $f : X \to Y$ be a morphism of monoid schemes and let k be a field. The morphism $f_k : X_k \to Y_k$ satisfies the valuative criterion of properness (resp., separatedness) if and only if f satisfies the valuative criterion of properness (resp., separatedness).*

Proof. By Theorem 5.2, for any local k-algebra R, there is a natural adjunction isomorphism

$$\text{Hom}_k(\text{Spec}(R), X_k) \cong F_X(\text{Spec } R) = \text{Hom}_{\text{MSch}}(\text{MSpec}(R, \times), X).$$

Now suppose R is a valuation ring with field of fractions F. Then $V = (R, \times)$ is a valuation monoid with $V^+ = (F, \times)$. Since R and F are local, a commutative square of the form

$$\begin{array}{ccc} \text{Spec } F & \longrightarrow & Y_k \\ \downarrow & & \downarrow \\ \text{Spec } R & \longrightarrow & X_k \end{array}$$

corresponds via adjunction to a commutative square of monoid schemes given by the solid arrows in the diagram

$$\begin{array}{ccc} \text{MSpec}(V^+) & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow \\ \text{MSpec}(V) & \longrightarrow & X. \end{array} \tag{8.10}$$

If $Y \to X$ satisfies the valuative criterion of properness (resp., separatedness), there exists a unique (resp., at most one) morphism of monoid schemes $\text{MSpec}(V) \to Y$ represented by the dotted arrow above that causes both triangles to commute. Again by adjunction, this gives a unique map $\text{Spec}(R) \to Y_k$ causing both triangles to commute in the first square.

Conversely, say a square (8.10) is given. By Construction 8.7, there is a valuation ring R with field of fractions $F = k(\Gamma)$ and morphisms

$$\mathrm{MSpec}(R, \times) \to \mathrm{MSpec}\, V \quad \text{and} \quad \mathrm{MSpec}(F, \times) \to \mathrm{MSpec}\, V^{+}$$

fitting into a commutative diagram

$$(8.11) \qquad \begin{array}{ccccc} \mathrm{MSpec}(F,\times) & \longrightarrow & \mathrm{MSpec}(V^{+}) & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathrm{MSpec}(R,\times) & \longrightarrow & \mathrm{MSpec}(V) & \longrightarrow & X. \end{array}$$

By Proposition 8.8, the left-hand square is a pushout square in the category of monoid schemes. Using adjunction as above, if $Y_k \to X_k$ satisfies the valuative criterion of properness (resp., separatedness), there exists a unique (resp., at most one) map represented by the dotted arrow in (8.11) that causes the outer two triangles to commute. Since the left-hand square is a pushout, it follows immediately that there exists a unique (resp., at most one) arrow $\mathrm{MSpec}(V) \to Y$ causing both triangles in (8.11) to commute. □

Corollary 8.12. *For any field k, a morphism between monoid schemes of finite type $Y \to X$ is proper if and only if $Y_k \to X_k$ is proper.*

Proof. Merely observe that Y_k and X_k are noetherian, and apply the valuative criterion of the properness theorem [20, Theorem II.4.7]. □

Remark 8.12.1. Say $f : Y \to X$ satisfies the valuative criterion of properness. If Y_k is quasi-compact, [15, Proposition 7.2.1] implies that f_k is proper.

Corollary 8.13. *A morphism between monoid schemes of finite type is proper if and only if it satisfies the valuative criterion of properness of Definition* 8.4 *for all discrete valuation monoids.*

Proof. If $f : X \to Y$ satisfies the criterion of Definition 8.4 for all discrete valuation monoids, then, for any field k, its k-realization $f_k : X_k \to Y_k$ satisfies the valuative criterion of properness for all DVRs. This follows, using adjunction, from the fact that $\mathrm{MSpec}(R, \times)$ is a discrete valuation monoid if R is a DVR. Since X_k and Y_k are noetherian and f_k has finite type, it follows that f_k is proper (see [20, Example II.4.11]). The result now follows from Corollary 8.12. □

Corollary 8.14. *A projective morphism $Y \to X$ between monoid schemes of finite type is proper. In particular, if X is a monoid scheme of finite type and X_Z is the blow-up along an equivariant closed subscheme Z, then the map $X_Z \to X$ is proper.*

Proof. Using Proposition 5.11 and Remark 7.1.1, we see that if k is a field, then the map $Y_k \to X_k$ is a projective morphism of k-schemes and hence is proper. For the second assertion, recall that $X_Z \to X$ is projective and X_Z has finite type. □

Remark 8.15. In fact, a projective morphism of arbitrary monoid schemes satisfies the valuative criterion of properness. We sketch the proof of this fact. First one observes that, by Corollary 8.6, it suffices to check that for any monoid scheme X and $n \geq 1$ the projection $\mathbb{P}^n_X \to X$ satisfies the criterion. Second, one reduces further to showing that if V is a valuation monoid, then any section $\mathrm{MSpec}(V^+) \to \mathbb{P}^n_{V^+}$ of the canonical projection extends to a section $\mathrm{MSpec}(V) \to \mathbb{P}^n_V$ of $\mathbb{P}^n_V \to \mathrm{MSpec}(V)$. Third, one observes that for an affine scheme $\mathrm{MSpec}\, A$ a section of $\mathbb{P}^n_A \to \mathrm{MSpec}\, A$ is determined by an equivalence class of n-tuples $(b_0, \dots, b_n)$ of elements of A such that at least one of the b_i is nonzero, modulo the coordinate-wise action of $U(A)$. Finally, one proves that if $b = (b_0, \dots, b_n)$ determines a section $\mathrm{MSpec}(V^+) \to \mathbb{P}^n_{V^+}$ as above, then multiplying the b_i by an appropriate power of a uniformizing parameter we obtain an equivalent tuple b' with $b'_i \in V$ for all i. Thus the section extends to $\mathrm{MSpec}(V)$.

Recall from Definition 4.1 that a monoid scheme of finite type is toric if it is separated, connected, torsionfree and normal. By Theorem 4.4, there is a faithful functor from fans to toric monoid schemes.

Corollary 8.16. *Let $\phi : (N', \Delta') \to (N, \Delta)$ be a morphism of fans. Then the associated morphism of toric monoid schemes $X' \to X$ is proper if and only if ϕ has the property that for each $\sigma \in \Delta$, $\phi_{\mathbb{R}}^{-1}(\sigma)$ is a union of cones in Δ'.*

Proof. This follows from the well-known fact that if k is a field, then $X'_k \to X_k$ is proper if and only if ϕ has the stated property (see [10, p. 39]). □

Corollary 8.17. *Every proper map between monoid schemes of finite type is separated.*

Proof. From Theorem 8.9 and the Valuative Criterion of Separatedness Theorem for noetherian schemes, it follows that the k-realization of a proper map between monoid schemes of finite type is separated if k is a field. Now use Proposition 5.13. □

9. Partially cancellative torsion free monoid schemes

A monoid A is *pctf* if it is isomorphic to a monoid of the form B/I where B is a cancellative torsion free monoid (i.e., a cancellative monoid whose group completion is torsion free) and I is an ideal. A monoid scheme is *pctf* if all of its stalks are.

Proposition 9.1. *We have:*

(1) *If a pctf monoid is finitely generated, then it is isomorphic to A/I where A is a finitely generated torsion free cancellative monoid.*

(2) *All submonoids and localizations of a pctf monoid are pctf. In particular, for a monoid A,* $\mathrm{MSpec}(A)$ *is pctf if and only if A is pctf.*

(3) *If A is a pctf monoid and $\mathfrak{p}$ is a prime ideal, then $A/\mathfrak{p}$ is a cancellative torsionfree monoid.*

(4) *An open subscheme of a pctf monoid scheme is pctf.*

(5) *An equivariant closed subscheme of a pctf monoid scheme is pctf.*

Proof. Say $A = B/I$ with B cancellative and torsion free. Pick elements $b_1, \dots, b_m$ in B that map to a generating set of A and let B' be the submonoid of B they generate. Then $A = B'/(I \cap B')$, proving the first assertion.

For the second, say $A = C/I$ with C cancellative and torsion free. If B is a submonoid of A, let B' denote the inverse image of B in C and set $I' = I \cap B'$. Then $B = B'/I'$, and so B is pctf. The assertion concerning localizations holds since $S^{-1}(C/I) \cong S^{-1}C/S^{-1}I$. The remaining assertion of part (2) is clear.

If $A = B/I$, then $A/\mathfrak{p} = B/\mathfrak{p}'$ for some prime ideal of B, so (3) follows from the elementary observation that if A is cancellative and torsionfree then so is $A/\mathfrak{p}$.

Assertion (4) is local and follows from (2); hence (5) is local, and is then easy. □

Proposition 9.2. *The blow-up of a pctf monoid scheme along an equivariant closed subscheme is pctf.*

Proof. Let $Y \to X$ be the blow-up of a pctf monoid scheme X along an equivariant closed subscheme. Since the question is local on X, we may assume that X is affine, say $X = \text{MSpec}(A)$ with A pctf. Then Y is $\text{MProj}(A[It])$ for an ideal I. For each $s \in I$, we get an affine open subset of Y given by the monoid

$$\left\{ \frac{f}{s^n} : f \in I^n, n \geq 0 \right\}.$$

This is a submonoid of $A[\frac{1}{s}]$ and hence is pctf. The collection of such open subsets as s varies over all elements of I forms an open cover of Y. Thus Y is pctf. □

Proposition 9.3. *Let $X = (X, \mathcal{A})$ and $Y = (Y, \mathcal{B})$ be monoid schemes and suppose $f : Y \to X$ is a morphism. There is a unique closed subscheme Z of X which is minimal with respect to the property that f factors through $Z \subset X$.*

If $U \subset X$ is an affine open subscheme of X, then $Z \cap U$ is the affine scheme $\text{MSpec}(C)$*, where the monoid C is the image of $\mathcal{B}(U) \to \mathcal{A}(U \times_X Y)$. In particular, if X is of finite type, then so is Z.*

Proof. If f factors through two different closed subschemes W_1 and W_2 of X, then it factors through $W_1 \times_X W_2$, which is (canonically isomorphic to) a closed subscheme of X (see Example 3.2). So, we define Z to be the inverse limit taken over the partially ordered set of closed subschemes W of X such that f factors through W.

For the local description of Z, we may assume that $X = U = \text{MSpec}(B)$ is affine. Any closed subscheme of X has the form $W = \text{MSpec}(D)$ with $B \to D$ a surjection of monoids. Then f factors through W if and only if $B \to \mathcal{A}(Y)$ factors through D, that is, if and only if $B \to C$ factors as $B \to D \to C$; in other words, if and only if $Z \subseteq W$. □

Definition 9.4. The subscheme Z of Proposition 9.3 is called the *scheme-theoretic image* of f. If f is an open immersion, we write $\overline{Y}$ for Z and (by abuse) call it the *closure* of Y.

Proposition 9.5. *Let Y be a monoid scheme and suppose $U \subset Y$ is an open subscheme that is pctf. Then the scheme-theoretic image $\overline{U}$ of U in Y is pctf. Moreover, if Y is separated, then $\overline{U}$ is separated.*

Proof. The first assertion is local on Y and so we may assume $Y = \mathrm{MSpec}(B)$ for a monoid B and $U = \mathrm{MSpec}(S^{-1}B)$ for a multiplicative subset S. Then $\overline{U}$ is the affine scheme associated to the image $\overline{B}$ of $B \to S^{-1}B$. The monoid $S^{-1}B$ is pctf by assumption and Proposition 9.1 (4), and hence so is $\overline{B}$ by Proposition 9.1 (2).

The second assertion is just the observation that a closed subscheme of a separated scheme is also separated by Lemma 3.4. □

10. Birational morphisms

A morphism $p : Y \to X$ of monoid schemes is *birational* if there is an open dense subscheme U of X such that $p^{-1}(U)$ is dense in Y and p induces an isomorphism from $p^{-1}(U)$ to U.

Proposition 10.1 (Birational maps). *Let $p : (Y, B) \to (X, A)$ be a map between monoid schemes of finite type. Then p is birational if and only if the following conditions hold:*

(1) *p maps the generic points of Y bijectively onto the generic points of X.*

(2) *A point $y \in Y$ is generic if (and only if) $p(y) \in X$ is generic.*

(3) *For each generic point $y \in Y$ the induced map $A(p(y)) \to B(y)$ on stalks is an isomorphism.*

Proof. If p is birational and U is as in the definition above, then U contains all of the generic points of X and $p^{-1}(U)$ contains all the generic points of Y as well as every point of y that maps to a generic point of X. The conditions are then clearly satisfied.

Conversely, take U to be the (dense open) set of generic points of X. By hypothesis, $p^{-1}(U)$ is the set of generic points of Y and the map $p : p^{-1}(U) \to U$ is bijective. Hence $p^{-1}(U)$ is open and dense. Since the map $p^{-1}(U) \to U$ is bijective and induces an isomorphism on all stalks, it is an isomorphism. □

Corollary 10.2. *If $p : X' \to X$ is a proper map of toric monoid schemes that is birational, then p is given by a map of fans $\phi : (N', \Delta') \to (N, \Delta)$ such that*

$$\phi : N' \xrightarrow{\cong} N$$

and the image of Δ' under the isomorphism $\phi_{\mathbb{R}}$ is a subdivision of Δ. Conversely, any such map ϕ induces a proper birational map of monoid schemes.

Proof. From Theorem 4.4 (2), p comes from a morphism of fans such that $\phi : N' \xrightarrow{\cong} N$, and such a morphism is a subdivision by Corollary 8.16. Conversely, if p is induced by a morphism of fans $\phi : (N', \Delta') \to (N, \Delta)$ such that $\phi_{\mathbb{R}}$ is a subdivision of Δ, then p_k is proper by [10, Section 2.4]; hence p is proper by Theorem 8.9. □

Example 10.3. If X is a monoid scheme of finite type, let X_η denote the equivariant closure of a generic point η (in the sense of Lemma 2.9). Then each X_η has a unique generic point, namely η. If X is pctf, then each X_η is cancellative and torsionfree by Proposition 9.1 (3), and hence pctf. If X is reduced, the morphism $\coprod_\eta X_\eta \to X$ is birational.

Proposition 10.4. *If $Y \to X$ is a birational map and $X' \to X$ is a morphism such that X' is of finite type and every generic point of X' maps to a generic point of X, then the pullback $Y \times_X X' \to X'$ is birational.*

Proof. The poset underlying $Y \times_X X'$ is given by the pullback of the underlying posets (by Proposition 3.1). Since $Y \to X$ is birational, a point (y, x') in $Y \times_X X'$ is generic if and only if x' is a generic point of X', and in this case y and x' map to the same point x of X, which is generic. Hence the map $Y \times_X X' \to X'$ is a bijection on sets of generic points. Writing A', A and B for the stalk functors of X', X and Y, the map on generic stalks is of the form

$$A'(x') \to A'(x') \wedge_{A(x)} B(y).$$

This is an isomorphism, since the map $A(x) \to B(y)$ is an isomorphism. □

Define the *height* of a point x in a monoid scheme X to be the dimension of $\mathcal{A}_x$, i.e., it is the largest integer n such that there exists a strictly decreasing chain $x = x_n > \cdots > x_0$ in the poset underlying X. We write this as $\mathrm{ht}(x)$ or $\mathrm{ht}_X(x)$.

For example, if $X = X(N, \Delta)$ is the monoid scheme associated to a fan, then

$$\mathrm{ht}(\sigma) = \dim(\sigma)$$

for each cone $\sigma \in \Delta$. Here $\dim(\sigma)$ refers to the dimension of the real vector subspace of $N_{\mathbb{R}}$ spanned by σ.

Lemma 10.5. *Suppose $p : Y \to X$ is a proper, birational map of separated pctf schemes of finite type. Then for any $y \in Y$, we have*

$$\mathrm{ht}_Y(y) \le \mathrm{ht}_X\big(p(y)\big).$$

Proof. Suppose $\mathrm{ht}_Y(y) = m$, so that we have a chain of points $y = y_m > \cdots > y_0$ in Y. Clearly y_0 must be minimal, and thus generic. Let $\eta = p(y_0)$, and define X_η to be the equivariant closure of $\{\eta\}$ in X. As pointed out in Example 10.3, X_η is cancellative and torsionfree. The pullback $Y_\eta = X_\eta \times_X Y$ is an equivariant closed subscheme of Y containing y_0 as its unique generic point, and hence each y_i. By Proposition 9.1 (5), Y_η is also pctf, and $Y_\eta \to X_\eta$ is birational by Proposition 10.4.

Let Y' denote the equivariant closure of y_0 in Y_η. By Example 10.3, $Y' \to Y_\eta$ is birational, Y' contains all the y_i and Y' is cancellative and torsionfree. Replacing X and Y by X_η and Y', we may assume that both X and Y are connected, cancellative and torsionfree. Hence the normalization maps $X_{\mathrm{nor}} \to X$ and $Y_{\mathrm{nor}} \to Y$ exist and are homeomorphisms (by Remark 1.6.1), and both X_{nor} and Y_{nor} are torsionfree. Since $Y \to X$ is birational, it induces a birational morphism $Y_{\mathrm{nor}} \to X_{\mathrm{nor}}$. The map $Y_{\mathrm{nor}} \to Y$ is finite by Lemma 1.7 and hence proper by Proposition 8.5. Thus $Y_{\mathrm{nor}} \to X$ and hence $Y_{\mathrm{nor}} \to X_{\mathrm{nor}}$ are proper. Thus we may assume that X and Y are separated, normal and torsionfree.

By Proposition 4.5 and Corollary 4.6, we have reduced to the case where $Y \to X$ is a proper birational map of toric monoid schemes, given by a map of fans $\phi : (N', \Delta') \to (N, \Delta)$. The birational hypothesis means that $\phi : N' \to N$ is an isomorphism. By Corollary 10.2, the proper hypothesis means that Δ' is a subdivision of Δ. Since $\phi(\sigma)$ is the smallest cone in Δ containing the image of σ under $\phi_{\mathbb{R}}$ and since height corresponds to dimension of cones, the result is now clear. □

11. Resolutions of singularities for toric varieties

The purpose of this section is to establish some properties for monoid schemes that are analogous to those known to hold for arbitrary varieties in characteristic zero. These properties will be used in Section 12 to prove that certain presheaves of spectra satisfy the analogue of "smooth cdh descent" for monoid schemes.

Theorem 11.1. *Let X be a separated cancellative pctf monoid scheme of finite type. Then there is a birational proper morphism $Y \to X$ such that Y is smooth.*

Proof. We may assume that X is connected. Since the normalization map is proper birational by Propositions 6.3 and 8.5, we may assume that X is normal. Since X is pctf, it is torsionfree by Proposition 9.1 (3). By Proposition 4.5, X is toric and $X \cong X(\Delta)$ for some fan Δ. There exists a subdivision Δ' of Δ such that $X(\Delta')$ is smooth, and it follows from Corollary 10.2 that the morphism $X(\Delta') \to X(\Delta)$ is proper birational. □

Let N be a free abelian group of finite rank. Recall (from [10, p. 34], e.g.) that a cone in $N_{\mathbb{R}}$ is called *simplicial* if it is generated by linearly independent vectors, and that a fan is simplicial if every cone in it is simplicial. We will need the notion of the barycentric subdivision of a simplicial fan Δ in $N_{\mathbb{R}}$: For a simplicial cone σ in $N_{\mathbb{R}}$ of dimension d, let $v_1, \dots, v_d$ be the minimal lattice points along the one-dimensional faces of σ, also called the rays of σ. For each nonempty subset S of $\{1, \dots, d\}$, let

$$v_S = \sum_{i \in S} v_i .$$

The *barycentric subdivision* of σ, which we write as $\sigma^{(1)}$, is defined as the collection of 2^d cones given as the span of vectors of the form $v_{S_1}, \dots v_{S_e}$, where $0 \le e \le d$ and $S_1 \subset \cdots \subset S_e$ is a chain of proper subsets of $\{1, \dots, d\}$. It is clear that if τ is a face of σ, then the set of cones in $\sigma^{(1)}$ that are contained in τ form the fan $\tau^{(1)}$. It follows that

$$\Delta^{(1)} := \{\sigma^{(1)} : \sigma \in \Delta\}$$

is again a simplicial fan. We inductively define

$$\Delta^{(i)} = (\Delta^{(i-1)})^{(1)}$$

for $i \ge 2$.

Lemma 11.2. *If Δ' is any subdivision of a simplicial fan Δ in $N_{\mathbb{R}}$, then for $i \gg 0$, the fan $\Delta^{(i)}$ is a subdivision of Δ'.*

Proof. It suffices to show that any ray of Δ', that is, any 1-dimensional cone of Δ', is a ray of some $\Delta^{(i)}$. Given a positive integer combination $v = \sum n_i v_i$ of the vertices in a cone, we may reorder the vertices to assume the n_i are in decreasing order. Then v is in the cone of $\Delta^{(1)}$ spanned by the v_{S_i}, where $S_i = \{1, \dots, i\}$, and (if $v \ne v_1$) we can write

$$v = \sum_i n'_i v_{S_i}$$

with $\sum n'_i < \sum n_i$. The result follows by induction on $\sum n_i$. □

Lemma 11.3. *If Δ is a smooth fan, then for all $i \geq 1$, the toric monoid scheme $X(\Delta^{(i)})$ is obtained from $X(\Delta)$ via a sequence of blow-ups along smooth centers.*

Proof. We may assume $i = 1$. If Δ is smooth already, then $\Delta^{(1)}$ is also smooth. In general, the fan $\Delta^{(1)}$ is obtained from Δ via a series of steps of the following sort: starting with a smooth fan Δ, we form a subdivision Δ' by picking a cone σ, letting $v_1, \dots, v_d$ be the minimal lattice points along its rays, and defining Δ' to be the subdivision of Δ given by insertion of the ray spanned by $v_1 + \cdots + v_d$. By Example 7.5, $X(\Delta') \to X(\Delta)$ is the blow-up along the smooth, closed equivariant subscheme defined by $x_1 = \cdots = x_d = 0$. □

Theorem 11.4. *For a morphism $\pi : Y \to X$ between separated cancellative pctf monoid schemes of finite type, assume X is smooth and $\pi : Y \to X$ is proper and birational. Then there exists a sequence of blow-ups along smooth closed equivariant centers,*

$$X^n \to \cdots \to X^1 \to X_0 = X,$$

such that $X^n \to X$ factors through $\pi : Y \to X$.

Proof. By Theorem 11.1, there is a proper birational morphism $Z \to Y$ with Z smooth. We may therefore assume that Y is smooth. We may also assume that X and Y are connected, so that they have unique generic points.

Thus, by Corollary 10.2, $Y \to X$ is given by a morphism $(N', \Delta') \to (N, \Delta)$ of fans that is an isomorphism of lattices and such that Δ' is a subdivision of Δ. Lemmas 11.2 and 11.3 complete the proof. □

12. cd-structures on monoid schemes.

Let $\mathcal{M}_{\text{pctf}}$ denote the category of monoid schemes of finite type that are separated and pctf. In this section, we will be concerned with cartesian squares of the form

(12.1)
$$\begin{array}{ccc} D & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ C & \xrightarrow{e} & X. \end{array}$$

Definition 12.2. An *abstract blow-up* is a cartesian square of monoid schemes of finite type of the form (12.1) such that p is proper, e is an equivariant closed immersion, and p maps the open complement $Y \setminus D$ isomorphically onto $X \setminus C$. The square with $Y = \emptyset$ and $C = X_{\text{red}}$ is such a square.

Proposition 12.3. *If X is of finite type, C is an equivariant closed subscheme of X and $p : Y \to X$ is the blow-up of X along C, then the resulting cartesian square is an abstract blow-up. If X belongs to $\mathcal{M}_{\text{pctf}}$, so do Y, C and D.*

Proof. By Corollary 8.14, p is proper. As noted in Definition 7.4, p maps $Y \setminus D$ isomorphically to $X \setminus C$ (because $D = C \times_X Y$). The second assertion follows from Propositions 9.1 and 9.2. □

Proposition 12.4. *Suppose an abstract blow-up square* (12.1) *is given with X in $\mathcal{M}_{\text{pctf}}$. Let $\bar{Y}$ be the scheme-theoretic image of $Y \setminus D$ in Y, and define $\bar{D} = C \times_X \bar{Y}$. Then*

$$\begin{array}{ccc} \overline{D} & \longrightarrow & \overline{Y} \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ C & \xrightarrow{e} & X \end{array}$$

is an abstract blow-up square in $\mathcal{M}_{\text{pctf}}$.

Proof. By Proposition 9.1, $X \setminus C$ and hence $Y \setminus D$ is pctf, and so by Proposition 9.5, $\overline{Y}$ is pctf as well. Since equivariant closed subschemes of pctf schemes are pctf, C and $\overline{D}$ also belong to $\mathcal{M}_{\text{pctf}}$. The map $\overline{Y} \to X$ is a composition of proper maps and hence is proper. Finally, $\bar{Y} \setminus \bar{D} = Y \setminus D$. □

Recall from [30, Definition 2.1] that a cd-structure on a category $\mathcal{C}$ is a collection of distinguished commutative squares in $\mathcal{C}$. If $\mathcal{C}$ has an initial object $\emptyset$, any cd-structure defines a topology: the smallest Grothendieck topology such that for each distinguished square (12.1) the sieve generated by $\{p, e\}$ is a covering sieve (and the empty sieve is a covering of the initial object). The coverings $\{p, e\}$ are called *elementary*.

Definition 12.5. The *blow-up* cd-*structure* on $\mathcal{M}_{\text{pctf}}$ is given by the collection of all abstract blow-up squares with X, Y, C, D all belonging to $\mathcal{M}_{\text{pctf}}$. The *Zariski* cd-*structure* on $\mathcal{M}_{\text{pctf}}$ is given by all cartesian squares associated to a covering of X by two open subschemes.

The cdh topology on $\mathcal{M}_{\text{pctf}}$ is the topology generated by the union of these two cd-structures.

Following [30, Definition 2.3 and Lemma 2.4], we say that a cd-structure is *complete* if $\mathcal{C}$ has an initial object $\emptyset$ and any pullback of an elementary covering contains a sieve which can be obtained by iterating elementary coverings. We say that a cd-structure is *regular* (see [30, Definition 2.10]) if each distinguished square (12.1) is a pullback, e is a monomorphism and the morphism of sheaves

$$\rho(D) \times_{\rho(C)} \rho(D) \amalg \rho(Y) \to \rho(Y) \times_{\rho(X)} \rho(Y) \tag{12.6}$$

is onto, where $\rho(T)$ denotes the sheafification of the presheaf represented by T.

Theorem 12.7. *The blow-up and Zariski* cd-*structures on $\mathcal{M}_{\text{pctf}}$ are complete and regular.*

Proof. The completeness property for Zariski squares is clear since they are preserved by pullback, and the regularity property is even clearer. For the blow-up cd-structure, consider an abstract blow-up square

$$\begin{array}{ccc} D & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ C & \xrightarrow{e} & X. \end{array}$$

Let $X' \to X$ be any morphism in $\mathcal{M}_{\text{pctf}}$ and consider the square involving X', C', Y' and D' formed by pullback. The scheme Y' might not belong to $\mathcal{M}_{\text{pctf}}$, but the scheme-theoretic image Y'' of $Y' \setminus D'$ in Y' does by Proposition 12.4. The resulting square involving C', X', Y'' and

$D'' := C' \times_{X'} Y''$ is an abstract blow-up by the same result, and hence by [30, Lemma 2.4] the blow-up cd-structure is complete.

For the regularity property, we need to show that (12.6) is onto. Every object admits a covering in this topology by affine, cancellative monoids, and it suffices to prove surjectivity of the map given by the underlying presheaves evaluated at such an affine cancellative U. That is, say $f : U \to Y$, $g : U \to Y$ are given with $p \circ f = p \circ g$. We need to prove either $f = g$ or they both factor through D and coincide as maps to C. Let u be the unique generic point of U. If either $f(u)$ or $g(u)$ lands in $Y \setminus D$, then they both must land there. Since $Y \setminus D \cong X \setminus C$, it follows that f and g coincide generically. But since U is cancellative, it follows $f = g$ on all of U. (To see this, one may work locally: If $h, l : A \to B$ are two maps of monoids with B cancellative and the compositions of h, l with the inclusion $B \hookrightarrow B^+$ coincide, then $h = l$.) Otherwise, we have that the generic point, and hence every point, of U is mapped by both f and g to points in the closed subset D of Y. Again using that U is cancellative, it follows that f, g factor through $D \hookrightarrow Y$. (This is also proven by working locally.) Finally, the compositions of these maps $f, g : U \to D$ with $D \to C$ coincide since $C \to X$ is a closed immersion. □

We define the *standard density structure* on $\mathcal{M}_{\mathrm{pctf}}$ as follows: The set $\mathcal{D}_i(X)$ consists of those open immersions $U \subset X$ such that every point in $X \setminus U$ has height at least i. It is clear that this satisfies the axioms required of a density structure of finite dimension in [30, Definition 2.20].

A cd-structure is said to be *bounded* for a given density structure if any distinguished square has a refinement which is reducing for the density structure in the sense of [30, Definition 2.21].

Theorem 12.8. *The blow-up and Zariski* cd-*structures on* $\mathcal{M}_{\mathrm{pctf}}$ *are both bounded for the standard density structure.*

Proof. To see that the blow-up cd-structure is bounded, we need to show that any abstract blow-up square (12.1) in $\mathcal{M}_{\mathrm{pctf}}$ has a refinement that is reducing for $\mathcal{D}_*$. Consider the square obtained by replacing Y by the monoid scheme-theoretic image of $Y \setminus D$ (in the sense of Definition 9.4), and D by the pullback. This is also an abstract blow-up square, and it refines (12.1). This refinement has the features that $p^{-1}(X \setminus C)$ is dense in Y, Y maps birationally onto the scheme-theoretic image of $X \setminus C$ in X, and D does not contain any generic points of Y.

To show that this square is reducing, we assume given $C_0 \in \mathcal{D}_i(C), Y_0 \in \mathcal{D}_i(Y)$ and $D_0 \in \mathcal{D}_{i-1}(D)$. Define X' to be the open subscheme $X \setminus Z$ of X, where $Z \subset X$ is the equivariant closure (in the sense of Lemma 2.9) of the union of the images of each of $C \setminus C_0$, $D \setminus D_0$ and $Y \setminus Y_0$ in X. We need to show that X' belongs to $\mathcal{D}_i(X)$ and that the pullback of the original square (12.1) along $X' \hookrightarrow X$ gives an abstract blow-up square.

If $y \in Y$ is a point of height at least i, then $p(y)$ has height at least i in the scheme-theoretic image of $X \setminus C$, by Lemma 10.5. Hence $p(y)$ has height at least i in X itself (since a closed immersion is an injection on underlying posets). If $d \in D$ has height at least $i-1$, then its height in Y is at least i (since D contains no generic points of Y) and hence its image in X has height at least i too. Since C is an equivariant closed subscheme, if $c \in C$ has height at least i, it has height at least i in X.

Recall that $Z \subset X$ is the equivariant closure of the union of the images of each of $C \setminus C_0$, $D \setminus D_0$ and $Y \setminus Y_0$ in X. Each of these images consists of points of height at least i and hence every point in Z has height at least i in X by Remark 2.9.1. Therefore X' belongs to $\mathcal{D}_i(X)$ and the pullback of the above square along $X' \hookrightarrow X$ gives an abstract blow-up square that proves our original square is reducing.

The argument in the previous paragraphs applies *mutatis mutandis* to show that every Zariski square is reducing. □

Corollary 12.9. *Let $\mathscr{S}$ be a presheaf of abelian groups on $\mathcal{M}_{\text{pctf}}$; let t be either the Zariski or the* cdh *topology, and write $a_t\mathscr{S}$ for the sheafification with respect to t. If $X \in \mathcal{M}_{\text{pctf}}$ is of dimension d, then*

$$H^n_t(X, a_t\mathscr{S}) = 0 \quad \textit{for } n > d.$$

Proof. Immediate from Theorem 12.8 and [30, Theorem 2.26]. □

The category of spectra we use in this paper will not be critical. In order to minimize technical issues, we will use the terminology that a *spectrum* E is a sequence E_n of simplicial sets together with bonding maps $b_n : E_n \to \Omega E_{n+1}$. We say that E is an Ω*-spectrum* if all bonding maps are weak equivalences. A map of spectra is a strict map. We will use the model structure on the category of spectra defined in [3]. Note that in this model structure, every fibrant spectrum is an Ω-spectrum. Given a Grothendieck topology, the category of contravariant functors $\mathcal{F}$ from $\mathcal{M}_{\text{pctf}}$ to spectra (*presheaves* of spectra) has a closed model structure, in which a morphism $\phi : \mathcal{F} \to \mathcal{F}'$ is a cofibration when $\mathcal{F}(X) \to \mathcal{F}'(X)$ is a cofibration for every monoid scheme X in $\mathcal{M}_{\text{pctf}}$; ϕ is a weak equivalence if it induces isomorphisms between the sheaves of stable homotopy groups (see [23, 24]). We write $\mathbb{H}_{\text{cdh}}(-, \mathcal{F})$ for the fibrant replacement of $\mathcal{F}$ using this model structure for the cdh topology, as in [7].

A presheaf of spectra $\mathcal{F}$ on $\mathcal{M}_{\text{pctf}}$ satisfies the *Mayer–Vietoris property* for some family $\mathcal{C}$ of cartesian squares if $\mathcal{F}(\emptyset) = *$ and the application of $\mathcal{F}$ to each member of the family gives a homotopy cartesian square of spectra.

Proposition 12.10. *Let $\mathcal{F}$ be a presheaf of spectra on $\mathcal{M}_{\text{pctf}}$. Then the canonical map $\mathcal{F}(X) \to \mathbb{H}_{\text{cdh}}(X, \mathcal{F})$ is a weak equivalence of spectra for all X if and only if it has the Mayer–Vietoris property for every abstract blow-up square and every Zariski square of pctf monoid schemes.*

Proof. By Theorems 12.7 and 12.8, the cdh cd-structure is complete, regular and bounded. Now the assertion follows from [7, Theorem 3.4]. □

Given Proposition 12.10, the definition of cdh descent given in [7, Terminology 3.5] becomes:

Definition 12.11. Let $\mathcal{F}$ be a presheaf of spectra on $\mathcal{M}_{\text{pctf}}$. We say that $\mathcal{F}$ satisfies cdh *descent* if the canonical map $\mathcal{F}(X) \to \mathbb{H}_{\text{cdh}}(X, \mathcal{F})$ is a weak equivalence of spectra for all X.

Remark 12.11.1. Writing $\mathbb{H}_{\text{zar}}(-, \mathcal{F})$ for the fibrant replacement with respect to the model structure for the Zariski topology, we obtain the notion of Zariski descent. The proof of Proposition 12.10 applies to show that $\mathcal{F}$ satisfies Zariski descent if and only if it has the Mayer–Vietoris property for every Zariski square. It follows that cdh descent implies Zariski descent.

It is useful to restrict to the full subcategory $\mathcal{S}$ of smooth monoid schemes (see Definition 6.4). By Proposition 6.5, these are the cancellative, torsionfree, separated monoid schemes of finite type whose k-realizations are smooth for any commutative ring k. (This condition is independent of k, by Proposition 6.5.)

Definition 12.12. We define the *smooth blow-up* cd-*structure* on $\mathcal{S}$ to consist of squares (12.1) such that X is smooth, e is the inclusion of an equivariant, smooth closed subscheme and Y is the blow-up of X along C. (These assumptions ensure, by (7.7), that Y and D are also smooth.)

The Zariski cd-structure is given by all cartesian squares in $\mathcal{S}$ associated to a covering of X by two open subschemes.

We define the scdh topology on $\mathcal{S}$ to be the Grothendieck topology associated to the union of the smooth blow-up cd-structure and the Zariski cd-structure on $\mathcal{S}$. For a presheaf of spectra on $\mathcal{S}$, we define $\mathbb{H}_{\mathrm{scdh}}(-,\mathcal{F})$ just as $\mathbb{H}_{\mathrm{cdh}}$ was defined above. We say such a presheaf $\mathcal{F}$ satisfies scdh descent if the canonical fibrant replacement map

$$\mathcal{F}(X) \to \mathbb{H}_{\mathrm{scdh}}(X,\mathcal{F})$$

is a weak equivalence for all $X \in \mathcal{S}$.

Proposition 12.13. *The smooth blow-up* cd-*structure and the Zariski* cd-*structure on $\mathcal{S}$ are regular, bounded, and complete. Consequently, a presheaf of spectra defined on $\mathcal{S}$ satisfies* scdh *descent if and only if it has the Mayer–Vietoris property for each smooth blow-up square and each Zariski square in $\mathcal{S}$.*

Proof. That the smooth blow-up cd-structure is complete can be proved exactly as Voevodsky did for smooth k-schemes in [31, Lemma 4.3], replacing resolution of singularities by our Theorem 11.4. Regularity is proved exactly as in Theorem 12.7 for the non-smooth case. The proof that the smooth blow-up cd-structure is bounded works exactly as in Theorem 12.8, keeping in mind that open subschemes of smooth monoid schemes are smooth. The proof that the Zariski cd-structure is complete, regular and bounded is again the same as in the non-smooth category. It follows that the scdh topology is generated by a complete, regular, bounded cd-structure and so [7, Theorem 3.4] applies to prove the second assertion. □

Proposition 12.14. *For any $X \in \mathcal{S}$ and any presheaf of spectra $\mathcal{F}$ defined on $\mathcal{M}_{\mathrm{pctf}}$, we have a weak equivalence*

$$\mathbb{H}_{\mathrm{cdh}}(X,\mathcal{F}) \xrightarrow{\sim} \mathbb{H}_{\mathrm{scdh}}(X,\mathcal{F}|_{\mathcal{S}}).$$

Proof. In this proof we write $\mathcal{F}_{\mathrm{cdh}}$ for the restriction of the presheaf $\mathbb{H}_{\mathrm{cdh}}(-,\mathcal{F})$ to $\mathcal{S}$. By Proposition 12.10, $\mathcal{F}_{\mathrm{cdh}}$ satisfies the Mayer–Vietoris property for smooth blow-up and Zariski squares. Therefore $\mathcal{F}_{\mathrm{cdh}}$ satisfies $scdh$ descent (Definition 12.12).

By Theorems 11.1 and 11.4, every covering sieve for the cdh topology on $\mathcal{M}_{\mathrm{pctf}}$ has a refinement containing a sieve generated by a cover consisting of objects of $\mathcal{S}$. It follows that $\mathcal{F}|_{\mathcal{S}} \to \mathcal{F}_{\mathrm{cdh}}$ is an scdh-local weak equivalence. Therefore

$$\mathbb{H}_{\mathrm{scdh}}(-,\mathcal{F}|_{\mathcal{S}}) \to \mathbb{H}_{\mathrm{scdh}}(-,\mathcal{F}_{\mathrm{cdh}})$$

is an objectwise weak equivalence (see [7, p. 561]). Together, the two objectwise weak equivalences exhibited in the proof give the assertion. □

13. Weak cdh_k descent

Throughout this section, we fix a commutative ring k.

Definition 13.1. Let X_k be a scheme of finite type over k and assume $Z_k \subset X_k$ is a closed subscheme. We say Z_k is *regularly embedded* in X_k if the sheaf of ideals defining Z_k is locally generated by a regular sequence – that is, if for all $x \in Z_k$, the kernel I_x of $\mathcal{O}_{X_k,x} \to \mathcal{O}_{Z_k,x}$ is generated by a $\mathcal{O}_{X_k,x}$-regular sequence of elements.

Definition 13.2. A presheaf of spectra $\mathcal{F}$ defined on $\mathcal{M}_{\mathrm{pctf}}$ has *weak* cdh_k *descent* if $\mathcal{F}$ has the Mayer–Vietoris property for each cartesian square

$$\begin{array}{ccc} D & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ C & \xrightarrow{e} & X \end{array}$$

in $\mathcal{M}_{\mathrm{pctf}}$ satisfying one of the following conditions:

(1) It is a member of the Zariski cd-structure.

(2) It is a finite abstract blow-up – i.e., it is a member of the abstract blow up cd-structure having the additional property that p is a finite morphism.

(3) C is an equivariant closed subscheme, $Y \to X$ is the blow-up of X along C, and C_k is a regularly embedded closed subscheme of X_k.

Remark 13.2.1. Theorems 13.3 and 14.3 below suggest (but do not prove) that the definition of weak cdh_k descent is actually independent of the choice of k.

Since a smooth blow-up square is an example of a blow-up along a regularly embedded subscheme, Propositions 12.13 and 12.14 imply the following theorem.

Theorem 13.3. *If $\mathcal{F}$ is a presheaf of spectra on $\mathcal{M}_{\mathrm{pctf}}$ that satisfies weak* cdh_k *descent, then $\mathcal{F}$ satisfies* scdh *descent. That is, the canonical map*

$$\mathcal{F}(X) \to \mathbb{H}_{\mathrm{cdh}}(X, \mathcal{F})$$

is a weak equivalence for every smooth monoid scheme X.

The main goal of this paper, realized in the next section, is to establish a partial generalization of Theorem 13.3 to all schemes in $\mathcal{M}_{\mathrm{pctf}}$. The goal of the rest of this section is to establish some technical properties needed in the next. We first introduce a slightly stronger notion than that of weak cdh_k descent.

Recall from [16, Section 6.10.1] that given a closed subscheme C_k of a k-scheme X_k, defined by an ideal sheaf $\mathcal{I}$, X_k is said to be *normally flat* along C_k if the restriction of each $\mathcal{I}^n/\mathcal{I}^{n+1}$ to C_k is flat.

Remark 13.3.1. Here is a monoid-theoretic condition on a sheaf I of ideals on a monoid scheme (X, A) which guarantees that, for all k, the k-realization of X is normally flat along the k-realization of the equivariant closed submonoid C defined by I: at each point x of C, under the natural action of the monoid A_x/I_x on each of the pointed sets $L_n = I_x^n/I_x^{n+1}$, each L_n is a bouquet of copies of A_x/I_x. We do not know if this condition is necessary.

We will say that a cartesian square of schemes in $\mathcal{M}_{\text{pctf}}$,

$$\begin{array}{ccc} D & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{e} & X, \end{array}$$

is a *nice blow-up square* if C is an equivariant closed subscheme of X, Y is the blow-up of X along C and there exists a cartesian square in $\mathcal{M}_{\text{pctf}}$ of the form

$$\text{(13.4)} \qquad \begin{array}{ccc} C & \xrightarrow[e]{} & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Z \end{array}$$

such that Z is cancellative, $X \to Z$ is the normalization of Z and B is an equivariant closed smooth subscheme of Z such that Z_k is normally flat along B_k.

Definition 13.5. A presheaf of spectra on $\mathcal{M}_{\text{pctf}}$ satisfies *weak+nice* cdh_k *descent* provided it satisfies weak cdh_k descent and, in addition, it has the Mayer–Vietoris property for all nice blow-up squares in $\mathcal{M}_{\text{pctf}}$.

Proposition 13.6. *If $\mathcal{F}$ is a presheaf of spectra on $\mathcal{M}_{\text{pctf}}$ that satisfies* cdh *descent, then $\mathcal{F}$ satisfies weak+nice* cdh_k *descent for any commutative ring k.*

Proof. This is immediate from Proposition 12.10, since each of the squares appearing in the definition of weak+nice cdh_k descent is a member of the cdh cd-structure. □

We will need the following technical result about local domains. Recall that if I is an ideal in a commutative ring R, then an ideal $J \subseteq I$ is called a *reduction* of I if $JI^{n-1} = I^n$ for some $n > 0$; a *minimal reduction* of I is a reduction which contains no other reduction of I.

Lemma 13.7. *Let R be a noetherian local domain with infinite residue field k, let $\mathfrak{p}$ be a prime ideal, and assume R is normally flat along $R/\mathfrak{p}$. Let J be a minimal reduction of $\mathfrak{p}$ that is generated by $h := \text{ht}(\mathfrak{p}) = \text{ht}(J)$ elements. (Given R and $\mathfrak{p}$ with these properties, such a J exists by* [19, Theorem 5.2 and Lemma 5.3]*.) Let $\tilde{R}$ be the normalization of R and assume $\tilde{R}$ is Cohen–Macaulay. Then $J\tilde{R}$ is a reduction of $\mathfrak{p}\tilde{R}$ generated by h elements and* $\text{Spec}(\tilde{R}/J\tilde{R})$ *is regularly embedded in* $\text{Spec}(\tilde{R})$.

Proof. We have that $J\mathfrak{p}^{n-1}\tilde{R} = \mathfrak{p}^n\tilde{R}$, and so the first assertion is clear.

Since $R \hookrightarrow \tilde{R}$ is an integral extension of domains, we have $h = \text{ht}(J) = \text{ht}(J\tilde{R})$. For any maximal ideal $\tilde{\mathfrak{m}}$ of $\tilde{R}$, we have that $J\tilde{R}_{\tilde{\mathfrak{m}}}$ is a height h ideal generated by h elements in the local ring $\tilde{R}_{\tilde{\mathfrak{m}}}$. Since $\tilde{R}_{\tilde{\mathfrak{m}}}$ is Cohen–Macaulay by assumption, these generators necessarily form a regular sequence. □

The following is the evident analogue of the notion of weak cdh_k descent for presheaves of spectra on the category of k-schemes.

Definition 13.8. For a commutative ring k, let Sch/k be the category of separated schemes essentially of finite type over k. A presheaf of spectra defined on Sch/k satisfies *weak* cdh *descent* if it has the Mayer–Vietoris property for each cartesian square

$$\begin{array}{ccc} D & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ C & \xrightarrow{e} & X \end{array}$$

of schemes satisfying one of the following conditions:

(1) e and p are open immersions whose images cover X.

(2) It is a finite abstract blow-up – i.e., e is a closed immersion, p is finite, and p maps $Y \setminus D$ isomorphically onto $X \setminus C$

(3) e is a regular closed immersion and p is the blow-up of X along C.

Lemma 13.9. *Assume k is a commutative regular noetherian domain containing an infinite field and $\mathcal{G}_k$ is a presheaf of spectra on* Sch/k *that satisfies weak* cdh *descent. Let $\mathcal{G}$ be the presheaf of spectra on $\mathcal{M}_{\mathrm{pctf}}$ defined by*

$$\mathcal{G}(X) := \mathcal{G}_k(X_k).$$

Then $\mathcal{G}$ satisfies weak+nice cdh_k *descent on $\mathcal{M}_{\mathrm{pctf}}$.*

Proof. Since the k-realizations of the squares involved in the definition of weak cdh_k descent for $\mathcal{M}_{\mathrm{pctf}}$ (Definition 13.2) are squares involved in the definition of weak cdh descent for Sch/k (Definition 13.8), it follows that $\mathcal{G}$ satisfies weak cdh_k descent. Say X, Y, C, D, Z, and B are as in the definition of a nice blow-up square. We need to prove that the square

$$\begin{array}{ccc} \mathcal{G}_k(X_k) & \longrightarrow & \mathcal{G}_k(C_k) \\ \downarrow & & \downarrow \\ \mathcal{G}_k(Y_k) & \longrightarrow & \mathcal{G}_k(D_k) \end{array} \qquad (13.10)$$

is homotopy cartesian.

Let R be any local ring of Z_k and let $\mathfrak{p}$ be the prime ideal of R cutting out B_k locally. Let $V = \mathrm{Spec}(\tilde{R}_{\tilde{\mathfrak{m}}})$, where $\tilde{R}$ is the normalization of R and $\tilde{\mathfrak{m}}$ is any of the maximal ideals of $\tilde{R}$. Then, since X_k is the normalization of Z_k by Proposition 6.1, V is the spectrum of a local ring of X_k, and for various choices of R and $\tilde{\mathfrak{m}}$, every local ring of X_k arises in this manner.

By Corollary 5.4, $C_k = X_k \times_{Z_k} B_k$, so the closed subscheme $V \times_{X_k} C_k$ of V is cut out by $\mathfrak{q} = \mathfrak{p}\tilde{R}_{\tilde{\mathfrak{m}}}$. As X is the normalization of the separated cancellative, torsionfree monoid scheme Z, Proposition 6.1 implies that X_k is a toric variety. By [22], all toric schemes over k are Cohen–Macaulay; hence so are X_k and V.

By Lemma 13.7, $\mathfrak{q} = \mathfrak{p}\tilde{R}_{\tilde{\mathfrak{m}}}$ admits a reduction $I \subset \mathfrak{q}$ such that $\mathrm{Spec}(\tilde{R}_{\tilde{\mathfrak{m}}}/I) \hookrightarrow V$ is a regular embedding. Since $V \times_{X_k} Y_k$ is the blow-up of V_k along $V \times_{X_k} C_k$ (by Proposition 7.7), and the exceptional divisor is $V \times_{X_k} D_k$ (by Corollary 5.4), the proof of [19, Proposition 5.6] (with $\mathcal{K}H$ replaced by $\mathcal{G}$) gives that

$$\begin{array}{ccc} \mathcal{G}_k(V) & \longrightarrow & \mathcal{G}_k(V \times_{X_k} C_k) \\ \downarrow & & \downarrow \\ \mathcal{G}_k(V \times_{X_k} Y_k) & \longrightarrow & \mathcal{G}_k(V \times_{X_k} D_k) \end{array}$$

is homotopy cartesian. Since $\mathcal{G}_k$ satisfies the Mayer–Vietoris property for Zariski covers and the V occurring here is an arbitrary local scheme of X_k, the proof of [19, Theorem 5.7] (with $\mathcal{K}H$ replaced by $\mathcal{G}_k$) shows that (13.10) is homotopy cartesian. □

Example 13.11. Let $\mathcal{K}H$ denote Weibel's homotopy algebraic K-theory [32]. We may view $\mathcal{K}H$ as a presheaf of spectra on Sch/k. By abuse of notation, we also write $\mathcal{K}H$ for the presheaf of spectra on $\mathcal{M}_{\mathrm{pctf}}$ defined by $\mathcal{K}H(X) = \mathcal{K}H(X_k)$.

By [29], [32, Proposition 4.9] and [28], $\mathcal{K}H$ satisfies weak cdh descent on Sch/k (Definition 13.8); by Lemma 13.9, $\mathcal{K}H$ satisfies weak+nice cdh_k descent on $\mathcal{M}_{\mathrm{pctf}}$.

14. Main theorem

In this section, we prove our main theorem (Theorem 14.3), which gives a condition for $\mathcal{F}$ to satisfy cdh descent on $\mathcal{M}_{\mathrm{pctf}}$. We will need the Bierstone–Milman Theorem, which we extract from the embedded version [1, Theorem 1.1].

Theorem 14.1. *Let X be a separated cancellative torsionfree monoid scheme of finite type, embedded as a closed subscheme (see Definition* 2.5*) in a smooth toric monoid scheme M (see Definition* 4.1*). For any commutative ring k containing a field, there is a sequence of blow-ups along smooth equivariant centers $Z_i \subset X_i$, $0 \le i \le n-1$,*

$$Y = X_n \to \cdots \to X_0 = X$$

such that Y is smooth, and each $(X_i)_k$ is normally flat along $(Z_i)_k$.

Proof. Since normal flatness is stable under flat extension of the base, and k is flat over a field, we may assume that k is a field. Let $\bar{k}$ denote the algebraic closure of k, and let T be the torus acting on $M_{\bar{k}}$. The Bierstone-Milman Theorem ([1, Theorem 1.1]) tells us that we can find a sequence of blow-ups $M_n \to \cdots \to M_0 = M_{\bar{k}}$ of smooth toric $\bar{k}$-varieties, the blow-up of M_i being taken along a smooth T-invariant center N_i, with the following properties. Setting $X'_0 = X_{\bar{k}}$, we inductively define $Z'_i = N_i \cap X'_i$; then Z'_i is a smooth equivariant $\bar{k}$-variety, X'_i is normally flat along Z'_i, and X'_{i+1} is the strict transform of X'_i.

The $\bar{k}$-realization functor from fans to (normal) toric $\bar{k}$-varieties (and equivariant morphisms) is well known to be an equivalence. It follows that each of the N_i and M_i and the morphisms between them come from fans, and hence by Theorem 4.4 are $\bar{k}$-realizations of toric monoid schemes (which by abuse of notation, we will call N_i and M_i), and morphisms of such.

Inductively we define monoid schemes X_i and Z_i, starting from

$$X_0 = X \quad \text{and} \quad Z_0 = N_0 \cap X,$$

to be the blow-up of the monoid scheme X_{i-1} along Z_{i-1} in the sense of Definition 7.4. By Proposition 7.7 and Corollary 5.4, $Z_i' = (Z_i)_{\bar{k}}$ and $X_i' = (X_i)_{\bar{k}}$. In particular, $(X_n)_{\bar{k}} = Y$ is a smooth toric variety and therefore the monoid scheme X_n is smooth by Proposition 6.5. Finally, faithfully flat descent implies that $(X_i)_k$ is normally flat along $(Z_i)_k$ if and only if $(X_i)_{\bar{k}}$ is normally flat along $(Z_i)_{\bar{k}}$. □

Theorem 14.2. *Suppose $\mathcal{G}$ is a presheaf of spectra on $\mathcal{M}_{\mathrm{pctf}}$ satisfying weak+nice* cdh$_k$ *descent for some commutative ring k containing a field. If $\mathcal{G}(X) \simeq *$ for all X in $\mathcal{S}$, then $\mathcal{G}(X) \simeq *$ for all X in $\mathcal{M}_{\mathrm{pctf}}$.*

Proof. We proceed by induction on the dimension of X. Given X, let $x_1, \dots, x_l$ be its generic points, and let $Y_i = \overline{\{x_i\}}^{\mathrm{eq}}$ be their equivariant closures (see Lemma 2.9). We have a cover $X = Y_1 \cup \dots \cup Y_l$ by equivariant closed subschemes each of which is cancellative by Example 10.3. Moreover, each $Y_i \times_X Y_j$ is equivariant and closed, hence pctf. Since $\mathcal{G}$ has the Mayer–Vietoris property for closed covers, and $\mathcal{G}$ vanishes on the $Y_i \times_X Y_j$ for all $i \neq j$ by the induction hypothesis, we get

$$\mathcal{G}(X) = \prod_i \mathcal{G}(Y_i).$$

We may thus assume that X is cancellative. (This also establishes the base case $\dim(X) = 0$, since in that case the Y_i are in $\mathcal{S}$.)

Since $\mathcal{G}$ satisfies Mayer–Vietoris for open covers, we may assume X is affine. In particular, we may assume X can be embedded in a smooth toric monoid scheme, for example, by choosing a surjection from a free abelian monoid onto A where $X = \mathrm{MSpec}(A)$. This will allow us to apply the Bierstone–Milman Theorem 14.1 to obtain a sequence of blow-ups along smooth monoid schemes Z_i,

$$Y = X_n \to \cdots \to X_0 = X.$$

We claim that $\mathcal{G}(X_i) \simeq \mathcal{G}(X_{i+1})$ for all i. Since $\mathcal{G}(Y) \simeq *$, this will finish the inductive step and hence the proof of the theorem. To simplify the notation, fix i and write Z for $Z_i \subset X_i$ and X_Z for X_{i+1}, the blow-up of X_i along Z, so that our goal is to prove that $\mathcal{G}(X_i) \to \mathcal{G}(X_Z)$ is a weak equivalence. Let $\tilde{X}$ denote the normalization $(X_i)_{\mathrm{nor}}$ of X_i and set $\tilde{Z} = Z \times_{X_i} \tilde{X}$. Write $\tilde{X}_{\tilde{Z}}$ for the blow-up of $\tilde{X}$ along $\tilde{Z}$. By naturality of blow-ups (see Definition 7.4), there is a commutative square

$$\begin{array}{ccc} \tilde{X}_{\tilde{Z}} & \longrightarrow & X_Z \\ \big\downarrow & & \big\downarrow \\ \tilde{X} & \longrightarrow & X_i \end{array}$$

(that need not be cartesian). Since the map $\tilde{X} \to X_i$ is finite, the map $\tilde{X}_{\tilde{Z}} \to X_Z$ is also finite, by Lemma 7.6. Applying $\mathcal{G}$ gives a commutative square of spectra

$$\begin{array}{ccc} \mathcal{G}(\tilde{X}_{\tilde{Z}}) & \longleftarrow & \mathcal{G}(X_Z) \\ \big\uparrow & & \big\uparrow \\ \mathcal{G}(\tilde{X}) & \longleftarrow & \mathcal{G}(X_i). \end{array}$$

To prove that the right-hand vertical arrow is a weak equivalence, it suffices to prove the other three are.

The finite map $\tilde{X} \to X_i$ is an isomorphism on the generic points. Consider the equivariant closure $E \subset X_i$ of the finitely many height 1 points of X_i; by Remark 2.9.1, every point in E has height ≥ 1 in X_i, so E is the complement of the generic point of X_i. Since E is pctf, $\mathcal{G}(E) \simeq *$ by our inductive assumption. Since the pullback $\tilde{E} := E \times_{X_i} \tilde{X}$ is an equivariant closed subscheme of $\tilde{X}$, it is pctf by Proposition 9.1, and hence $\mathcal{G}(\tilde{E}) \simeq *$ as well, by induction. Using the finite abstract blow-up square involving X_i, $\tilde{X}$, E and $\tilde{E}$, we have a weak equivalence

$$\mathcal{G}(X_i) \xrightarrow{\simeq} \mathcal{G}(\tilde{X}).$$

The map $\tilde{X}_{\tilde{Z}} \to X_Z$ is also finite and birational, and so the same argument shows

$$\mathcal{G}(X_Z) \xrightarrow{\simeq} \mathcal{G}(\tilde{X}_{\tilde{Z}})$$

is a weak equivalence. Finally, observe that

$$\begin{array}{ccc} \tilde{Z} \times_{\tilde{X}} \tilde{X}_{\tilde{Z}} & \longrightarrow & \tilde{X}_{\tilde{Z}} \\ \downarrow & & \downarrow \\ \tilde{Z} & \longrightarrow & \tilde{X} \end{array}$$

is a nice blow-up square, because the bottom row may be compared with $Z \to X_i$ and $(X_i)_k$ is normally flat along Z_k. Because $\mathcal{G}$ has descent for nice blow-up squares, and

$$\mathcal{G}(\tilde{Z}) \simeq \mathcal{G}(\tilde{Z} \times_{\tilde{X}} \tilde{X}_{\tilde{Z}}) \simeq *$$

by the induction hypothesis, we get a weak equivalence

$$\mathcal{G}(\tilde{X}) \xrightarrow{\simeq} \mathcal{G}(\tilde{X}_{\tilde{Z}}).$$

It follows that $\mathcal{G}(X_i) \simeq \mathcal{G}(X_Z)$, as claimed. This completes the proof. □

We now state and prove the main theorem of this paper, which gives a partial generalization of Theorem 13.3 to all objects in the category $\mathcal{M}_{\text{pctf}}$.

Theorem 14.3. *Let $\mathcal{F}_k$ be a presheaf of spectra on* Sch$/k$ *for some commutative regular noetherian ring k containing an infinite field, and define $\mathcal{F}$ to be the presheaf of spectra on $\mathcal{M}_{\text{pctf}}$ defined by*

$$\mathcal{F}(X) = \mathcal{F}_k(X_k).$$

If $\mathcal{F}_k$ satisfies weak cdh *descent on* Sch$/k$*, then $\mathcal{F}$ satisfies* cdh *descent on $\mathcal{M}_{\text{pctf}}$.*

Proof. Let $\mathcal{G}$ be the homotopy fiber of $\mathcal{F} \to \mathbb{H}_{\text{cdh}}(-, \mathcal{F})$ – i.e., for all X in $\mathcal{M}_{\text{pctf}}$, $\mathcal{G}(X)$ is the homotopy fiber of $\mathcal{F}(X) \to \mathbb{H}_{\text{cdh}}(X, \mathcal{F})$. By Lemma 13.9 and Proposition 13.6, both $\mathcal{F}$ and $\mathbb{H}_{\text{cdh}}(-, \mathcal{F})$ satisfy weak+nice cdh$_k$ descent, and hence $\mathcal{G}$ satisfies weak+nice cdh$_k$ descent too. Theorem 13.3 gives that $\mathcal{G}(X) \simeq *$ for all $X \in \mathcal{S}$. Now we apply Theorem 14.2 to conclude $\mathcal{G}(X) \simeq *$ for all X in $\mathcal{M}_{\text{pctf}}$. □

The following corollary is the theorem announced in the introduction.

Corollary 14.4. *Assume k is a commutative regular noetherian ring containing an infinite field and let $\mathcal{F}_k$ be a presheaf of spectra on* Sch/k *that satisfies the Mayer–Vietoris property for Zariski covers, finite abstract blow-up squares, and blow-ups along regularly embedded subschemes. Then $\mathcal{F}_k$ satisfies the Mayer–Vietoris property for all abstract blow-up squares of toric k-schemes obtained from subdividing a fan.*

Proof. By Definition 13.8, $\mathcal{F}_k$ satisfies weak cdh descent on Sch/k. By Theorem 14.3, $\mathcal{F}$ satisfies cdh descent in $\mathcal{M}_{\mathrm{pctf}}$. Now use Proposition 12.10. □

Corollary 14.5. *Let k be a commutative regular noetherian ring containing a field. The presheaf of spectra $\mathcal{K}H$ on $\mathcal{M}_{\mathrm{pctf}}$, defined as $\mathcal{K}H(X) = \mathcal{K}H(X_k)$, satisfies* cdh *descent. Moreover, both natural maps*

$$\mathcal{K}H(X) \to \mathbb{H}_{\mathrm{cdh}}(X, \mathcal{K}H) \leftarrow \mathbb{H}_{\mathrm{cdh}}(X, \mathcal{K})$$

are weak equivalences for all X in $\mathcal{M}_{\mathrm{pctf}}$.

Proof. We first reduce to the case when k is of finite type over a field. We can express k as a filtered colimit of rings k_i, all regular of finite type over a field (by Popescu's theorem [27, Theorem 2.5]). The functor $\mathcal{K}H$ is the homotopy colimit of the corresponding functors defined by k_i-realization. By Proposition 12.10, we can check descent by showing that certain squares of monoid schemes are transformed by $\mathcal{K}H$ into homotopy co-cartesian squares of spectra (a square of spectra is homotopy cartesian if and only if it is homotopy co-cartesian); since homotopy colimits of homotopy co-cartesian squares are homotopy co-cartesian, we may assume that k is of finite type over its field of constants.

Now if the regular ring k does not contain an infinite field, it is smooth over the (perfect) field of constants it contains and hence stays regular under base change from its field of constants to any algebraic extension. We can therefore apply the standard transfer argument and may assume that k contains an infinite field.

By Example 13.11 and Theorem 14.3, $\mathcal{K}H$ satisfies cdh descent on $\mathcal{M}_{\mathrm{pctf}}$. For any X in $\mathcal{M}_{\mathrm{pctf}}$, consider the commutative square of spectra:

$$\begin{array}{ccc} \mathcal{K}(X) & \longrightarrow & \mathcal{K}H(X) \\ \downarrow & & \downarrow \\ \mathbb{H}_{\mathrm{cdh}}(X, \mathcal{K}) & \longrightarrow & \mathbb{H}_{\mathrm{cdh}}(X, \mathcal{K}H), \end{array}$$

where $\mathcal{K}$ is algebraic K-theory, regarded as a presheaf of spectra on Sch/k and hence on $\mathcal{M}_{\mathrm{pctf}}$. Since $\mathcal{K}H$ satisfies cdh descent, the right-hand vertical map is a weak equivalence for all X. This is the first assertion of the corollary.

If X is smooth, then the top horizontal map is a weak equivalence by [32] (since X_k is smooth over k hence regular by Proposition 6.5). By fibrant replacement and Proposition 12.14, the bottom map is also a weak equivalence for all X in $\mathcal{S}$. By induction on $\dim(X)$ and Theorem 11.1, this implies that $\mathbb{H}_{\mathrm{cdh}}(-, \mathcal{K}) \to \mathbb{H}_{\mathrm{cdh}}(-, \mathcal{K}H)$ is a local weak equivalence and, as observed (for any site) in [7, p. 561], this implies that $\mathbb{H}_{\mathrm{cdh}}(X, \mathcal{K}) \to \mathbb{H}_{\mathrm{cdh}}(X, \mathcal{K}H)$ is a weak equivalence for all X in $\mathcal{M}_{\mathrm{pctf}}$. □

Remark 14.6. It follows from Corollaries 12.9 and 14.5 and a cdh descent argument that if $X \in \mathcal{M}_{\mathrm{pctf}}$ is of dimension d and k is a commutative regular ring containing a field, then $KH_n(X_k) = 0$ for $n < -d$ (cf. [19, Theorem 8.19]). The analogous statement for K-theory is also true, at least if X is cancellative and torsionfree. Indeed, for affine X, $K_n(X_k) = 0$ for $n < 0$, by [18, Theorem 1.3]; the general case follows from this by a Zariski descent argument, using Corollary 12.9.

In order to apply Corollary 14.5 to the relation between K-theory and topological cyclic homology, we need to recall some terms. Fix a prime p and a commutative regular ring k of characteristic p. To each scheme X essentially of finite type over k, there is a pro-spectrum $\{TC^\nu(X, p)\}_{\nu=0}^{\infty}$ and the cyclotomic trace is a compatible family of morphisms

$$\mathrm{tr}^\nu : \mathcal{K}(X) \to TC^\nu(X, p).$$

Define $\mathcal{F}_k^\nu$ to be the presheaf of spectra on Sch/k given as the homotopy fiber of

$$\mathcal{K}(X) \to TC^\nu(X, p).$$

Then Geisser and Hesselholt observe in the proof of [12, Theorem B] that each $\mathcal{F}_k^\nu$ takes elementary Nisnevich squares and regular blow-up squares to homotopy cartesian squares of pro-spectra.

Following Geisser–Hesselholt [12], a strict map of pro-spectra $\{X^\nu\} \to \{Y^\nu\}$ is said to be a *weak equivalence* if for every q the induced map $\{\pi_q(X^\nu)\} \to \{\pi_q(Y^\nu)\}$ is an isomorphism of pro-abelian groups. A square diagram of strict maps of pro-spectra is said to be *homotopy cartesian* if the canonical map from the upper left pro-spectrum to the level-wise homotopy limit of the other terms is a weak equivalence.

Given a class $\mathcal{C}$ of squares we will say that a pro-presheaf of spectra satisfies the pro-analogue of $\mathcal{C}$ descent if it sends each square in $\mathcal{C}$ to a homotopy cartesian square of pro-spectra.

Define $\{\mathcal{F}^\nu\}$ to be the pro-presheaf of spectra on $\mathcal{M}_{\mathrm{pctf}}$ given as the family of homotopy fibers of the maps $\mathcal{K}(-) \to TC^\nu(-, p)$. That is, $\mathcal{F}^\nu(X) = \mathcal{F}_k^\nu(X_k)$ is the homotopy fiber of $\mathcal{K}(X_k) \to TC^\nu(X_k, p)$ for each X and ν.

Proposition 14.7. *Assume k is a commutative regular noetherian ring containing an infinite field of characteristic $p > 0$. Then $\{\mathcal{F}^\nu\}$ satisfies* cdh *descent on $\mathcal{M}_{\mathrm{pctf}}$ in the sense that $\{\mathcal{F}^\nu\} \to \{\mathbb{H}(-, \mathcal{F}^\nu)\}$ is a weak equivalence of pro-spectra.*

Proof. Fix ν and let $\mathcal{G}^\nu$ be the homotopy fiber of $\mathcal{F}^\nu \to \mathbb{H}_{\mathrm{cdh}}(-, \mathcal{F}^\nu)$. It suffices to prove that for each X and q the pro-abelian group $\{\pi_q \mathcal{G}^\nu(X)\}$ is pro-zero. We will do so by modifying the proof of Theorem 14.3.

For each ν, $\mathbb{H}_{\mathrm{cdh}}(-, \mathcal{F}^\nu)$ satisfies weak+nice cdh_k descent by Proposition 13.6. By [11, Theorem 1] and [13, Theorems B and D], $\{\mathcal{F}_k^\nu\}$ sends finite abstract blow-up squares to homotopy cartesian squares of pro-spectra. Thus $\{\mathcal{F}_k^\nu\}$ satisfies the pro-analogue of weak cdh descent (Definition 13.8). In the proof of Lemma 13.9, the reduction ideals used are reduction ideals on affine neighborhoods of the maximal ideal $\mathfrak{m}$ of R. By the argument used in the proof of [12, Theorem 1.1], the proof of our Lemma 13.9 now applies mutatis mutandis to show that the pro-presheaf of spectra $\mathcal{F}^\nu$ satisfies the Mayer–Vietoris property for nice blow-up squares. It now follows that $\{\mathcal{G}^\nu\}$ satisfies the pro-analogue of weak+nice cdh_k descent.

For each ν, $\mathcal{F}^\nu$ satisfies Zariski descent and also has the Mayer–Vietoris property for regular blow-ups, so $\mathcal{F}^\nu$ satisfies scdh descent by 12.13. By definition, this means that for each smooth X the spectrum $\mathcal{G}^\nu(X)$ is contractible. Now the proof of Theorem 14.2 applies verbatim to finish the proof. □

Corollary 14.8. *Assume k is any commutative regular noetherian ring of characteristic $p > 0$. For any monoid scheme X in $\mathcal{M}_{\mathrm{pctf}}$, the following square of pro-spectra is homotopy cartesian:*

$$\begin{array}{ccc} \mathcal{K}(X) & \longrightarrow & \mathcal{K}H(X) \\ \downarrow & & \downarrow \\ \{TC^\nu(X,p)\} & \longrightarrow & \{\mathbb{H}_{\mathrm{cdh}}(X, TC^\nu(-,p))\}. \end{array}$$

Proof. By a standard transfer argument as in Corollary 14.5, we may assume that k contains an infinite field. By Proposition 14.7, the homotopy fiber $\{\mathcal{F}^\nu(X)\}$ of the left vertical map is weakly equivalent to $\{\mathbb{H}_{\mathrm{cdh}}(X, \mathcal{F}^\nu)\}$. By Corollary 14.5, this coincides up to weak equivalence with the homotopy fiber of the right vertical map. □

Remark 14.9. As explained in Remark 14.6, if k is any commutative regular ring containing a field, and $X \in \mathcal{M}_{\mathrm{pctf}}$ is cancellative and torsionfree, then we have $K_n(X_k) = 0$ for $n < -\dim X$. To extend this result to all $X \in \mathcal{M}_{\mathrm{pctf}}$ it would suffice to prove that the bottom horizontal map in the diagram in Corollary 14.8 induces an isomorphism (resp., an epimorphism) of homotopy groups in degrees $n < -\dim(X)$ (resp., $n = -\dim(X)$). Geisser and Hesselholt proved the analogue statement for schemes essentially of finite type over a field of positive characteristic which admits resolution of singularities ([12, Theorem C]). Adapting their methods to our situation seems rather hard.

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References

[1] *E. Bierstone* and *P. Milman*, Desingularization of toric and binomial varieties, J. Algebraic Geom. **15** (2006), 443–486.
[2] *B. A. Blander*, Local projective model structures on simplicial presheaves, K-Theory **24** (2001), 283–301.
[3] *A. K. Bousfield* and *E. M. Friedlander*, Homotopy theory of Γ-spaces, spectra, and bisimplicial sets, in: Geometric applications of homotopy theory. II (Evanston 1977), Lecture Notes in Math. **658**, Springer-Verlag, Berlin (1978), 80–130.
[4] *A. Connes* and *C. Consani*, Schemes over $\mathbb{F}_1$ and zeta functions, Compos. Math. **146** (2010), 1383–1415.
[5] *A. Connes* and *C. Consani*, On the notion of geometry over $\mathbb{F}_1$, J. Algebraic Geom. **20** (2011), 525–557.
[6] *A. Connes*, *C. Consani* and *M. Marcolli*, Fun with $\mathbb{F}_1$, J. Number Theory **129** (2009), 1532–1561.
[7] *G. Cortiñas*, *C. Haesemeyer*, *M. Schlichting* and *C. Weibel*, Cyclic homology, cdh-cohomology and negative K-theory, Ann. of Math. (2) **167** (2008), 549–573.
[8] *A. Deitmar*, $\mathbb{F}_1$-schemes and toric varieties, Beitr. Algebra Geom. **49** (2008), 517–525.
[9] *D. Eisenbud* and *J. Harris*, The geometry of schemes, Grad. Texts in Math. **197**, Springer-Verlag, New York 2000.
[10] *W. Fulton*, Introduction to toric varieties, Ann. of Math. Stud. **131**, Princeton University Press, Princeton 1993.
[11] *T. Geisser* and *L. Hesselholt*, Bi-relative algebraic K-theory and topological cyclic homology, Invent. Math. **166** (2006), 359–395.

[12] *T. Geisser* and *L. Hesselholt*, On the vanishing of negative K-groups, Math. Ann. **348** (2010), 707–736.
[13] *T. Geisser* and *L. Hesselholt*, On relative and bi-relative algebraic K-theory of rings of finite characteristic, J. Amer. Math. Soc. **24** (2011), 29–49.
[14] *R. Gilmer*, Commutative semigroup rings, The University of Chicago Press, Chicago 1984.
[15] *A. Grothendieck*, Élements de géométrie algébrique. II: Étude globale élémentaire de quelques classe de morphismes, Publ. Math. Inst. Hautes Études Sci. **8** (1961), 1–222.
[16] *A. Grothendieck*, Élements de géométrie algébrique. IV: Étude locale des schemas et des morphismes de schemas (2^{eme} partie), Publ. Math. Inst. Hautes Études Sci. **24** (1964), 1–231.
[17] *A. Grothendieck* and *J. Dieudonné*, Élements de géométrie algébrique. I, Grundlehren Math. Wiss. **166**, Springer-Verlag, Berlin 1971.
[18] *J. Gubeladze*, K-theory of affine toric varieties, Homology, Homotopy Appl. **1** (1999) 135–145.
[19] *C. Haesemeyer*, Descent properties of homotopy K-theory, Duke Math. J. **125** (2004), 589–620.
[20] *R. Hartshorne*, Algebraic geometry, Grad. Texts in Math. **52**, Springer-Verlag, New York 1977.
[21] *A. Hattori*, Theory of multi-fans, Osaka J. Math. **40** (2003), 1–68.
[22] *M. Hochster*, Rings of invariants of tori, Cohen–Macaulay rings generated by monomials, and polytopes, Ann. of Math. (2) **96** (1972), 318–337.
[23] *J. F. Jardine*, Simplicial presheaves, J. Pure Appl. Algebra **47** (1987), 35–87.
[24] *J. F. Jardine*, Generalized étale cohomology theories, Birkhäuser-Verlag, Basel 1997.
[25] *K. Kato*, Toric singularities, Amer. J. Math. **116** (1994), 1073–1099.
[26] *J. López Peña* and *O. Lorscheid*, Mapping $\mathbb{F}_1$-land: An overview of geometries over the field with one element, in: Noncommutative geometry, arithmetic, and related topics (Baltimore 2009), Johns Hopkins University Press, Baltimore (2011), 241–265.
[27] *D. Popescu*, General Néron desingularization and approximation, Nagoya Math. J. **104** (1986), 85–115.
[28] *R. W. Thomason*, Les K-groupes d'un schéma éclaté et une formule d'intersection excédentaire, Invent. Math. **112** (1993), 195–215.
[29] *R. W. Thomason* and *T. Trobaugh*, Higher algebraic K-theory of schemes and of derived categories, in: The Grothendieck Festschrift, vol. III, Progr. Math. **88**, Birkhäuser-Verlag, Boston (1990), 247–436.
[30] *V. Voevodsky*, Homotopy theory of simplicial sheaves in completely decomposable topologies, J. Pure Appl. Algebra **214** (2010), 1384–1398.
[31] *V. Voevodsky*, Unstable motivic homotopy categories in Nisnevich and cdh-topologies, J. Pure Appl. Algebra **214** (2010), 1399–1406.
[32] *C. Weibel*, Homotopy algebraic K-theory, in: Algebraic K-theory and algebraic number theory (Honolulu 1987), Contemp. Math. **83**, American Mathematical Society, Providence (1989), 461–488.
[33] *C. Weibel*, The negative K-theory of normal surfaces, Duke Math. J. **108** (2001), 1–35.
[34] *O. Zariski* and *P. Samuel*, Commutative algebra, vol. II, reprint of the 1958–1960 Van Nostrand edition, Grad. Texts in Math. **29**, Springer-Verlag, New York 1975.

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