Toroidal Embeddings and Polyhedral Divisors

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Abstract. Given an effective action of an $(n-1)$-dimensional torus on an n-dimensional normal affine variety, Mumford constructs a toroidal embedding, while Altmann and Hausen give a description in terms of a polyhedral divisor on a curve. We compare the fan of the toroidal embedding with this polyhedral divisor.

Mathematics Subject Classification: Primary 14M25, Secondary 14L30

Introduction

Suppose X is an n-dimensional normal affine variety over the complex numbers with an effective action by the $(n-1)$-dimensional torus T. With $T \cong (\mathbb{C}^*)^{n-1}$, we associate the lattice $M \cong \mathbb{Z}^{n-1}$ of characters and the dual lattice $N = \mathrm{Hom}(M, \mathbb{Z})$ of one-parameter subgroups. The action defines the weight cone ω in M generated by the degrees of semi-invariant functions on X and the dual cone σ in N. Effectivity of the action translates to the fact that ω is full-dimensional and σ is pointed.

Notation. A cone δ "in" a lattice N is really a subset of the vector space $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$. The toric variety associated with this cone will be denoted by $\mathrm{TV}(\delta)$.

Our goal is to compare two sets of combinatorial data associated with X. Mumford [3, Chapter 4, §1] takes a rational quotient map p from X to a complete nonsingular curve C. He defines X'' to be the normalization of the graph of p and shows that for certain open subsets U of C, we obtain a toroidal embedding $(U \times T, X'')$. This determines a combinatorial datum, namely the toroidal fan $\Delta(X, U)$. It is a collection of cones in *different* lattices $\mathbb{Z} \times N$, one for each point $P \in C \setminus U$, glued along their common face in $(0, N)$.

Altmann and Hausen [1] construct a divisor $\mathcal{D}$ with polyhedral coefficients on a nonsingular curve Y; this divisor determines a T-variety $\widetilde{X}$, affine over Y, which contracts to X. Here, $\mathcal{D}$ is of the form $\Sigma_{P \in Y} \Delta_P \otimes P$, where the Δ_P are polyhedra in $N_{\mathbb{Q}}$ with tail cone σ, only finitely many nontrivial.

To compare these data, we note that the curve Y is an open subset of C, namely the image of the map $\pi \colon X'' \to C$. In fact, the varieties $\widetilde{X}$ and X'' agree, which allows us to describe $\Delta(X, U)$ in terms of $\mathcal{D}$. Defining the homogenization of a polyhedron $\Delta \subset N_{\mathbb{Q}}$ with tail σ to be the cone in $\mathbb{Z} \times N$ generated by $(1, \Delta)$ and $(0, \sigma)$, we obtain the following result.

Theorem 1. *The toroidal fan $\Delta(X, U)$ is equal to the fan obtained by gluing the homogenizations of the coefficient polyhedra Δ_P of points $P \in Y \setminus U$ along their common face $(0, \sigma)$.*

In Section 1, we recall relevant facts about toroidal embeddings and summarize the construction of the embedding $(U \times T, X'')$. Section 2 contains some details about polyhedral divisors on curves. Finally, we present the proof of Theorem 1 in Section 3.

1. Toroidal interpretation

Toroidal embeddings. A toroidal embedding [3, Chapter 2] is a pair (U, X) of a normal variety X and an open subset $U \subset X$ such that for each point $x \in X$, there exists a toric variety (H, Z) with embedded torus $H \subset Z$ which is locally formally isomorphic at some point $z \in Z$ to (U, X) at x. We will further assume that the components $E_1, \ldots, E_r$ of $X \setminus U$ are normal, i.e., that all toroidal embeddings are "without self-intersection".

The components of the sets $\cap_{i \in I} E_i \setminus \cup_{i \notin I} E_i$ for all subsets $I \subset \{1, \ldots, r\}$ give a stratification of X. The *star* of a stratum Y is defined to be the union of strata Z with $Y \subset \overline{Z}$. Given a stratum Y, we have the lattice M_Y of Cartier divisors on the star of Y with support in the complement of U. The submonoid of effective divisors is dual to a polyhedral cone σ_Y in the dual lattice N_Y.

If $Z \subset \mathrm{star}(Y)$ is a stratum, its cone σ_Z is a face of σ_Y. The *toroidal fan* of the embedding (U, X) is the union of the cones σ_Y glued along common faces.

Remark 1. A toroidal fan differs from a conventional fan only in that it lacks a global embedding into a lattice.

Below, we will use the fact that an étale map $(U, \mathrm{star}(Y)) \to (H, \mathrm{TV}(\delta))$ induces an isomorphism $\sigma_Y \xrightarrow{\sim} \delta$ of lattice cones.

Toroidal embeddings for torus actions. We return to the T-variety X and summarize Mumford's description [3, Chapter 4, §1]. There is a canonically defined rational quotient map $p \colon X \dashrightarrow C$ to a complete nonsingular curve C. Sufficiently small invariant open sets $W \subset X$ split as $W \cong U \times T$ for some open set $U \subset C$, where the first projection $U \times T \to U$ corresponds to p. We will identify $U \times T$ with W.

We define X' to be the closure of the graph of the rational map p in $X \times C$, and X'' to be its normalization. The action of T on X lifts to X''. We may consider $U \times T$ as an open subset of X''; the projection to U now extends to a regular map $\pi \colon X'' \to C$.

After possibly replacing U by an open subset, we are in the following situation: Let $P \in C \setminus U$ be a point in the complement of U. The sets U, $U' = U \cup \{P\}$ and $\pi^{-1}(U')$ are affine with coordinate rings R, R' and S, respectively. We may regard S as a subring of $R \otimes \mathbb{C}[M]$ which is generated by homogeneous elements with respect to the M-grading. Denoting by s a local parameter at $P \in C$, the ring S is generated over R' by a finite number of monomials $s^k \chi^u$.

The corresponding semigroup in $\mathbb{Z} \times M$ and its dual cone δ_P in $\mathbb{Z} \times N$ define a toric variety $Z = \mathrm{TV}(\delta_P)$. The monomial generators of S define an étale map $\pi^{-1}(U') \to Z$ which shows that the embedding $(U \times T, \pi^{-1}(U'))$ is toroidal with cone isomorphic to δ_P. By considering all points $P \in C \setminus U$, we see that $(U \times T, X'')$ is a toroidal embedding.

Theorem A ([3, Chapter 4, §1]). *The embedding $(U \times T, X'')$ is toroidal. Its fan $\Delta(X, U)$ consists of the cones δ_P glued along the common face $\delta_P \cap (0, N_\mathbb{Q})$.*

Remark 2. This common face is $\sigma \subset N_\mathbb{Q}$ and corresponds to $\pi^{-1}(U)$, an open subset of each $\pi^{-1}(U')$. For points P that lie outside the image of π, we have $\pi^{-1}(U') = \pi^{-1}(U)$, hence the cone δ_P is equal to $(0, \sigma)$.

Remark 3. Given $U \subset C$, the constructed toroidal fan $\Delta(X, U)$ is independent of the choice of equivariant isomorphism $U \times T \cong W$. It does however depend on the choice of U.

If we don't require that there be an étale model for the whole of $\pi^{-1}(U')$, we can enlarge U to form a canonical embedding $(V \times T, \widetilde{X})$. Here, V is obtained by adding to any U as above all points P with a toric model that splits as $Z = \mathbb{A}^1 \times F$, where $F = \mathrm{TV}(\sigma)$ is the generic fiber of π. That is, the points P with cone δ_P isomorphic to $\sigma \times \mathbb{Q}_{\geq 0}$.

Example. The affine threefold $X = \mathrm{SL}(2, \mathbb{C}) = \mathbb{C}[a, b, c, d]/(ad - bc - 1)$ admits a two-dimensional torus action by defining

$$(t_1, t_2) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} t_1 a & t_2 b \\ t_2^{-1} c & t_1^{-1} d \end{pmatrix}.$$

It admits a quotient morphism $\pi \colon X \to \mathbb{A}^1 = \mathrm{Spec}\,\mathbb{C}[s]$ with $s \mapsto ad$. Let W be the open subset of matrices with no vanishing entries. With $U = \mathbb{A}^1 \setminus \{0, 1\}$, we get an isomorphism $W \cong U \times T$ by mapping $t_1 \mapsto a$ and $t_2 \mapsto b$.

We consider $P = 0$, so $U' = \mathbb{A}^1 \setminus \{1\}$. The coordinate ring of $\pi^{-1}(U')$ is generated over $\mathbb{C}[s]_{s(s-1)}$ by t_1, st_1^{-1} and $t_2^{\pm 1}$. Thus δ_0 is generated by $(1, 0, 0)$ and $(1, 1, 0)$. Similarly, δ_1 is generated by $(1, 0, 0)$ and $(1, 0, 1)$, as shown in Figure 1. The fan $\Delta(X, U)$ is obtained by gluing these two cones at the vertex.

2. Polyhedral divisors on curves

We turn to the construction and relevant properties of proper polyhedral divisors on curves, restating results of Altmann and Hausen [1] in the setting of codimension one actions.

$$\delta_0 \qquad\qquad \delta_1 \qquad\qquad \Delta_1,\ \Delta_2$$
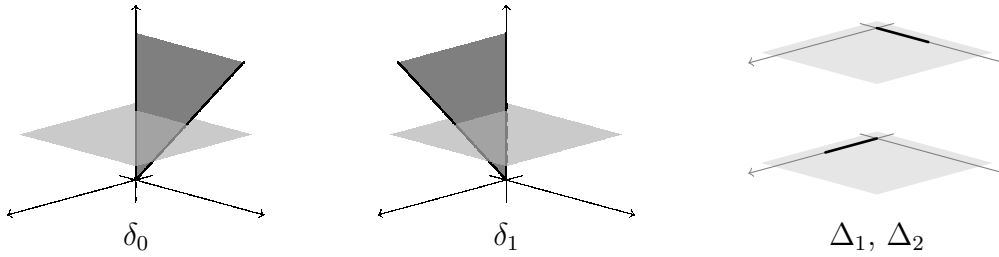
FIGURE 1. Cones and polyhedra for $\mathrm{SL}(2, \mathbb{C})$

Given a cone σ in N, the set of polyhedra with tail cone σ

$$\mathrm{Pol}_\sigma^+ = \{\Delta \subset N_{\mathbb{Q}} \mid \Delta = \Pi + \sigma \text{ for some compact polytope } \Pi\}$$

forms a semigroup under Minkowski addition. It is embedded in the group of differences Pol_σ; the neutral element is σ. A divisor $\mathcal{D} \in \mathrm{Pol}_\sigma \otimes \mathrm{CaDiv}(Y)$ on a smooth curve Y is called a *polyhedral divisor*. Under certain positivity assumptions ($\sum \Delta_P \subsetneq \sigma$ is almost the right condition, see [1, Example 2.12]), $\mathcal{D}$ is called *proper*. We may express it as

$$\mathcal{D} = \sum \Delta_P \otimes P,$$

where the sum ranges over all prime divisors of Y, and all but finitely many of the polyhedra Δ_P are equal to σ.

A proper polyhedral divisor defines an affine T-variety. Each weight u in the weight monoid $\omega \cap M$ gives a $\mathbb{Q}$-divisor $\mathcal{D}(u)$ on Y by

$$\mathcal{D}(u) = \sum \min\langle u, \Delta_P \rangle \cdot P.$$

This allows us to define an M-graded sheaf $\mathcal{A}$ of $\mathcal{O}_Y$-algebras by setting $\mathcal{A}_u = \mathcal{O}_Y(\mathcal{D}(u))$. We denote by $\widetilde{X}$ the relative spectrum $\mathrm{Spec}_Y(\mathcal{A})$ and by $X = \mathcal{X}(\mathcal{D})$ its affine contraction $\mathrm{Spec}\,\Gamma(Y, \mathcal{A})$.

We summarize the relevant results on proper polyhedral divisors.

Theorem B ([1, Theorem 3.4]). *Given a T-variety X as above, there is a curve Y and a proper polyhedral divisor $\mathcal{D}$ on Y such that the associated T-variety $\mathcal{X}(\mathcal{D})$ is equivariantly isomorphic to X.*

Theorem C ([1, Theorem 3.1]). *Let X and $\widetilde{X}$ be given by a proper polyhedral divisor on the curve Y.*

  (i) *The contraction map $\widetilde{X} \to X$ is proper and birational.*
 (ii) *The map $\pi \colon \widetilde{X} \to Y$ is a good quotient for the T-action on $\widetilde{X}$; in particular, it is affine.*
(iii) *There is an affine open subset $U \subset Y$ such that the contraction map restricts to an isomorphism on $\pi^{-1}(U)$.*

Example. A polyhedral divisor for the torus action on $X = \mathrm{SL}(2, \mathbb{C})$ is computed easily by considering the closed embedding in the toric variety $\mathrm{Mat}(2 \times$

$2, \mathbb{C}) \cong \mathbb{A}^4$. The toric computation [1, Section 11] shows that $\mathbb{A}^4$ with the induced $(\mathbb{C}^*)^2$-action may be described by the divisor $\mathcal{D}' = \Delta_1 \otimes D_1 + \Delta_2 \otimes D_2$ on $\mathbb{A}^2$, where $D_i = \operatorname{div}(x_i)$ are the coordinate axes and $\Delta_i = \operatorname{conv}\{0, e_i\}$. The image of X in $\mathbb{A}^2$ is the line through $(1, 0)$ and $(0, 1)$. Hence, $\mathcal{D}'$ restricts to the divisor $\mathcal{D} = \Delta_1 \otimes [0] + \Delta_2 \otimes [1]$ on $\mathbb{A}^1$.

3. Comparison

Now to compare the toroidal and polyhedral data associated with a T-variety X. By Theorem B, we may assume X is given by a polyhedral divisor $\mathcal{D}$ on a curve Y, contained in the complete curve C. As above, we have the T-variety $\widetilde{X}$ with the quotient map π to Y and the contraction to X.

Denote the open subset of points P with trivial coefficient $\Delta_P = \sigma$ by V. Then for any open subset $U \subset V$, we have

$$\pi^{-1}(U) = \operatorname{Spec}_U \mathcal{O}_U \otimes \mathbb{C}[\omega \cap M] = U \times \operatorname{TV}(\sigma).$$

In particular, $U \times T$ is an open subset of $\widetilde{X}$. By part (iii) of Theorem C, we may regard $U \times T$ as a subset of X after possibly shrinking U. The projection to U gives the required rational quotient map $X \dashrightarrow C$.

We get varieties X' and X'' as before and note the following fact.

Lemma 1. $\widetilde{X}$ *is canonically isomorphic to* X''.

Proof. It follows from the construction of X'' that the maps $\widetilde{X} \to X$ and $\widetilde{X} \to C$ factor through a map $\varphi \colon \widetilde{X} \to X''$. Since both maps to X are proper, so is φ. Since both maps to C are affine, so is φ. Since φ is also birational, it is an isomorphism. □

Now for suitable U, we saw above that $(U \times T, \widetilde{X})$ is a toroidal embedding with fan $\Delta(X, U)$. We recall the statement of our claim.

Theorem 1. *The toroidal fan* $\Delta(X, U)$ *is equal to the fan obtained by gluing the homogenizations of the coefficient polyhedra* Δ_P *of points* $P \in Y \setminus U$ *along their common face* $(0, \sigma)$.

To see this, consider $P \in Y \setminus U$ and $U' = U \cup \{P\}$ with local parameter s at P. Since $\mathcal{D}_{|U}$ is trivial, we have $\mathcal{D}_{|U'} = \Delta_P \otimes P$. The graded parts of $\mathcal{A} = \bigoplus_{u \in \omega \cap M} \mathcal{A}_u$ are thus

$$\mathcal{A}_u = \mathcal{O}_{U'}\big(\mathcal{D}_{|U'}(u)\big) = \mathcal{O}_{U'}\big(\min\langle u, \Delta_P\rangle \cdot P\big) = \mathcal{O}_{U'}\big(\lfloor\min\langle u, \Delta_P\rangle\rfloor \cdot P\big).$$

Hence, we can express the graded parts of the coordinate ring S of $\pi^{-1}(U')$ as

$$S_u = \Gamma\big(U', \mathcal{O}_{U'}(\mathcal{D}(u))\big) = R' \cdot s^{-\lfloor\min\langle u, \Delta_P\rangle\rfloor}.$$

It follows that the monomial semigroup of the toric model consists of the pairs $(k, u) \in \mathbb{Z} \times M$ with $k \geq -\min\langle u, \Delta_P\rangle$. By Lemma 2 below, we see that δ_P is the homogenization of Δ_P. As Remark 2 implies that points in the complement of Y don't contribute to $\Delta(X, U)$, the proof is complete.

Lemma 2. *Let Δ be a polyhedron in N with tail cone σ. Let δ in $\mathbb{Z} \times N$ be its homogenization, i.e., $\delta = \mathrm{pos}\{(0, \sigma), (1, \Delta)\}$. Then the dual cone $\delta^\vee$ consists of those pairs $(r, u) \in \mathbb{Q} \times M_\mathbb{Q}$ with $u \in \sigma^\vee$ and $r \geq -\min\langle u, \Delta \rangle$.*

Proof. By definition, we have $(r, u) \in \delta^\vee$ if and only if (r, u) is non-negative on both $(0, \sigma)$ and $(1, \Delta)$. The first condition is equivalent to $u \in \sigma^\vee$. The second condition means that $r \geq -\langle u, v \rangle$ for any $v \in \Delta$, that is, $r \geq -\min\langle u, \Delta \rangle$. $\quad\square$

Example. For the example of $\mathrm{SL}(2, \mathbb{C})$, clearly the homogenizations of the segments $\mathrm{conv}\{0, e_i\}$ give the cones δ_0, δ_1 generated by $(1, 0)$ and $(1, e_i)$. This is illustrated in Figure 1.

Remark 4. Both descriptions generalize to the non-affine case. Mumford treats this directly, while the polyhedral approach involves the fans of polyhedral divisors developed by Altmann, Hausen and Süß [2]. It should be straightforward to carry this result over.

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