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Toroidal magnetic fields in type II superconducting neutron stars

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ABSTRACT

We determine constraints on the form of axisymmetric toroidal magnetic fields dictated by hydrostatic balance in a type II superconducting neutron star with a barotropic equation of state. Using Lagrangian perturbation theory, we find the quadrupolar distortions due to such fields for various models of neutron stars with type II superconducting and normal regions. We find that the star becomes prolate and can be sufficiently distorted to display precession with a period of the order of years. We also study the stability of such fields using an energy principle, which allows us to extend the stability criteria established by R. J. Tayler for normal conductors to more general media with magnetic free energy that depends on density and magnetic induction, such as type II superconductors. We also derive the growth rate and instability conditions for a specific instability of type II superconductors, first discussed by P. Muzikar, C. J. Pethick and P. H. Roberts, using a local analysis based on perturbations around a uniform background.

Key words: dense matter – MHD – stars: magnetic fields – stars: neutron.

1 INTRODUCTION

Timing residuals varying on time-scales of the order of months to years have been detected in several pulsars, most spectacularly in PSR B1828−11, where several cycles of nearly periodic variation have been reported (Stairs, Lyne & Shemar 2000; Stairs et al. 2003). For PSR B1828−11, the precession period is $P_{\rm p} \approx 500\,{\rm d} \approx 4.3 \times 10^7$ s and the spin period is $P_\star \approx 0.405$ s; interpreting the long-term timing residuals as rigid body precession then implies a stellar distortion $\epsilon \approx P_\star/P_{\rm p} \approx 9.4 \times 10^{-9}$. Precession affects arrival times in two ways (Cordes 1993; Akgün, Link & Wasserman 2006). (i) Geometrical residuals arise because the pulsar beam crosses the plane formed by the angular momentum of the star and the line of sight to the observer at times that vary periodically over the precession cycle. (ii) Variations in the angle between the spin and magnetic axes result in a periodic variation of the pulsar spindown torque, causing pulse arrival times to vary periodically as well. Precession models that combine these two effects describe the data from PSR B1828−11 adequately (Jones & Andersson 2001; Link & Epstein 2001; Akgün et al. 2006).

Problems with these models remain, however. One is the observation by Shaham (1977, 1986) that vortex line pinning can prevent long-period precession, substituting instead precession with very short periods (of the order of 10–100 spin periods, rather than 10^8) that damps out after perhaps 10^4 cycles, contrary to observations (Sedrakian, Wasserman & Cordes 1999). Although Link & Cutler (2002) showed that the precession amplitude in PSR B1828−11 may be large enough to unpin all vortex lines in the crystalline stellar crust, Link (2003) argued that the interaction of (magnetized) core superfluid vortex lines with the flux tubes in type II superconducting regions would also prevent long-period precession. One way out is that the core neutrons are not superfluid, an idea that gets some support from comparing theoretical models for cooling neutron stars with observations (e.g. Yakovlev & Pethick 2004, and references therein).

Even if vortex line pinning is not an issue, the required stellar distortion is problematic. Although the rotational distortion of a fluid star is substantial, $\epsilon_{\rm rot} \approx E_{\rm rot}/E_{\rm grav} \approx 7 \times 10^{-8}\, R_6^3/M_{1.4}\, P_\star^2$ (for uniform density), where $R_\star = 10^6 R_6$ cm and $M_\star = 1.4 M_{1.4}\, {\rm M}_\odot$ are the radius and mass, and $P_\star$ is the spin period in seconds, the bulge in a slowly rotating, self-gravitating fluid is always axisymmetric about the angular momentum axis, and cannot result in precession. Only the solid crust of a neutron star can support distortions that are fixed in the rotating frame of the star, as are needed for precession. However, the crust of a neutron star is not very rigid: its shear modulus is only about 0.01 times the crustal pressure. Consequently, $\epsilon \ll \epsilon_{\rm rot}$ if the crustal distortion is 'relaxed' at the current rotational frequency of the star (Baym & Pines 1971; Cutler, Ushomirsky & Link 2003). For PSR B1828−11, agreement between the observed and calculated precession frequencies would require that the crustal deformation be relaxed at a rotation frequency of about 40 Hz, compared with the present frequency of about 2.5 Hz (Cutler et al. 2003).

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An alternative explanation for the precession frequency is that it is due to stellar distortions resulting from magnetic stresses. The idea that a rotating magnetic star must precess goes back about 50 yr (e.g. Spitzer 1958). If the magnetic field and rotational axes are not lined up, then the moment of inertia of the star is the sum of two axisymmetric contributions that are misaligned: the rotational distortion, estimated above, and a magnetic distortion of the order of $\epsilon_{\rm mag} = E_{\rm mag}/E_{\rm grav}$. In such a case, the star will precess about the magnetic axis with a frequency proportional to the magnetic distortion (Mestel & Takhar 1972; Mestel et al. 1981; Nittmann & Wood 1981).

For the typical inferred dipole magnetic fields of neutron stars, the magnetic deformation is far too small, and the resulting precession period is far too long: $\epsilon_{\rm mag} \sim 10^{-12} B_{12}^2 R_6^4 M_{1.4}^{-2}$ for a dipole magnetic field strength $B = 10^{12} B_{12}$ G. However, substantial internal toroidal fields (e.g. $B_{12} \sim 100$) could lead to large enough magnetic distortions to account for the precession frequency of PSR B1828−11 (e.g. Ioka 2001; Cutler 2002).

Larger magnetic deformations could also result from type II superconductivity in the neutron star's core for a given magnetic induction strength in the superconductor (e.g. Jones 1975; Easson & Pethick 1977; Cutler 2002; Wasserman 2003). In this paper, we shall examine the distortions of a fluid neutron star induced by the enhanced magnetic stresses associated with type II superconductivity. Here we focus on primarily toroidal fields, partly because they are easier to treat, but also because they lead to *prolate* stellar distortions, which the data on PSR B1828−11 seem to favour at least weakly (Wasserman 2003; Akgün et al. 2006). We will include a weaker poloidal component that can leak into the stellar magnetosphere, as is required for the pulsar to be active. Differential rotation within a newborn neutron star most likely amplifies the toroidal component of the field (Thompson & Duncan 2001), but stable configurations will require some poloidal field as well (e.g. Braithwaite & Nordlund 2006). We have developed the (more complicated) formalism needed to treat purely poloidal fields in a compressible type II superconductor (Akgün 2007), and will present those calculations elsewhere.

As a result of ${}^1S_0$ pairing via strong interactions, the protons in the interior of a neutron star are expected to form a type II superconductor at baryon number densities between ~0.1–0.6 $\rm fm^{-3}$ (e.g. Baym, Pethick & Pines 1969; Baym & Pethick 1975; Easson & Pethick 1977; Elgarøy et al. 1996; Jones 2006; Baldo & Schulze 2007). Magnetic flux penetrates the superconducting region in the neutron star in the form of quantized magnetic flux tubes. Typically, in a neutron star the critical field is $H_{\rm c1} \sim 10^{15}$ G, and the magnetic induction is $B \sim 10^{12}\,{\rm G} \ll H_{\rm c1}$, so the magnetic field is $H \approx H_{\rm c1}$ and is approximately a function of baryon density (e.g. Easson & Pethick 1977).

In the neutron star crust, which exists at densities below $\sim 2 \times 10^{14}\,{\rm g\,cm^{-3}}$ (Baym, Bethe & Pethick 1971; Lorenz, Ravenhall & Pethick 1993), protons are bound in nuclei, and as a result, superconductivity is suppressed. Magnetic stresses in a type II superconductor are $\sim HB/4\pi \approx H_{\rm c1}B/4\pi$, and consequently will be about $H_{\rm c1}/B \sim 10^3$ times larger than those in a normal conductor with the same B, which scale as $B^2/8\pi$ (Jones 1975; Easson & Pethick 1977). Stresses of this magnitude are capable of distorting the neutron star sufficiently to cause precession of the star with a period of the order of a year (Cutler 2002; Wasserman 2003). However, we note that hydrostatic equilibrium requires approximate continuity of HB throughout the star, so the induction $B_{\rm n}$ in the normal region is much larger than the induction $B_{\rm s}$ in the superconducting region: $B_{\rm n} \propto (HB_{\rm s})^{1/2} \gg B_{\rm s}$. Configurations with large discontinuities in stress are unstable, so it is unrealistic to embed a superconducting region with an anomalously large stress inside a star with otherwise much smaller stress.

The magnetic force in a type II superconductor is inherently different than in a normal conductor. The difference results from the fact that the magnetic free energy in a type II superconductor depends both on the magnetic induction, B (or equivalently, $u_{\rm mag} = B^2/8\pi$) and on the proton number density, $n_{\rm p}$. The proton number density is a function of the baryon number density, and consequently can be expressed as a function of total mass density, ρ. A good approximation is to take $n_{\rm p} \propto \rho$ (Easson & Pethick 1977). On the other hand, in a normal conductor the magnetic free energy is a function of magnetic induction alone.

The purpose of this paper is to determine magnetic field configurations in neutron stars with type II superconductors, *consistent with hydrostatic balance*, and assess their stability. We assume that the magnetic deformations are small, which enables a perturbative treatment. We neglect rotational deformations, slow fluid motions and associated viscous effects, which can be included at a later stage (extending methods laid out by Mestel & Takhar 1972; Mestel et al. 1981; Nittmann & Wood 1981). With these solutions we can determine the magnetic distortion explicitly (cf. Cutler 2002, who expressed the distortions in terms of averages over unspecified field configurations).

Assuming (cold nuclear) matter with a barotropic equation of state $p(\rho)$ imposes significant constraints on the possible variation of the magnetic induction $B(r, \theta)$ in the star. This is because Euler's equation of magnetohydrostatic balance requires that the magnetic force per unit mass be a total gradient (a result well known for normal magnetic equilibria; see e.g. Prendergast 1956; Monaghan 1965). The fact that $H \approx H_{\rm c1}(\rho)$ is a function of r alone to lowest order further restricts the range of possible $B(r, \theta)$. With these constraints, we can evaluate the quadrupolar deformation of the star in hydrostatic balance (as well as other multipoles, which are uninteresting for precession). In practice, we only calculate these for the $\gamma = 2$ polytropic equation of state $p = \kappa\rho^2$, where κ is a constant, but the formalism can be applied to any $p(\rho)$. Moreover, although we only present examples for which $H = H_{\rm c1}(\rho)$, our formalism applies to any magnetic free energy $F(\rho, B)$, hence $H = 4\pi\partial F/\partial B = H(\rho, B)$.

Even with the restrictions imposed by hydrostatic balance in a barotropic fluid, and the density dependence of H, many possible $B(r, \theta)$ are permitted, even when we trim the set of solutions by obvious requirements such as regularity. Stability ought to weed out even more possibilities. To examine this question, we use the energy principle that has proved fruitful for normal magnetic substances (e.g. Bernstein et al. 1958; Tayler 1973), extended to superconductors in which the magnetic free energy (and consequently H) has arbitrary dependencies on ρ and B. (Roberts 1981 examined this problem for $H \propto \rho$.) From this stability criterion, we show that the most pernicious axisymmetric instability is the interchange instability (just as in normal conductors), and we show how the list of candidate field configurations can be winnowed further by requiring immunity against it. The interchange instability can be viewed as a magnetic buoyancy mode. Our detailed treatment of perturbations is applied specifically to one-component fluids. Buoyancy due to multifluid composition, which arises as a result

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of the density dependence of the number density of charged particles in chemical equilibrium, will introduce new modes (Reisenegger & Goldreich 1992), and may change the interchange instability conditions (Ferrière, Zimmer & Blanc 1999, 2001). We postpone a complete consideration of these effects to a later paper, but in Section 4.3 we argue that stability constraints on the toroidal field shape remain the same.

For non-axisymmetric perturbations, the character of the energy principle is markedly different in the superconducting case. From it we find a specific stability criterion for what we will refer to as the Muzikar–Pethick–Roberts (MPR) instability first discussed by Muzikar & Pethick (1981) and Roberts (1981), who showed that for sufficiently weak magnetic induction $B \lesssim 10^{13}$ G, the density dependence of H promotes the formation of domains with and without magnetic flux. From a local stability analysis, we show that this instability only acts for $m > 0$ (non-axisymmetric) modes and only on very small scales perpendicular to the field, corresponding to wavenumbers $\sim 10^4/R_\star$. We estimate the growth time of the instability on these scales to be of the order of 10^3 s for typical parameters, i.e. longer than typical Alfvén wave crossing times. Although this is a distinctive mode associated with type II superconductors, the fact that it only acts on small length-scales may cause it to be suppressed by small viscous effects. Moreover, since the instability is local it is likely to be present in a rotating star as well. Preliminary calculations suggest that while the stability condition is altered by buoyancy, the unstable MPR mode persists and has the same growth rate as in a one-component fluid.

In this treatment, we neglect rotation and internal fluid motions. Our primary goal is to understand the effects of the density dependence of the magnetic free energy $F(\rho, B)$ on equilibrium and stability. This case has been previously treated by Roberts (1981), who considered poloidal fields in a completely type II superconducting star of uniform density and magnetic field $H \propto \rho$. Here we extend these considerations to barotropic equations of state and magnetic fields of the form $H(\rho, B)$ in fluid stars with type II superconducting shells. We will be concerned with toroidal magnetic fields in this paper, deferring the detailed treatment of poloidal fields to future work. We then calculate explicitly the extent of stellar deformation due to the magnetic field. Spitzer (1958) and Mestel & Takhar (1972) argued that, to lowest order, the rotational and magnetic deformations can be calculated separately. Then, a misalignment in the rotational and magnetic deformations leads to precession, as mentioned above.

In addition to the proton superconductor, there may be a commingled neutron superfluid in the core of a neutron star. If so, the two superfluids are coupled via entrainment. One consequence is that the vortices in the neutron superfluid acquire magnetic flux and therefore couple to the magnetic flux tubes in the proton superconductor. This interaction is expected to impede precession (Link 2003). The long-term periodicity observed in PSR B1828−11 may require this interaction to be of limited scale, perhaps implying that there is no commingling of the two fluids. Moreover, theoretical models for cooling neutron stars suggest that there is no compelling observational evidence for core neutron superfluid (Yakovlev & Pethick 2004). Although gap calculations generally support the existence of a $^1S_0$ crustal neutron superfluid and a core proton superconductor, the theory is less certain about the $^3P_2$ core neutron superfluid. (Elgarøy et al. 1996; Baldo & Schulze 2007). Here, we assume that there is no core neutron superfluid overlapping with regions of proton superconductivity. This simplifies the problem, as the behaviour of a mixed superfluid–superconductor system can be very complex (Glampedakis, Andersson & Jones 2007). Moreover, for the reasons given above, this may even be justified.

Here, we are primarily concerned with the equilibrium structure of the magnetic field. Although we will also discuss the stability from an energy principle point of view, we will not delve into the more comprehensive treatment of modes which should also include rotation, internal velocity fields, multifluid components and the elastic crust, as well as dissipation, mutual friction and entrainment effects, which would arise in a superfluid–superconductor mixture. In particular, dissipation is strongly dependent on whether the neutrons are superfluid or not. Moreover, there will be friction on the magnetic flux tubes which is especially important if they coexist with neutron vortices. Stability of rotating stars is known to be affected by normal magnetic fields (Glampedakis & Andersson 2007), and we expect the same to be true in the presence of superconductivity. Therefore, our work is only a first step towards a more complete treatment of the neutron star interior, where we highlight features arising from the density dependence of the magnetic free energy.

The outline of this paper is as follows. In Section 2, we discuss the magnetic stress tensor and force in a type II superconductor. In Section 3, we determine the form of the toroidal magnetic fields in the normal and superconducting regions, consistent with the boundary conditions at the stellar surface and internal boundaries. We then proceed with the calculation of the hydrostatic equilibrium in the presence of such magnetic fields in various neutron star models with type II and normal regions. We calculate the density and gravitational potential perturbations and determine the moments of inertia of the perturbed star. In Section 4, we discuss the stability of toroidal fields in the normal and superconducting cases. We show that the interchange instability is the worst axisymmetric instability, and derive the MPR instability conditions and relevant time and length-scales from a local analysis. In Section 5, we discuss the possibility of adding a small poloidal component to help stabilize the toroidal fields. We derive the form of this poloidal field that is consistent with the requirements that the magnetic force be a gradient and that the magnetic induction be divergenceless.

2 MAGNETIC FORCE IN A TYPE II SUPERCONDUCTOR

The magnetic stress tensor in a type II superconductor is given as (Easson & Pethick 1977)

$$\sigma_{ij} = \left[F - \rho\frac{\partial F}{\partial \rho} - B\frac{\partial F}{\partial B}\right]\delta_{ij} + \frac{H_i B_j}{4\pi}. \tag{1}$$

The magnetic free energy $F(\rho, B)$ is a function of mass density, ρ and magnetic induction, B. In isotropic media the magnetic field H_i and induction B_i are parallel, so that $\sigma_{ij} = \sigma_{ji}$. In general, the relation between the magnetic field and induction is given through (Josephson

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1966),

$$H = 4\pi \frac{\partial F}{\partial B}. \tag{2}$$

In a normal conducting medium we have $H = B$, i.e. the magnetic field is independent of density, and the free energy is equal to the magnetic energy $F = B^2/8\pi$. Thus, the stress tensor in this case reduces to

$$\sigma_{ij} = -\frac{B^2}{8\pi}\delta_{ij} + \frac{B_i B_j}{4\pi}. \tag{3}$$

On the other hand, the magnetic field in a strongly type II superconducting medium, such as the proton superconductor in a neutron star, is $H \approx H_{c1} \gg B$, and depends most sensitively on the proton number density n_p and the superconducting energy gap Δ (Tinkham 1975; Easson & Pethick 1977), which are functions of baryon density ρ (Elgarøy et al. 1996; Baldo & Schulze 2007); therefore, $H \approx H(\rho)$ and $F \approx HB/4\pi$. In this case, the magnetic stress tensor reduces to

$$\sigma_{ij} = -\rho \frac{\partial F}{\partial \rho}\delta_{ij} + \frac{H_i B_j}{4\pi}. \tag{4}$$

The stress tensor used by Roberts (1981) is of this form, with $H \propto \rho$.

In general, the gradient of the free energy is given as

$$\nabla_i F = \frac{\partial F}{\partial \rho}\nabla_i \rho + \frac{\partial F}{\partial B}\nabla_i B. \tag{5}$$

From equation (2) it follows that

$$B\nabla_i \frac{\partial F}{\partial B} = \frac{B_k \nabla_i H_k}{4\pi}. \tag{6}$$

Making use of these relations as well as the fact that $\nabla \cdot \boldsymbol{B} = 0$, the magnetic force density can be calculated from equation (1) as

$$f_i = \nabla_j \sigma_{ij} = -\rho\nabla_i \frac{\partial F}{\partial \rho} - B\nabla_i \frac{\partial F}{\partial B} + \frac{B_j \nabla_j H_i}{4\pi} = \frac{[(\nabla \times \boldsymbol{H}) \times \boldsymbol{B}]_i}{4\pi} - \rho\nabla_i \frac{\partial F}{\partial \rho}. \tag{7}$$

This is the form of the force in a type II superconductor. (In fact, it is true in any magnetic medium where the free energy is a function of density and magnetic induction.) This is inherently different from the force in a normal conducting medium, which can be retrieved by setting $H = B$ and $F = B^2/8\pi$.

In hydrostatic balance,

$$\nabla p + \rho\nabla\phi = \boldsymbol{f}_{\mathrm{mag}}, \tag{8}$$

where p is pressure, ρ is mass density, ϕ is gravitational potential and $\boldsymbol{f}_{\mathrm{mag}}$ is the magnetic force density (equation 7). In barotropic equations of state, pressure is a function of density and we can define $\mathrm{d}h(\rho) = \rho^{-1}\mathrm{d}p(\rho)$; then,

$$\rho\nabla(h + \phi) = \boldsymbol{f}_{\mathrm{mag}}. \tag{9}$$

This equation requires the magnetic force per unit mass to be a gradient of a potential, i.e. $\boldsymbol{f}_{\mathrm{mag}} = -\rho\nabla\psi$. We will express the magnetic potential as the sum of two terms,

$$\psi = \psi_{\mathrm{I}} + \psi_{\mathrm{II}}, \tag{10}$$

where, we define,

$$\frac{(\nabla \times \boldsymbol{H}) \times \boldsymbol{B}}{4\pi} = \frac{\boldsymbol{J} \times \boldsymbol{B}}{c} = -\rho\nabla\psi_{\mathrm{I}} \quad \text{and} \quad \psi_{\mathrm{II}} = \frac{\partial F}{\partial \rho}. \tag{11}$$

$\boldsymbol{J}$ is the current density, ψ_{I} is the magnetic potential for a normal conductor, and ψ_{II} is present only for a type II superconductor. The second term in the magnetic force (equation 7) is already a gradient. On the other hand, note that the requirement for the first term to be a gradient can be expressed alternatively as

$$\nabla \times \left(\frac{\boldsymbol{J} \times \boldsymbol{B}}{\rho c}\right) = 0. \tag{12}$$

This equation needs to be satisfied for both the normal and type II superconducting cases, and imposes a severe restriction on the form of the magnetic fields, which are also required to satisfy $\nabla \cdot \boldsymbol{B} = 0$. The normal conducting case is discussed, for example, in Prendergast (1956) and Monaghan (1965). For the strongly type II case and $H \propto \rho$, Roberts (1981) found poloidal field configurations for uniformly dense stars, and Akgün (2007) found poloidal field configurations for $\gamma = 2$ polytropes.

3 TOROIDAL FIELDS

The current density for a toroidal field $\boldsymbol{H} = H(r, \theta)\hat{\phi}$ is

$$\frac{4\pi\boldsymbol{J}}{c} = \nabla \times \boldsymbol{H} = \nabla(Hr\sin\theta) \times \frac{\hat{\phi}}{r\sin\theta}. \tag{13}$$

Taking the induction to be $\boldsymbol{B} = B(r, \theta)\hat{\phi}$, we get

$$\frac{\boldsymbol{J} \times \boldsymbol{B}}{\rho c} = \frac{(\nabla \times \boldsymbol{H}) \times \boldsymbol{B}}{4\pi\rho} = -\frac{B\nabla(Hr\sin\theta)}{4\pi\rho r\sin\theta}. \tag{14}$$

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This is clearly a total gradient, as required by equation (12), for magnetic inductions of the form

$$B(r,\theta) = 4\pi\rho r \sin\theta\, f(Hr\sin\theta), \tag{15}$$

where f is an arbitrary function of $\zeta = Hr\sin\theta$. The factor of 4π is included so that defining a new function through $f(\zeta) = g'(\zeta)$ gives, using the definitions in equation (11),

$$\frac{\boldsymbol{J}\times\boldsymbol{B}}{\rho c} = -\nabla g(\zeta), \quad \text{i.e.} \quad \psi_{\rm I}(r,\theta) = g(\zeta). \tag{16}$$

This is valid for any $H(r, \theta)$. However, for a strongly type II superconductor $H \approx H(r)$, and we have (equation 11),

$$\psi_{\rm II} = \frac{B}{4\pi}\frac{\mathrm{d}H}{\mathrm{d}\rho} = \frac{\mathrm{d}\ln H}{\mathrm{d}\ln\rho}\zeta g'(\zeta). \tag{17}$$

For a normal conductor $H = B$, and equation (15) implies that the magnetic induction is now given through the form,

$$B(r,\theta) = \frac{h(\rho r^2\sin^2\theta)}{r\sin\theta}, \tag{18}$$

where h is an arbitrary function of $\xi = \rho r^2 \sin^2\theta$. It then follows that

$$\frac{\boldsymbol{J}\times\boldsymbol{B}}{\rho c} = -\nabla\psi = -\frac{\nabla h^2(\xi)}{8\pi\xi}, \quad \text{i.e.} \quad \psi'(\xi) = \frac{h(\xi)h'(\xi)}{4\pi\xi}. \tag{19}$$

Note that, for a uniform density, the magnetic induction is a function of the cylindrical radius, $\varpi = r\sin\theta$.

3.1 Star with a superconducting shell

Consider the case of a strongly type II superconducting region confined to a spherical shell between radii r_1 and r_2 (where $r_2 > r_1$). Let the magnetic field be $B_{\rm c}$ inside the normal core, H inside the superconducting shell (with a corresponding magnetic induction $B_{\rm s}$), and $B_{\rm n}$ inside the normal outer layer (as depicted in Fig. 1). Since the fields have no radial components in this case, they need not be continuous across the boundaries, and there will be surface currents.

In fact, it turns out that in the toroidal case it is not possible to have a continuous magnetic field across the boundaries, if $H = H(\rho)$ in the superconducting region. Consider one of the boundaries of the superconducting shell, located at $r = r_{\rm b}$. For the present discussion, it is immaterial whether the normal region lies on the inside or the outside of the boundary. In the absence of surface currents, the boundary condition that follows from Maxwell's equations requires the continuity of the tangential magnetic field,

$$\hat{\boldsymbol{r}}\times\boldsymbol{H} = \hat{\boldsymbol{r}}\times\boldsymbol{B}_{\rm n}. \tag{20}$$

Since H is a function of radius in a strongly type II superconductor, for this equation to be satisfied everywhere on the surface of a spherical boundary, the magnetic field $B_{\rm n}$ inside the normal region (given by equation 18) would have to be a function of only radius at the boundary as well. This implies that we must choose a function $h(\xi) \propto \xi^{1/2}$, so that $B_{\rm n}(r, \theta) \propto \rho^{1/2}(r)$. However, in this case, the magnetic potential becomes $\psi_{\rm n}(\xi) \propto \ln\xi$ (equation 19), which diverges whenever $\xi = \rho\, r^2 \sin^2\theta$ is zero. In other words, it diverges at the centre of the star ($r \to 0$), at the surface ($\rho \to 0$), and along the symmetry axis ($\theta \to 0$). We also note that when the magnetic induction $B_{\rm s}$ inside the superconducting region (given by equation 15) is chosen so that it is angle independent [i.e. $f(\zeta) \propto 1/\zeta$], the corresponding potential is also logarithmic, $\psi_{\rm I}(\zeta) \propto \ln\zeta$.

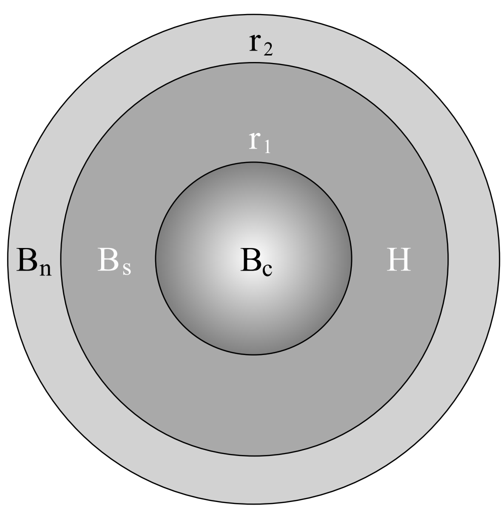

Figure 1. A star with a normal core, superconducting shell, and a surrounding normal layer. The radius of the core is r_1 and the outer radius of the superconducting shell is r_2.

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We therefore conclude that continuous toroidal fields, or more generally, angle-independent magnetic inductions, are inconsistent under the assumption that $H = H(\rho)$ holds up to the boundaries of the superconducting region. In a more realistic treatment, $H(\rho, B)$ should be allowed to decrease smoothly to about B_s near the boundaries, which would remove the need for surface currents.

3.2 Boundary conditions

Hydrostatic equilibrium for a fluid with a barotropic equation of state, in the absence of magnetic fields, is spherically symmetric and is given by (from equation 9),

$$\nabla(h + \phi) = 0. \tag{21}$$

When a magnetic force that is small in comparison to pressure and gravity is applied, the equilibrium quantities are changed by small amounts δp, $\delta\rho$, δh and $\delta\phi$, where δ denotes Eulerian changes. Writing the magnetic force in terms of the magnetic potential, $\boldsymbol{f}_{\mathrm{mag}} = -\rho\nabla\psi$, the equation for the perturbations around the background equilibrium can be written as

$$\nabla(\delta h + \delta\phi + \psi) = 0. \tag{22}$$

From here it follows that

$$\delta h = \frac{\mathrm{d}h}{\mathrm{d}\rho}\delta\rho = \mathfrak{B}_{\mathrm{o}} - \delta\phi - \psi. \tag{23}$$

$\mathfrak{B}_{\mathrm{o}}$ is Bernoulli's constant and is the same for the entire star. This can be understood by treating the entire star as a single fluid region, with a magnetic potential that varies continuously throughout the interior, but that has steep changes in some small intervals corresponding to the boundaries.

While the background quantities p, ρ and ϕ are continuous throughout the star, their perturbations are not. Only $\delta\phi$ and its gradient are required to be continuous, since there cannot be delta functions in mass. This implies that there will be a density perturbation jump at a boundary, given by (from equation 23),

$$\frac{\mathrm{d}h}{\mathrm{d}\rho}(\delta\rho_{\mathrm{s}} - \delta\rho_{\mathrm{n}}) = -\psi_{\mathrm{s}} + \psi_{\mathrm{n}}. \tag{24}$$

Here the subscripts s and n refer to the superconducting and normal regions, respectively.

There must be substantial surface currents at the boundaries of the superconducting shell, and therefore, the magnetic field is discontinuous across them. Otherwise, as discussed before, the magnetic potentials become singular. From the continuity of stress, it follows that

$$n_j \Sigma_{ij,s} = n_j \Sigma_{ij,n}. \tag{25}$$

Σ_{ij} is the total stress tensor and n_j is the normal unit vector of the boundary, which in this case is simply the radial unit vector $\hat{\boldsymbol{r}}$. Thus, we require the rr, $r\theta$ and $r\phi$ components of the stress tensor to be continuous. The last two vanish identically for fluids with toroidal fields.

The total stress is

$$\Sigma_{ij} = -\delta p\, \delta_{ij} + \sigma_{ij}, \tag{26}$$

and from equation (25), we have

$$-\delta p_{\mathrm{s}} + \sigma_{rr,s} = -\delta p_{\mathrm{n}} + \sigma_{rr,n}. \tag{27}$$

Using the fact that for a polytrope $p = \kappa\rho^{\gamma}$, we have $\mathrm{d}h/\mathrm{d}\rho = \gamma p/\rho^2$ and $\delta p = (\gamma p/\rho)\delta\rho$, we can combine this result with equation (24) to get

$$\frac{\gamma p}{\rho}(\delta\rho_{\mathrm{s}} - \delta\rho_{\mathrm{n}}) = -\rho(\psi_{\mathrm{s}} - \psi_{\mathrm{n}}) = \sigma_{rr,s} - \sigma_{rr,n}. \tag{28}$$

The components of the stress tensor inside the normal and superconducting regions are given by (equations 3 and 4),

$$\sigma_{rr,n} = -\frac{B_{\mathrm{n}}^2}{8\pi} \quad \text{and} \quad \sigma_{rr,s} = -\rho\frac{\partial F}{\partial\rho} = -\rho\psi_{\mathrm{II}}. \tag{29}$$

Using $\psi_{\mathrm{s}} = \psi_{\mathrm{I}} + \psi_{\mathrm{II}}$ (equation 10), we thus obtain

$$-\psi_{\mathrm{I}} = -\psi_{\mathrm{n}} + \frac{B_{\mathrm{n}}^2}{8\pi\rho}. \tag{30}$$

This equation needs to be satisfied by the magnetic fields at the boundary. Note that since $\psi_{\mathrm{I}} \propto HB_{\mathrm{s}}/\rho$ and $\psi_{\mathrm{n}} \propto B_{\mathrm{n}}^2/\rho$, this equation implies that $B_{\mathrm{n}} \propto (HB_{\mathrm{s}})^{1/2}$. If we take $H \gg B_{\mathrm{s}}$ to hold at the boundaries of the superconductor as well as its interior, then the boundary condition clearly requires $B_{\mathrm{n}} \gg B_{\mathrm{s}}$. Taking a more general $H(\rho, B)$, varying continuously from $H_{\mathrm{c1}}(\rho)$ to B_{s} through a thin boundary layer, would result in a smooth but similar growth in the magnetic induction between the strongly type II and normal regions. (Surface currents would be smoothed out over this boundary layer.) For entirely normal conductors, the corresponding boundary condition simply implies the continuity of magnetic fields.

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In a more sophisticated treatment of the transitions from superconducting to normal and/or fluid to crust, two dimensionless ratios characterize the superconducting state. One is

$$\kappa = \frac{\lambda}{\xi} \approx \frac{8.2\Delta\,(\mathrm{MeV})}{(n_{p,37})^{5/6}}, \tag{31}$$

where λ is the London penetration depth, ξ is the coherence length in the proton superconductor, $n_\mathrm{p} = 10^{37} n_{p,37}\,\mathrm{cm}^{-3}$ is the proton number density and Δ is the proton superconducting gap. The other is

$$\frac{a}{\lambda} \approx 68 B_{12}^{-1/2} (n_{p,37})^{1/2}, \tag{32}$$

where a is the spacing between flux tubes (Tinkham 1975). In a type II superconductor, $\kappa > 1/\sqrt{2}$.

At the crust-core boundary, n_p falls dramatically, and a/λ drops, which means that interactions between flux tubes become important. As a result, our approximation that $H \approx H_{\mathrm{c}1}(\rho)$ must fail, and must be replaced by a more general (and complicated) function of both ρ and B.

At the inner boundary of the superconducting layer, Δ ultimately disappears, and κ falls below $1/\sqrt{2}$. In this regime, we expect a boundary layer of a type I superconductor to form. In fact, it is also possible for such a layer to form at the crust-core boundary, since the gap depends exponentially on the density of states near the proton Fermi surface, which falls with proton density. Thus, at both boundaries, we expect the magnetic field to decrease rapidly from $H \sim 10^{15}$ G to $B_\mathrm{n} \sim (HB_\mathrm{s})^{1/2}$.

3.3 Derivation of the magnetic fields

We will assume a simple power-law relation between the magnetic field in the superconducting region and mass density,

$$H = H_\mathrm{c} \left(\frac{\rho}{\rho_\mathrm{c}}\right)^{\sigma}, \tag{33}$$

where H_c and ρ_c stand for the central values of the corresponding quantities. When the superconducting region is confined to a shell, we can take H_c to be the extrapolated field strength at the centre. In reality, in a strongly type II superconductor, H depends on the superconducting energy gap Δ, in addition to the proton number density n_p (Tinkham 1975; Easson & Pethick 1977). Both n_p and Δ are functions of baryon density ρ (Elgarøy et al. 1996; Baldo & Schulze 2007). Δ vanishes at sufficiently high densities, and protons become normal. At low densities, superconductivity is suppressed since protons are bound in the nuclei in the neutron star crust. In both cases, the transition from superconducting to normal state may be sharp and we take the form given by equation (33) in superconducting regions.

In this case, equations (16) and (17) imply $\psi_\mathrm{I} = g(\zeta)$ and $\psi_\mathrm{II} = \sigma\zeta g'(\zeta)$, where $\zeta = Hr\sin\theta$. Consider a power-law function of the form $g(\zeta) = N\zeta^n$, where N is a constant; then $\psi_\mathrm{I} = N\zeta^n$ and $\psi_\mathrm{II} = n\sigma\, N\zeta^n$, so that the total magnetic potential becomes

$$\psi_\mathrm{s} = \psi_\mathrm{I} + \psi_\mathrm{II} = (n\sigma + 1)N\zeta^n. \tag{34}$$

We exclude $n = 0$ since that corresponds to zero magnetic induction and force. On the other hand, for $n < 0$ the magnetic potential diverges when either $r \to 0$ or $\theta \to 0$. Moreover, the magnetic force diverges in the same limits in the interval $0 < n < 1$. Therefore, the only non-singular choices are $n \geqslant 1$. The magnetic induction inside the superconductor is (equation 15),

$$B_\mathrm{s}(r,\theta) = B_\mathrm{o} \left(\frac{\rho}{\rho_\mathrm{c}}\right)^{\sigma(n-1)+1} \left(\frac{r}{r_\mathrm{o}}\right)^{n} \sin^n\theta \quad \text{where} \quad B_\mathrm{o} = 4\pi n N \rho_\mathrm{c} H_\mathrm{c}^{n-1} r_\mathrm{o}^{n}. \tag{35}$$

The constant r_o will be defined later. The corresponding magnetic potential can be written as

$$\psi_\mathrm{s}(r,\theta) = \Psi_\mathrm{o} \left(\frac{\rho}{\rho_\mathrm{c}}\right)^{n\sigma} \left(\frac{r}{r_\mathrm{o}}\right)^{n} \sin^n\theta \quad \text{where} \quad \Psi_\mathrm{o} = \frac{(n\sigma+1)H_\mathrm{c}B_\mathrm{o}}{4\pi n\rho_\mathrm{c}}. \tag{36}$$

Inside the normal region we have, from equations (18) and (19), defining $\xi = \rho r^2 \sin^2\theta$,

$$B_\mathrm{n}(r,\theta) = \frac{h(\xi)}{r\sin\theta} \quad \text{and} \quad \psi_\mathrm{n}'(\xi) = \frac{h(\xi)h'(\xi)}{4\pi\xi}. \tag{37}$$

We will assume a power law for the arbitrary function, $h(\xi) = M\xi^m$, where M is a constant. Then,

$$\frac{B_\mathrm{n}^2}{8\pi\rho} = \frac{M^2\xi^{2m-1}}{8\pi} \quad \text{and} \quad \psi_\mathrm{n} = \frac{mM^2\xi^{2m-1}}{4\pi(2m-1)}. \tag{38}$$

The boundary condition (equation 30) gives, after some rearrangement,

$$N\zeta^n = \frac{M^2\xi^{2m-1}}{8\pi(2m-1)}. \tag{39}$$

In order to satisfy this equation for all values of θ at the boundary (which we will assume to be located at some radius $r = r_\mathrm{b}$) we must have

$$n = 4m - 2 \quad \text{whence} \quad M = \left[\frac{4\pi n N H^n(r_\mathrm{b})}{\rho^{n/2}(r_\mathrm{b})}\right]^{1/2}. \tag{40}$$

Then the magnetic field in the normal region is

$$B_\mathrm{n}(r,\theta) = \hat{B}_\mathrm{o} \left(\frac{\rho}{\rho_\mathrm{c}}\right)^{(n+2)/4} \left(\frac{r}{r_\mathrm{o}}\right)^{n/2} \sin^{n/2}\theta \quad \text{where} \quad \hat{B}_\mathrm{o} = M\rho_\mathrm{c}^{(n+2)/4} r_\mathrm{o}^{n/2}. \tag{41}$$

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Note that B_s and B_n must have different angular dependencies in order for the potentials ψ_s and ψ_n to be consistent. Moreover,

$$\hat{B}_o = (H_c B_o)^{1/2} \left[\frac{\rho(r_b)}{\rho_c}\right]^{n(2\sigma-1)/4}, \tag{42}$$

so that the magnetic fields in the normal regions are moderately strong. The magnetic potential in the normal region is

$$\psi_n(r,\theta) = \hat{\Psi}_o \left(\frac{\rho}{\rho_c}\right)^{n/2} \left(\frac{r}{r_o}\right)^n \sin^n\theta \quad \text{where} \quad \hat{\Psi}_o = \frac{(n+2)\hat{B}_o^2}{8\pi n \rho_c}. \tag{43}$$

Thus, it follows that $\hat{\Psi}_o \propto \Psi_o$,

$$\frac{\hat{\Psi}_o}{\Psi_o} = \frac{n+2}{2(n\sigma+1)} \frac{\hat{B}_o^2}{H_c B_o} = \frac{n+2}{2(n\sigma+1)} \left[\frac{\rho(r_b)}{\rho_c}\right]^{n(2\sigma-1)/2}. \tag{44}$$

As in the superconducting case, we need to have $n \geqslant 1$ in order to avoid any divergences in the potentials or forces.

3.4 The $n = 1$ case

In a later section, we will show that toroidal fields by themselves are unstable, and that the $n = 1$ case is the closest to being stable. We will be concerned particularly with cases where $H \propto \rho$, i.e. $\sigma = 1$. This corresponds to taking the proton number density to be proportional to the baryon density, $n_p \propto \rho$, and neglecting logarithmic dependencies in H, which is a good first-order approximation (Easson & Pethick 1977; Muzikar & Pethick 1981). The magnetic potentials in the superconducting and normal regions become, from equations (36) and (43),

$$\psi_s = \Psi_o \left(\frac{\rho}{\rho_c}\right)\left(\frac{r}{r_o}\right)\sin\theta \quad \text{and} \quad \psi_n = \hat{\Psi}_o \left(\frac{\rho}{\rho_c}\right)^{1/2}\left(\frac{r}{r_o}\right)\sin\theta, \tag{45}$$

where, from equation (44), we have,

$$\Psi_o = \frac{H_c B_o}{2\pi\rho_c} \quad \text{and} \quad \frac{\hat{\Psi}_o}{\Psi_o} = \frac{3}{4}\left[\frac{\rho(r_b)}{\rho_c}\right]^{1/2}. \tag{46}$$

The angular part of the potentials can be expanded in Legendre polynomials,

$$\sin\theta = \sum_{\ell=0}^{\infty} \Theta_\ell P_\ell(\cos\theta). \tag{47}$$

Only even ℓ remain in the series and the coefficients are

$$\Theta_\ell = \frac{2\ell+1}{2}\int_{-1}^{1} \sin\theta P_\ell(\cos\theta)\mathrm{d}(\cos\theta) = \frac{(2\ell+1)\pi^2}{2(\ell+2)(1-\ell)\Gamma^2(\ell/2+1)\Gamma^2(1/2-\ell/2)}. \tag{48}$$

In particular, $\Theta_0 = \pi/4$. Subsequent terms in the expansion have the ratio

$$\frac{\Theta_{\ell+2}}{\Theta_\ell} = \frac{(2\ell+5)(\ell+1)(\ell-1)}{(2\ell+1)(\ell+4)(\ell+2)}. \tag{49}$$

Clearly, $\Theta_{\ell+2}/\Theta_\ell \to 1$ as $\ell \to \infty$. The result can also be expressed in terms of the spherical harmonics which are related to the Legendre polynomials through

$$Y_\ell(\theta) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta). \tag{50}$$

Then, for even ℓ,

$$\sin\theta = \sum_{\ell=0}^{\infty} \tilde{\Theta}_\ell Y_\ell(\theta) \quad \text{where} \quad \tilde{\Theta}_\ell = \sqrt{\frac{4\pi}{2\ell+1}}\Theta_\ell. \tag{51}$$

We will consider a $\gamma = 2$ polytrope for which the equation of state is $p = \kappa\rho^2$, where κ is a constant. In this case, the background density is of the form $\rho = \rho_c \sin x/x$, in terms of the dimensionless variable $x = r/r_o$, where $r_o = \sqrt{\kappa/2\pi G}$. The stellar radius is $R_\star = \pi r_o$, and the stellar mass is $M_\star = \pi M_o$, where $M_o = 4\pi\rho_c r_o^3$. The central density is given by $\rho_c = \pi M_\star/4R_\star^3$. For a neutron star with $M_\star \approx 1.4\,\mathrm{M}_\odot$ and $R_\star \approx 10^6$ cm, we have $\rho_c \approx 2.2 \times 10^{15}\,\mathrm{g\,cm^{-3}}$.

As noted before, superconductivity exists only within a certain range of densities, or equivalently, a range of radii, which we will denote by $x_1 < x < x_2$. In particular, it is suppressed in the crust where the protons become bound in nuclei. The crust exists at densities below $\rho \approx 2 \times 10^{14}\,\mathrm{g\,cm^{-3}}$ (Baym et al. 1971; Lorenz et al. 1993), corresponding to an outer radius of $x_2 \approx 0.9\,\pi$. On the other hand, the proton pairing gap vanishes at higher densities. This cut-off for superconductivity is not as well established and estimates range from $\rho \approx 5 \times 10^{14}\,\mathrm{g\,cm^{-3}}$ to $10^{15}\,\mathrm{g\,cm^{-3}}$ (Elgarøy et al. 1996; Baldo & Schulze 2007). Thus, the inner boundary of the superconducting shell ranges from $x_1 \approx 0.8\pi$ to 0.6π, respectively.

The magnetic potential for the $n = 1$ case in a three-component star consisting of a type II superconducting shell surrounded by normal regions (as depicted in Fig. 1) is shown in Fig. 2. Note that the potential within the superconducting shell (which is taken to be in the interval $0.6 < x/\pi < 0.9$) is larger than those in the normal regions.

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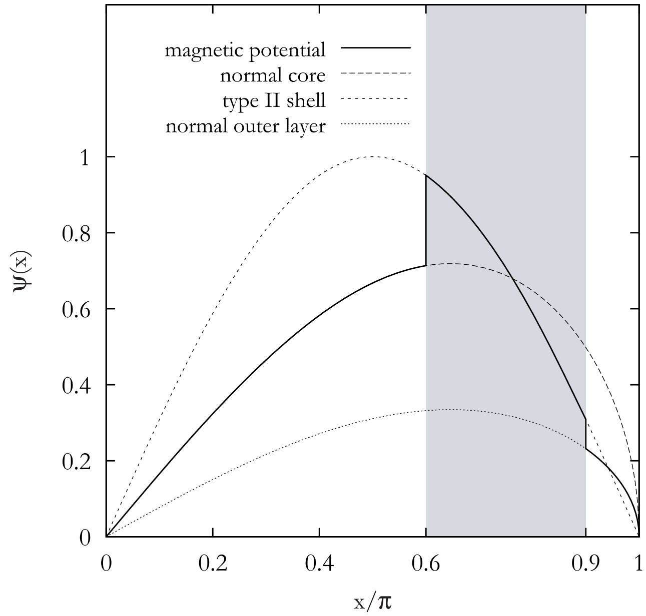

Figure 2. Magnetic potential profile for a three-component star with a normal core, type II superconducting shell, and surrounding normal layer. The potential is shown for the $n = \sigma = 1$ case for the magnetic field (equation 45), and a $\gamma = 2$ polytropic equation of state. The superconducting shell lies between $x_1 = 0.6\pi$ and $x_2 = 0.9\pi$, and is shown shaded. The potential is shown along the equator of the star, i.e. $\sin\theta = 1$, in units of Ψ_o defined in equation (46). The profiles for the potentials within each region are shown extended over the whole star for comparison.

3.5 Calculation of the gravitational potential perturbation

The gravitational potential perturbations are given by the perturbed Poisson's equation,

$$\nabla^2 \delta\phi = 4\pi G \delta\rho. \tag{52}$$

For a $\gamma = 2$ polytrope, we have $\mathrm{d}h/\mathrm{d}\rho = p'(\rho)/\rho = 2\kappa$, and equation (23) becomes $2\kappa\,\delta\rho = \mathfrak{B}_o - \delta\phi - \psi$. Expanding the perturbations in spherical harmonics as $\delta\phi(x, \theta) = \phi_\ell(x) Y_\ell(\theta)$ and so on, Poisson's equation gives

$$\frac{1}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \frac{\mathrm{d}\phi_\ell}{\mathrm{d}x}\right) + \left[1 - \frac{\ell(\ell+1)}{x^2}\right]\phi_\ell = \mathfrak{B}_o \delta_{\ell 0} - \psi_\ell. \tag{53}$$

The complete solution of this equation is the sum of a homogeneous solution and a particular solution. The homogeneous solution is given in terms of the spherical Bessel functions, $\phi_h(x) = A_\ell j_\ell(x) + B_\ell y_\ell(x)$, and the particular solution can be found by the method of variation of parameters, $\phi_p(x) = \tilde{A}_\ell(x) j_\ell(x) + \tilde{B}_\ell(x) y_\ell(x)$. Thus, the gravitational potential perturbations in the three regions (core, superconducting shell and outer normal layer, as depicted in Fig. 1) are

$$\begin{aligned}
\phi_{c,\ell}(x) &= [A_\ell + \tilde{A}_\ell(x)] j_\ell(x) + [B_\ell + \tilde{B}_\ell(x)] y_\ell(x) + \mathfrak{B}_o \delta_{\ell 0}, \\
\phi_{s,\ell}(x) &= [C_\ell + \tilde{C}_\ell(x)] j_\ell(x) + [D_\ell + \tilde{D}_\ell(x)] y_\ell(x) + \mathfrak{B}_o \delta_{\ell 0}, \\
\phi_{n,\ell}(x) &= [E_\ell + \tilde{E}_\ell(x)] j_\ell(x) + [F_\ell + \tilde{F}_\ell(x)] y_\ell(x) + \mathfrak{B}_o \delta_{\ell 0},
\end{aligned} \tag{54}$$

where A_ℓ through F_ℓ are constants, and we define

$$\tilde{A}_\ell(x) = -\int_x^\pi t^2 \psi_{c,\ell}(t) y_\ell(t)\,\mathrm{d}t \quad \text{and} \quad \tilde{B}_\ell(x) = -\int_0^x t^2 \psi_{c,\ell}(t) j_\ell(t)\,\mathrm{d}t. \tag{55}$$

Here $\psi_{c,\ell}$ refers to the ℓth component of the spherical harmonic expansion of the potential ψ_c. The remaining coefficients are defined in an analogous fashion. Note that the integration boundaries can be arbitrarily adjusted, which amounts to a redefinition of the constants A_ℓ through F_ℓ above. The particular choice made here makes sure there are no singularities, but is otherwise immaterial.

Since there can be no gravitational forces in the centre the gradient of the gravitational potential must vanish there. This implies that as $x \to 0$ we must have $\phi_\ell \to$ constant for $\ell = 0$, and $\phi_\ell \to 0$ and $\phi'_\ell \to 0$ for $\ell \neq 0$. As $x \to 0$, the limiting values of the spherical Bessel functions are $j_\ell \propto x^\ell$ and $y_\ell \propto x^{-\ell-1}$. It therefore follows that $B_\ell = 0$ for all values of ℓ. The remaining five coefficients A_ℓ, C_ℓ, D_ℓ, E_ℓ and F_ℓ, and Bernoulli's constant $\mathfrak{B}_o$ are to be determined from the continuity of the potentials and their derivatives across the shell boundaries, which we will take to be located at x_1 and x_2, such that $x_1 < x_2$,

$$\begin{aligned}
&\phi_{c,\ell}(x_1) = \phi_{s,\ell}(x_1) \quad \text{and} \quad \phi'_{c,\ell}(x_1) = \phi'_{s,\ell}(x_1) \\
&\phi_{s,\ell}(x_2) = \phi_{n,\ell}(x_2) \quad \text{and} \quad \phi'_{s,\ell}(x_2) = \phi'_{n,\ell}(x_2)
\end{aligned} \tag{56}$$

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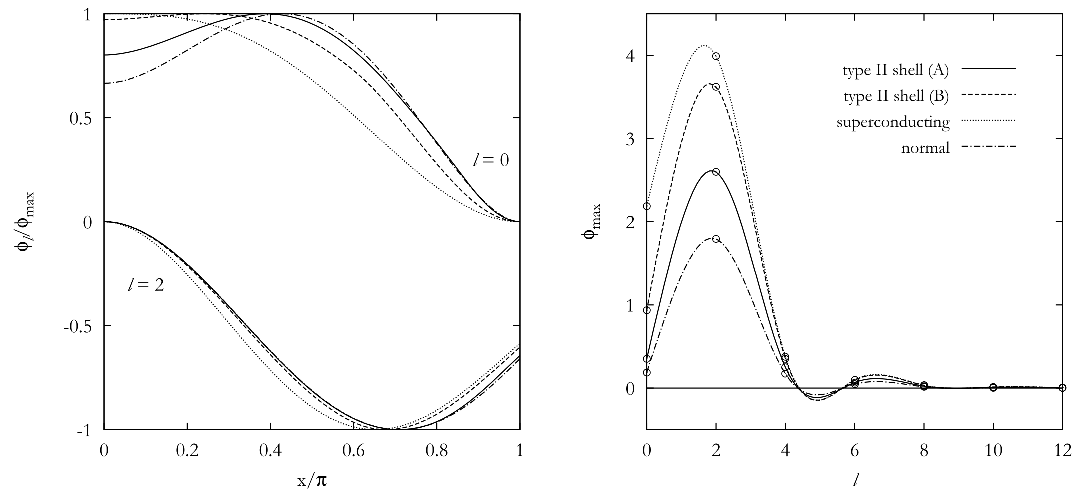

Figure 3. Gravitational potential perturbation for a fluid star with toroidal fields, expanded in spherical harmonics for the $n = 1$ case (equation 45). The potentials are shown for four sample models: type II superconducting shell between $x_1 = 0.8\pi$ and $x_2 = 0.9\pi$ (case A) and between $x_1 = 0.6\pi$ and $x_2 = 0.9\pi$ (case B), completely superconducting star ($x_1 = 0$ and $x_2 = \pi$), and completely normal star ($x_1 = x_2 = 0.9\pi$). The left-hand plot shows the first two harmonics ϕ_ℓ (for $\ell = 0$ and $\ell = 2$) scaled by the maximum value of the potential, $\phi_{\max}$. The right-hand plot shows $\phi_{\max}$ for the first few ℓ, in units of Ψ_o defined in equation (46). The points for different values of ℓ (shown with circles) are connected by a cubic spline curve. The amplitude of ϕ_ℓ decreases sharply with ℓ.

and from the boundary conditions at the stellar surface, which is located at $x = \pi$,

$$\begin{aligned} &\pi\phi'_{n,\ell}(\pi) + (\ell+1)\phi_{n,\ell}(\pi) = 0 \quad \text{for} \quad \ell \neq 0, \\ &\phi'_{n,\ell}(\pi) = \phi_{n,\ell}(\pi) = 0 \qquad\qquad \text{for} \quad \ell = 0. \end{aligned} \tag{57}$$

The surface boundary conditions follow from the multipole expansion of the gravitational potential, which implies that $\phi_\ell \propto x^{-\ell-1}$, and the conservation of mass, which additionally implies $\phi_\ell = 0$ for $\ell = 0$.

Making use of various relations between spherical Bessel functions,[1] the continuity conditions at the shell boundaries (equation 56) yield

$$\begin{aligned} &A_\ell + \tilde{A}_\ell(x_1) = C_\ell + \tilde{C}_\ell(x_1) \quad \text{and} \quad \tilde{B}_\ell(x_1) = D_\ell + \tilde{D}_\ell(x_1) \\ &C_\ell + \tilde{C}_\ell(x_2) = E_\ell + \tilde{E}_\ell(x_2) \quad \text{and} \quad D_\ell + \tilde{D}_\ell(x_2) = F_\ell + \tilde{F}_\ell(x_2) \end{aligned} \tag{58}$$

and the surface boundary conditions (equation 57) give, since $\tilde{E}_\ell(\pi) = 0$,

$$\begin{aligned} &E_\ell j_{\ell-1}(\pi) + \left[F_\ell + \tilde{F}_\ell(\pi)\right] y_{\ell-1}(\pi) = 0 \text{ for } \quad \ell \neq 0, \\ &\mathfrak{B}_o = \frac{E_\ell}{\pi^2 y_1(\pi)} = -\frac{F_\ell + \tilde{F}_\ell(\pi)}{\pi^2 j_1(\pi)} \qquad \text{for} \quad \ell = 0. \end{aligned} \tag{59}$$

Special cases can be considered. For instance, for $x_1 = 0$ and $x_2 = \pi$ we retrieve the completely superconducting star. In this case $\tilde{B}_\ell(x_1) = \tilde{D}_\ell(x_1) = 0$ so that $D_\ell = 0$. Since $\tilde{C}_\ell(x_2) = 0$ as well, the surface boundary conditions reduce to

$$\begin{aligned} &C_\ell j_{\ell-1}(\pi) + \tilde{D}_\ell(\pi) y_{\ell-1}(\pi) = 0 \quad \text{for} \quad \ell \neq 0, \\ &\mathfrak{B}_o = \frac{C_\ell}{\pi^2 y_1(\pi)} = -\frac{\tilde{D}_\ell(\pi)}{\pi^2 j_1(\pi)} \quad \text{for} \quad \ell = 0. \end{aligned} \tag{60}$$

On the other hand, letting $x_1 \to 0$ while keeping $x_2 < \pi$ we retrieve the case of a superconducting core surrounded by a normal region. When $x_1 = x_2$ the star is completely normal conducting. All such cases are equivalent, up to a scaling determined by the magnitude of the magnetic potential (which is given through equation 44). Sample models are shown in Figs 3 and 4 for the $n = 1$ case discussed before (equation 45).

3.6 Density perturbation

The density perturbation within each region can be calculated through equation (23), which for a $\gamma = 2$ polytrope becomes

$$2\kappa\,\delta\rho = \mathfrak{B}_o - \delta\phi - \psi. \tag{61}$$

Sample plots of density perturbations for the $n = 1$ case are shown in Fig. 5. The density jump at a boundary is then given through

$$2\kappa\,\Delta_\rho = 2\kappa(\delta\rho_{\text{in}} - \delta\rho_{\text{out}}) = \psi_{\text{out}} - \psi_{\text{in}}. \tag{62}$$

[1] In particular, letting f_ℓ denote either j_ℓ or y_ℓ, we have $j_\ell(x)y'_\ell(x) - j'_\ell(x)y_\ell(x) = x^{-2}$, $x f'_\ell(x) = x f_{\ell-1}(x) - (\ell+1)f_\ell(x)$ and $(2\ell+1)f'_\ell(x) = \ell f_{\ell-1}(x) - (\ell+1)f_{\ell+1}(x)$.

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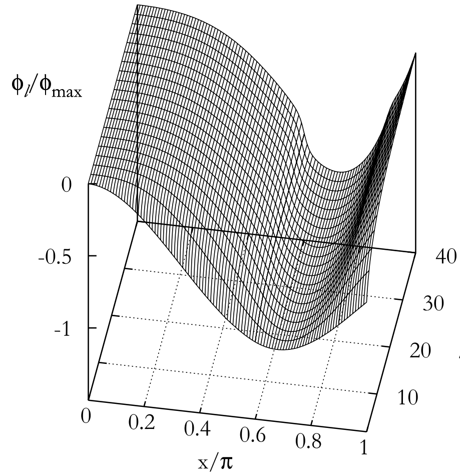

Figure 4. Gravitational potential perturbation for a fluid star as a function of ℓ. The potential is shown for the $n = 1$ case of a three-component star with a superconducting shell between $x_1 = 0.6\pi$ and $x_2 = 0.9\pi$. The same scaling is used as in Fig. 3, and only $\ell > 0$ are shown.

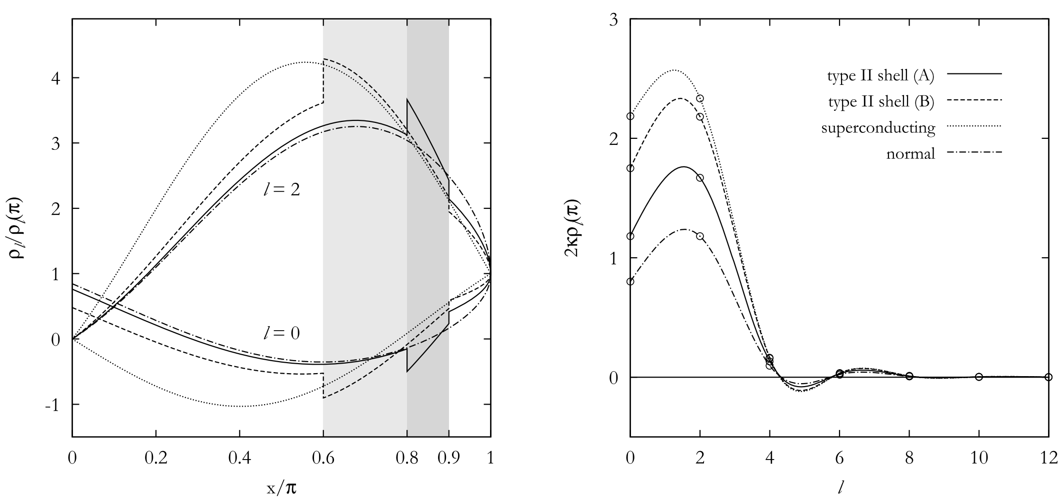

Figure 5. Density perturbations for a fluid star with toroidal fields, expanded in spherical harmonics. Plots are shown for the same four sample cases considered in Fig. 3. The left-hand plot shows the first two harmonics ρ_ℓ (for $\ell = 0$ and $\ell = 2$) scaled by the surface value of the density perturbation $\rho_\ell(\pi)$. The shaded regions indicate the position of the superconducting shell. The right-hand plot shows $2\kappa\rho_\ell(\pi)$ for the first few ℓ, in units of Ψ_o defined in equation (46).

In particular, consider the density jump when going from a normal region into a superconducting region at a boundary $r = r_b$. Using equations (36) and (43), we get

$$2\kappa\,\Delta_\rho = 2\kappa(\delta\rho_n - \delta\rho_s) = \psi_s - \psi_n = \frac{n(2\sigma - 1)}{2(n\sigma + 1)}\psi_s(r_b, \theta) \quad \text{where} \quad n \geqslant 1. \tag{63}$$

Note that $\Delta_\rho \geqslant 0$ for $\sigma > 1/2$. In other words, the density perturbation *decreases* when going from a normal region into a superconducting region, and vice versa. Also note that the jump goes to zero at the poles, i.e. $\Delta_\rho \to 0$ as $\theta \to 0$, since the magnetic potentials vanish there.

The relation between the Eulerian density perturbation and the Lagrangian displacement is given through

$$\delta\rho = -\nabla \cdot (\rho\boldsymbol{\xi}) = -\rho\nabla \cdot \boldsymbol{\xi} - \rho'\xi_r. \tag{64}$$

Normally, the term $\nabla \cdot \boldsymbol{\xi}$ inside the fluid is undetermined. However, at the surface $\rho = 0$, so that we can calculate the radial displacement, which determines the shape of the perturbed stellar surface,

$$\xi_r = -\delta\rho/\rho'. \tag{65}$$

For a $\gamma = 2$ polytrope we have $\rho = \rho_c \sin x/x$, so that at the surface $\rho'(\pi) = -\rho_c/\pi$ and $\xi_r = \pi\delta\rho/\rho_c$. The $\ell = 0$ term in the spherical harmonic expansion of ξ_r defines a spherically symmetric expansion (or compression) of the star, while higher order ℓ determine the deformation of the surface as a function of the polar angle, θ.

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Table 1. Values of $\phi_2(R_\star)$ for the cases considered in Fig. 3. The negative signs signify the fact that the models considered here are prolate, i.e. $\delta I_1 > \delta I_3$.

Case	$\phi_2(R_\star)/\Psi_o$
Type II shell (A)	−1.67
Type II shell (B)	−2.18
Superconducting	−2.33
Normal	−1.18

3.7 Quadrupolar distortion

The moment of inertia of the unperturbed star is given by

$$I_{ij} = \int_V \rho(r^2\delta_{ij} - r_i r_j)\, \mathrm{d}^3 r. \tag{66}$$

Since the star is initially spherically symmetric we have $I_{xx} = I_{yy} = I_{zz}$. For a $\gamma = 2$ polytrope the density profile is given through $\rho = \rho_c \sin x/x$, so that the moment of inertia becomes

$$I_o \equiv I_{xx} = \int_V \rho r^2 \left(1 - \sin^2\theta \cos^2\varphi\right) \mathrm{d}^3 r = \frac{8(\pi^2 - 6)\rho_c R_\star^5}{3\pi^3}. \tag{67}$$

Here $R_\star$ is the stellar radius, which corresponds to $x = R_\star/r_o = \pi$.

The application of the magnetic perturbation renders the star axisymmetric ($I_1 = I_2 \neq I_3$). In this case the moments of inertia become $I_1 = I_o + \delta I_1$ around an axis that lies in the equatorial plane, and $I_3 = I_o + \delta I_3$ around the axis of symmetry which passes through the poles. We will define the star to be *oblate* when $\delta I_3 > \delta I_1$ and *prolate* when $\delta I_3 < \delta I_1$. In other words, when more of the mass is distributed towards the equator the star is oblate, and when more of the mass is closer to the poles the star is prolate. The difference between the moments of inertia is related to the *gravitational quadrupole moment*, which in turn is related to the $\ell = 2$ harmonic of the gravitational potential at the stellar surface,

$$Q_{20} = \int_V \rho r^2 Y_2(\theta) \mathrm{d}^3 r = -\sqrt{\frac{5}{4\pi}}(\delta I_3 - \delta I_1) = -\frac{5R_\star^3 \phi_2(R_\star)}{4\pi G}. \tag{68}$$

Thus,

$$\phi_2(R_\star) = \sqrt{\frac{4\pi}{5}} \frac{G(\delta I_3 - \delta I_1)}{R_\star^3}. \tag{69}$$

Therefore, the sign of ϕ_2 at the surface determines whether the star is prolate or oblate. Note that for all the cases shown in Fig. 3, $\phi_2(R_\star)$ is negative and consequently the star is prolate. The precession frequency of an axisymmetric star is $\sim\epsilon\Omega_\star$, where $\Omega_\star$ is the angular velocity and ϵ is a dimensionless constant defined through

$$\epsilon = \frac{I_3 - I_1}{I_1} \approx \frac{\delta I_3 - \delta I_1}{I_o} = \frac{3\pi^2\sqrt{5\pi}\phi_2(R_\star)}{16(\pi^2 - 6)G\rho_c R_\star^2}. \tag{70}$$

For the $n = 1$ case, the gravitational potential perturbations are measured in units of $\Psi_o = H_c B_o/2\pi\rho_c$ (equation 46). The central density for a $\gamma = 2$ polytrope is $\rho_c = \pi M_\star/4R_\star^3$. Thus, we can rewrite the above equation as

$$\epsilon = 0.945 \times 10^{-9} \left(\frac{\phi_2(R_\star)}{\Psi_o}\right) \left(\frac{H_c}{10^{15}\,\mathrm{G}}\right) \left(\frac{B_o}{10^{12}\,\mathrm{G}}\right) \left(\frac{R_\star}{10\,\mathrm{km}}\right)^4 \left(\frac{M_\star}{1.4\,\mathrm{M}_\odot}\right)^{-2}. \tag{71}$$

Sample values of $\phi_2(R_\star)$ are listed in Table 1, and $\phi_2(R_\star)$ as a function of superconducting shell width in a three-component star is plotted in Fig. 6. Note that the values of ϵ for the various models are very similar. This should not be surprising, as the magnetic fields in all cases are of similar magnitude.

In particular, the normal case considered here (in Figs 3 and 5, and in Table 1) is for a magnetic field of strength $\hat{B}_o = (H_c B_o)^{1/2}[\rho(x_2)/\rho_c]^{1/4} \approx 1.8 \times 10^{13}$ G (equation 42). This is simply the limiting value of the normal field as the superconducting shell vanishes, $x_1 \to x_2$. In the normal case, the magnetic potential is given in units of $\hat{\Psi}_o = 3\hat{B}_o^2/8\pi\rho_c$ (equation 43), which can be evaluated for different choices of $\hat{B}_o$.

4 STABILITY OF MAGNETIC FIELDS

In this section, we will discuss the stability of toroidal fields in neutron stars. We will follow the *energy principle* considerations outlined in Bernstein et al. (1958) and Tayler (1973). An extensive review is also given in Freidberg (1982). The formalism that is developed in this section is valid for any $H(\rho, B)$ and is applicable to both normal and superconducting neutron stars. For the purpose of this section, we will treat the entire star as either normal or superconducting, and therefore will not worry about internal boundaries.

We also ignore rotation, and thus do not need to pay attention to 'trivial' displacements discussed by Friedman & Schutz (1978). In magnetic stars, trivial modes are defined by the requirements that $\delta\rho = 0$ and $\delta \boldsymbol{B} = 0$. Since we will express the energy of the perturbations in

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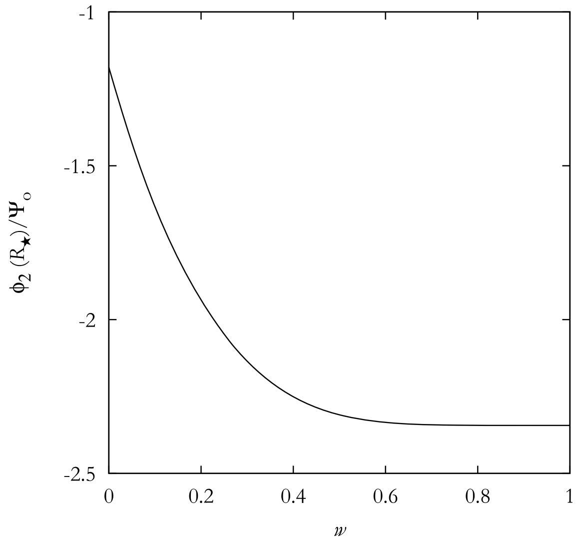

Figure 6. $\phi_2(R_\star)$ as a function of the width $w = (x_2 - x_1)/x_2$ of the superconducting shell in a three-component star. The outer radius of the shell is fixed at $x_2 = 0.9\pi$. The type II shell models (cases A and B) listed in Table 1 are retrieved by setting $x_1 = 0.8\pi$ $(w = 1/9)$ and $x_1 = 0.6\pi$ $(w = 1/3)$, respectively. When $x_1 = x_2$ $(w = 0)$ the star becomes normal.

terms of $\delta\rho$ and $\delta\boldsymbol{B}$, trivial displacements will have no effect on it (see equation B60 in Friedman & Schutz 1978 and footnote 3 in Glampedakis & Andersson 2007). However, in a rotating star, trivial displacements will have to be taken into consideration.

Glampedakis & Andersson (2007) emphasize the importance of the magnetic field for rotating stars by showing that sufficiently strong fields can stabilize inertial modes that would otherwise be unstable. The same will be true for type II superconducting stars. We will not treat rotation-induced instabilities here. Instead, we emphasize the effects of the magnetic free energy $F(\rho, B)$ in a type II superconductor. Energy conditions presume zero dissipation. Moreover, we consider a single fluid, which in reality consists of at least three fluids: neutrons, protons and electrons. There will be additional buoyant modes which may or may not alter the stability conditions we derive.

Assuming small oscillatory perturbations about equilibrium, we have, from equation (8),

$$-\rho\frac{\mathrm{d}^2\boldsymbol{\xi}}{\mathrm{d}t^2} = \rho\omega^2\boldsymbol{\xi} = \delta\left(\nabla p + \rho\nabla\phi - \boldsymbol{f}_{\mathrm{mag}}\right) = -\boldsymbol{\mathcal{F}}(\boldsymbol{\xi}). \tag{72}$$

The force operator $\boldsymbol{\mathcal{F}}$ is self-adjoint, which implies that the eigenvalues ω^2 are real. One condition for stability is that all frequencies ω be real, so that there are no growing modes. Alternatively, the variation in the total potential energy due to the perturbations should always be positive,

$$\delta W = -\frac{1}{2}\int \boldsymbol{\xi}\cdot\boldsymbol{\mathcal{F}}(\boldsymbol{\xi})\,\mathrm{d}V > 0. \tag{73}$$

To lowest order, the integration is carried over the equilibrium volume. The Lagrangian and Eulerian pressure perturbations are given by $\Delta p = (\gamma p/\rho)\Delta\rho = -\gamma p\,\nabla\cdot\boldsymbol{\xi}$ and $\delta p = \Delta p - \boldsymbol{\xi}\cdot\nabla\,p = -\gamma p\,\nabla\cdot\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla p$. Here γ is for the perturbations, and in general may differ from the background polytropic index. The difference gives rise to buoyancy terms, which will not be considered in this paper; however, we will comment on their effects on stability briefly.

Integrating by parts, we get

$$\begin{aligned}
\delta W &= \delta W_{\mathrm{p}} + \delta W_{\mathrm{mag}},\\
\delta W_{\mathrm{p}} &= \frac{1}{2}\int[\gamma p(\nabla\cdot\boldsymbol{\xi})^2 + (\boldsymbol{\xi}\cdot\nabla p)(\nabla\cdot\boldsymbol{\xi}) - (\boldsymbol{\xi}\cdot\nabla\phi)(\nabla\cdot\rho\boldsymbol{\xi}) + \rho\boldsymbol{\xi}\cdot\nabla\delta\phi]\,\mathrm{d}V\\
&\quad -\frac{1}{2}\oint \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{\xi}\,[\gamma p\nabla\cdot\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\nabla p]\,,\\
\delta W_{\mathrm{mag}} &= -\frac{1}{2}\int \boldsymbol{\xi}\cdot\delta\boldsymbol{f}_{\mathrm{mag}}\,\mathrm{d}V.
\end{aligned} \tag{74}$$

We will refer to the two parts in the energy as the hydrostatic part δW_{p}, which includes the contributions from pressure and gravity, and the magnetic part δW_{mag}. In equilibrium, the pressure and density are related through a polytropic equation of state and consequently they both go to zero at the surface. Therefore, the surface integral vanishes.

We now turn our attention to the calculation of the magnetic energy variation. Faraday's law gives the variation in the magnetic field in a perfect conductor as

$$\delta\boldsymbol{B} = \nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}). \tag{75}$$

We next discuss the normal and superconducting cases separately.

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4.1 Normal conducting star

In a normal conducting medium, the force is given as

$$\boldsymbol{f}_{\rm mag} = \frac{\boldsymbol{J} \times \boldsymbol{B}}{c} = \frac{(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}}{4\pi}. \tag{76}$$

The perturbed force becomes

$$\delta \boldsymbol{f}_{\rm mag} = \frac{\delta \boldsymbol{J} \times \boldsymbol{B}}{c} + \frac{\boldsymbol{J} \times \delta \boldsymbol{B}}{c} \quad \text{where} \quad \frac{\delta \boldsymbol{J}}{c} = \frac{\nabla \times \delta \boldsymbol{B}}{4\pi}. \tag{77}$$

Integrating the first term in $\delta W_{\rm mag}$, given through equation (74), by parts and rearranging, we thus have

$$\delta W_{\rm mag} = -\frac{1}{2}\int \boldsymbol{\xi} \cdot \delta \boldsymbol{f}_{\rm mag}\, {\rm d}V = \frac{1}{2}\int \left[\frac{|\delta \boldsymbol{B}|^2}{4\pi} - \frac{\boldsymbol{J} \cdot \delta \boldsymbol{B} \times \boldsymbol{\xi}}{c}\right] {\rm d}V + \frac{1}{8\pi}\oint {\rm d}\boldsymbol{S} \cdot [\boldsymbol{\xi}(\boldsymbol{B} \cdot \delta \boldsymbol{B}) - \boldsymbol{B}(\boldsymbol{\xi} \cdot \delta \boldsymbol{B})]\,. \tag{78}$$

The first surface integral vanishes when $\mathrm{d}\boldsymbol{S} \cdot \boldsymbol{B} = 0$, i.e. when the magnetic field is perpendicular to the surface, as is the case for a toroidal field. On the other hand, the second surface integral vanishes when the field vanishes at the surface. This form of the energy variation is the same as that given by Bernstein et al. (1958) for $\mathrm{d}\boldsymbol{S} \cdot \boldsymbol{B} = 0$. The surface integrals may be relevant, for instance, in the case of poloidal fields. However, we will not need to worry about these as we will be considering toroidal fields that vanish at the surface.

4.2 Type II superconducting star

The magnetic force for a type II superconductor is given by equation (7):

$$\boldsymbol{f}_{\rm mag} = \frac{\boldsymbol{J} \times \boldsymbol{B}}{c} - \rho \nabla \psi_{\rm II}, \tag{79}$$

where $\psi_{\rm II} = \partial F/\partial \rho$, from equation (11). The current density is now given through $4\pi \boldsymbol{J}/c = \nabla \times \boldsymbol{H}$. The magnetic free energy F is a function of ρ and B and is related to the magnetic field through equation (2), $H = 4\pi \partial F/\partial B$. The perturbation of the force gives

$$\delta \boldsymbol{f}_{\rm mag} = \frac{\delta \boldsymbol{J} \times \boldsymbol{B}}{c} + \frac{\boldsymbol{J} \times \delta \boldsymbol{B}}{c} - \delta\rho \nabla \psi_{\rm II} - \rho \nabla \delta \psi_{\rm II}. \tag{80}$$

Consider the energy due to the first term of the magnetic force. Following the same procedure as in the derivation of equation (78), we get

$$\begin{aligned} \frac{1}{c}\int \boldsymbol{\xi} \cdot \delta \boldsymbol{J} \times \boldsymbol{B}\, {\rm d}V &= -\frac{1}{4\pi}\int \boldsymbol{\xi} \times \boldsymbol{B} \cdot (\nabla \times \delta \boldsymbol{H})\, {\rm d}V \\ &= \frac{1}{4\pi}\oint {\rm d}\boldsymbol{S} \cdot [\boldsymbol{B}(\boldsymbol{\xi} \cdot \delta \boldsymbol{H}) - \boldsymbol{\xi}(\boldsymbol{B} \cdot \delta \boldsymbol{H})] - \frac{1}{4\pi}\int \delta \boldsymbol{H} \cdot \delta \boldsymbol{B}\, {\rm d}V. \end{aligned} \tag{81}$$

When B vanishes on the surface we can drop the surface integral. On the other hand, note that we can rewrite the last two terms in the magnetic energy variation as

$$\int (\delta\rho\, \boldsymbol{\xi} \cdot \nabla \psi_{\rm II} + \rho \boldsymbol{\xi} \cdot \nabla \delta \psi_{\rm II})\, {\rm d}V = \int (\delta\rho\, \boldsymbol{\xi} \cdot \nabla \psi_{\rm II} + \delta\rho\, \delta\psi_{\rm II})\, {\rm d}V = \int \delta\rho \Delta \psi_{\rm II}\, {\rm d}V. \tag{82}$$

Here, we have made use of the relation $\Delta = \delta + \boldsymbol{\xi} \cdot \nabla$, between Lagrangian and Eulerian perturbations. Thus, the magnetic energy variation for a type II superconductor becomes, from equation (74),

$$\begin{aligned} \delta W_{\rm mag} &= -\frac{1}{2}\int \boldsymbol{\xi} \cdot \delta \boldsymbol{f}_{\rm mag}\, {\rm d}V \\ &= \frac{1}{2}\int \left[\frac{\delta \boldsymbol{H} \cdot \delta \boldsymbol{B}}{4\pi} - \frac{\boldsymbol{J} \cdot \delta \boldsymbol{B} \times \boldsymbol{\xi}}{c} + \delta\rho \Delta \psi_{\rm II}\right] {\rm d}V + \frac{1}{8\pi}\oint {\rm d}\boldsymbol{S} \cdot [\boldsymbol{\xi}(\boldsymbol{B} \cdot \delta \boldsymbol{H}) - \boldsymbol{B}(\boldsymbol{\xi} \cdot \delta \boldsymbol{H})]\,. \end{aligned} \tag{83}$$

This is to be contrasted with the magnetic energy for the normal case given by equation (78). In particular, the first two terms in the volume integrals are of the same form, with a $\boldsymbol{B}$ in the normal case replaced by an $\boldsymbol{H}$ in the superconducting case. The same is true for the surface integral terms. However, in the superconducting case there is also an additional term that arises from the potential $\psi_{\rm II}$, that has no analogue in the normal case.

In the strongly type II superconducting case the magnetic field is a function of density only, $H = H(\rho)$. On the other hand, in the normal case we have $H = B$. In general, H, $\psi_{\rm II}$ and F will all be functions of ρ and B. Using the definition of the potential $\psi_{\rm II}$ from equation (11), we get

$$\Delta \psi_{\rm II} = \frac{\partial^2 F}{\partial \rho^2}\Delta\rho + \frac{\partial^2 F}{\partial \rho \partial B}\Delta B. \tag{84}$$

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We will assume that the form of $\delta\boldsymbol{B}$ given through equation (75) is still valid for the superconducting case. Also note the following relations which will be of use:

$$\delta\hat{\boldsymbol{B}} = \frac{\delta\boldsymbol{B}}{B} - \frac{\delta B\hat{\boldsymbol{B}}}{B},$$
$$\delta B = \hat{\boldsymbol{B}}\cdot\delta\boldsymbol{B},$$
$$\delta\boldsymbol{H} = \delta H\hat{\boldsymbol{B}} + H\delta\hat{\boldsymbol{B}},$$
$$\delta H = \frac{\partial H}{\partial\rho}\delta\rho + \frac{\partial H}{\partial B}\delta B. \tag{85}$$

Note that $\hat{\boldsymbol{B}} \perp \delta\hat{\boldsymbol{B}}$, which also follows from $\delta(\hat{\boldsymbol{B}}\cdot\hat{\boldsymbol{B}}) = 0$. Using the above relations we have

$$\delta\boldsymbol{H}\cdot\delta\boldsymbol{B} = \delta H\delta B + \frac{H}{B}\left[\delta\boldsymbol{B}\cdot\delta\boldsymbol{B} - (\delta B)^2\right]. \tag{86}$$

Using equation (2) which relates H and F, the perturbation in the magnetic field can be written as

$$\delta H = 4\pi\left(\frac{\partial^2 F}{\partial\rho\partial B}\delta\rho + \frac{\partial^2 F}{\partial B^2}\delta B\right). \tag{87}$$

This allows us to express the energy in terms of derivatives of F.

For a strongly type II superconductor $H \propto \rho$, equation (83) reduces to (Roberts 1981; Akgün 2007)

$$\delta W_{\rm mag} = \frac{1}{8\pi}\int [\delta\boldsymbol{H}\cdot\delta\boldsymbol{B} - \delta\boldsymbol{B}\cdot\boldsymbol{\xi}\times(\nabla\times\boldsymbol{H}) - (\boldsymbol{H}\cdot\delta\boldsymbol{B})(\nabla\cdot\boldsymbol{\xi}) + \delta\boldsymbol{H}\cdot(\boldsymbol{\xi}\cdot\nabla\boldsymbol{B}) - \delta\boldsymbol{B}\cdot(\boldsymbol{\xi}\cdot\nabla\boldsymbol{H})]\,{\rm d}V$$
$$+\frac{1}{8\pi}\oint {\rm d}\boldsymbol{S}\cdot[\boldsymbol{\xi}(\delta\boldsymbol{H}\cdot\boldsymbol{B} + \boldsymbol{H}\cdot\delta\boldsymbol{B}) - \boldsymbol{B}(\boldsymbol{\xi}\cdot\delta\boldsymbol{H})]. \tag{88}$$

4.3 Stability criteria

Tayler (1973) derives stability conditions for toroidal fields in a normal star in cylindrical coordinates using the energy principle given by equation (78). The equivalent conditions in spherical coordinates are given by Goossens & Veugelen (1978). We will now proceed to derive stability criteria for toroidal fields in a type II superconducting star, along the same lines. We will take the magnetic field to be given as a function of density and magnetic induction, $H = H(\rho, B)$. This will allow us to consider both the strongly type II superconducting case and the normal case simultaneously. We will closely follow the notation of Goossens & Veugelen (1978) in order to facilitate comparisons.

It is clearly sufficient for stability to show that the integrand of the energy of the perturbations is positive throughout the region of integration,

$$\delta W = \frac{1}{2}\int \mathcal{E}{\rm d}V > 0 \quad \text{if} \quad \mathcal{E} > 0. \tag{89}$$

Even if $\mathcal{E}$ becomes negative in a small region the system is unstable. Define $\mathcal{E}_{\rm p}$ and $\mathcal{E}_{\rm mag}$ as the integrands of $\delta W_{\rm p}$ and $\delta W_{\rm mag}$, i.e. $\mathcal{E} = \mathcal{E}_{\rm p} + \mathcal{E}_{\rm mag}$. As in previous works (Bernstein et al. 1958; Tayler 1973; Goossens & Veugelen 1978; Roberts 1981) we will drop the gravitational potential perturbation term in $\mathcal{E}_{\rm p}$. The hydrostatic and magnetic parts of the energy are then given through equations (74) and (83), respectively,

$$\mathcal{E}_{\rm p} = \gamma p(\nabla\cdot\boldsymbol{\xi})^2 + (\boldsymbol{\xi}\cdot\nabla p)(\nabla\cdot\boldsymbol{\xi}) - (\boldsymbol{\xi}\cdot\nabla\phi)(\nabla\cdot\rho\boldsymbol{\xi}),$$
$$\mathcal{E}_{\rm mag} = \frac{1}{4\pi}[\delta\boldsymbol{H}\cdot\delta\boldsymbol{B} - \delta\boldsymbol{B}\cdot\boldsymbol{\xi}\times(\nabla\times\boldsymbol{H})] + \delta\rho\Delta\psi_{\rm II}. \tag{90}$$

The azimuthal angle φ does not explicitly appear in any of the coefficients in these equations, so that we can expand the components of the Lagrangian displacement as

$$\xi_r = R(r,\theta){\rm e}^{{\rm i}m\varphi}, \quad \xi_\theta = S(r,\theta){\rm e}^{{\rm i}m\varphi} \quad \text{and} \quad \xi_\phi = {\rm i}T(r,\theta){\rm e}^{{\rm i}m\varphi}. \tag{91}$$

Here m is an integer. Since only the real parts are significant, the scalar multiplications and vector dot products are to be treated as $Z\cdot Z^*$ where Z^* stands for complex conjugate. It will be of great notational convenience to define an operator Λ of a scalar argument $u = u(r,\theta)$,

$$\Lambda(u) \equiv R\partial_r u + \frac{S\partial_\theta u}{r}. \tag{92}$$

This is simply the directional derivative along the Lagrangian displacement, $\boldsymbol{\xi}\cdot\nabla\, u = \Lambda(u)\,{\rm e}^{{\rm i}m\varphi}$. We will find it convenient to redefine the φ component of the Lagrangian displacement as

$$\hat{T} = \frac{mT}{r\sin\theta}. \tag{93}$$

Also define

$$D = \frac{\partial_r(r^2R)}{r^2} + \frac{\partial_\theta(S\sin\theta)}{r\sin\theta} - \hat{T} = D_0 - \hat{T}, \tag{94}$$

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which is simply the divergence of the Lagrangian displacement, $\nabla\cdot\boldsymbol{\xi}=D\mathrm{e}^{\mathrm{i}m\varphi}$. Note that D_0 is independent of $\hat{T}$. Using these definitions, we can express the hydrostatic part given by equation (90) as

$$\mathcal{E}_{\mathrm{p}}=\gamma pD^2+[\Lambda(p)-\rho\Lambda(\phi)]\,D-\Lambda(\rho)\Lambda(\phi). \tag{95}$$

The equations of equilibrium for the unperturbed background state are given by equation (8):

$$\partial_r p+\rho\partial_r\phi=-\frac{B}{r}\partial_r\left(r\frac{\partial F}{\partial B}\right)-\rho\partial_r\left(\frac{\partial F}{\partial\rho}\right),$$
$$\partial_\theta p+\rho\partial_\theta\phi=-\frac{B}{\sin\theta}\partial_\theta\left(\sin\theta\frac{\partial F}{\partial B}\right)-\rho\partial_\theta\left(\frac{\partial F}{\partial\rho}\right). \tag{96}$$

Note the notational convention for partial derivatives that we will employ for the remainder of this section: derivatives with respect to coordinates x will be shortened as ∂_x, while derivatives of the magnetic free energy F with respect to ρ and B will be explicitly written. Using these equations we can eliminate the pressure gradient in $\mathcal{E}_{\mathrm{p}}$ and rewrite it in terms of the gravitational and magnetic forces. Using the definition of the operator Λ from equation (92), we have

$$\Lambda(p)=-\rho\Lambda(\phi)-\rho\Lambda\left(\frac{\partial F}{\partial\rho}\right)-B\Lambda\left(\frac{\partial F}{\partial B}\right)-B\frac{\partial F}{\partial B}\left(\frac{R+S\cot\theta}{r}\right). \tag{97}$$

Next, consider the magnetic part of the integrand given by equation (90). Using equation (86) for $\delta\boldsymbol{H}\cdot\delta\boldsymbol{B}$, we have

$$\mathcal{E}_{\mathrm{mag}}=\frac{1}{4\pi}\left[\delta H\delta B+\frac{H}{B}\left(|\delta\boldsymbol{B}|^2-(\delta B)^2\right)-\delta\boldsymbol{B}\cdot\boldsymbol{\xi}\times(\nabla\times\boldsymbol{H})\right]+\delta\rho\Delta\psi_{\mathrm{II}}. \tag{98}$$

$\Delta\psi_{\mathrm{II}}$ and δH are given through equations (84) and (87), respectively. We can also express the magnetic field in terms of the free energy through equation (2), $H=4\pi\partial F/\partial B$. The various terms in $\mathcal{E}_{\mathrm{mag}}$ can be evaluated using the relations given in equation (85). In particular,

$$\frac{|\delta\boldsymbol{B}|^2-(\delta B)^2}{B^2}=\frac{m^2(R^2+S^2)}{r^2\sin^2\theta}\quad\text{and}\quad\frac{\delta\boldsymbol{B}\cdot\boldsymbol{\xi}\times(\nabla\times\boldsymbol{H})}{HB}=\hat{X}\hat{Y}+\hat{T}\hat{Y}, \tag{99}$$

where we define the following auxiliary quantities:

$$\hat{X}=D_0+\frac{\Lambda(B)}{B}-\frac{R+S\cot\theta}{r}\quad\text{and}\quad\hat{Y}=\frac{\Lambda(H)}{H}+\frac{R+S\cot\theta}{r}. \tag{100}$$

The magnetic part can then be written as

$$\mathcal{E}_{\mathrm{mag}}=B\frac{\partial F}{\partial B}\left[\frac{m^2(R^2+S^2)}{r^2\sin^2\theta}-\hat{X}\hat{Y}-\hat{T}\hat{Y}\right]+\frac{\partial^2F}{\partial\rho\partial B}\delta\rho\delta B+\frac{\partial^2F}{\partial B^2}(\delta B)^2+\frac{\partial^2F}{\partial\rho\partial B}\delta\rho\Delta B+\frac{\partial^2F}{\partial\rho^2}\delta\rho\Delta\rho, \tag{101}$$

where

$$\frac{\delta B}{B}=-\hat{X}\mathrm{e}^{\mathrm{i}m\varphi},$$
$$\frac{\Delta B}{B}=-\left[D_0-\frac{R+S\cot\theta}{r}\right]\mathrm{e}^{\mathrm{i}m\varphi},$$
$$\frac{\delta\rho}{\rho}=-\left[D+\frac{\Lambda(\rho)}{\rho}\right]\mathrm{e}^{\mathrm{i}m\varphi},$$
$$\frac{\Delta\rho}{\rho}=-D\mathrm{e}^{\mathrm{i}m\varphi}. \tag{102}$$

We will next consider the $m=0$ and $m\neq 0$ cases separately.

4.3.1 *The $m=0$ case*

In this case $\hat{T}=0$ from equation (93) and the total energy can be written as, using equations (95) and (101) for $\mathcal{E}_{\mathrm{p}}$ and $\mathcal{E}_{\mathrm{mag}}$, respectively,

$$\mathcal{E}=\mathcal{E}_{\mathrm{p}}+\mathcal{E}_{\mathrm{mag}}=\mathcal{K}_0{D_0}^2+\mathcal{K}_1D_0+\mathcal{K}_2, \tag{103}$$

where D_0 is defined in equation (94). We have, in terms of the operator Λ defined by equation (92),

$$\mathcal{K}_0=\gamma p+B^2\frac{\partial^2F}{\partial B^2}+2\rho B\frac{\partial^2F}{\partial\rho\partial B}+\rho^2\frac{\partial^2F}{\partial\rho^2},$$
$$\mathcal{K}_1=-2\rho\,\Lambda(\phi)-2\left[B\frac{\partial F}{\partial B}+B^2\frac{\partial^2F}{\partial B^2}+\rho B\frac{\partial^2F}{\partial\rho\partial B}\right]\frac{R+S\cot\theta}{r},$$
$$\mathcal{K}_2=-\Lambda(\rho)\Lambda(\phi)-\left[\Lambda(B)\frac{\partial F}{\partial B}+B\Lambda(B)\frac{\partial^2F}{\partial B^2}+B\Lambda(\rho)\frac{\partial^2F}{\partial\rho\partial B}\right]\frac{R+S\cot\theta}{r}+\left[B\frac{\partial F}{\partial B}+B^2\frac{\partial^2F}{\partial B^2}\right]\left(\frac{R+S\cot\theta}{r}\right)^2. \tag{104}$$

All derivatives of R and S are included in D_0. By completing the square we get

$$\mathcal{E}=\mathcal{K}_0\left(D_0+\frac{\mathcal{K}_1}{2\mathcal{K}_0}\right)^2+\mathcal{K}_2-\frac{\mathcal{K}_1^2}{4\mathcal{K}_0}. \tag{105}$$

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The first term is non-negative and the remaining terms form a quadratic in R and S, which is also the minimum value of $\mathcal{E}$ with respect to D_0,

$$\mathcal{K}_2 - \frac{\mathcal{K}_1^2}{4\mathcal{K}_0} = a_0 R^2 + b_0 RS + c_0 S^2. \tag{106}$$

The subscripts in the coefficients stand for $m = 0$. Define the following auxiliary quantities:

$$\begin{aligned}
U_0 &= \frac{1}{r}\left(B\frac{\partial F}{\partial B} + B^2\frac{\partial^2 F}{\partial B^2} + \rho B\frac{\partial^2 F}{\partial\rho\partial B}\right),\\
U_1 &= \frac{1}{r}\left(\partial_r B\frac{\partial F}{\partial B} + B\partial_r B\frac{\partial^2 F}{\partial B^2} + B\partial_r\rho\frac{\partial^2 F}{\partial\rho\partial B}\right),\\
U_2 &= \frac{1}{r^2}\left(\partial_\theta B\frac{\partial F}{\partial B} + B\partial_\theta B\frac{\partial^2 F}{\partial B^2} + B\partial_\theta\rho\frac{\partial^2 F}{\partial\rho\partial B}\right),\\
U_3 &= \frac{1}{r^2}\left(B\frac{\partial F}{\partial B} + B^2\frac{\partial^2 F}{\partial B^2}\right).
\end{aligned} \tag{107}$$

We then find that the coefficients in the quadratic are given by

$$\begin{aligned}
a_0 &= -\partial_r\rho\,\partial_r\phi - U_1 + U_3 - \frac{1}{\mathcal{K}_0}\left(\rho\partial_r\phi + U_0\right)^2,\\
b_0 &= -\frac{\partial_r\rho\,\partial_\theta\phi}{r} - \frac{\partial_\theta\rho\,\partial_r\phi}{r} - U_1\cot\theta - U_2 + 2U_3\cot\theta - \frac{2}{\mathcal{K}_0}\left(\rho\partial_r\phi + U_0\right)\left(\frac{\rho\partial_\theta\phi}{r} + U_0\cot\theta\right),\\
c_0 &= -\frac{\partial_\theta\rho\,\partial_\theta\phi}{r^2} - U_2\cot\theta + U_3\cot^2\theta - \frac{1}{\mathcal{K}_0}\left(\frac{\rho\partial_\theta\phi}{r} + U_0\cot\theta\right)^2.
\end{aligned} \tag{108}$$

A sufficient condition for stability is that the quadratic form be always positive throughout the integration region. This corresponds to the following conditions, which are not all independent,

$$a > 0, \quad c > 0 \quad \text{and} \quad b^2 < 4ac. \tag{109}$$

When these conditions are satisfied the star is stable, therefore these are *sufficient* conditions for stability. If we can show that the star is unstable as soon as one of these conditions is violated, then we will have shown that the conditions are also *necessary* for stability. For the $m = 0$ case it can be shown that the interchange instability sets in when these conditions fail, as will be proven in a later section. Therefore, these conditions are necessary and sufficient conditions for the $m = 0$ case. However, the same will not be true in general for the $m \neq 0$ case, as will be discussed later.

One way of deriving these conditions is to consider the minimum value of the quadratic form $Q = aR^2 + bRS + cS^2$ with respect to S (or equivalently, R). For a minimum we need $\mathrm{d}Q/\mathrm{d}S = 0$ and $\mathrm{d}^2Q/\mathrm{d}S^2 > 0$. Substituting the value of S that minimizes Q and requiring that $Q > 0$ we get the condition $b^2 < 4ac$, while the second requirement gives $c > 0$. These two conditions then imply the third, $a > 0$.

We can now consider special cases. In the strongly type II superconducting case the magnetic field is a function of density, $H = H(\rho)$ and the magnetic free energy is given by equation (2) as $F = HB/4\pi$. In particular, consider a power law of the form $H \propto \rho^\sigma$. From equations (104) and (107), we have

$$\mathcal{K}_0 = \gamma p + \frac{\sigma(\sigma+1)HB}{4\pi}, \quad U_0 = \frac{(\sigma+1)HB}{4\pi r}, \quad U_1 = \frac{\partial_r(HB)}{4\pi r}, \quad U_2 = \frac{\partial_\theta(HB)}{4\pi r^2} \quad \text{and} \quad U_3 = \frac{HB}{4\pi r^2}, \tag{110}$$

so that the coefficients become

$$\begin{aligned}
a_0 &= -\partial_r\rho\,\partial_r\phi - \frac{\partial_r(HB)}{4\pi r} + \frac{HB}{4\pi r^2} - \frac{1}{\mathcal{K}_0}\left[\rho\partial_r\phi + \frac{(\sigma+1)HB}{4\pi r}\right]^2,\\
b_0 &= -\frac{\partial_r\rho\,\partial_\theta\phi}{r} - \frac{\partial_\theta\rho\,\partial_r\phi}{r} - \frac{\partial_r(HB)}{4\pi r}\cot\theta - \frac{\partial_\theta(HB)}{4\pi r^2} + \frac{HB}{2\pi r^2}\cot\theta\\
&\quad - \frac{2}{r\mathcal{K}_0}\left[\rho\partial_r\phi + \frac{(\sigma+1)HB}{4\pi r}\right]\left[\rho\partial_\theta\phi + \frac{(\sigma+1)HB}{4\pi}\cot\theta\right],\\
c_0 &= -\frac{\partial_\theta\rho\,\partial_\theta\phi}{r^2} - \frac{\partial_\theta(HB)}{4\pi r^2}\cot\theta + \frac{HB}{4\pi r^2}\cot^2\theta - \frac{1}{r^2\mathcal{K}_0}\left[\rho\partial_\theta\phi + \frac{(\sigma+1)HB}{4\pi}\cot\theta\right]^2.
\end{aligned} \tag{111}$$

On the other hand, in the normal conducting case the magnetic field and induction are equal $H = B$, and the free energy is $F = B^2/8\pi$, so that from equations (104) and (107), we have

$$\mathcal{K}_0 = \gamma p + \frac{B^2}{4\pi}, \quad U_0 = \frac{B^2}{2\pi r}, \quad U_1 = \frac{B\partial_r B}{2\pi r}, \quad U_2 = \frac{B\partial_\theta B}{2\pi r^2} \quad \text{and} \quad U_3 = \frac{B^2}{2\pi r^2}, \tag{112}$$

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and the coefficients are given by

$$a_0 = -\partial_r\rho\,\partial_r\phi - \frac{B\partial_r B}{2\pi r} + \frac{B^2}{2\pi r^2} - \frac{1}{\mathcal{K}_0}\left(\rho\partial_r\phi + \frac{B^2}{2\pi r}\right)^2,$$

$$b_0 = -\frac{\partial_r\rho\,\partial_\theta\phi}{r} - \frac{\partial_\theta\rho\,\partial_r\phi}{r} - \frac{B\partial_r B}{2\pi r}\cot\theta - \frac{B\partial_\theta B}{2\pi r^2} + \frac{B^2}{\pi r^2}\cot\theta$$
$$- \frac{2}{r\mathcal{K}_0}\left(\rho\partial_r\phi + \frac{B^2}{2\pi r}\right)\left(\rho\partial_\theta\phi + \frac{B^2}{2\pi}\cot\theta\right),$$

$$c_0 = -\frac{\partial_\theta\rho\,\partial_\theta\phi}{r^2} - \frac{B\partial_\theta B}{2\pi r^2}\cot\theta + \frac{B^2}{2\pi r^2}\cot^2\theta - \frac{1}{r^2\mathcal{K}_0}\left(\rho\partial_\theta\phi + \frac{B^2}{2\pi}\cot\theta\right)^2. \tag{113}$$

These are the same as the results given by Goossens & Veugelen (1978). [2]

4.3.2 *The $m \neq 0$ case*

When $m \neq 0$, the hydrostatic and magnetic parts of the energy are given by equations (95) and (101), respectively. In this case, the integrand $\mathcal{E} = \mathcal{E}_{\rm p} + \mathcal{E}_{\rm mag}$ is quadratic in the rescaled φ component of the Lagrangian displacement $\hat{T}$, defined by equation (93), and does not contain any derivatives of it. Therefore, we can write the energy as

$$\mathcal{E} = \mathcal{E}_{\rm o} + \alpha\hat{T}^2 + \beta\hat{T} + B\frac{\partial F}{\partial B}\left[\frac{m^2(R^2+S^2)}{r^2\sin^2\theta}\right], \tag{114}$$

where $\mathcal{E}_{\rm o}$ is the energy for the $m = 0$ case, given by equation (103), and we define

$$\alpha = \gamma p + \rho^2\frac{\partial^2 F}{\partial\rho^2},$$

$$\beta = -2\left(\gamma p + \rho B\frac{\partial^2 F}{\partial\rho\partial B} + \rho^2\frac{\partial^2 F}{\partial\rho^2}\right)D_0 + 2\rho B\frac{\partial^2 F}{\partial\rho\partial B}\left(\frac{R + S\cot\theta}{r}\right) + 2\rho\Lambda(\phi). \tag{115}$$

$\mathcal{E}_{\rm o}$ is independent of $\hat{T}$. We therefore have $\mathrm{d}^2\mathcal{E}/\mathrm{d}\hat{T}^2 = 2\alpha$. The $\gamma\,p$ term in α will be the dominant term for the cases of interest to us, so that $\mathrm{d}^2\mathcal{E}/\mathrm{d}\hat{T}^2 > 0$, and consequently $\mathcal{E}$ can be minimized with respect to $\hat{T}$. Setting $\mathrm{d}\mathcal{E}/\mathrm{d}\hat{T} = 0$ we get the value that minimizes the energy, $\hat{T} = -\beta/2\alpha$. Substituting this back into the energy we find the minimum as

$$\mathcal{E} = \mathcal{E}_{\rm o} - \frac{\beta^2}{4\alpha} + B\frac{\partial F}{\partial B}\left[\frac{m^2(R^2+S^2)}{r^2\sin^2\theta}\right]. \tag{116}$$

As was done in equation (103) for the $m = 0$ case, we can once again group together terms of different order in D_0, defined by equation (94),

$$\mathcal{E} = \mathcal{L}_0 {D_0}^2 + \mathcal{L}_1 D_0 + \mathcal{L}_2. \tag{117}$$

For notational convenience, define a set of auxiliary quantities:

$$V_0 = \alpha^{-1/2}\left(\gamma p + \rho B\frac{\partial^2 F}{\partial\rho\partial B} + \rho^2\frac{\partial^2 F}{\partial\rho^2}\right),$$

$$V_1 = \alpha^{-1/2}\left(\rho\partial_r\phi + \frac{\rho B}{r}\frac{\partial^2 F}{\partial\rho\partial B}\right),$$

$$V_2 = \alpha^{-1/2}\left(\frac{\rho\partial_\theta\phi}{r} + \frac{\rho B\cot\theta}{r}\frac{\partial^2 F}{\partial\rho\partial B}\right) \tag{118}$$

and

$$W_1 = -\rho\partial_r\phi - U_0 + V_0V_1,$$
$$W_2 = -\frac{\rho\partial_\theta\phi}{r} - U_0\cot\theta + V_0V_2, \tag{119}$$

where α is defined in equation (115), and U_0 is defined in equation (107). Also invoking the definitions of $\mathcal{K}_i$ from equation (104), we have

$$\mathcal{L}_0 = \mathcal{K}_0 - {V_0}^2 = B^2\frac{\partial^2 F}{\partial B^2} - \frac{1}{\alpha}\left(\rho B\frac{\partial^2 F}{\partial\rho\partial B}\right)^2,$$

$$\mathcal{L}_1 = \mathcal{K}_1 + 2V_0(V_1R + V_2S) = 2(W_1R + W_2S),$$

$$\mathcal{L}_2 = \mathcal{K}_2 - (V_1R + V_2S)^2 + B\frac{\partial F}{\partial B}\left[\frac{m^2(R^2+S^2)}{r^2\sin^2\theta}\right]. \tag{120}$$

[2] Note that there is a typo in equation (13) of Goossens & Veugelen (1978).

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Rearranging the terms we get

$$\mathcal{E} = \mathcal{L}_0 \left(D_0 + \frac{\mathcal{L}_1}{2\mathcal{L}_0} \right)^2 + \mathcal{L}_2 - \frac{\mathcal{L}_1^2}{4\mathcal{L}_0}. \tag{121}$$

Note that $\mathcal{L}_0$ is not necessarily positive, so unlike in the $m = 0$ case, it is not obvious that the first term is positive definite. In fact, for the strongly type II case where the free energy is of the form $F = H(\rho)B/4\pi$, we have $\mathcal{L}_0 < 0$. On the other hand, for the normal case $F = B^2/8\pi$, so that $\mathcal{L}_0 > 0$. For negative $\mathcal{L}_0$ the system is unstable since we can find displacement fields with sufficiently large derivatives which will make the D_0 term dominant in the energy. Therefore, for stability we must require $\mathcal{L}_0 > 0$, or using the definitions of equation (120),

$$B^2 \frac{\partial^2 F}{\partial B^2} > \left(\rho B \frac{\partial^2 F}{\partial \rho \partial B} \right)^2 \Big/ \left(\gamma p + \rho^2 \frac{\partial^2 F}{\partial \rho^2} \right). \tag{122}$$

This is a necessary but not sufficient condition for stability. This is related to what we will refer to as the MPR instability (Muzikar & Pethick 1981; Roberts 1981), which we will discuss in more detail in a later section.

Another way of looking at equation (121) is to say that when $\mathcal{L}_0 > 0$, the energy can be minimized with respect to D_0. The minimum is a quadratic in R and S, just like equation (106) for the $m = 0$ case,

$$\mathcal{L}_2 - \frac{\mathcal{L}_1^2}{4\mathcal{L}_0} = a_m R^2 + b_m RS + c_m S^2. \tag{123}$$

The coefficients are given as, using the definitions of U_i, V_i and W_i made in equations (107), (118) and (119),

$$\begin{aligned}
a_m &= -\partial_r \rho\, \partial_r \phi - U_1 + U_3 - {V_1}^2 + \frac{m^2 B}{r^2 \sin^2\theta} \frac{\partial F}{\partial B} - \frac{W_1^2}{\mathcal{L}_0}, \\
b_m &= -\frac{\partial_r \rho\, \partial_\theta \phi}{r} - \frac{\partial_\theta \rho\, \partial_r \phi}{r} - U_1 \cot\theta - U_2 + 2U_3 \cot\theta - 2V_1 V_2 - \frac{2W_1 W_2}{\mathcal{L}_0}, \\
c_m &= -\frac{\partial_\theta \rho\, \partial_\theta \phi}{r^2} - U_2 \cot\theta + U_3 \cot^2\theta - {V_2}^2 + \frac{m^2 B}{r^2 \sin^2\theta} \frac{\partial F}{\partial B} - \frac{W_2^2}{\mathcal{L}_0}.
\end{aligned} \tag{124}$$

This quadratic is positive if the coefficients satisfy the conditions listed in equation (109). However, the system will be definitely stable only when $\mathcal{L}_0 > 0$. On the other hand, if these conditions are violated, i.e. if the quadratic is negative, then the system is unstable regardless of the sign of $\mathcal{L}_0$. Also note that, clearly, the $|m| = 1$ case is the worst instability, as noted previously for the normal case by Tayler (1973) and Goossens & Veugelen (1978). On the other hand, when $\mathcal{L}_0 < 0$ the energy is maximized with respect to D_0, and it is always possible to find a Lagrangian displacement field with sufficiently large derivatives that will make the system unstable.

The coefficients for the strongly type II case can be obtained by setting $F = HB/4\pi$. On the other hand, for the normal case we have $F = B^2/8\pi$, and the coefficients reduce to

$$\begin{aligned}
a_m &= -\partial_r \rho\, \partial_r \phi - \frac{(\rho \partial_r \phi)^2}{\gamma p} - \frac{B \partial_r B}{2\pi r} - \frac{B^2}{2\pi r^2} + \frac{m^2 B^2}{4\pi r^2 \sin^2\theta}, \\
b_m &= -\frac{\partial_r \rho\, \partial_\theta \phi}{r} - \frac{\partial_\theta \rho\, \partial_r \phi}{r} - \frac{2\rho^2 \partial_r \phi\, \partial_\theta \phi}{\gamma p r} - \frac{B \partial_r B}{2\pi r} \cot\theta - \frac{B \partial_\theta B}{2\pi r^2} - \frac{B^2}{\pi r^2} \cot\theta, \\
c_m &= -\frac{\partial_\theta \rho\, \partial_\theta \phi}{r^2} - \frac{(\rho \partial_\theta \phi)^2}{\gamma p r^2} - \frac{B \partial_\theta B}{2\pi r^2} \cot\theta - \frac{B^2}{2\pi r^2} \cot^2\theta + \frac{m^2 B^2}{4\pi r^2 \sin^2\theta}.
\end{aligned} \tag{125}$$

These are the same as the results given by Goossens & Veugelen (1978).

In the next two sections we will consider the special cases of the completely normal conducting star and the strongly type II superconducting star with $H \propto \rho$. The coefficients a, b and c (given by equations 108 and 124) have hydrostatic terms that are of the form $\partial_r \rho$ and $\partial_\theta \rho$, and magnetic terms of the order of the magnetic free energy F. The radial dependence of the background quantities arises from the much stronger hydrostatic forces, while the θ dependence arises as a result of magnetic forces. Therefore, $\partial_\theta \rho \sim F \ll \partial_r \rho$. We will calculate the coefficients to first order in the magnetic energy, which is much smaller than the hydrostatic terms. We will assume that the perturbations and the background state have the same index, thus neglecting buoyancy effects. If we include buoyancy, then to leading order, the coefficient a will be a buoyant term, c will be a purely magnetic term, and b will be the product of a buoyant term and a magnetic term. Thus, $b^2 \ll 4ac$, and the stability conditions (given by equation 109) will reduce to $a > 0$ and $c > 0$. The first condition is necessary for stability to buoyancy, and the second is the same condition on the magnetic field as without buoyancy. We will consider the effects of multifluid composition in more detail in future work.

4.4 Stability criteria for a normal star

We will now examine the stability of a particular magnetic field configuration in a normal star. The equilibrium equations in this case are, from equation (96),

$$\begin{aligned}
\partial_r p + \rho \partial_r \phi &= -\frac{B \partial_r (Br)}{4\pi r} \\
\partial_\theta p + \rho \partial_\theta \phi &= -\frac{B \partial_\theta (B \sin\theta)}{4\pi \sin\theta}.
\end{aligned} \tag{126}$$

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Let p_o, ρ_o and ϕ_o refer to the hydrostatic equilibrium in the absence of magnetic fields. This equilibrium is spherically symmetric and is simply given through

$$\partial_r p_o + \rho_o \partial_r \phi_o = 0. \tag{127}$$

The difference between p_o, ρ_o and ϕ_o and the corresponding quantities p, ρ and ϕ in the presence of magnetic fields is of the order of the magnetic pressure $\sim B^2$, which we assume to be small compared to the hydrostatic pressure. Therefore, using the equations of equilibrium we can expand equation (113) for $m = 0$ to lowest order in B^2,

$$\begin{aligned}
a_0 &\approx \frac{B^2}{4\pi r^2}\left(\frac{\mathrm{d}\ln\rho_o}{\mathrm{d}\ln r}\right)^2 + \left[\frac{3B^2}{4\pi r^2} - \frac{B\partial_r B}{4\pi r}\right]\frac{\mathrm{d}\ln\rho_o}{\mathrm{d}\ln r} - \frac{B\partial_r B}{2\pi r} + \frac{B^2}{2\pi r^2},\\
b_0 &\approx \left[\frac{3B^2}{4\pi r^2}\cot\theta - \frac{B\partial_\theta B}{4\pi r^2}\right]\frac{\mathrm{d}\ln\rho_o}{\mathrm{d}\ln r} - \frac{B\partial_r B}{2\pi r}\cot\theta - \frac{B\partial_\theta B}{2\pi r^2} + \frac{B^2}{\pi r^2}\cot\theta,\\
c_0 &\approx -\frac{B\partial_\theta B}{2\pi r^2}\cot\theta + \frac{B^2}{2\pi r^2}\cot^2\theta.
\end{aligned} \tag{128}$$

On the other hand, for $m = 1$, we have, from equation (125),

$$\begin{aligned}
a_m &\approx -\left(2 + \frac{\mathrm{d}\ln\rho_o}{\mathrm{d}\ln r}\right)\left(\frac{B^2}{4\pi r^2} + \frac{B\partial_r B}{4\pi r}\right) + \frac{B^2}{4\pi r^2\sin^2\theta},\\
b_m &\approx -\left(2 + \frac{\mathrm{d}\ln\rho_o}{\mathrm{d}\ln r}\right)\left(\frac{B^2}{4\pi r^2}\cot\theta + \frac{B\partial_\theta B}{4\pi r^2}\right) - \frac{B\partial_r B}{2\pi r}\cot\theta - \frac{B^2}{2\pi r^2}\cot\theta,\\
c_m &\approx -\frac{B\partial_\theta B}{2\pi r^2}\cot\theta - \frac{B^2}{2\pi r^2}\cot^2\theta + \frac{B^2}{4\pi r^2\sin^2\theta}.
\end{aligned} \tag{129}$$

We will now consider a specific example. Let the equation of state be given by a $\gamma = 2$ polytrope, where the background density profile is $\rho = \rho_c \sin x/x$, in terms of the dimensionless radial coordinate $x = r/r_o$. Assume a magnetic field of the form given by equation (41),

$$B(r,\theta) = \hat{B}_o\left(\frac{\rho}{\rho_c}\right)^{(n+2)/4}\left(\frac{r}{r_o}\right)^{n/2}\sin^{n/2}\theta = \hat{B}_o x^{(n-2)/4}\sin^{(n+2)/4}x\,\sin^{n/2}\theta, \tag{130}$$

where $n \geqslant 1$. Then, for $m = 0$, the coefficients become, from equation (128),

$$\begin{aligned}
a_0 &\approx \frac{\hat{B}_o^2}{16\pi r_o^2}(2-n)(1 + x\cot x)^2 x^{(n-6)/2}\sin^{(n+2)/2}x\,\sin^n\theta,\\
b_0 &\approx \frac{\hat{B}_o^2}{4\pi r_o^2}(2-n)(1 + x\cot x)x^{(n-6)/2}\sin^{(n+2)/2}x\,\sin^{n-1}\theta\cos\theta,\\
c_0 &\approx \frac{\hat{B}_o^2}{4\pi r_o^2}(2-n)x^{(n-6)/2}\sin^{(n+2)/2}x\,\sin^{n-2}\theta\cos^2\theta.
\end{aligned} \tag{131}$$

Note that $b_0^2 = 4a_0c_0$, so that the quadratic forms a complete square, i.e. $a_0R^2 + b_0RS + c_0S^2 = a_0(R + b_0S/2a_0)^2$. However, for $n > 2$, we have $a_0 < 0$ and $c_0 < 0$, and the conditions for stability (equation 109) are violated. Thus, only fields with $1 \leqslant n \leqslant 2$ are marginally stable for $m = 0$.

On the other hand, for $m = 1$, we have, from equation (129),

$$\begin{aligned}
a_m &\approx \frac{\hat{B}_o^2}{16\pi r_o^2}\left[4 - (n+2)(1 + x\cot x)^2\sin^2\theta\right]x^{(n-6)/2}\sin^{(n+2)/2}x\,\sin^{n-2}\theta,\\
b_m &\approx -\frac{\hat{B}_o^2}{4\pi r_o^2}(n+2)(1 + x\cot x)x^{(n-6)/2}\sin^{(n+2)/2}x\,\sin^{n-1}\theta\cos\theta,\\
c_m &\approx \frac{\hat{B}_o^2}{4\pi r_o^2}\left[1 - (n+2)\cos^2\theta\right]x^{(n-6)/2}\sin^{(n+2)/2}x\,\sin^{n-2}\theta.
\end{aligned} \tag{132}$$

Since a_m and c_m become negative in some regions, they violate the stability conditions given by equation (109). Consequently, the normal magnetic field is unstable for $m = 1$. Thus, we might expect $n = 1$ models with both normal and superconducting regions to be unstable. Poloidal fields may stabilize the star, as in normal conductors (Tayler 1973; Wright 1973; Braithwaite & Nordlund 2006), and we consider adding them in a following section.

4.5 Stability criteria for a superconducting star with $H \propto \rho$

We will now consider the strongly type II superconducting case with $H \propto \rho$ (i.e. $\sigma = 1$) in more detail. In this case $F = HB/4\pi$, and the equations of equilibrium (equation 96) explicitly give

$$\begin{aligned}
\partial_r p + \rho\partial_r\phi &= -\frac{B\partial_r(Hr)}{4\pi r} - \frac{H\partial_r B}{4\pi},\\
\partial_\theta p + \rho\partial_\theta\phi &= -\frac{HB}{4\pi}\cot\theta - \frac{H\partial_\theta B}{4\pi}.
\end{aligned} \tag{133}$$

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Using these equations as well as the equation of equilibrium in the absence of magnetic fields (equation 127), we can expand the coefficients for $m = 0$ (equation 111) to lowest order in *HB*,

$$a_0 \approx \frac{HB}{2\pi r^2}\left(\frac{\mathrm{d}\ln\rho_\mathrm{o}}{\mathrm{d}\ln r}\right)^2 + \left[\frac{3HB}{4\pi r^2} - \frac{\partial_r(HB)}{4\pi r}\right]\frac{\mathrm{d}\ln\rho_\mathrm{o}}{\mathrm{d}\ln r} - \frac{\partial_r(HB)}{4\pi r} + \frac{HB}{4\pi r^2},$$

$$b_0 \approx \left[\frac{3HB}{4\pi r^2}\cot\theta - \frac{\partial_\theta(HB)}{4\pi r^2}\right]\frac{\mathrm{d}\ln\rho_\mathrm{o}}{\mathrm{d}\ln r} - \frac{\partial_r(HB)}{4\pi r}\cot\theta - \frac{\partial_\theta(HB)}{4\pi r^2} + \frac{HB}{2\pi r^2}\cot\theta,$$

$$c_0 \approx -\frac{\partial_\theta(HB)}{4\pi r^2}\cot\theta + \frac{HB}{4\pi r^2}\cot^2\theta. \tag{134}$$

For a $\gamma = 2$ polytrope we have $\rho = \rho_\mathrm{c}\sin x/x$. Consider a magnetic field of the form given by equation (35), for $\sigma = 1$,

$$B(r,\theta) = B_\mathrm{o}\left(\frac{\rho}{\rho_\mathrm{c}}\right)^n\left(\frac{r}{r_\mathrm{o}}\right)^n \sin^n\theta = B_\mathrm{o}\sin^n x\sin^n\theta, \tag{135}$$

where $n \geqslant 1$. In particular, we get, from equation (134),

$$c_0 \approx \frac{H_\mathrm{c}B_\mathrm{o}}{4\pi r_\mathrm{o}^2}(1-n)x^{-3}\sin^{n+1}x\sin^{n-2}\theta\cos^2\theta. \tag{136}$$

For all $n > 1$ this is negative, thus immediately violating one of the conditions for stability (equation 109). For $n = 1$ all three coefficients vanish to lowest order in *HB*, implying that the magnetic field is marginally stable. In Appendix A, we show that this result is true for any $H(\rho, B)$.

For $m \neq 0$, we have (equation 120)

$$\mathcal{L}_0 = -\frac{1}{\gamma p}\left(\frac{HB}{4\pi}\right)^2 < 0, \tag{137}$$

which implies that even if the conditions given in equation (109) are met the system will still be unstable. This is the MPR instability and will be discussed in a following section in more detail.

4.6 Interchange instability

In this section we will show that the $m = 0$ stability conditions correspond to the stability criteria for the interchange of two magnetic flux tubes, as demonstrated for the normal case by Tayler (1973). Consider two axisymmetric flux tubes located at coordinates r, θ and at $r + \delta r$, $\theta + \delta\theta$, and having volumes V and $V + \delta V$ and corresponding cross-sections A and $A + \delta A$, respectively. We will assume that the interchange is adiabatic so that the mass ρV, magnetic flux BA and pV^γ are all conserved.

Let the pressure, density and magnetic induction of the two tubes initially be

$$\begin{array}{llll} \text{at } r,\ \theta: & p & \rho & B \\ \text{at } r+\delta r,\ \theta+\delta\theta: & p+\delta p & \rho+\delta\rho & B+\delta B \end{array} \tag{138}$$

After the interchange the corresponding quantities are, defining a cylindrical radius by $\varpi = r\sin\theta$,

$$\begin{array}{llll} \text{at } r,\ \theta: & \dfrac{(p+\delta p)(V+\delta V)^\gamma}{V^\gamma} & \dfrac{(\rho+\delta\rho)(V+\delta V)}{V} & \dfrac{(B+\delta B)(V+\delta V)\varpi}{V(\varpi+\delta\varpi)} \\ \text{at } r+\delta r,\ \theta+\delta\theta: & \dfrac{pV^\gamma}{(V+\delta V)^\gamma} & \dfrac{\rho V}{V+\delta V} & \dfrac{BV(\varpi+\delta\varpi)}{(V+\delta V)\varpi} \end{array} \tag{139}$$

The total energy is the sum of internal, magnetic and gravitational energies. Without loss of generality, we can take the zero of the gravitational potential to be at r, θ. Prior to the interchange, the energy is

$$E_i = \frac{pV}{\gamma-1} + \frac{(p+\delta p)(V+\delta V)}{\gamma-1} + F(\rho,B)V + F(\rho+\delta\rho, B+\delta B)(V+\delta V) + (\rho+\delta\rho)(V+\delta V)\delta\phi. \tag{140}$$

Here F is the magnetic free energy. After the interchange, we have

$$E_f = \frac{(p+\delta p)(V+\delta V)^\gamma}{(\gamma-1)V^{\gamma-1}} + \frac{pV^\gamma}{(\gamma-1)(V+\delta V)^{\gamma-1}} + F(\rho_1,B_1)V + F(\rho_2,B_2)(V+\delta V) + \rho V\delta\phi. \tag{141}$$

Here ρ_1 and B_1 are the new density and induction at r, θ, and ρ_2 and B_2 are the corresponding quantities at $r + \delta r$, $\theta + \delta\theta$ (equation 139). We need to calculate the energy difference resulting from the interchange to second order:

$$\Delta E = E_f - E_i = \Delta E_\mathrm{p} + \Delta E_\mathrm{m}. \tag{142}$$

Here for notational convenience we denote by ΔE_p the change in the internal and gravitational energies, and ΔE_m is the change in the magnetic energy. To second order we have

$$\Delta E_\mathrm{p} \approx \gamma p\frac{(\delta V)^2}{V} + (\delta p - \rho\delta\phi)\delta V - V\delta\rho\delta\phi. \tag{143}$$

Using the equations of equilibrium (equation 96) we have

$$\delta p = -\rho\delta\phi - B\frac{\partial F}{\partial B}\frac{\delta\varpi}{\varpi} - B\frac{\partial^2 F}{\partial\rho\partial B}\delta\rho - \rho\frac{\partial^2 F}{\partial\rho\partial B}\delta B - B\frac{\partial^2 F}{\partial B^2}\delta B - \rho\frac{\partial^2 F}{\partial\rho^2}\delta\rho. \tag{144}$$

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The magnetic term in the energy change is lengthy. First, note that

$$F(\rho+\delta\rho, B+\delta B) \approx F(\rho, B) + \frac{\partial F}{\partial \rho}\delta\rho + \frac{\partial F}{\partial B}\delta B + \frac{1}{2}\frac{\partial^2 F}{\partial \rho^2}(\delta\rho)^2 + \frac{\partial^2 F}{\partial\rho\partial B}\delta\rho\delta B + \frac{1}{2}\frac{\partial^2 F}{\partial B^2}(\delta B)^2. \quad (145)$$

We can write the magnetic terms in E_f (equation 141) as $F(\rho_i, B_i) = F(\rho+\delta\rho_i, B+\delta B_i)$, so they can be expanded in a similar fashion. Here we have, to second order,

$$\delta\rho_1 = \rho_1 - \rho = \frac{(\rho+\delta\rho)(V+\delta V)}{V} - \rho = \rho\left[\frac{\delta\rho}{\rho} + \frac{\delta V}{V} + \frac{\delta\rho}{\rho}\frac{\delta V}{V}\right],$$

$$\delta\rho_2 = \rho_2 - \rho = \frac{\rho V}{V+\delta V} - \rho \approx \rho\left[-\frac{\delta V}{V} + \left(\frac{\delta V}{V}\right)^2\right],$$

$$\delta B_1 = B_1 - B = \frac{(B+\delta B)(V+\delta V)\varpi}{V(\varpi+\delta\varpi)} - B$$

$$\approx B\left[\frac{\delta B}{B} + \frac{\delta V}{V} - \frac{\delta\varpi}{\varpi} + \frac{\delta B}{B}\frac{\delta V}{V} - \frac{\delta\varpi}{\varpi}\frac{\delta B}{B} - \frac{\delta\varpi}{\varpi}\frac{\delta V}{V} + \left(\frac{\delta\varpi}{\varpi}\right)^2\right],$$

$$\delta B_2 = B_2 - B = \frac{BV(\varpi+\delta\varpi)}{(V+\delta V)\varpi} - B \approx B\left[\frac{\delta\varpi}{\varpi} - \frac{\delta V}{V} - \frac{\delta\varpi}{\varpi}\frac{\delta V}{V} + \left(\frac{\delta V}{V}\right)^2\right]. \quad (146)$$

Using these and equations (143) and (144) we can write the energy change as

$$\frac{\Delta E}{V} \approx \mathcal{M}_0\left(\frac{\delta V}{V}\right)^2 + \mathcal{M}_1\frac{\delta V}{V} + \mathcal{M}_2, \quad (147)$$

where

$$\mathcal{M}_0 \approx \gamma p + B^2\frac{\partial^2 F}{\partial B^2} + 2\rho B\frac{\partial^2 F}{\partial\rho\partial B} + \rho^2\frac{\partial^2 F}{\partial\rho^2},$$

$$\mathcal{M}_1 \approx -2\rho\delta\phi - 2\left(B\frac{\partial F}{\partial B} + B^2\frac{\partial^2 F}{\partial B^2} + \rho B\frac{\partial^2 F}{\partial\rho\partial B}\right)\frac{\delta\varpi}{\varpi},$$

$$\mathcal{M}_2 \approx -\delta\rho\delta\phi - \left(\frac{\partial F}{\partial B}\delta B - B\frac{\partial^2 F}{\partial B^2}\delta B - B\frac{\partial^2 F}{\partial\rho\partial B}\delta\rho\right)\frac{\delta\varpi}{\varpi} + \left(B\frac{\partial F}{\partial B} + B^2\frac{\partial^2 F}{\partial B^2}\right)\left(\frac{\delta\varpi}{\varpi}\right)^2. \quad (148)$$

Since $\mathcal{M}_0 > 0$ for the cases of interest, the change in energy can be minimized with respect to $\delta V/V$. The minimum of the energy becomes

$$\frac{\Delta E}{V} \approx \mathcal{M}_2 - \frac{\mathcal{M}_1^2}{4\mathcal{M}_0}. \quad (149)$$

The small quantities need to be expanded only to first order:

$$\delta\varpi = \delta(r\sin\theta) = \delta r\sin\theta + r\delta\theta\cos\theta,$$

$$\delta\rho = \delta r\partial_r\rho + \delta\theta\partial_\theta\rho, \quad (150)$$

and similarly for B and ϕ. The energy can then be written as

$$\frac{\Delta E}{V} \approx a_0(\delta r)^2 + b_0 r\delta r\delta\theta + c_0 r^2(\delta\theta)^2. \quad (151)$$

a_0, b_0 and c_0 are the same as in equation (108) and the conditions for stability are the same as in equation (109). In fact, the same conclusion could have been drawn by comparing equation (148) to (104). Thus, we have shown that the $m = 0$ stability conditions are the same as the conditions for stability under the interchange of magnetic flux tubes. In other words, the interchange is the worst instability for $m = 0$.

4.7 The MPR instability

In this section we will derive the criteria for the instability discussed by Muzikar & Pethick (1981) and Roberts (1981). Using equation (2) we can write the magnetic stress tensor as (equation 1),

$$\sigma_{ij} = \left(F - \rho F_{,\rho} - BF_{,B}\right)\delta_{ij} + BF_{,B}\hat{B}_i\hat{B}_j. \quad (152)$$

Consider perturbations around a state of uniform density ρ and uniform magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$. The Lagrangian displacement associated with these perturbations is

$$\boldsymbol{\xi}(\boldsymbol{r}, t) = \boldsymbol{\xi}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r} - \mathrm{i}\omega t). \quad (153)$$

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In this case, we have

$$\delta \boldsymbol{B} = \nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}) = \mathrm{i}B(k_z\boldsymbol{\xi} - \boldsymbol{k}\cdot\boldsymbol{\xi}\hat{\boldsymbol{z}}),$$
$$\delta B = \hat{\boldsymbol{B}}\cdot\delta\boldsymbol{B} = -\mathrm{i}B\boldsymbol{k}_\perp\cdot\boldsymbol{\xi}_\perp,$$
$$\delta\hat{\boldsymbol{B}} = B^{-1}(\delta\boldsymbol{B} - \delta B\hat{\boldsymbol{B}}) = \mathrm{i}k_z\boldsymbol{\xi}_\perp,$$
$$\delta\rho = -\nabla\cdot(\rho\boldsymbol{\xi}) = -\mathrm{i}\rho\boldsymbol{k}\cdot\boldsymbol{\xi}. \tag{154}$$

Here $\perp$ means perpendicular to $\hat{\boldsymbol{z}}$.

The magnetic force density is, from equation (152),

$$\boldsymbol{f}_m = \nabla\cdot\boldsymbol{\sigma} = -\left(\rho F_{,\rho\rho} + BF_{,\rho B}\right)\nabla\rho - \left(\rho F_{,\rho B} + BF_{,BB}\right)\nabla B + \boldsymbol{B}\cdot\nabla(F_{,B}\hat{\boldsymbol{B}}). \tag{155}$$

Since the background quantities are constant the perturbation in the magnetic force becomes

$$\delta\boldsymbol{f}_m = -\left(\rho^2F_{,\rho\rho} + \rho BF_{,\rho B}\right)\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{\xi}) - \left(\rho BF_{,\rho B} + B^2F_{,BB}\right)\boldsymbol{k}(\boldsymbol{k}_\perp\cdot\boldsymbol{\xi}_\perp)$$
$$+\hat{\boldsymbol{z}}k_z\left[\rho BF_{,\rho B}(\boldsymbol{k}\cdot\boldsymbol{\xi}) + B^2F_{,BB}(\boldsymbol{k}_\perp\cdot\boldsymbol{\xi}_\perp)\right] - BF_{,B}k_z^2\boldsymbol{\xi}_\perp. \tag{156}$$

Since the background state is symmetric with respect to $\hat{\boldsymbol{z}}$ we can choose $\boldsymbol{k} = \hat{\boldsymbol{z}}k_z + \hat{\boldsymbol{x}}k_x$. With this choice equation (154) becomes

$$\delta\boldsymbol{B} = \mathrm{i}B(k_z\xi_x\hat{\boldsymbol{x}} + k_z\xi_y\hat{\boldsymbol{y}} - k_x\xi_x\hat{\boldsymbol{z}}),$$
$$\delta B = -\mathrm{i}Bk_x\xi_x,$$
$$\delta\hat{\boldsymbol{B}} = \mathrm{i}k_z(\xi_x\hat{\boldsymbol{x}} + \xi_y\hat{\boldsymbol{y}}),$$
$$\delta\rho = -\mathrm{i}\rho(k_x\xi_x + k_z\xi_z). \tag{157}$$

The components of the magnetic force become

$$(\delta f_m)_x = -\xi_x\left[k_x^2\left(\rho^2F_{,\rho\rho} + 2\rho BF_{,\rho B} + B^2F_{,BB}\right) + k_z^2BF_{,B}\right] - \xi_zk_xk_z\left(\rho^2F_{,\rho\rho} + \rho BF_{,\rho B}\right),$$
$$(\delta f_m)_y = -\xi_yk_z^2BF_{,B},$$
$$(\delta f_m)_z = -\xi_xk_xk_z\left(\rho^2F_{,\rho\rho} + \rho BF_{,\rho B}\right) - \xi_zk_z^2\rho^2F_{,\rho\rho}. \tag{158}$$

In addition, there is a pressure restoring force, $\delta\boldsymbol{f}_\mathrm{p} = -\nabla\delta p = -\gamma p\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{\xi})$, or in components,

$$(\delta f_\mathrm{p})_x = -\gamma p(k_x^2\xi_x + k_xk_z\xi_z),$$
$$(\delta f_\mathrm{p})_y = 0,$$
$$(\delta f_\mathrm{p})_z = -\gamma p(k_xk_z\xi_x + k_z^2\xi_z). \tag{159}$$

We will neglect gravitational forces, so that the equations for the perturbations become

$$-\rho\omega^2\boldsymbol{\xi} = \delta\boldsymbol{f}_\mathrm{p} + \delta\boldsymbol{f}_m. \tag{160}$$

From equations (158) and (159) it follows that the equation for ξ_y completely decouples from the equations for ξ_x and ξ_z,

$$\rho\omega^2\xi_y = k_z^2BF_{,B}\xi_y. \tag{161}$$

This implies that one pair of modes has $\xi_x = \xi_z = 0$ and $\xi_y \neq 0$ with $\omega^2 = k_z^2BF_{,B}/\rho$. These modes are the generalization of the Alfvén modes. The remaining modes are given through

$$\rho\omega^2\xi_x = \xi_x\left[k_x^2\left(\gamma p + \rho^2F_{,\rho\rho} + 2\rho BF_{,\rho B} + B^2F_{,BB}\right) + k_z^2BF_{,B}\right] + \xi_zk_xk_z\left(\gamma p + \rho^2F_{,\rho\rho} + \rho BF_{,\rho B}\right),$$
$$\rho\omega^2\xi_z = \xi_xk_xk_z\left(\gamma p + \rho^2F_{,\rho\rho} + \rho BF_{,\rho B}\right) + \xi_zk_z^2\left(\gamma p + \rho^2F_{,\rho\rho}\right). \tag{162}$$

From these two equations we get the characteristic equation for the modes, after some rearrangement,

$$\rho^2\omega^4 - \rho\omega^2\mathcal{E}_0 + \mathcal{E}_1 = 0, \tag{163}$$

where, defining $k^2 = k_x^2 + k_z^2$,

$$\mathcal{E}_0 = k^2\gamma p + k_x^2\left(\rho^2F_{,\rho\rho} + 2\rho BF_{,\rho B} + B^2F_{,BB}\right) + k_z^2\left(BF_{,B} + \rho^2F_{,\rho\rho}\right),$$
$$\mathcal{E}_1 = k_x^2k_z^2\left(\gamma pB^2F_{,BB} + \rho^2B^2F_{,\rho\rho}F_{,BB} - \rho^2B^2F_{,\rho B}^2\right) + k_z^4BF_{,B}\left(\gamma p + \rho^2F_{,\rho\rho}\right). \tag{164}$$

In the absence of magnetic fields, we have, defining $\gamma p = \rho c_\mathrm{s}^2$,

$$\rho^2\omega^4 - \rho^2\omega^2k^2c_\mathrm{s}^2 = 0, \tag{165}$$

which has two roots: $\omega^2 = 0$ and $\omega^2 = k^2c_\mathrm{s}^2$. The latter corresponds to sound waves. In the cases of interest, the magnetic terms will be much smaller in comparison to the pressure terms, so that one of the roots will have $\omega^2 \approx k^2c_\mathrm{s}^2$ and therefore will be definitely positive. Since $\mathcal{E}_1$ is the product of the two roots, the condition for stability is $\mathcal{E}_1 > 0$, which for $k_z \neq 0$ becomes

$$k_x^2\left(\gamma pB^2F_{,BB} + \rho^2B^2F_{,\rho\rho}F_{,BB} - \rho^2B^2F_{,\rho B}^2\right) + k_z^2BF_{,B}\left(\gamma p + \rho^2F_{,\rho\rho}\right) > 0. \tag{166}$$

For sufficiently large k_x, or more precisely when $k_x^2BF_{,BB} \gg k_z^2F_{,B}$, this reduces to

$$F_{,BB} > \frac{\rho^2F_{,\rho B}^2}{\gamma p + \rho^2F_{,\rho\rho}} \approx \frac{\rho^2F_{,\rho B}^2}{\gamma p}. \tag{167}$$

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This is exactly the same condition for stability as in equation (122). When pressure dominates, it is also of the same form as the condition given by Roberts (1981). From equation (163) it follows that the potentially unstable modes are given through

$$\rho\omega^2 \approx \frac{\mathcal{E}_1}{k^2\gamma p} \approx \frac{k_x^2 k_z^2}{k^2}\left(B^2 F_{,BB} - \frac{\rho^2 B^2 F_{,\rho B}^2}{\gamma p}\right) + \frac{k_z^4}{k^2} B F_{,B}. \tag{168}$$

The magnetic free energy in the strongly type II case ($H \gg B$) can be written as (Tinkham 1975; Muzikar & Pethick 1981)

$$F = \frac{H(\rho)B}{4\pi} + \sqrt{\frac{3}{32\pi^3}}\frac{\Phi_o^2}{\lambda^4}\left(\frac{\lambda}{a}\right)^{5/2}\exp\left(-\frac{a}{\lambda}\right). \tag{169}$$

Here $\Phi_o = hc/2e$ is the flux quantum ($n_\Phi = B/\Phi_o$ is the flux line density per unit area), $\lambda = (m_p c^2/4\pi n_p e^2)^{1/2}$ is the London penetration depth, n_p is the number density of protons and a is the distance between flux lines in a triangular lattice,

$$a = \left(\frac{4}{3}\right)^{1/4}\left(\frac{\Phi_o}{B}\right)^{1/2}. \tag{170}$$

The magnetic field strength in this case is (Tinkham 1975; Easson & Pethick 1977),

$$H \simeq H_{c1} = \frac{\Phi_o \ln(\lambda/\xi)}{4\pi\lambda^2}, \tag{171}$$

where $\xi = \hbar^2 k_F/\pi m_p \Delta$ is the coherence length, $\xi \ll \lambda$; $k_F = (3\pi^2 n_p)^{1/3}$ is the Fermi wavenumber of protons, and Δ is the superconducting energy gap. The first term in equation (169) is the energy of an isolated flux line, and the second term arises due to the interaction between flux lines. Note that only a depends on B and only the interaction term contributes to $F_{,BB}$. Also note that $\lambda^2 \propto 1/\rho$ when the proton number density is proportional to the baryon number density, as suggested by Baym et al. (1971). Defining a new variable by $u = a/\lambda$ we have (equation 169),

$$F = \frac{H(\rho)B}{4\pi} + E(\rho)u^{-5/2}\mathrm{e}^{-u} \quad \text{where} \quad E(\rho) = \sqrt{\frac{3}{32\pi^3}}\frac{\Phi_o^2}{\lambda^4}. \tag{172}$$

Then, introducing an auxiliary function $f(u)$,

$$B^2 F_{,BB} = \frac{E(\rho)}{4}\left(u^{-1/2} + 2u^{-3/2} + \frac{5}{4}u^{-5/2}\right)\mathrm{e}^{-u} = E(\rho) f(u). \tag{173}$$

Only the first term needs to be retained when $u \gg 1$, i.e. when the spacing between flux lines is large compared to the penetration depth. In the same limit, we can also approximate

$$\rho B F_{,\rho B} \approx \frac{\rho B}{4\pi}\frac{\mathrm{d}H}{\mathrm{d}\rho} = \frac{\sigma H B}{4\pi} \quad \text{where} \quad \sigma = \frac{\mathrm{d}\ln H}{\mathrm{d}\ln\rho}. \tag{174}$$

Using these equations, we can write the condition for *instability* as, from equation (167),

$$u^4 f(u) < \sqrt{\frac{2}{27\pi}}\frac{\sigma^2 H^2}{\gamma p}. \tag{175}$$

Note than when $\sigma = 0$, i.e. when H is independent of ρ, there is no instability. Thus, it does not arise in a normal medium. Moreover, $\sigma > 0$ is not required in order to have an instability, contrary to the conclusions of Muzikar & Pethick (1981).

We take the magnetic field strength to be $H \sim 10^{15}$ G, and the typical density in the superconducting region to be $\rho \sim 3 \times 10^{14}\,\mathrm{g\,cm^{-3}}$, which corresponds to a pressure $p \sim 4 \times 10^{33}\,\mathrm{erg\,cm^{-3}}$, for a $\gamma = 2$ polytrope and a radius $R_\star \approx 10$ km. We also take $\sigma = 1$. From equation (175) it follows that instabilities arise for $u > u_o$ where $u_o \simeq 20$. Using equation (170) and the definitions of λ and Φ_o, we can find the largest magnetic induction which is unstable,

$$B < \frac{4\pi h e n_p}{\sqrt{3} m_p c u_o^2} = 1.15 \times 10^{13}\left(\frac{n_p}{0.01\,\mathrm{fm}^{-3}}\right)\left(\frac{u_o}{20}\right)^{-2}\ \mathrm{G}. \tag{176}$$

The proton number density n_p is a function of the baryon number density, and for $n_b \sim 0.2\,\mathrm{fm}^{-3}$, we have $n_p \sim 0.01\,\mathrm{fm}^{-3}$ (Elgarøy et al. 1996; Zuo et al. 2004).

For toroidal fields $\hat{z}$ is along the $\hat{\phi}$ direction, so that for modes we have $\exp(\mathrm{i}k_z z) = \exp(\mathrm{i}m\phi)$. We can take $k_z \sim m/R_\star$ for a star of radius $R_\star$. The condition given in equation (167) can lead to instabilities when the perpendicular wave vector k_x is sufficiently larger than k_z. Using equations (172) and (173), we get

$$\frac{k_z^2}{k_x^2} \ll \frac{B F_{,BB}}{F_{,B}} \approx \frac{4\pi E(\rho) f(u)}{H(\rho)B} = \frac{3\sqrt{2\pi}\, u^2 f(u)}{\ln(\lambda/\xi)} \lesssim \frac{\sigma^2 H B}{4\pi\gamma p}, \tag{177}$$

where the last inequality follows from the condition for instability (equation 175). The length-scale of the instabilities is small compared to the size of the star; for a $\gamma = 2$ polytrope,

$$L_x = k_x^{-1} \ll \frac{R_\star}{m}\sqrt{\frac{\sigma^2 H B}{4\pi\gamma p}} \approx 3.1 \times 10^2 \frac{\sqrt{\sigma^2 H_{15} B_{12}}}{m\rho_{14}}\ \mathrm{cm}. \tag{178}$$

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Here $H_{15} = H/10^{15}$ G, $B_{12} = B/10^{12}$ G, and $\rho_{14} = \rho/10^{14}$ g cm^{-3}. From equation (168) we can estimate the growth rate of the instability, using $\gamma p = \rho c_s^2$,

$$\tilde{\omega} = \sqrt{-\omega^2} \sim \left| \frac{k_z B F_{,\rho B}}{c_s} \right| \sim \frac{m|\sigma| H B}{4\pi\rho c_s R_\star}. \tag{179}$$

The corresponding growth time-scale is

$$\frac{1}{\tilde{\omega}} \approx 3.7 \times 10^3 \frac{\rho_{14}^{3/2} R_6^2}{m|\sigma| H_{15} B_{12}} \text{ s}. \tag{180}$$

Here $R_6 = R_\star/10^6$ cm. Note that $m = 0$ is stable. The unstable modes will be dissipated if the kinematic viscosity of the fluid is

$$\eta > \frac{\tilde{\omega}}{k_x^2} \approx 26 \frac{H_{15}^2 B_{12}^2}{m\rho_{14}^{7/2} R_6^2} \text{ cm}^2 \text{ s}^{-1}. \tag{181}$$

This value is well below the estimated values of the viscosity in a neutron star, which are typically in the range $\eta \sim 10^{4-5}$ cm^2 s^{-1} (for a review see Andersson, Comer & Glampedakis 2005).

Note that similar results will hold for poloidal fields, except that in this case $k_z \gtrsim 1/R_\star$ will depend on both the number of radial nodes and the angular momentum quantum number of the mode. Simple linear analysis along the lines outlined by Hide (1971) reveals that the growth rate of the MPR mode will not be strongly affected by buoyancy, but the condition for stability will be modified. Moreover, due to the local nature of the mode, it is likely to be unaffected by rotation.

5 NEARLY TOROIDAL FIELDS

In normal conducting stars, the presence of poloidal components in addition to toroidal components may help stabilize the magnetic fields (Tayler 1973; Wright 1973), which has also been confirmed by recent numerical simulations (Braithwaite & Nordlund 2006). Moreover, pulsar observations reveal the presence of a dipole-like field in the neutron star magnetosphere, implying that a poloidal component of the magnetic field must exist. The treatment of fully poloidal fields is considerably more complicated and will be discussed in a subsequent paper. The complication arises as a result of the fact that in the poloidal case the direction of the magnetic field is not known, and must be computed numerically (Roberts 1981).

In this section, we will consider the case when there is a small poloidal component in addition to the much larger toroidal field. We will evaluate the constraints on the shape of the poloidal component that result from the restrictions that the magnetic force per unit mass be expressible as a gradient of a potential and that $\nabla \cdot \boldsymbol{B} = 0$. We will then consider the boundary conditions that must also be satisfied. Let us assume that the direction of the field is given by

$$\hat{\boldsymbol{n}} = \hat{\boldsymbol{\phi}} + \boldsymbol{\varepsilon}, \tag{182}$$

where $\boldsymbol{\varepsilon}$ is a poloidal vector and $|\boldsymbol{\varepsilon}| \ll 1$. In what follows, we will retain only the first-order terms in $|\boldsymbol{\varepsilon}|$.

The form of the magnetic field inside the superconductor is $\boldsymbol{H} = H(r,\theta)(\hat{\boldsymbol{\phi}} + \boldsymbol{\varepsilon})$ and the current density can be written as the sum of toroidal and poloidal components, so instead of equation (13), we now have

$$\begin{aligned} \boldsymbol{J} &= \boldsymbol{J}_{\rm tor} + \boldsymbol{J}_{\rm pol}, \\ \frac{4\pi \boldsymbol{J}_{\rm tor}}{c} &= \nabla \times H\hat{\boldsymbol{\phi}} = \frac{\nabla(Hr\sin\theta) \times \hat{\boldsymbol{\phi}}}{r\sin\theta}, \\ \frac{4\pi \boldsymbol{J}_{\rm pol}}{c} &= \nabla \times H\boldsymbol{\varepsilon}. \end{aligned} \tag{183}$$

Note that $\boldsymbol{J}_{\rm tor}$ (due to the toroidal magnetic field) is a poloidal field and $\boldsymbol{J}_{\rm pol}$ (due to the poloidal magnetic field) is a toroidal field, i.e. $\boldsymbol{J}_{\rm tor} \perp \hat{\boldsymbol{\phi}}$ and $\boldsymbol{J}_{\rm pol} \parallel \hat{\boldsymbol{\phi}}$. Taking the induction to be $\boldsymbol{B} = B(r,\theta)(\hat{\boldsymbol{\phi}} + \boldsymbol{\varepsilon})$, the first term in the force density, given by equation (7), becomes

$$\frac{\boldsymbol{J} \times \boldsymbol{B}}{c} = \frac{\boldsymbol{J}_{\rm tor} \times B\hat{\boldsymbol{\phi}}}{c} + \frac{\boldsymbol{J}_{\rm pol} \times B\hat{\boldsymbol{\phi}}}{c} + \frac{\boldsymbol{J}_{\rm tor} \times B\boldsymbol{\varepsilon}}{c}. \tag{184}$$

The first term is due to the toroidal field, and the second and third terms are due to the presence of the small poloidal component. Since $\boldsymbol{J}_{\rm pol}$ is a toroidal field the second term vanishes. On the other hand, the third term is a cross-product of two poloidal fields, and therefore is a toroidal field. However, we require the toroidal force density to be zero, so it must vanish. This means that $\boldsymbol{\varepsilon} \parallel \boldsymbol{J}_{\rm tor}$, or equivalently, in terms of an arbitrary function λ,

$$B(r,\theta)\boldsymbol{\varepsilon} = \lambda(r,\theta)\boldsymbol{J}_{\rm tor}. \tag{185}$$

Thus, the force is of the same form as in the purely toroidal case, and in order for it to be a gradient, the induction B must still be of the form given by equation (15). We get a condition on the unknown function λ from $\nabla \cdot \boldsymbol{B} = \boldsymbol{J}_{\rm tor} \cdot \nabla\lambda = 0$,

$$\frac{4\pi \boldsymbol{J}_{\rm tor} \cdot \nabla\lambda}{c} = \frac{\hat{\boldsymbol{\phi}} \cdot \nabla\lambda \times \nabla(Hr\sin\theta)}{r\sin\theta} = 0. \tag{186}$$

This equation is satisfied by functions of the form

$$\lambda(r,\theta) = \lambda(Hr\sin\theta). \tag{187}$$

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Thus, the poloidal vector ε is given by equation (185), using equation (15) for B and equation (183) for $\boldsymbol{J}_{\rm tor}$,

$$\varepsilon = \frac{\lambda(Hr\sin\theta)\boldsymbol{J}_{\rm tor}}{4\pi\rho r\sin\theta f(Hr\sin\theta)} = \frac{H^2}{\rho}\nabla\tilde{\lambda}(Hr\sin\theta)\times\hat{\boldsymbol{\phi}}. \tag{188}$$

In a normal conductor, we have, setting $H = B$ and using equation (18) for B,

$$\varepsilon = \frac{\mu(Br\sin\theta)\boldsymbol{J}_{\rm tor}}{B} = \nabla\tilde{\mu}(Br\sin\theta)\times\hat{\boldsymbol{\phi}}. \tag{189}$$

Here μ and $\tilde{\mu}$ are arbitrary functions.

5.1 Boundary conditions

Neglecting second-order terms in the small quantity $|\varepsilon|$ in the magnetic stress tensors for the normal and superconducting regions (equations 3 and 4), the boundary conditions for the continuity of stress (equation 25) become

$$-\delta p_{\rm s} + \sigma_{rr,s} = -\delta p_{\rm n} + \sigma_{rr,n} \quad \text{and} \quad \sigma_{r\phi,s} = \sigma_{r\phi,n}. \tag{190}$$

The *rr* components of the magnetic stress tensors are the same as in the purely toroidal case (equation 29), so the first equation is the same as before (equation 27). However, we now have the second equation, which explicitly gives, using equations (3) and (4) for the stress tensors,

$$(\hat{\boldsymbol{\phi}}\cdot\boldsymbol{H})(\hat{\boldsymbol{r}}\cdot\boldsymbol{B}_{\rm s}) = (\hat{\boldsymbol{\phi}}\cdot\boldsymbol{B}_{\rm n})(\hat{\boldsymbol{r}}\cdot\boldsymbol{B}_{\rm n}). \tag{191}$$

We also have the additional boundary condition on the continuity of the normal component of the poloidal magnetic induction, which follows from Maxwell's equations,

$$\hat{\boldsymbol{r}}\cdot\boldsymbol{B}_{\rm s} = \hat{\boldsymbol{r}}\cdot\boldsymbol{B}_{\rm n}. \tag{192}$$

The last two equations imply that we must have

$$\hat{\boldsymbol{\phi}}\cdot\boldsymbol{H} = \hat{\boldsymbol{\phi}}\cdot\boldsymbol{B}_{\rm n}, \quad \text{i.e.} \quad H = B_{\rm n}. \tag{193}$$

This is equivalent to the requirement for the continuity of the $\hat{\phi}$ component of the magnetic field in the absence of surface currents (equation 20). However, as was previously discussed, this is inconsistent with our assumption that H is a function of radius up to the boundaries of the superconductor. This assumption now requires the presence of a discontinuity in the $\hat{\phi}$ component of the magnetic force, although the forces within the superconducting and normal regions have no such components. This is an artefact of the incomplete description of the transition boundary, which we have treated as discontinuous. A more realistic treatment should impose zero toroidal force everywhere.

Incidentally, note that we cannot simply assume that the radial components of the poloidal vectors vanish at the boundary, which would also satisfy the above equations (equations 191 and 192). This would imply that the functions $\tilde{\lambda}$ and $\tilde{\mu}$ in equations (188) and (189) are constants, which in turn would cause the poloidal vectors to vanish everywhere within the normal and superconducting regions.

6 CONCLUSION

Our main goal in this paper has been to compute the distortion of a neutron star due to a toroidal magnetic field in its interior, assuming that the star is either partly or entirely a type II superconductor. Previous authors have estimated the order of magnitude of this distortion (Jones 1975; Easson & Pethick 1977; Cutler 2002), finding that it is enhanced by a factor H/B for given magnetic induction B and magnetic field H compared with the normal case (where $H = B$). In the strongly type II regime, $H \sim 10^{15}$ G, so that $H/B \sim 10^3/B_{12}$ (Jones 1975; Easson & Pethick 1977). Such large enhancements could result in magnetic distortions $\epsilon \sim 10^{-9}$ to 10^{-8}, which are large enough to be important for neutron star precession (Wasserman 2003) and possibly for gravitational radiation emission (Cutler 2002). These earlier works did not compute the structure of the magnetic field in detail.

Here, we have paid closer attention to the requirements of hydrostatic balance and stability. The assumption of a barotropic equation of state, $p = p(\rho)$, which ought to apply to a cold neutron star, severely constrains the structure of the toroidal field. Similar restrictions have been known for a long time for normal conductors (e.g. Prendergast 1956; Monaghan 1965). The restrictions arise because the magnetic acceleration must be a total gradient in hydrostatic balance. Under the assumption that the magnetic free energy F is a function of (matter or baryon) density ρ and magnetic induction B, we find that, for toroidal fields, we must require (equation 15)

$$B(r,\theta) \propto \rho r\sin\theta f(Hr\sin\theta), \tag{194}$$

where f is an arbitrary function. Given this function, and $F(\rho, B)$, we can compute $H(\rho, B) = 4\pi\partial F/\partial B$ (equation 2). Equation (194) is then an implicit equation that can be used to find $B(r, \theta)$ (assuming axisymmetry). Similar constraints can be derived for poloidal magnetic fields, but are more complicated since the field direction must be solved for (e.g. Roberts 1981 for superconducting, uniform density stars; we will consider superconducting, barotropic stars in a future paper).

Our calculations have concentrated on neutron stars with a strongly type II regime where H is independent of B; our models allow for as many as two normal regimes interior or exterior to the superconductor. The main result of these calculations is equation (71) for the magnetic distortion,

$$\epsilon = 0.945\times10^{-9}\left(\frac{\phi_2(R_\star)}{\Psi_{\rm o}}\right)\left(\frac{H_{\rm c}}{10^{15}\,{\rm G}}\right)\left(\frac{B_{\rm o}}{10^{12}\,{\rm G}}\right)\left(\frac{R_\star}{10\,{\rm km}}\right)^4\left(\frac{M_\star}{1.4\,{\rm M}_\odot}\right)^{-2}, \tag{195}$$

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with $\phi_2(R_\star)/\Psi_o \approx -2$ in all cases, as is summarized in Table 1. These results were computed for an equation of state $p = \kappa\rho^2$ and $H \propto \rho$ (e.g. Easson & Pethick 1977). Calculations can be done in a similar way for other $p(\rho)$ and $H(\rho, B)$.

Although we have separated the star into strongly type II and normal sectors for computing the deformations due to a toroidal field, we have noted that this assumption, while mathematically well defined, leads to sudden jumps in density and magnetic induction at the boundaries of the superconductor. In effect, we have assumed that the magnetic free energy changes discontinuously from $F = H(\rho)B/4\pi$ in the type II superconductor to $F = B^2/8\pi$ in the normal conductor. However, our formalism can be applied more generally to $F(\rho, B)$ that varies smoothly from type II to normal, probably with intermediate domains of type I superconductivity. Such models ought to be free of discontinuities in ρ and B, but will still have rapid variations in radially thin domains. In particular, we expect magnetic stresses to be approximately continuous across boundaries, so the magnetic induction B_n in the normal regions will be larger than the induction B_s in the superconductor, $B_\mathrm{n} \sim (HB_\mathrm{s})^{1/2} \gg B_\mathrm{s}$. Strong toroidal fields $B_\mathrm{n} \sim 10^{13.5}$ G (corresponding to $H \sim 10^{15}$ G and $B_\mathrm{s} \sim 10^{12}$ G) are needed for large distortions; toroidal fields $B_\mathrm{n} \sim 10^{12}$ G imply $B_\mathrm{s} \sim 10^9$ G and therefore will lead to $\epsilon \sim 10^{-12}$. We have postponed considering models with realistic $F(\rho, B)$, which would be more intricate mathematically, to later work.

A toroidal field can be produced as a result of the winding up of the magnetic field early in the history of a neutron star (Thompson & Duncan 2001). The resulting field could be stronger than 10^{12} G. When the star has cooled down sufficiently, the superconducting shell forms. This would produce a large stress within the superconductor and the star would become dynamically unstable. This, in turn, would lead to a lowering of the induction inside the superconductor until stability can be restored. In equilibrium, the stresses within the superconductor and the normal regions will be comparable. In other words, the amplitude of the magnetic stress may be fixed by the original amplification of the toroidal field. The superconductor adjusts to the requirement of approximately continuous stress by lowering B_s. In this sense, the superconductor does not really amplify the stress.

Magnetic fields not only need to be in magnetohydrostatic equilibrium, but they must also be stable with respect to perturbations. We have derived stability criteria from an energy principle for generic $F(\rho, B)$. This is more general than the treatment of Roberts (1981), who assumed $H \propto \rho$, and it also includes the normal case treated previously by Tayler (1973) as the special case $H = B$. In a completely type II superconducting star with $H \propto \rho$ and $B \propto \sin^n\theta$ (equation 35), we find that only $n = 1$ is stable to $m = 0$ (axisymmetric) perturbations. In fact, as we show in Appendix A, this is true for any magnetic field of the form $H(\rho, B)$. For $m \neq 0$ all field configurations in a type II star are prone to the MPR instability, found by Muzikar & Pethick (1981) and Roberts (1981), when $B \lesssim 10^{13}$ G. There is also a minimum wavenumber for instability, and it is very large: the MPR instability is a small scale instability. From a linear perturbation analysis around a uniform background, we find that the instability has a length-scale $\sim 10^{-4}\, R_\star$, where $R_\star$ is the stellar radius, and a time-scale $\sim 10^3$ s. This time-scale is relatively long compared to an Alfvén crossing time $t_\mathrm{A} = R_\star(4\pi\rho/HB)^{1/2} \approx 3.5\, R_6(\rho_{15}/H_{15}B_{12})^{1/2}$ s, but short compared to a typical precession period of the order of a year. We have also argued that the MPR instability cannot occur for $m = 0$ in toroidal fields: our linear analysis implies zero growth rate for modes with wave vectors entirely orthogonal to the unperturbed magnetic field. Because of the large wavenumbers required for the instability, viscous effects, which cannot be studied via stability analyses from energy principles, could prevent it from occurring altogether. Our estimate is that a kinematic viscosity of $\sim$10–100 cm^2 s^{-1} would be enough to shut off the instability; this value is smaller than most estimates of the kinematic viscosity in neutron star matter (Andersson et al. 2005).

We find that normal toroidal fields with $B \propto \sin^{n/2}\theta$ (equation 41) are unstable for $m = 1$. Therefore, toroidal fields in a star with normal and superconducting regions will be unstable. Poloidal fields may help stabilize the stellar magnetic field, as has been found for normal conductors (e.g. Tayler 1973; Wright 1973; Braithwaite & Nordlund 2006). Moreover, the emission from radio pulsars additionally requires exterior, poloidal fields. Consequently, we have also considered nearly toroidal fields in which the field direction is $\hat{\boldsymbol{\phi}} + \boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon} \perp \hat{\boldsymbol{\phi}}$ and $|\boldsymbol{\varepsilon}| \ll 1$. Here, too, the form of $\boldsymbol{\varepsilon}$ is not completely arbitrary: to maintain hydrostatic balance and eliminate toroidal forces, we find the requirement (equation 188),

$$\boldsymbol{\varepsilon} = \frac{H^2}{\rho}\nabla\tilde{\lambda}(Hr\sin\theta) \times \hat{\boldsymbol{\phi}}, \tag{196}$$

where $\tilde{\lambda}$ is an arbitrary function. We derived equation (196) for type II regimes, but it holds elsewhere (in particular, in normal regions). We have seen, though, that when we assume discontinuous transitions in the magnetic free energy between type II and normal regions, there are discontinuities in the $r\phi$ component of the magnetic stress tensor, implying a surface toroidal force. A more complete treatment with continuously varying $F(\rho, B)$ would not have such surface forces since equation (196) would then guarantee vanishing toroidal forces everywhere.

The results found here can be applied directly to precession of neutron stars. For fluid stars, Spitzer (1958) argued that precession is inevitable if the magnetic and rotational axes are misaligned; Mestel & Takhar (1972) showed that the star precesses about its magnetic symmetry axis with a period $P_\mathrm{p} = P_\star/3\epsilon_\mathrm{mag}\cos\chi$ where χ is the misalignment angle. For a radio pulsar, there would be no effect on the arrival times of pulses if the pulsar beam is along the magnetic axis of the star. Wasserman (2003) showed that crustal distortions with a symmetry axis that is also misaligned with the magnetic axis would lead to periodically varying timing residuals. For PSR B1828−11, spindown can enhance the effect considerably, and the data can be accounted for with $B \sim 10^{12}$–10^{13} G, $\chi \sim 1$ rad, and a modest permanent crustal distortion $\sim$0.01 times the magnetic distortion. (Perhaps fortuitously, this is close to the crustal distortion found by Cutler et al. 2003 for relaxation near the actual rotation frequency of PSR B1828−11.) The model favours prolate figures (see also Akgün et al. 2006), as would be expected from (predominantly) toroidal fields. Why the magnetic and spin axes are misaligned remains unexplained. Moreover, the effects of the slow, time

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variable fluid motions that would be required in such a model (e.g. Mestel & Takhar 1972; Mestel et al. 1981; Nittmann & Wood 1981) have yet to be computed.

In this paper, we have not examined the effects of rotation, internal velocity fields, multifluid components, drag and dissipation. These will likely introduce new modes and will alter the properties of modes of non-rotating stars.

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APPENDIX A: STABILITY CRITERIA FOR A MAGNETIC FIELD $H(\rho, B)$

The coefficients for $m = 0$ for a magnetic free energy $F(\rho, B)$ are given by equation (108), where the various quantities are defined in equations (104) and (107). The hydrostatic equilibrium in the absence of magnetic fields is spherically symmetric, $\partial_r p_o + \rho_o \partial_r \phi_o = 0$. In the presence of magnetic fields, the equilibrium is given by equation (96). Using these equations, we can rewrite the coefficients as, to lowest order in F,

$$a_0 \approx T_0 \left(\frac{\mathrm{d}\ln\rho}{\mathrm{d}r}\right)^2 + T_1 \frac{\mathrm{d}\ln\rho}{\mathrm{d}r} - U_1 + U_3,$$
$$b_0 \approx T_2 \frac{\mathrm{d}\ln\rho}{\mathrm{d}r} - U_1 \cot\theta - U_2 + 2U_3 \cot\theta,$$
$$c_0 \approx -U_2 \cot\theta + U_3 \cot^2\theta, \tag{A1}$$

where

$$T_0 = B^2 F_{,BB} + 2\rho B F_{,\rho B} + \rho^2 F_{,\rho\rho},$$
$$T_1 = 2U_0 - \frac{BF_{,B}}{r} - B\partial_r B F_{,BB} - B\partial_r \rho F_{,\rho B} - \rho\partial_r B F_{,\rho B} - \rho\partial_r\rho F_{,\rho\rho},$$
$$T_2 = 2U_0 \cot\theta - \frac{1}{r}(BF_{,B}\cot\theta + B\partial_\theta B F_{,BB} + B\partial_\theta\rho F_{,\rho B} + \rho\partial_\theta B F_{,\rho B} + \rho\partial_\theta\rho F_{,\rho\rho}). \tag{A2}$$

Consider the case of a magnetic field $H(\rho, B)$. In this case, the magnetic free energy $F(\rho, B)$ is given through $H = 4\pi F_{,B}$ (equation 2). To lowest order in F, the density is a function of radius, $\rho(r)$. Therefore, partial derivatives of ρ with respect to the angle θ can be dropped. Then, equation (A1) can be written equivalently as

$$a_0 \approx \frac{1}{r^2}\left[Q_1 \left(\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln r}\right)^2 + Q_2 \frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln r} + Q_3\right],$$
$$b_0 \approx \frac{\cot\theta}{r^2}\left[Q_4 \frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln r} + Q_3 + Q_5\right],$$
$$c_0 \approx \frac{\cot^2\theta}{r^2} Q_5, \tag{A3}$$

where, we define

$$Q_0 = BF_{,B} + B^2 F_{,BB} \qquad Q_2 = Q_0 + Q_1\left(1 - \frac{\partial\ln B}{\partial\ln r}\right) \qquad Q_4 = Q_0 + Q_1\left(1 - \frac{\partial\ln B}{\partial\ln\sin\theta}\right)$$
$$Q_1 = B^2 F_{,BB} + \rho B F_{,\rho B} \quad Q_3 = Q_0\left(1 - \frac{\partial\ln B}{\partial\ln r}\right) \qquad Q_5 = Q_0\left(1 - \frac{\partial\ln B}{\partial\ln\sin\theta}\right) \tag{A4}$$

The magnetic induction is given by equation (15), which we can rewrite in terms of a new arbitrary function g as

$$B(r, \theta) = \frac{\rho g(Hr\sin\theta)}{H}. \tag{A5}$$

Let $\zeta = Hr\sin\theta$ be the argument of the function g, and define

$$\eta = \frac{\mathrm{d}\ln g}{\mathrm{d}\ln\zeta}, \quad \xi = \frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln r}, \quad \sigma_\rho = \frac{\partial\ln H}{\partial\ln\rho} \quad \text{and} \quad \sigma_B = \frac{\partial\ln H}{\partial\ln B}. \tag{A6}$$

Then, after some algebra it follows that

$$\frac{\partial\ln B}{\partial\ln r} = \frac{\xi(1-\sigma_\rho) + \eta(1+\xi\sigma_\rho)}{1+\sigma_B(1-\eta)} \quad \text{and} \quad \frac{\partial\ln B}{\partial\ln\sin\theta} = \frac{\eta}{1+\sigma_B(1-\eta)}. \tag{A7}$$

Using $H = 4\pi F_{,B}$, we also get

$$Q_0 = \frac{HB}{4\pi}(1+\sigma_B) \quad \text{and} \quad Q_1 = \frac{HB}{4\pi}(\sigma_\rho + \sigma_B). \tag{A8}$$

Then, the coefficients become (from equation A3)

$$a_0 \approx \frac{HB}{4\pi r^2}\frac{[1+\sigma_B+\xi(\sigma_\rho+\sigma_B)]^2(1-\eta)}{1+\sigma_B(1-\eta)},$$
$$b_0 \approx \frac{HB\cot\theta}{2\pi r^2}\frac{[1+\sigma_B+\xi(\sigma_\rho+\sigma_B)](1+\sigma_B)(1-\eta)}{1+\sigma_B(1-\eta)},$$
$$c_0 \approx \frac{HB\cot^2\theta}{4\pi r^2}\frac{(1+\sigma_B)^2(1-\eta)}{1+\sigma_B(1-\eta)}. \tag{A9}$$

Since $b_0^2 = 4a_0c_0$, one of the stability conditions is immediately marginally satisfied. The other two conditions give

$$\frac{1-\eta}{1+\sigma_B(1-\eta)} > 0. \tag{A10}$$

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Using equation (A7) we can rewrite this condition as, for $1 + \sigma_B > 0$,

$$\frac{\partial \ln B}{\partial \ln \sin\theta} < 1. \tag{A11}$$

Thus, the magnetic fields are marginally stable for $B \propto \sin\theta$.

For a strongly type II superconducting star $H = H(\rho)$, so that $\sigma_B = 0$, and equation (A10) reduces to $\eta < 1$. For a normal conducting star $H = B$, so that $\sigma_B = 1$, and we get $(1 - \eta)/(2 - \eta) > 0$. This condition can be expressed in an alternative way by noting that equation (A5) for a normal conductor is $B = \rho g(Br \sin\theta)/B$. Thus, B is given as a function of itself. This equation can be rewritten as $B = h(\rho r^2 \sin^2\theta)/r \sin\theta$, and the magnetic free energy is given by $F = B^2/8\pi = \rho f(\rho r^2 \sin^2\theta)$, where h and f are arbitrary functions. From here and from equation (A7) it follows that, defining $\varpi = \rho r^2 \sin^2\theta$,

$$\frac{\partial \ln B}{\partial \ln \sin\theta} = \frac{\mathrm{d} \ln f}{\mathrm{d} \ln \varpi} = \frac{\eta}{2 - \eta}. \tag{A12}$$

The same result is obtained by considering the derivative of B with respect to r, though it involves more algebra. Thus, the stability condition for the normal conducting case is better expressed as

$$\frac{\mathrm{d} \ln f}{\mathrm{d} \ln \varpi} < 1. \tag{A13}$$

For a normal conducting star, the field is marginally stable for $f \propto \varpi$, i.e. $B \propto \rho r \sin\theta$, as noted in Section 4.4. Similarly, for a strongly type II superconducting star, the field is marginally stable for $g \propto \zeta$, i.e. $B \propto \rho r \sin\theta$, as noted in Section 4.5.

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