TOROIDAL WAVE FUNCTIONS*

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Abstract. The Helmholtz equation is solved in toroidal coordinates. A complete set of solutions is obtained representing radiations from a ring source.

Introduction. Up to now the Helmholtz equation has been solved only for separable coordinate systems. This paper presents solutions for a non-separable coordinate system, the toroidal. The main interest will be with continuous, single-valued solutions satisfying the radiation condition and possessing a ring singularity. Exact expressions for each of the wave functions will be obtained in the form of a series expansion and integral over finite range. The series expansions are uniformly convergent everywhere in space except, of course, at the ring singularity.

The time dependence will be of the form exp $(-i\omega t)$.

1. Series solution of the Helmholtz equation in toroidal coordinates. The relation between toroidal and cartesian coordinates systems is given by [5, p. 151]

$$
\begin{aligned}
x &= \frac{d \sinh \xi \cos \phi}{\cosh - \cos \eta}, \\
y &= \frac{d \sinh \xi \sin \phi}{\cosh - \cos \eta}, \\
z &= \frac{d \sin \eta}{\cosh \xi - \cos \eta}.
\end{aligned}
\tag{1.1}
$$

Domains of the coordinates are $0 \leq \eta \leq 2\pi$, $0 \leq \phi \leq 2\pi$, $0 \leq \xi \leq \infty$ where $\xi = \xi_0$ defines a torus

$$z^2 + (\rho - d \coth \xi_0)^2 = d^2 \operatorname{csch}^2 \xi_0$$

and $\eta = \eta_0$ defines a sphere

$$(z - d \cot \eta_0)^2 + \rho^2 = d^2 \csc \eta_0^2$$

where $\rho = (x^2 + y^2)^{1/2}$.

The metric coefficients are given by the following relations

$$
\begin{aligned}
h_\xi = h_\eta &= \frac{d}{\cosh \xi - \cos \eta}, \\
h_\phi &= \frac{d \sinh \xi}{\cosh \xi - \cos \eta}.
\end{aligned}
\tag{1.2}
$$

From now on, in order to facilitate analysis, the variable s will be used instead of ξ where the two are related by the equation $\cosh \xi = s$.

Now it has been shown [7], that for a certain class of non-separable rotational coordinates (u_1, u_2, ϕ), there are solutions of

$$\nabla^2 \psi + k^2 \psi = 0 \tag{1.3}$$

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given by

$$\psi(u_1, u_2, \phi) = e^{i\mu\phi} \sum_r a_r(u_i)[h_3(u_1, u_2)]^r$$
$$\psi(u_1, u_2, \phi) = e^{i\mu\phi}B(u_j) \sum_r b_r(u_i)[h_3(u_1, u_2)]^r \tag{1.4}$$

where $i \neq 3$, and $i \neq j \neq 3$, provided that the metric coefficients h_3 and h_i, and the function $B(u_j)$ satisfy certain conditions. The coefficients $a_r(u_i)$ and $b_r(u_i)$ of the above series satisfy a recurrence set of ordinary differential equations.

In particular, the toroidal coordinates (ξ, η, ϕ) belong to this rotational class and the solutions are such that the power series given by (1.4) have a lower termination. If the power series are expressed in the variable $(s - \cos \eta)^{-1}$ instead of h_ϕ the differential equations involving the coefficients of the power series take a simpler form. The function $B(\eta)$ for toroidal coordinates is $\sin \eta$. Solutions of (1.3) are given by

$$\psi_e(s, \eta, \phi) = e^{i\mu\phi} \sum_{r=T}^{\infty} A_r(s)\,(s - \cos \eta)^{-r} \tag{1.5}$$

and

$$\psi_0(s, \eta, \phi) = e^{i\mu\phi} \sin \eta \sum_{r=T'}^{\infty} B_r(s)\,(s - \cos \eta)^{-r}. \tag{1.6}$$

The coefficients must satisfy the differential equations

$$(s^2 - 1)A_r'' + 2sA_r' - A_r\left[r(r+1) + \frac{\mu^2}{s^2 - 1}\right] + k^2 d^2 A_{r-2}$$
$$- (2r - 1)[(s^2 - 1)A_{r-1}' - s(r - 1)A_{r-1}] = 0, \tag{1.7}$$

$$(s^2 - 1)B_r'' + 2sB_r' - B_r\left[r(r-1) + \frac{\mu^2}{s^2 - 1}\right] + k^2 d^2 B_{r-2}$$
$$- (2r - 1)[(s^2 - 1)B_{r-1}' - s(r - 2)B_{r-1}] = 0, \tag{1.8}$$

where the prime denotes differentiation with respect to s. The numbers T and T' are determined from the boundary conditions. The problem of solving a partial differential equation in three variables in which only one variable is separable is reduced to solving a recurrence set of ordinary differential equations in one variable.

The homogeneous equation corresponding to the equation (1.7) is

$$(s^2 - 1)\omega'' + 2s\omega' - r(r + 1)\omega - \frac{\mu^2}{s^2 - 1}\omega = 0 \tag{1.9}$$

of which, the associated Legendre functions $P_r^\mu(s)$ and $Q_r^\mu(s)$ are solutions. The function $\omega_r^\mu(s)$ will be used to represent both solutions of (1.9).

It is immediately evident that for $r = T$ in (1.7) and $r = T'$ in (1.8), the non-homogeneous portion of the equations vanish. Hence one has $A_T = \omega_T^\mu(s)$ and $B_{T'} = \omega_{T'-1}^\mu(s)$. Since the solutions of the corresponding homogeneous equations of (1.7) and (1.8) are known, the equations can be solved easily, resulting in the following relations,

$$A_r(s) = \omega_r^\mu(s) \int_{c_1}^{s} \frac{dz}{(1 - z^2)[\omega_r^\mu(z)]^2} \int_{c_2}^{z} F_r(x)\omega_r^\mu(x)\, dx \tag{1.10}$$

and

$$B_r(s) = \omega^{\mu}_{r-1}(s) \int_{c_1'}^{s} \frac{dz}{(1 - z^2)[\omega^{\mu}_{r-1}(z)]^2} \int_{c_2'}^{z} G_r(x)\omega^{\mu}_{r-1}(x)\, dx, \tag{1.11}$$

where

$$F_r(s) = k^2 d^2 A_{r-2} - (2r - 1)[(s^2 - 1)A'_{r-1} - s(r - 1)A_{r-1}] \tag{1.12}$$

$$G_r(s) = k^2 d^2 B_{r-2} - (2r - 1)[(s^2 - 1)B'_{r-1} - s(r - 2)B_{r-1}]. \tag{1.13}$$

The constants c_1, c_2, c_1' and c_2' are determined from the boundary conditions.

Define a basic set of solutions $\psi^{\mu}_{eT}(P)$ and $\psi^{\mu}_{eT}(Q)$ such that $A_T(s)$ is equal to $P^{\mu}_T(s)$ and $Q^{\mu}_T(s)$ respectively and the constants of integration in the expression for $A_r(s)$ are taken as fixed numbers.

Let $\psi^{\mu}_{eT}(s, \eta, \phi)$ be a solution of Eq. (1.3) of the form (1.5) with arbitrary constants of integration and with $A_T(s) = a_1 P^{\mu}_T(s) + b_1 Q^{\mu}_T(s)$. Then

$$\psi^{\mu}_{eT}(s, \eta, \phi) - a_1\psi^{\mu}_{eT}(P) - b_1\psi^{\mu}_{eT}(Q)$$

is a solution of Eq. (1.3), with the coefficient of $(s - \cos\eta)^{-T}$ vanishing. Thus we have

$$\psi^{\mu}_{eT}(s, \eta, \phi) - a_1\psi^{\mu}_{eT}(P) - b_1\psi^{\mu}_{eT}(Q) = \psi^{\mu}_{eT+1}(s, \eta, \phi),$$

where $\psi^{\mu}_{eT+1}(s, \eta, \phi)$ is a solution of Eq. (1.3) of the form (1.5) where the lower limit of summation is $T + 1$. The coefficient of $(s - \cos\eta)^{-T-1}$ in $\psi^{\mu}_{eT+1}(s, \eta, \phi)$ must be of the form $a_2 P^{\mu}_{T+1}(s) + b_2 Q^{\mu}_{T+1}(s)$. Hence in a similar manner we have

$$\psi^{\mu}_{eT}(s, \eta, \phi) - a_1\psi^{\mu}_{eT}(P) - a_2\psi^{\mu}_{eT+1}(P) - b_1\psi^{\mu}_{eT}(Q) - b_2\psi^{\mu}_{eT+1}(Q) = \psi^{\mu}_{eT+2}(s, \eta, \phi).$$

By mathematical induction we see that any solution $\psi^{\mu}_{eT}(s, \eta, \phi)$ with arbitrary constants of integration in the expressions for $A_r(s)$, is just a linear combination of $\psi^{\mu}_{ep}(P)$ and $\psi^{\mu}_{ep}(Q)$ where $p = T, T + 1, T + 2, \cdots$. A similar discussion follows for the solutions of the form $\psi^{\mu}_{0T'}(s, \eta, \phi)$. So no restriction is placed if we take the constants in the integrals (1.10) and (1.11) as pre-determined fixed numbers, thus allowing us to obtain explicit expressions for a set of solutions of (1.3).

Any solutions of the form $\{\psi^{\mu}_{eT}(s, \eta, \phi) + \psi^{\mu}_{0T'}(s, \eta, \phi)\}$ with arbitrary constants of integration can be expressed as a linear combination of the explicit solutions $\psi^{\mu}_{ep}(P)$, $\psi^{\mu}_{ep}(Q)$, $\psi^{\mu}_{0p'}(P)$ and $\psi^{\mu}_{0p'}(Q)$ where $p = T, T + 1, \cdots$ and $p' = T', T' + 1, \cdots$.

From now on we shall be interested in solutions periodic in the angle ϕ. Hence we set $\mu = m$ where $m = 0, \pm 1, \pm 2, \cdots$.

2. Obtaining explicit expressions for ψ^{m}_{eT}(P) and $\psi^{m}_{0T'}$(P). The wave functions $\psi^{m}_{eT}(P)$ and $\psi^{m}_{oT'}(P)$ are defined as those solutions satisfying (1.5) and (1.6) respectively where the coefficients $A_r(s)$ and $B_r(s)$ are given by (1.10) and (1.11) with the constants c_1, c_2, c_1', c_2' all set equal to unity, and $A_T(s)$ and $B_{T'}(s)$ equal to $P^{-|m|}_T(s)$ and $P^{-|m|}_{T'-1}(s)$ respectively. The negative superscript is taken so that our results include the case in which T and $T' - 1$ are integers such that $|T| < |m|$ and $|T' - 1| < |m|$ when the associated Legendre functions $P^{|m|}_T(s)$ and $P^{|m|}_{T'-1}$ do not exist.

For convenience the symbol M will be used to signify $|m|$, where m is a positive or negative integer or zero, i.e.

$$M = |m| = 0, 1, 2, 3, \cdots \tag{2.1}$$

The integral operator $I(M, r)$ operating on the function $F(s)$ is defined as follows:

$$I(M, r)F(s) = \omega_r^M(s) \int_1^s \frac{dz}{(1 - z^2)[\omega_r^M(z)]^2} \int_1^z \omega_r^M(x)F(x)\, dx. \tag{2.2}$$

Hence the coefficients $A_r(s)$ and $B_r(s)$ for $\psi_{eT}^m(P)$ and $\psi_{0T'}^m(P)$ satisfy the relations

$$A_r(s) = I(M, r)F_r(s), \tag{2.3}$$

$$B_r(s) = I(M, r - 1)G_r(s), \tag{2.4}$$

where $F_r(s)$ and $G_r(s)$ are defined by (1.12) and (1.13). To calculate $A_r(s)$ and $B_r(s)$ the following lemma* is needed:

Lemma: If

$$G(M, X, X - R) = (s^2 - 1)^{X/2} P_{X-R-1}^{-M-X}(s), \tag{2.5}$$

where X is a positive, and M a non-negative integer and if

$$F(s) = a\left[(s^2 - 1)\frac{d}{dx} G(M, X - 1, X - R) - s(R - 1)G(M, X - 1, X - R)\right] + bG(M, X - 1, X - R + 1), \tag{2.6}$$

then

$$I(M, R)F(s) = -[a(M + 2X - R - 1) + b]G(M, X, X - R)[2X]^{-1}. \tag{2.7}$$

Now $A_T(s) = P_T^{-M}(s) = P_{-T-1}^{-M}(s) = G(M, 0, -T)$ hence setting $r = T + 1$ in (1.13) and using the fact that $A_{T-1}(s) \equiv 0$, one has

$$F_{T+1}(s) = -(2T + 1)\left[(s^2 - 1)\frac{d}{ds} G(M, 0, -T) - sTG(M, 0, -T)\right]. \tag{2.8}$$

But

$$A_{T+1}(s) = I(M, T + 1)F_{T+1}(s). \tag{2.9}$$

Now $F_{T+1}(s)$ corresponds to expression (2.6) where

$$a = -(2T + 1), \qquad b = 0, \qquad X = 1, \qquad R = T + 1,$$

hence the lemma gives A_{T+1} from (2.9) and (2.7), resulting in the following relation

$$\begin{aligned} A_{T+1} &= -2^{-1}[-(2T + 1)(M - T)]G(M, 1, -T) \\ &= (T + \tfrac{1}{2})(M - T)(s^2 - 1)^{1/2} P_{-T-1}^{-M-1}(s). \end{aligned} \tag{2.10}$$

To obtain $A_{T+2}(s)$ one must break the operation

$$A_{T+2}(s) = I(M, T + 2)F_{T+2}(s) \tag{2.11}$$

into two separate integrals, i.e. $F_{T+2}(s)$ must be broken up into the following two expressions

$$-(2T + 3)[(s^2 - 1)A'_{T+1} - s(T + 1)A_{T+1}] \tag{2.12}$$

*This is a consequence of Lemma 2, page 13, [6].

and

$$k^2 d^2 A_T . \tag{2.13}$$

Expression (2.12) reduces to the form given by (2.6) where

$$a = -(2T + 3)(T + \tfrac{1}{2})(M - T), \qquad b = 0, \qquad X = 2, \qquad R = T + 2$$

and (2.13) reduces to the form given by (2.6) where

$$a = 0, \qquad b = k^2 d^2, \qquad X = 1, \qquad R = T + 2.$$

Hence using (2.11) and (2.7) one obtains

$$\begin{aligned} A_{T+2} &= -[-(2T + 3)(T + \tfrac{1}{2})(M - T)(M - T + 1)]G(M, 2, -T)4^{-1} \\ &\qquad - k^2 d^2 G(M, 1, -T - 1)2^{-1} \\ &= \tfrac{1}{2}(T + 3/2)(T + \tfrac{1}{2})(M - T)(M - T + 1)(s^2 - 1)P_{-T-1}^{-M-2}(s) \\ &\qquad - \frac{k^2 d^2}{2}(s^2 - 1)^{1/2}P_{-T-2}^{-M-1}(s). \end{aligned} \tag{2.14}$$

Rather than calculate the remaining $A_r(s)$ in a similar manner it is better to obtain them through mathematical induction. Assume that

$$A_r(s) = \sum_{t=0}^{[(r-T)/2]} (k\,d)^{2t} A_r^t G(M, r - T - t, -T - t), \tag{2.15}$$

where A_r^t are constants. In the expression for the upper limit of summation, the following notation is used: $[x]$ is the integer such that $x - 1 < [x] \le x$. The above expression obviously holds for $r = T, T + 1$ and $T + 2$.

Assume that the expression holds for $r - 1$ and $r - 2$, hence in order for it to hold for r one must have

$$A_r(s) = I(M, r)F_r(s), \tag{2.16}$$

where from (1.13) and (2.15), $F_r(s)$ becomes

$$\begin{aligned} F_r(s) = k^2 d^2 & \sum_{t=0}^{[(r-T-2)/2]} (k\,d)^{2t} A_{r-2}^t G(M, r - 2 - T - t, -T - t) \\ & - (2r - 1)\left[(s^2 - 1)\frac{d}{ds} - s(r - 1)\right] \\ & \sum_{t=0}^{[(r-T-1)/2]} (k\,d)^{2t} A_{r-1}^t G(M, r - 1 - T - t, -T - t). \end{aligned}$$

The coefficient of $(kd)^{2t}$ in $F_r(s)$ is

$$\begin{aligned} & A_{r-2}^{t-1} G(M, r - 1 - T - t, -T - t + 1) \\ & \qquad - (2r - 1)\left[(s^2 - 1)\frac{d}{ds} - s(r - 1)\right]A_{r-1}^t G(M, r - 1 - T - t, -T - t). \end{aligned}$$

Hence the coefficient of $(kd)^{2t}$ in $I(M, r)F_r(s)$ is, on setting $a = -(2r - 1)A_{r-1}^t$, $b = A_{r-2}^{t-1}$, $X = r - T - t$, $R = r$ in expression (2.6) and using (2.7)

$$[(M - 2T - 2t + r - 1)(2r - 1)A_{r-1}^t - A_{r-2}^{t-1}]\frac{G(M, r - T - t, -T - t)}{2(r - T - t)}. \tag{2.17}$$

Equating coefficients of $(kd)^{2t}$ in (2.16) one obtains the relation

$$A_r^t G(M, r - T - t, - T - t) = [(M - 2T - 2t + r - 1)(2r - 1)A_{r-1}^t - A_{r-2}^{t-1}] \frac{G(M, r - T - t, - T - t)}{2(r - T - t)} \tag{2.18}$$

which shows that the expression (2.15) holds for r if it holds for $r - 1$ and $r - 2$, provided that the constants A_r^t satisfy the equality

$$A_r^t = [(M - 2t - 2T + r - 1)(2r - 1)A_{r-1}^t - A_{r-2}^{t-1}]2^{-1}(r - T - t)^{-1}. \tag{2.19}$$

Thus one obtains

$$A_r^t = \frac{(-1)^{t+r-T}\Gamma(- T - t + \frac{1}{2})\Gamma(- 2T + r - 2t + M)}{2^t(r - T - 2t)!(t)!\Gamma(- r + t + \frac{1}{2})\Gamma(M - T)}. \tag{2.20}$$

This expression for the constants A_r^t holds for $r = T, T + 1, T + 2$ as is seen when comparing the values given by (2.20) for the cases $r = T + 1, t = 0; r = T + 2, t = 0;$ and $r = T + 2, t = 1$ with values of constants in equations (2.10) and (2.14). Since the expression (2.15) holds for $r = T, T + 1, T + 2$ and it has been shown that if it holds for $r - 1, r - 2$ then it will hold for r, one can, by mathematical induction, conclude that (2.15) holds for every r.

Hence one can immediately write

$$\psi_{eT}^m(P) = e^{im\phi} \sum_{r=T}^{\infty} (s - \cos \eta)^{-r} \sum_{t=0}^{[(r-T)/2]} (k\,d)^{2t} A_r^t(s^2 - 1)^{(r-T-t)/2} P_{-T-t-1}^{-M-r+T+t}(s). \tag{2.21}$$

Noting that $B_{T'}(s) = P_{-T'}^{-M}(s)$ we can obtain in a manner similar to the above the following explicit expression for $\psi_{0T'}^m(P)$

$$\psi_{0T'}^m(P) = e^{im\phi} \sin \eta \sum_{r=T'}^{\infty} (s - \cos \eta)^{-r} \sum_{t=0}^{[(r-T')/2]} (k\,d)^{2t} \cdot B_r^t(s^2 - 1)^{(r-T'-t)/2} P_{-T'-t}^{-M-r+T'+t}(s), \tag{2.22}$$

where the constants B_r^t are given by

$$B_r^t = \frac{(-1)^{t+r-T'}\Gamma(- T' - t + .5)\Gamma(- 2T' + r - 2t + M + 1)}{2^t(t)!(r - T' - 2t)!\Gamma(- r + t + .5)\Gamma(M - T' + 1)}. \tag{2.23}$$

The expressions (2.21) and (2.22) can be placed in a neater form. In (2.21), interchange the order of summation of r and t, obtaining

$$\sum_{r=T}^{\infty} \sum_{t=0}^{[(r-T)/2]} = \sum_{t=0}^{\infty} \sum_{r=2t+T}^{\infty}.$$

(The validity of this operation will be shown below.) Then replace the summation over r by summation over σ where $\sigma = - 2t + (r - T)$. One then has

$$\psi_{eT}^m(P) = e^{im\phi} \sum_{t=0}^{\infty} \sum_{\sigma=0}^{\infty} \frac{(k\,d)^{2t} a_\sigma^t(s^2 - 1)^{(\sigma+t)/2}}{(s - \cos \eta)^{2t+\sigma+T}} P_{T+t}^{-M-t-\sigma}(s), \tag{2.24}$$

where

$$a_\sigma^t = \frac{(-1)^t(M - T)_\sigma(T + t + .5)_\sigma}{2^t(t)!(\sigma)!}. \tag{2.25}$$

Similarly $\psi_{0T'}^m(P)$ reduces to the form

$$\psi_{0T'}^m(P) = e^{im\phi} \sin \eta \sum_{t=0}^{\infty} \sum_{\sigma=0}^{\infty} \frac{(k\,d)^{2t} b_\sigma^t (s^2 - 1)^{(\sigma+t)/2}}{(s - \cos \eta)^{2t+\sigma+T'}} P_{T'+t-1}^{-M-t-\sigma}(s), \tag{2.26}$$

where

$$b_\sigma^t = \frac{(-1)^t(M + 1 - T')_\sigma(T' + t + .5)_\sigma}{2^t(t)!(\sigma)!}. \tag{2.27}$$

To show convergence of the above series we require the following inequality

$$P_\nu^{-M}(s) \leq (s - 1)^{M/2}(s + 1)^{-M/2}[s \pm (s^2 - 1)^{1/2}]^\nu/\Gamma(1 + M), \tag{2.28}$$

where the positive and negative signs are taken when $\nu \geq 0$ and $0 > \nu$ respectively. We shall represent the series expansion in (2.24) by the following

$$\psi_{eT}^m(P) = e^{im\phi} \sum_{t=0}^{\infty} \sum_{\sigma=0}^{\infty} C_{t\sigma}(s, \eta). \tag{2.29}$$

Using (2.28) and the inequality

$$(s - 1)(s - \cos \eta)^{-1} \leq 1$$

which holds for $1 \leq s \leq \infty$ and $0 \leq \eta \leq 2\pi$ we can show that the series $\sum_{\sigma=0}^{\infty} | C_{t\sigma}(s, \eta) |$ is dominated by the absolutely convergent series $D_t(s, \eta) \sum_{\sigma=0}^{\infty} | d_{t\sigma} |$ where

$$| d_{t\sigma} | = \left| \frac{(M - T)_\sigma(T + t + .5)_\sigma}{(\sigma)!(1 + M + t)_\sigma} \right| \tag{2.30}$$

and

$$D_t(s, \eta) = \left(\frac{s - 1}{s + 1}\right)^{M/2} \frac{k^{2t}d^{2t}(s - 1)^t[s \pm (s^2 - 1)^{1/2}]^{t+T}}{2^t(t)!\Gamma(1 + M + t)(s - \cos \eta)^{2t+T}},$$

where the positive sign is taken when $t + T \geq 0$, negative sign otherwise.

For large t, $(t + T + \frac{1}{2} > 0)$, we have

$$\sum_{\sigma=0}^{\infty} | d_{t\sigma} | \leq \frac{\Gamma(1 + M + t)(\pi)^{1/2}}{\Gamma(1 + t + T) \mid \Gamma(.5 + M - T) \mid} + K\left[1 + 0\left(\frac{1}{t}\right)\right], \tag{2.31}$$

where K is a constant which vanishes if $M \geq T$.

Using (2.31) it can be shown that the series $\sum_{t=0}^{\infty} \sum_{\sigma=0}^{\infty} D_t(s, \eta) \mid d_{t\sigma} \mid$ is absolutely convergent for every value of kd and every value of s and η in the ranges $0 \leq s \leq \infty$ and $0 \leq \eta \leq 2\pi$. Since the series $\sum \sum \mid c_{t\sigma} \mid$ is dominated by $\sum \sum D_t(s, \eta) \mid d_{t\sigma} \mid$ we can say that the original series given by (2.24) is uniformly and absolutely convergent for $1 \leq s \leq \infty$ and $0 \leq \eta \leq 2\pi$. Hence the change of order of summation above is valid.

A similar discussion follows for the series (2.26) except that the region of uniform convergence $1 \leq s \leq \infty$ and $0 \leq \eta \leq 2\pi$, holds only if $T' - 1 - M$ is a non-negative integer. For other values of $T' - 1 - M$ the region of uniform convergence is $1 \leq s \leq \infty$ and $0 < \eta < 2\pi$, the wave functions $\psi_{0T'}^m(P)$ being discontinuous along the lines $\eta = 0$ and $\eta = 2\pi$.

3. Determination of T, T′ through continuity conditions. We will be concerned with solutions of the Helmholtz equation for the exterior problem, that is, solutions which are non-singular and continuous in the region external to the torus $s = s_0$, where by external we mean $s_0 \geq s \geq 1$. The region $s_0 \leq s \leq \infty$ describes the surface and interior of a torus. The limiting torus $s = \infty$ describes a ring with radius d and centre, the origin. As is seen from (1.1), a portion of the surface of the torus $s = 1$ is the z-axis. The rest of the surface extends throughout infinity in all directions, i.e. if R is the spherical polar coordinate, then $R = \infty$ for torus $s = 1$.

We shall eventually consider solutions satisfying the radiation condition, but before doing so, we must consider the effect of applying the conditions of continuity and non-singularity in the region $s_0 \geq s \geq 1$. Since $Q_T^m(s)$ is singular on the surface $s = 1$, solutions involving the associated Legendre functions $Q_T^m(s)$ will be singular there. Hence it seen that solutions which are non-singular in region $s_0 \geq s \geq 1$ can be formed only from $\psi_{eT}^m(P)$ and $\psi_{0T'}^m(P)$ solutions. The condition of continuity shall be applied to the $\psi_{eT}^m(P)$ and $\psi_{0T'}^m(P)$ solutions.

As was mentioned above $\psi_{0T'}^m(P)$ is discontinuous at $\eta = 0$ unless $T' - 1 - M$ is a non-negative integer. Even though the other wave functions are discontinuous at $\eta = 0$, it is possible that there exists a linear combination of them which is continuous at $\eta = 0$.

We will be concerned with solutions odd in the variable η, of the form

$$\psi_0^m(s, \eta, \phi) = \sum_p \alpha_p \psi_{0p}^m(P), \tag{3.1}$$

where p increases by integral values only. The coefficients α_p functions of kd only, are normalized such that there exists at least one coefficient which does not vanish when k is zero. Let

$$[\psi_0^m(s, \eta, \phi)]_{k=0} = \sum_{p=T'}^{T'+N'} \alpha_p [\psi_{0p}^m(P)]_{k=0}\ , \tag{3.2}$$

where N' is an non-negative integer. We require $\psi_0^m(s, \eta, \phi)$ to be continuous at $\eta = 0$ for every s and kd in the ranges $1 \leq s \leq \infty$ and $0 \leq kd \leq \infty$. Hence a necessary condition for continuity is that the Laplace portion of $\psi_0^m(s, \eta, \phi)$ must be continuous at $\eta = 0$. Since $\psi_0^m(s, \eta, \phi)$ is an odd function of η, the following must hold for η approaching zero

$$[\psi_0^m(s, \eta, \phi)]_{k=0} \sim 0(\eta) \qquad 1 \leq s \leq \infty. \tag{3.3}$$

Now it can be shown that $[\psi_{0T'}^m(P)]_{k=0}$ has the value as η approaches $+0$

$$[\psi_{0T'}^m(P)]_{k=0} \sim \left(\frac{s-1}{s+1}\right)^{M/2} \frac{(2\pi)^{1/2}(s-1)^{1/2-T'}}{\Gamma(-T'+1+M)\Gamma(T'+.5)} + 0(\eta). \tag{3.4}$$

The term independent of η in (3.4) is zero only if $T' - 1 - M$ or $-T' - \frac{1}{2}$ is a non-negative integer. Because of the factor $(s-1)^{-T'}$ there is no linear combination of $\psi_{0T'}^m(P)$ which will satisfy (3.3) unless $T' - 1 - M$ or $-T' - \frac{1}{2}$ is a non-negative integer. Hence in (3.2) the lower limits of summation may have the values $M + 1, M + 2, \cdots$ or $-\frac{1}{2}, -3/2, \cdots$. In the latter case the upper limit of summation is $-\frac{1}{2}$.

The even solutions of η, $\psi_{eT}^m(P)$ have all been shown to be uniformly convergent for the region $1 \leq s \leq \infty$ and $0 \leq \eta \leq 2\pi$, and hence are continuous everywhere in the region. However, if we differentiate term by term the series given by (2.24) with respect to the variable η, the resulting series can be shown to be uniformly convergent in the region $1 \leq s \leq \infty$ and $0 < \eta < 2\pi$ and discontinuous at $\eta = 0$ or 2π, except when $T - M$ is a non-negative integer. In this case the differential series is continuous at $\eta = 0$ and $\eta = 2\pi$.

We will be concerned with obtaining a necessary condition for continuity of the partial derivative with respect to η for the even solutions $\psi_e^m(s, \eta, \phi)$ where

$$\psi_e^m(s, \eta, \phi) = \sum_p \beta_p \psi_{ep}^m(P) \tag{3.5}$$

and which has the value when k vanishes

$$[\psi_e^m(s, \eta, \phi)]_{k=0} = \sum_{p=T}^{T+N} \beta_p [\psi_{ep}^m(P)]_{k=0} . \tag{3.6}$$

Using the asymptotic relation for $\eta \to +0$

$$\left[\frac{\partial}{\partial \eta} \psi_{eT}^m(P)\right]_{k=0} \sim -\left(\frac{s-1}{s+1}\right)^{M/2} \frac{(2\pi)^{1/2}(s-1)^{-1/2-T}}{\Gamma(-T+M)\Gamma(T+.5)} + 0(\eta) \tag{3.7}$$

we can deduce in a manner similar to the above that the lower limit of summation in (3.6) may be $M, M+1, M+2, \cdots$ or $-\frac{1}{2}, -3/2, \cdots$, and in the latter case the upper limit of summation is $-\frac{1}{2}$.

Imposing the above conditions of continuity on T and T' restricts the number of solutions.

To simplify further work $\psi_{eN}^{m*}(P)$ and $\psi_{0N}^{m*}(P)$ shall be defined by the relations

$$\psi_{eN}^{m*}(P) = (k\, d)^N \psi_{e(M+N)/2}^m(P), \tag{3.8}$$

$$\psi_{0N}^{m*}(P) = 2^{-1/2}(k\, d)^N \psi_{0(M+N+1)/2}^m(P), \tag{3.9}$$

where in (3.8) T is replaced by $(M+N)/2$ and in (3.9) T' is replaced by $(M+N+1)/2$.

Later on we will be concerned with wave functions of the form

$$\sum_{r=N}^{\infty} \alpha_r \psi_{0r}^*(P), \tag{3.10}$$

$$\sum_{r=N}^{\infty} \beta_r \psi_{er}^*(P), \tag{3.11}$$

where α_r and β_r are independent of kd. Substituting in the expressions (3.8) and (3.9) and normalizing we see that on employing the necessary conditions for continuity we obtain for (3.11)

$$N = M + 2l \quad \text{or} \quad -N - 1 = M + 2l$$

and for (3.10)

$$N = M + 2l + 1 \quad \text{or} \quad -N - 1 = M + 2l + 1,$$

where $l = 0, 1, 2, 3, \cdots$.

4. Asymptotic value of $\psi_{eN}^{m*}(P)$, $\psi_{0N}^{m*}(P)$ when $d \to 0$. The main problem is to find the linear combination of $\psi_{eN}^{m*}(P)$ and $\psi_{0N}^{m*}(P)$ which represents outgoing radiation from a ring source. A necessary condition for this is that the wave function represent radiation from a point source when the radius d of the ring approaches zero. Hence one must first consider the asymptotic values

$$\lim_{d\to 0} \psi_{eN}^{m*}(R, \theta, \phi),$$

$$\lim_{d\to 0} \psi_{0N}^{m*}(R, \theta, \phi),$$

where (R, θ, ϕ) are spherical polar coordinates and before taking the limit, the toroidal coordinates (s, η, ϕ) are replaced by (R, θ, ϕ).

Since

$$\rho = \frac{d(s^2 - 1)^{1/2}}{(s - \cos \eta)}, \qquad z = \frac{d \sin \eta}{(s - \cos \eta)}, \tag{4.1}$$

where (ρ, z, ϕ) are cylindrical polar coordinates, one obtains

$$\frac{R^2}{d^2} = \frac{(\rho^2 + z^2)}{d^2} = \frac{(s + \cos \eta)}{(s - \cos \eta)} \tag{4.2}$$

and

$$\operatorname{Tan} \theta = \frac{(s^2 - 1)^{1/2}}{\sin \eta}. \tag{4.3}$$

Eliminating η from (4.2) and (4.3), one can obtain an expression for s in terms of R and θ. Hence the following are obtained when d approaches zero

$$\left.\begin{aligned} (s^2 - 1)^{1/2} &\sim \frac{d}{R} 2 \sin \theta \\ s &\sim 1 + \frac{d^2}{R^2} 2 \sin^2 \theta \end{aligned}\right\}. \tag{4.4}$$

Similarly one obtains

$$\left.\begin{aligned} (s - \cos \eta) &\sim 2d^2R^{-2} \\ \sin \eta &\sim 2\, dR^{-1} \cos \theta \end{aligned}\right\}. \tag{4.5}$$

The asymptotic values given by (4.4) and (4.5) hold not only when R is fixed and d approaches zero, but when d is fixed and R approaches infinity.

Using the above the following limits can be calculated

$$\left.\begin{aligned} \lim_{d\to 0} \psi_{eN}^{m*}(R, \theta, \phi) &= H_N^m(R, \theta, \phi) \\ \lim_{d\to 0} \psi_{0N}^{m*}(R, \theta, \phi) &= H_N^m(R, \theta, \phi) \end{aligned}\right\} 0 < \theta < \frac{\pi}{2}, \tag{4.6}$$

where

$$H_N^m(R, \theta, \phi) = 2^{(M-N)/2} e^{im\phi} \sum_{t=0}^{\infty} \frac{(-1)^t (kR)^{2t+N}}{2^t (t)!} \sin \theta^t P_{N+t}^{-M-t} (\cos \theta). \tag{4.7}$$

For present purposes the limit is only required for the range $0 < \theta < \pi/2$.

5. Solutions of the wave equation representing radiations from a ring source. Having obtained the asymptotic values of $\psi_{eN}^{m*}(P)$ and $\psi_{0N}^{m*}(P)$ we now find the linear combination ψ_N^M representing a solution of the Helmholtz equation, satisfying the radiation condition, and possessing a ring singularity (i.e. singular at the limiting surface $s = \infty$). This is done by using the necessary condition that ψ_N^M represent radiation from a point source when $d \to 0$. No further restriction is placed if it is required that

$$\lim_{d\to 0} \psi_N^M = e^{im\phi} h_N^{(1)}(kR) P_N^M (\cos \theta), \tag{5.1}$$

where N is an integer.
Since

$$h_N^{(1)}(kR)P_N^M (\cos \theta) = j_N(kR)P_N^M (\cos \theta) + i(-1)^{N+1} j_{-N-1}(kR)P_{-N-1}^M (\cos \theta)$$

it can be seen that the desired linear combination has the form

$$\psi_N^M = \Psi_N^M + i(-1)^{N+1}\Psi_{-N-1}^M$$

where

$$\Psi_N^M = \sum_r C_r(N, M)\Psi_r^{m*}(P). \tag{5.2}$$

Thus it is seen that (5.1) can be replaced by

$$\lim_{d\to 0} \Psi_N^M = e^{im\phi} j_N(kR) P_N^M (\cos \theta). \tag{5.3}$$

Using (4.6) and (5.2) this reduces to

$$\sum_r C_r(N, M)H_r^m(R, \theta, \phi) = e^{im\phi} j_N(kR)P_N^M (\cos \theta) \qquad 0 < \theta < \frac{\pi}{2}. \tag{5.4}$$

Equation (5.4) determines the unknown constants $C_r(N, M)$. It is an identity in the variables (R, θ, ϕ). In solving for $C_r(N, M)$ we shall consider θ to be in the range $0 < \theta < \pi/2$. This will be sufficient for present purposes.

The right hand side of (5.4) is of the form

$$\frac{(\pi)^{1/2}}{2} \sum_{p=0}^{\infty} \frac{(-1)^p \left(\frac{kR}{2}\right)^{N+2p}}{(p)!\Gamma(p + N + 3/2)} P_N^M (\cos \theta) e^{im\phi}. \tag{5.5}$$

This is a power series in (kR) with lowest power N and with all terms in the power series of the same parity as N. Thus considering the expression for H_r^m given by (4.7), it is obvious that the left hand side of (5.4) must be of the form

$$\sum_{r=0}^{\infty} C_{N+2r}(N, M)H_{N+2r}^m(R, \theta, \phi) \tag{5.6}$$

and on substitution of the expression given by (4.12) this becomes

$$e^{im\phi} 2^{(M-N)/2}(kR)^N \sum_{p=0}^{\infty} \frac{(kR)^{2p}}{2^p} \sum_{r=0}^{p} \frac{C_{N+2r}(N, M)\cdot(-1)^{p-r}}{\Gamma(p - r + 1)} \sin \theta^{p-r} P_{N+p+r}^{-M-p+r} (\cos \theta). \tag{5.7}$$

Substitute the expression given by (5.5) and (5.7) in (5.4), divide out $(kR)^N e^{im\phi}$, and equate coefficients of $(kR)^{2p}$ to obtain

$$\frac{(\pi)^{1/2} P_N^M (\cos \theta)}{(p)!\Gamma(p + N + 3/2)2^{N+2p+1}} = 2^{(M-N)/2-p} \sum_{r=0}^{p} \frac{C_{N+2r}(N, M)\cdot(-1)^r}{\Gamma(p - r + 1)} \sin \theta^{p-r} \tag{5.8}$$

$$\times P_{N+p+r}^{-M-p+r} (\cos \theta), \quad \text{where} \quad p = 0, 1, 2, 3, \cdots$$

The constants $C_{N+2r}(N, M)$ are determined from the infinite set of equations (5.8). To solve for $C_{N+2r}(N, M)$ the following result is used

$$\frac{P_N^{-M}(\cos\theta)}{2^p(p)!\Gamma(p+N+3/2)} = \sum_{r=0}^{p} \frac{\sin\theta^{p-r} P_{N+p+r}^{-M-p+r}(\cos\theta)}{2^r(p-r)!(r)!\Gamma(N+3/2+r)}. \tag{5.9}$$

The proof of the above expression is found in [6].

Using the fact that

$$P_N^M(\cos\theta) = (-1)^M \frac{\Gamma(N+M+1)}{\Gamma(N-M+1)} P_N^{-M}(\cos\theta) \tag{5.10}$$

and the identity given by (5.9) one immediately obtains

$$C_{N+2r}(N, M) = (\pi)^{1/2} \frac{\Gamma(N+M+1)(-1)^{r+M}}{\Gamma(N-M+1)(r)!\Gamma(N+3/2+r)\, 2^{(M+N)/2+r+1}}. \tag{5.11}$$

So now the coefficients $C_r(N, M)$ in (5.2) have been found such that Eq. (5.3) holds. Since the limiting process of d approaching zero is independent of whether $\psi_{er}^{m*}(P)$ or $\psi_{0r}^{m*}(P)$ are used in (5.2), one has two separate solutions for Ψ_N^M, odd and even solutions in the variable η. Define the even solutions by S_N^M and the odd solutions by T_N^M as follows:

$$e^{im\phi} S_N^M = \sum_{r=0}^{\infty} C_{N+2r}(N, M) \psi_{eN+2r}^{m*}(P), \tag{5.12}$$

$$e^{im\phi} T_N^M = \sum_{r=0}^{\infty} C_{N+2r}(N, M) \psi_{0N+2r}^{m*}(P). \tag{5.13}$$

Applying the necessary condition for continuity of the function and its derivatives as given in Sec. (3), it is seen that in (5.12) the subscript N is such that

$$N = M + 2l \quad \text{or} \quad -N - 1 = M + 2l$$

and the subscript N in (5.13) is such that

$$N = M + 2l + 1 \quad \text{or} \quad -N - 1 = M + 2l + 1,$$

where $l = 0, 1, 2, \cdots$

S_N^M and T_N^M have the following properties

$$\left.\begin{aligned} \lim_{d\to 0} S_N^M &= j_N(kR)P_N^M(\cos\theta) \\ \lim_{d\to 0} T_N^M &= j_N(kR)P_N^M(\cos\theta) \end{aligned}\right\} 0 < \theta < \frac{\pi}{2}.$$

Hence the functions V_N^M and W_N^M defined as follows,

$$V_N^M = S_N^M + i(-1)^{N+1} S_{-N-1}^M, \tag{5.14}$$

$$W_N^M = T_N^M + i(-1)^{N+1} T_{-N-1}^M, \tag{5.15}$$

have the property

$$\left.\begin{aligned} \lim_{d\to 0} V_N^M &= h_N^{(1)}(kR)P_N^M(\cos\theta) \\ \lim_{d\to 0} W_N^M &= h_N^{(1)}(kR)P_N^M(\cos\theta) \end{aligned}\right\} 0 < \theta < \frac{\pi}{2}.$$

Thus we have a set of functions $e^{im\phi} V_N^M(s, \eta)$ and $e^{im\phi} W_N^M(s, \eta)$ which satisfy the necessary condition for outgoing radiation. Their analytic properties of continuity convergence, singularities etc., must be investigated.

The first thing that is required is to evaluate (5.12) and (5.13). From (5.12) and (5.11) it is seen that

$$e^{im\phi} S_N^M = \frac{\Gamma(N + M + 1)(\pi)^{1/2}(-1)^m}{\Gamma(N - M + 1)2^{(M+N)/2+1}} \sum_{r=0}^{\infty} \frac{(-1)^r \psi_{eN+2r}^{m*}(P)}{2^r(r)!\Gamma(N + 3/2 + r)}, \tag{5.16}$$

where by (3.8), $\psi_{eN+2r}^{m*}(P)$ is obtained from (2.24) by replacing T by $(N + M)/2 + r$ and multiplying the resulting series by $(kd)^{N+2r}$.

The expression given by (5.16) can be simplified and the following is obtained

$$S_N^M = \frac{\Gamma(N + M + 1)(\pi)^{1/2}(-1)^M(k\,d)^N}{\Gamma(N - M + 1)2^{(M+N)/2+1}(s - \cos\eta)^{(M+N)/2}} \sum_{p=0}^{\infty} \frac{(k\,d)^{2p}(-1)^p K_p^1}{(s - \cos\eta)^p 2^p}, \tag{5.17}$$

where

$$K_p^1 = \frac{1}{(p)!\Gamma(N + 3/2 + p)} \sum_{r=0} \frac{\left(\frac{M - N}{2}\right)_r \left(\frac{N + M + 1}{2}\right)_r (s^2 - 1)^{r/2}}{(r)!(s - \cos\eta)^r} \cdot P_{p+(N+M)/2}^{-M-r}(s). \tag{5.18}$$

Since N is specified for S_N^M such that $N = M + 2l$ or $-M - 2l - 1$, and l is a positive integer or zero, the series in (5.18) terminates when $r = l$.

Similarly one can reduce the expression for T_N^M to the following

$$T_N^M = \frac{\Gamma(N + M + 1)(\pi)^{1/2}(-1)^M \sin\eta(k\,d)^N}{\Gamma(N - M + 1)2^{(M+N+3)/2}(s - \cos\eta)^{(M+N+1)/2}} \sum_{p=0}^{\infty} \frac{(k\,d)^{2p}(-1)^p K_p^2}{2^p(s - \cos\eta)^p}, \tag{5.19}$$

where

$$K_p^2 = \frac{1}{(p)!\Gamma(p + 3/2 + N)} \sum_{r=0} \frac{\left(\frac{M - N + 1}{2}\right)_r \left(\frac{M + N}{2} + 1\right)_r (s^2 - 1)^{r/2}}{(r)!(s - \cos\eta)^r} \cdot P_{p+(M+N-1)/2}^{-M-r}(s). \tag{5.20}$$

Since N is specified for T_N^M to have the values $N = M + 2l + 1$ or $-M - 2l - 2$, the series in (5.20) terminates when $r = l$.

Hence we see that S_N^M and T_N^M are comprised of two series one finite and the other infinite. Hence for convergence, we only need to consider summation over p. Using the inequality (2.28) an absolutely convergent dominant series can be found for the series

$$\sum_p^{\infty} \frac{(kd)^{2p} P_{p+(N+M)/2}^{-M-r}(s)}{2^p(s - \cos\eta)^p(p)!\Gamma(N + 3/2 + p)}.$$

Hence it can be shown that the series expression in (5.17) is absolutely and uniformly convergent for the region $1 \leq s \leq \infty$, $0 \leq \eta \leq 2\pi$ and $0 \leq |kd| \leq \infty$. If the series (5.17) is differentiated term by term by either s or η, the resulting series is also uniformly convergent for the same region.

The same remarks hold true for the series expansion (5.19) given for T_N^M.

6. Integral representation of $V^M_{M+2l}(s, \eta)$ and $W^M_{M+2l+1}(s, \eta)$. Before the asymptotic values for large R can be obtained, we must obtain an integral representation for each of the functions.

Using the following integral expression for the associated Legendre function $P^\mu_n(s)$ where $\mu < \frac{1}{2}$

$$P^\mu_n(s) = \frac{2^\mu(s^2 - 1)^{-\mu/2}}{(\pi)^{1/2}\Gamma(\frac{1}{2} - \mu)} \int_0^\pi [s + (s^2 - 1)^{1/2} \cos t]^{n+\mu} (\sin t)^{-2\mu}\, dt \tag{6.1}$$

we obtain

$$\frac{(s^2 - 1)^{r/2} P_n^{-M-r}(s)}{(s - \cos \eta)^r} = \frac{2^{-M}(s^2 - 1)^{M/2}}{(\pi)^{1/2}\Gamma(\frac{1}{2} + M + r)} \int_0^\pi \frac{Z^r (\sin t)^{2M}\, dt}{[s + (s^2 - 1)^{1/2} \cos t]^{M-n}}, \tag{6.2}$$

where

$$Z = \frac{(s^2 - 1)(\sin t)^2}{2(s - \cos \eta)[s + (s^2 - 1)^{1/2} \cos t]}. \tag{6.3}$$

From (5.18) and (6.2) we thus obtain

$$K^1_p = \frac{2^{-M}(s^2 - 1)^{M/2}(\pi)^{-1/2}}{(p)!\Gamma(N + 3/2 + p)\Gamma(\frac{1}{2} + M)} \cdot \int_0^\pi \frac{{}_2F_1\left(\frac{M - N}{2}, \frac{N + M + 1}{2}; \frac{1}{2} + M; Z\right)(\sin t)^{2M}\, dt}{[s + (s^2 - 1)^{1/2} \cos t]^{(M-N)/2-p}}. \tag{6.4}$$

The range of Z is $0 \le Z \le 1$, Z being unity when $\cos \eta = 1$ and $s + (s^2 - 1)^{1/2} \cos t = 1$. Since we are considering the case where $N = M + 2l$ or $-N - 1 = M + 2l$ the hypergeometric function in the expression (6.4) is a polynomial with finite argument. Thus the interchange of order of summation and integration is valid. Now in the expression for S^M_N (5.17) substitute expression (6.4) for K^1_p. Interchange the order of summation and integration. Noting that

$$j_N(X) = \sum_{p=0}^{\infty} \frac{(-1)^p\left(\frac{x}{2}\right)^{2p+N}}{(p)!\Gamma(N + 3/2 + p)} \frac{(\pi)^{1/2}}{2} \tag{6.5}$$

$$= \frac{(\pi)^{1/2}(k\, d)^N 2^{-N/2-1}}{(s - \cos \eta)^{N/2}} \sum_{p=0} \frac{(-1)^p (k\, d)^{2p}[s + (s^2 - 1)^{1/2} \cos t]^{p+N/2}}{2^p(s - \cos \eta)^p(p)!\Gamma(N + 3/2 + p)}, \tag{6.6}$$

where

$$X = kd\, 2^{1/2}\left[\frac{s + (s^2 - 1)^{1/2} \cos t}{s - \cos \eta}\right]^{1/2} \tag{6.7}$$

it is seen that the infinite series in the integrand is uniformly convergent for $\infty \ge s \ge 1$ and $0 \le t \le \pi$. Hence the following holds

$$S^M_N = \frac{\Gamma(N + M + 1)(-2)^M(\pi)^{-1/2}}{\Gamma(N - M + 1)\Gamma(\frac{1}{2} + M)} \cdot \int_0^\pi {}_2F_1\left(\frac{M - N}{2}, \frac{N + M + 1}{2}; \frac{1}{2} + M; Z\right) \cdot j_N(X) Z^{M/2} (\sin t)^M\, dt \qquad 1 \le s < \infty. \tag{6.8}$$

When $N = M + 2l$ the integral in (6.8) will hold also for $s = \infty$. From (5.14) we have

$$V_{M+2l}^{M} = \frac{(2M+2l)!(-2)^{-M}(\pi)^{-1/2}}{(2l)!\Gamma(\frac{1}{2}+M)} \int_0^\pi {}_2F_1(-l, M+l+\tfrac{1}{2}; \tfrac{1}{2}+M; Z) h_{M+2l}^{(1)}(X) \times Z^{M/2} (\sin t)^M \, dt, \tag{6.9}$$

where $h_N^{(1)}(X)$ is the spherical Hankel function of the first kind. In a similar manner the following integral expression may be obtained for T_N^M where $N = M + 2l + 1$ or $-N - 1 = M + 2l + 1$,

$$T_N^M = \frac{\Gamma(N+M+1)(-2)^{-M} \sin \eta (s^2-1)^{-1/2}}{\Gamma(N-M+1)(\pi)^{1/2}\Gamma(M+\frac{1}{2})} \cdot \int_0^\pi {}_2F_1\left(\frac{M-N+1}{2}, \frac{M+N}{2}+1; \tfrac{1}{2}+M; Z\right) \cdot j_N(X) Z^{(M+1)/2} \, (\sin t)^{M-1} \, dt. \tag{6.10}$$

The integral expression in (6.10) holds in the ranges $1 \leq s \leq \infty$ for $N = M + 2l + 1$ and $1 \leq s < \infty$ for $N = -M - 2l - 2$. We also obtain the integral expression for W_{M+2l+1}^M which holds for $1 \leq s < \infty$

$$W_{M+2l+1}^{M} = \frac{(2M+2l+1)!(-2)^{-M} \sin \eta (s^2-1)^{-1/2}}{(2l+1)!(\pi)^{1/2}\Gamma(\frac{1}{2}+M)} \cdot \int_0^\pi {}_2F_1(-l, M+l+3/2; \tfrac{1}{2}+M; Z) h_{M+2l+1}^{(1)}(X) Z^{(M+1)/2} (\sin t)^{M-1} \, dt. \tag{6.11}$$

7. Asymptotic values of $V_{M+2l}^M(s, \eta)$ and $W_{M+2l+1}^M(s, \eta)$ for $R \to \infty$. Having obtained the integral formulation of the wave functions, we are now in a position to investigate their asymptotic values when R approaches ∞.

From (4.4) and (4.5) the following asymptotic values are obtained for R approaching ∞

$$\frac{(s^2-1)^{1/2}}{s} \sim \frac{d}{R}\left[2 \sin \theta + 0\left(\frac{1}{R}\right)\right], \tag{7.1}$$

$$\frac{s - \cos \eta}{s} \sim \frac{2d^2}{R^2}\left[1 + 0\left(\frac{1}{R^2}\right)\right]. \tag{7.2}$$

Hence the asymptotic values of Z and X for large R are

$$Z \sim \sin^2 t \sin^2 \theta + 0\left(\frac{1}{R}\right) \tag{7.3}$$

and

$$X \sim kR + kd \cos t \sin \theta + 0\left(\frac{1}{R}\right). \tag{7.4}$$

Insert the values (7.3) and (7.4) into (6.9) and using the fact that when R approaches ∞

$$h_{M+2l}^{(1)}(X) \sim (-i)^{M+2l+1} \frac{\exp(ikR + ikd \cos t \sin \theta)}{kR} \tag{7.5}$$

we have

$$V_{M+2l}^M \sim (-i)^{M+2l+1} \frac{e^{ikR}}{kR} R_{M+2l}^M (\cos \theta), \tag{7.6}$$

where

$$R^M_{M+2l}(\cos\theta) = \frac{(2M+2l)!(-1)^M(\pi)^{-1/2}}{(2l)!(2)^M\Gamma(M+.5)} \times \int_0^\pi {}_2F_1(-l, M+l+\tfrac{1}{2}; \tfrac{1}{2}+M; \sin t^2 \sin\theta^2)\sin\theta^M \sin t^{2M} e^{ikd\cos t\sin\theta}\, dt. \tag{7.7}$$

Expand the hypergeometric series in (7.7) and use the relation

$$\Gamma(n+\tfrac{1}{2})J_n(z) = \pi^{-1/2}\left(\frac{z}{2}\right)^n \int_0^\pi e^{iz\cos t}\sin t^{2n}\, dt \tag{7.8}$$

to obtain

$$R^M_{M+2l}(\cos\theta) = \frac{(2M+2l)!}{(2l)!(-kd)^M}\sum_{r=0}^{l}\frac{(-l)_r(M+l+\frac{1}{2})_r}{(r)!}\left(\frac{2\sin\theta}{kd}\right)^r J_{M+r}(kd\sin\theta). \tag{7.9}$$

To obtain the asymptotic expansion for $W^M_{M+2l+1}(s, \eta)$ the following result is needed [from (4.3)]

$$\sin\eta(s^2-1)^{-1/2} = \cot\theta. \tag{7.10}$$

In a manner similar to the above, the asymptotic value of $W^M_{M+2l+1}(s, \eta)$ can be obtained for R approaching ∞

$$W^M_{M+2l+1}(s, \eta) \sim (-i)^{M+2l+2}\frac{e^{ikR}}{kR}R^M_{M+2l+1}(\cos\theta), \tag{7.11}$$

where

$$R^M_{M+2l+1}(\cos\theta) = \frac{(2M+2l+1)!\cos\theta}{(2l+1)!(-kd)^M}\sum_{r=0}^{l}\frac{(-l)_r(M+l+3/2)_r}{(r)!}\left(\frac{2\sin\theta}{kd}\right)^r \cdot J_{M+r}(kd\sin\theta). \tag{7.12}$$

From (7.6) and (7.11) it is seen that the wave functions $e^{im\phi}V^M_{M+2l}(s, \eta)$ and $e^{im\phi}W^M_{M+2l+1}(s, \eta)$ will satisfy the radiation condition. We shall consider the case where d approaches zero. It can be shown that the asymptotic values for Z and X given by (7.3) and (7.4) will hold when d vanishes. Hence we have

$$\lim_{d\to 0} V^M_{M+2l} = h^{(1)}_{M+2l}(kR)\lim_{d\to 0} R^M_{M+2l}(\cos\theta). \tag{7.13}$$

But

$$\lim_{d\to 0} R^M_{M+2l}(\cos\theta) = \frac{(2M+2l)!(-1)^M}{(2l)!(M)!}\cdot\left(\frac{\sin\theta}{2}\right)^M {}_2F_1(-l, M+l+\tfrac{1}{2}; M+1; \sin^2\theta) \tag{7.14}$$

$$= \frac{(2M+2l)!(-1)^M}{(2l)!}P^{-M}_{M+2l}(\cos\theta) = P^M_{M+2l}(\cos\theta) \qquad 0 \le \theta \le \pi. \tag{7.15}$$

We see that $R^M_{M+2l}(\cos\theta)$ is identical with the associated Legendre function $P^M_{M+2l}(\cos\theta)$ when d vanishes. In a similar manner it can be shown that $R^M_{M+2l+1}(\cos\theta)$ becomes identical to $P^M_{M+2l+1}(\cos\theta)$ when d is zero. Hence the toroidal wave functions $e^{im\phi}V^M_{M+2l}(s, \eta)$ and $e^{im\phi}W^M_{M+2l+1}(s, \eta)$ are identical to the spherical polar wave functions when d vanishes. Thus when d vanishes they represent radiations from a point source.

The remaining detail to be considered is the nature of the singularities of the wave function at $s = \infty$.

8. Asymptotic behaviour of $V_{M+2l}^M(s, \eta)$, $W_{M+2l+1}^M(s, \eta)$ as $s \to \infty$. In order to investigate the asymptotic behaviour of $V_{M+2l}^M(s, \eta)$ and $W_{M+2l+1}^M(s, \eta)$ we must investigate the functions S_{M+2l}^M, $S_{-M-2l-1}^M$, T_{M+2l+1}^M and $T_{-M-2l-2}^M$ separately.

For S_{M+2l}^M and T_{M+2l+1}^M use the integral expressions. We have shown that the expressions hold for $1 \leq s \leq \infty$ and since the integrand is finite for $s = \infty$ we may use the integral expression to determine the asymptotic values.

From (6.3) and (6.7) the asymptotic values of Z and X become as s approaches ∞

$$Z \sim 2^{-1}(1 - \cos t), \tag{8.1}$$

$$X \sim kd\, 2^{1/2}(1 + \cos t)^{1/2}. \tag{8.2}$$

Hence we obtain the following for s approaching ∞

$$S_{M+2l}^M \sim \text{constant}, \tag{8.3}$$

$$T_{M+2l+1}^M \sim \frac{\sin \eta}{s} \text{ constant}. \tag{8.4}$$

It is more difficult to find the asymptotic values for $S_{-M-2l-1}^M$ and $T_{-M-2l-2}^M$. For simplification of work, the functions $F_p(s, \eta; l, M)$ and $G_p(s, \eta; l, M)$ are defined as follows

$$\left.\begin{aligned} F_p(s, \eta; l, M) &= (s - \cos \eta)^{l+1/2} \sum_{r=0}^{l} \frac{(-l)_r(M + l + \frac{1}{2})_r}{(r)!(s - \cos \eta)^r} (s^2 - 1)^{r/2} P_{p-l-1/2}^{-M-r}(s) \\ G_p(s, \eta; l, M) &= \sin \eta (s - \cos \eta)^{l+1/2} \sum_{r=0}^{l} \frac{(-l)_r(M + l + 3/2)_r}{(r)!(s - \cos \eta)^r} \\ &\qquad \cdot (s^2 - 1)^{r/2} P_{p-l-3/2}^{-M-r}(s) \end{aligned}\right\}. \tag{8.5}$$

Thus from (5.17), (5.18) and (8.5) it is seen that

$$S_{-M-2l-1}^M(s, \eta) = \frac{\Gamma(2M + 2l + 1)(\pi)^{1/2}(kd)^{-M-2l-1}}{\Gamma(2l + 1)2^{-l+1/2}(-1)^M} \cdot \sum_{p=0}^{\infty} \frac{(-1)^p(kd)^{2p}F_p(s, \eta; l, M)}{(p)!2^p\Gamma(-M - 2l + p + \frac{1}{2})(s - \cos \eta)^p} \tag{8.6}$$

and from (5.19) and (5.20)

$$T_{-M-2l-2}^M(s, \eta) = \frac{\Gamma(2M + 2l + 2)(\pi)^{1/2}(kd)^{-M-2l-2}}{\Gamma(2l + 2)2^{-l+1/2}(-1)^M} \cdot \sum_{p=0}^{\infty} \frac{(-1)^p(kd)^{2p}G_p(s, \eta; l, M)}{(p)!2^p\Gamma(-M - 2l + p - \frac{1}{2})(s - \cos \eta)^p}. \tag{8.7}$$

Now multiply $S_{-M-2l-1}^M(s, \eta)$ by $(kd)^{M+2l+1} e^{im\phi}$ and let k approach zero. The only term which is non-vanishing is the term involving $F_0(s, \eta; l, M)$. Thus we have the following

$$\{(kd)^{M+2l+1} e^{im\phi} S_{-M-2l-1}^M\}_{k=0} = \frac{\Gamma(2M + 2l + 1)(\pi)^{1/2}2^{l-1/2}(-1)^M e^{im\phi}}{\Gamma(2l + 1)\Gamma(-M - 2l + \frac{1}{2})} F_0(s, \eta; l, M). \tag{8.8}$$

But any solution of the wave equation when $k = 0$ is a solution of the Laplace equation. Thus $e^{im\phi} F_0(s, \eta; l, M)$ is a solution of the Laplace equation. Now $F_0(s, \eta; l, M)$ is a power series of $(s - \cos\eta)$ up to the lth power, multiplied by $(s - \cos\eta)^{1/2}$ and is non-singular when $s = 1$. Since the complete set of solutions of the Laplace equation which are non-singular at $s = 1$ are

$$\left.\begin{aligned} &e^{im\phi}(s - \cos\eta)^{1/2} P_{n-1/2}^{-M}(s) \cos n\eta \\ \text{and} \quad &e^{im\phi}(s - \cos\eta)^{1/2} P_{n-1/2}^{-M}(s) \sin n\eta \end{aligned}\right\} \tag{8.9}$$

then

$$F_0(s, \eta; l, M) = \sum_{r=0}^{l} a_r (s - \cos\eta)^{1/2} P_{r-1/2}^{-M}(s) \cos r\eta. \tag{8.10}$$

By a similar analysis it is seen that

$$G_0(s, \eta; l, M) = \sum_{r=1}^{l+1} b_r (s - \cos\eta)^{1/2} P_{r-1/2}^{-M}(s) \sin r\eta. \tag{8.11}$$

So the problem is to calculate a_r and b_r. To find the coefficients a_r multiply both sides of Eq. (8.10) by $(s^2 - 1)^{-M/2}$ and let s approach 1, using the fact that when s approaches 1

$$(s^2 - 1)^{(r-M)/2} P_n^{-M-r}(s) \sim \frac{2^{-M}(s-1)^r}{\Gamma(1 + M + r)},$$

we obtain the equation

$$\sum_{r=0}^{l} a_r \cos r\eta = (1 - \cos\eta)^l. \tag{8.12}$$

But on the expansion of $(1 - \cos\eta)^r$ in a Fourier series we see that

$$a_r = \frac{\epsilon_r (2l)!(-1)^r}{(l - r)!(l + r)!2^{l-1}}, \tag{8.13}$$

where

$$\left.\begin{aligned} \epsilon_r &= 1 \quad \text{when} \quad r = 1, 2, 3, \cdots \\ \epsilon_r &= \tfrac{1}{2} \quad \text{when} \quad r = 0. \end{aligned}\right\} \tag{8.14}$$

Now in a similar manner Eq. (8.11) may be reduced to the following

$$\sin\eta(1 - \cos\eta)^l = \sum_{r=1}^{l+1} b_r \sin r\eta. \tag{8.15}$$

Thus the coefficients b_r are derived from a Fourier analysis and b_r are

$$b_r = \frac{r(2l + 1)!(-1)^{r+1}}{(l + 1 + r)!(l + 1 - r)!2^{l-1}}. \tag{8.16}$$

Thus we obtain the following expressions for $F_0(s, \eta; l, M)$ and $G_0(s, \eta; l, M)$

$$F_0(s, \eta; l, M) = (s - \cos\eta)^{1/2} \sum_{r=0}^{l} \frac{\epsilon_r (2l)!(-1)^r}{(l - r)!(l + r)!2^{l-1}} P_{r-1/2}^{-M}(s) \cos r\eta, \tag{8.17}$$

$$G_0(s, \eta; l, M) = (s - \cos\eta)^{1/2} \sum_{r=1}^{l+1} \frac{r(2l + 1)!(-1)^{r+1}}{(l + 1 + r)!(l + 1 - r)!2^{l-1}} P_{r-1/2}^{-M}(s) \sin r\eta. \tag{8.18}$$

Using the following asymptotic values for the associated Legendre function

$$\left.\begin{aligned} &P_{-1/2}^{-M}(s) \sim \left(\frac{2}{\pi}\right)^{1/2} \frac{s^{-1/2}}{\Gamma(M + \frac{1}{2})} \{\log_e (2s) - \gamma - \psi(M + \tfrac{1}{2})\} \\ \text{where} \quad &\gamma = 0.5772156649\cdots \quad \text{and} \quad \psi(z) = \frac{d \log_e \Gamma(z)}{dz} \\ &P_n^{-M}(s) \sim \frac{2^n \Gamma(n + \frac{1}{2}) s^n}{(\pi)^{1/2} \Gamma(1 + n + M)} \qquad n > -\tfrac{1}{2} \end{aligned}\right\} \tag{8.19}$$

we see that the values of $F_0(s, \eta; l, M)$ and $G_0(s, \eta; l, M)$ become the following when s approaches ∞

$$F_0(s, \eta; l, M) \sim \frac{\epsilon_l (-1)^l}{2^{l-1}} s^{1/2} P_{l-1/2}^{-M}(s) \cos l\eta \,, \tag{8.20}$$

$$G_0(s, \eta; l, M) \sim \frac{(-1)^l s^{1/2}}{2^l} P_{l+1/2}^{-M}(s) \sin (l + 1)\eta. \tag{8.21}$$

It can be shown [6] that

$$(s - \cos \eta)^{-p} F_p(s, \eta; l, M) \leq 0(s^{l-1}) \qquad p = 1, 2, 3, \cdots$$

and hence the dominant term in expression (8.6) given for $S_{-M-2l-1}^M$ is $F_0(s, \eta; l, M)$ for $s \gg 1$ provided that $kd < s$. Similarly it can be shown that $G_0(s, \eta; l, M)$ is the dominant term in expression (8.7) given for $T_{-M-2l-1}^M$.

Thus from (8.6) and (8.20) we have when s approaches ∞

$$S_{-M-2l-1}^M(s, \eta) \sim \frac{(2M + 2l)!(2\pi)^{1/2}(-1)^{M+l}\epsilon_l}{(2l)!\Gamma(-M - 2l + \frac{1}{2})(kd)^{M+2l+1}} s^{1/2} P_{l-1/2}^{-M}(s) \cos l\eta \tag{8.22}$$

and from (5.14), (8.22) and (8.3) we have

$$V_{M+2l}^M(s, \eta) \sim \frac{i(-1)^{l+1}(2\pi)^{1/2}(2M + 2l)!\epsilon_l}{(kd)^{M+2l+1}(2l)!\Gamma(-M - 2l + \frac{1}{2})} s^{1/2} P_{l-1/2}^{-M}(s) \cos l\eta \,. \tag{8.23}$$

From (8.7) and (8.21) we have

$$T_{-M-2l-2}^M(s, \eta) \sim \frac{(2M + 2l + 1)!(-1)^{M+l}(\pi/2)^{1/2} s^{1/2}}{(2l + 1)!\Gamma(-M - 2l - \frac{1}{2})(kd)^{M+2l+2}} P_{l+1/2}^{-M}(s) \sin (l + 1)\eta \tag{8.24}$$

and hence

$$W_{M+2l+1}^M(s, \eta) \sim \frac{i(-1)^l(2M + 2l + 1)!(\pi/2)^{1/2} s^{1/2}}{(2l + 1)!\Gamma(-M - 2l - \frac{1}{2})(kd)^{M+2l+2}} P_{l+1/2}^{-M}(s) \sin (l + 1)\eta. \tag{8.25}$$

9. Orthogonality and general discussion. We have obtained a set of solutions of the Helmholtz equation in toroidal coordinates which satisfy the radiation condition, are continuous and convergent in all space and possess a ring singularity. When the radius d of the ring described by the coordinate $s = \infty$ approaches zero, the toroidal wave functions become identical with spherical polar wave functions. Hence the toroidal wave functions $e^{im\phi} V_{M+2l}^M(s, \eta)$ and $e^{im\phi} W_{M+2l+1}^M(s, \eta)$ form a complete set. That is, any solution which is continuous and single-valued outside the torus $s = s_0$ (i.e. region $1 \leq s \leq s_0$) and satisfies the radiation condition and arbitrary boundary conditions on the torus can be represented by a linear combination of the above toroidal wave functions.

Now the question of orthogonality arises. Are the functions $e^{im\phi}\ V^M_{M+2l}(s,\ \eta)$ and $e^{im\phi}\ W^M_{M+2l+1}(s,\ \eta)$, orthogonal over the surface of every torus?

If we choose a weight function $p(s,\ \eta)$ which is an even function of the variable η, then we obtain the following

$$\int_0^{2\pi}\int_0^{2\pi}[e^{im\phi}V^M_{M+2l}(s,\ \eta)][e^{-im'\phi}V^{M'}_{M+2l'}(s,\ \eta)]p(s,\ \eta)\ d\eta\ d\phi = 0, \qquad M \neq M' \tag{9.1}$$

$$\int_0^{2\pi}\int_0^{2\pi}[e^{im\phi}W^M_{M+2l+1}(s,\ \eta)][e^{-im'\phi}W^{M'}_{M'+2l'+1}(s,\ \eta)]p(s,\ \eta)\ d\eta\ d\phi = 0, \qquad M' \neq M \tag{9.2}$$

$$\int_0^{2\pi}\int_0^{2\pi}[e^{im\phi}V^M_{M+2l}(s,\ \eta)][e^{-im'\phi}W^{M'}_{M'+2l'+1}(s,\ \eta)]p(s,\ \eta)\ d\eta\ d\phi = 0, \tag{9.3}$$

which holds for every torus s = constant. What can be said about the integrals?

$$\int_0^{2\pi}\int_0^{2\pi}[e^{im\phi}V^M_{M+2l}(s,\ \eta)][e^{-im\phi}V^M_{M+2l'}(s,\ \eta)]p(s,\ \eta)\ d\eta\ d\phi, \qquad l \neq l' \tag{9.4}$$

$$\int_0^{2\pi}\int_0^{2\pi}[e^{im\phi}W^M_{M+2l+1}(s,\ \eta)][e^{-im\phi}W^M_{M+2l'+1}(s,\ \eta)]p(s,\ \eta)\ d\eta\ d\phi, \qquad l \neq l'. \tag{9.5}$$

It can be shown that if $p = h_\phi$, then the integrals given by (9.4) and (9.5) do not vanish. In fact for $p = h_\phi$, it can be shown that there is no set of linear combinations of the wave functions $e^{im\phi}\ V^M{}_{M+2l}$ and $e^{im\phi}\ W^M_{M+2l+1}$ which form a complete orthogonal set over the surface of every torus. For $p \neq h_\phi$ nothing at present can be said about the vanishing of the integrals given by (9.4) and (9.5).

As it stands now one can say that, the wave functions $e^{im\phi}\ V^M_{M+2l}(s,\ \eta)$ and $e^{im\phi}\ W^M_{M+2l+1}(s,\ \eta)$ form a partially orthogonal set over every torus. To complete the orthogonal set, one must at present use the Hilbert-Schmidt process for every torus $s = s_0$. That is, for each torus $s = s_0$ and each value of m one must form an orthogonal set from the functions $e^{im\phi}\ V^M_M(s_0\ ,\ \eta)$, $e^{im\phi}\ V^M_{M+2}(s_0\ ,\ \eta)$, $e^{im\phi}\ V^M_{M+4}(s_0\ ,\ \eta)$, $\cdots$ using the Hilbert-Schmidt process, and also form an orthogonal set from the functions $e^{im\phi}\ W^M_{M+1}(s_0\ ,\ \eta)$, $e^{im\phi}\ W^M_{M+3}(s_0\ ,\ \eta)$, $\cdots$. However the functions do form a complete orthogonal set over the surface of the limiting torus $s = s_0$ where $s_0 \gg 1$.

Apart from the difficulty involved in the problem of incomplete orthogonality which is a property of non-separability, the toroidal wave functions derived in this paper can be used to solve any problem of diffraction of acoustic waves by a torus or of a radiating torus. In fact they are useful in solving the vector wave equation, and already have been of practical value in electromagnetic problems.

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