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Abstract: We suggest that the absence of torsion in conventional gravity could in fact be dynamical. A gravitational Meissner effect might produce instanton-like vortices of nonzero torsion concentrated at four-dimensional points; such torsion vortices would be the gravitational analogs of magnetic flux vortices in a type II superconductor. Ordinary torsion-free spacetime would correspond to the field-free superconducting region of a superconductor; a dense phase of "torsion foam" with vanishing metric but well-defined affine connection might be the analog of a normal conductor.

INTRODUCTION

The discovery of the instanton solution [1] to Euclidean SU(2) Yang-Mills theory has led to a new understanding of the Yang-Mills path integral quantization [2] and a new nonperturbative picture of the vacuum [3]. The instanton is characterized by finite action and by a self-dual field strength which is concentrated at a four-dimensional point and falls off rapidly in all directions.

The interesting properties of the Yang-Mills instanton naturally led to a search for analogs in Einstein's theory of gravitation. The most promising gravitational instanton seems to be the metric II of Eguchi and Hanson [4]. This Euclidean metric has self-dual Riemannian curvature concentrated at the origin of the manifold, is asymptotically flat, and has vanishing action. Furthermore, it can be shown to contribute to the spin 3/2 axial anomaly [5]. However, the manifold of this metric is not asymptotically Euclidean (S^3 at ∞), but is only asymptotically locally Euclidean (ALE); the 3-manifold at ∞ is S^3 with opposite points identified [6]. Gibbons and Hawking [7] have now found an entire series of "multiple gravitational instanton" solutions which asymptotically describe S^3 modulo the cyclic group of order k; the curvatures for these solutions are self-dual, have maxima in the interior of the manifold, and fall off rapidly at ∞. Hitchin [8] has in fact suggested the existence of an even larger set of manifolds admitting self-dual

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metrics which asymptotically describe all possible lens spaces of S^3: S^3 modulo the cyclic group of order k, the dihedral group of order k, the tetrahedral group, the cubic group and the icosahedral group. These manifolds may provide a complete classification of asymptotically locally Euclidean gravitational instantons with self-dual curvature.

In summary, we list the parallels between the Yang-Mills instanton solution of ref. [1] and the Einstein metric of ref. [4]:

Yang-Mills [1]	Einstein [4]
Self-dual field strength	Self-dual curvature
$A_\mu \to$ pure gauge at ∞	$g_{\mu\nu} \to$ ALE at ∞
No singularities	Geodesically complete
Finite action	Zero action
Gives spin 1/2 anomaly	Gives spin 3/2 anomaly.

GRAVITATIONAL MEISSNER EFFECT

The concept of gravitational instanton which we have just discussed involves classical Euclidean vacuum Einstein solutions with localized bumps in the curvature. We would now like to explore the idea that other interesting instanton-like objects might occur in a more general Euclidean Einstein-Cartan theory of gravity. The Cartan structure equations for a manifold with a metric are

$$ds^2 = \sum_{a=0}^{3} e^a \, e_a = g_{\mu\nu}(x) dx^\mu \, dx^\nu$$

$$\text{torsion} = T^a = de^a + \omega^a{}_b \wedge e^b$$

$$\text{curvature} = R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \, , \tag{1}$$

where the e^a are vierbein one-forms and the $\omega^a{}_b$ are the connection one-forms. The standard Einstein theory is based upon the use of the Levi-Civita connection on a Riemannian manifold. The torsion-free condition and the metricity condition,

$$T^a = 0 \, , \qquad \omega^a{}_b = -\omega^b{}_a \, , \tag{2}$$

uniquely determine the Levi-Civita connection.

Now we would like to ask whether the condition that the torsion T^a vanish is necessarily fundamental. Is it possible that this could be a <u>dynamical effect</u> of a more general theory, much as the vanishing of the field in a superconductor is a dynamical effect of the Landau-Ginzberg theory?

In order to exhibit the possible parallels between gravity and superconductivity, we recall that the Landau-Ginzberg theory contains a Maxwell field coupled to a scalar field obeying the equation of motion

$$D_\mu D^\mu \varphi = g\, \varphi(|\varphi|^2 - \lambda^2). \tag{3}$$

Near the broken-symmetry vacuum, the magnitude of φ is constant,

$$|\varphi| = \lambda . \tag{4}$$

Far from walls and impurities, one finds in the static limit that φ is covariant constant,

$$D_\mu \varphi = (\partial_\mu + i\, e\, A_\mu)\varphi = 0. \tag{5}$$

Applying a second covariant differentiation to Eq. (5), we find

$$[D_\mu, D_\nu]\varphi = i\, e\, F_{\mu\nu}\, \varphi = 0 , \tag{6}$$

so we conclude that either φ or $F_{\mu\nu}$ must vanish. In a type II superconductor, we obtain the Meissner effect: $F_{\mu\nu} = 0$ almost everywhere with the exception of Abrikosov vortices. Therefore we have

$$\begin{aligned} A_\mu &= -\frac{i}{e}\partial_\mu \theta \\ \varphi &= \lambda\, e^{i\theta} , \end{aligned} \tag{7}$$

where θ is a phase. Now consider a circular path parametrized by $0 \leqslant \theta \leqslant 2\pi$. Such a circle S^1 necessarily encloses a vortex line since as the circle shrinks to zero, the map $S^1 \to U(1)$ changes homotopy type; this is possibly only if $\varphi = 0$ somewhere inside the circle.

A gravitational analog of the Meissner effect might therefore arise from some object which is a covariant constant in a "superconducting region" where Einstein's theory of gravity is valid. We suggest that the appropriate object is the vierbein one-form itself, whose covariant constancy implies vanishing torsion:

$$T^a = D\, e^a = 0 . \tag{8}$$

Requiring the exterior derivative of the torsion also to vanish results in the cyclic identity:

$$dT^a = 0 \rightarrow R^a{}_b \wedge e^b = 0 \rightarrow \epsilon_{ebcd} R^a{}_{bcd} = 0 . \qquad (9)$$

Hence ordinary spacetime, which is torsion-free and whose curvature satisfies the cyclic identity, would be analogous to the field-free region of a superconductor.

We are thus led to the following table showing the parallels between gravitation and superconductivity:

	Gravity	Superconductivity
Bundle Group	$SO(4) \subset GL(4, \mathbb{R})$	$U(1) \subset \mathbb{C}$
Connection 1-form	ω_{ab}	A
Field Strength	$R_{ab} = d\omega_{ab} + \omega_{ac} \wedge \omega_{cb}$	$F = dA$
Covariant Constant Object	$\left\{ e^a{}_\mu \in GL(4, \mathbb{R}) \right.$; $T^a = De^a = 0$	$\left\{ \varphi \in \mathbb{C} \right.$; $D\varphi = 0$
Meissner Effect	$R^a{}_b \wedge e^b = 0$	$F\varphi = 0$
Topological Object	Torsion Vortex (Pointlike)	Magnetic Vortex (Line)
Enclosing Manifold	S^3 = three-sphere	S^1 = circle
Inside Vortex	$T^a \neq 0$, $R^a{}_b \wedge e^b \neq 0$	$F \neq 0$
Vortex Map	Iwasawa Decomposition of $e^a{}_\mu$	$U(1)$: $\varphi = \lvert\varphi\rvert e^{i\theta}$
Topological Charge	Euler and Pontrjagin numbers	Quantized Flux of F
Fundamental Equation	?	Landau-Ginzberg

It is very important to realize that $T^a = 0$ is the condition which locks together the $SO(4)$ gauge group and the group of coordinate transformations. $T^a = 0$ determines a rigid relation between the $SO(4)$ principal bundle and the tangent bundle of the underlying manifold. In this case, the topological invariants of the $SO(4)$ bundle are identifiable with the Euler characteristic and Hirzebruch signature of the base manifold. The latter correspondence may no longer be true if $T^a \neq 0$; as we shall now show, near a (point) torsion vortex, the $SO(4)$ principal bundle is unglued from the tangent bundle and the topological invariants of the two bundles no longer coincide. A torsion vortex in a localized region of Euclidean spacetime would thus combine essential features of both gravitational and Yang-Mills instantons.

CONNECTIONS WITH TORSION

We can make a simple example of a system with torsion by modifying the Levi-Civita connection of a flat Euclidean metric. Let $\{x_\mu\}$ be coordinates on $\mathbb{R}^4$. We let $\rho^2 = x_\mu x_\mu$ and introduce flat-space vierbeins in a nonstandard coordinate system,

$$\begin{aligned} e^0 &= 2\, x_\mu\, dx_\mu = d(\rho^2) \\ e^1 &= x_0\, dx_1 - x_1\, dx_0 + x_2\, dx_3 - x_3\, dx_2 \\ &= \rho^2\, \sigma_x\,, \qquad \text{cyclic.} \end{aligned} \tag{10}$$

The one-forms e^a obey the structure equations

$$de^0 = 0, \qquad de^i = \rho^{-2}(e^0 \wedge e^i + \epsilon_{ijk}\, e^j \wedge e^k) \tag{11}$$

and the σ_k can be expressed in polar coordinates on S^3 if desired. We remark that our coordinate system was chosen to give vierbeins e^a which are C^∞ (have infinitely differentiable coefficients). The Levi-Civita connection one-forms obtained by applying Eqs. (1) and (2) to Eq. (10) are self-dual and singular at the origin:

$$\omega^i{}_0 = \omega^j{}_k = \rho^{-2}\, e^i\,, \qquad \text{cyclic.} \tag{12}$$

The torsion vanishes by construction and the curvature vanishes because the metric was in fact flat.

Now we may introduce torsion by choosing a new regularized connection,

$$\omega^i{}_0 = \omega^j{}_k = \varphi(\rho^2)\, e^i\,, \qquad \text{cyclic,} \tag{13}$$

where $\varphi(\rho^2)$ is some C^∞ function regular at the origin and falling like $1/\rho^2$ at infinity:

$$\varphi(\rho^2) \xrightarrow[\rho^2 \to 0]{} (\text{regular}), \qquad \varphi(\rho^2) \xrightarrow[\rho^2 \to \infty]{} 1/\rho^2\,. \tag{14}$$

The torsion two-forms become

$$T^{o} = 0, \qquad T^{i} = (-\varphi + 1/\rho^{2})[e^{o} \wedge e^{i} + \epsilon_{ijk}\, e^{j} \wedge e^{k}] \tag{15}$$

and so fall off rapidly away from the origin. The curvatures are

$$R^{1}{}_{0} = R^{2}{}_{3} = (\varphi' + \varphi/\rho^{2})e^{0} \wedge e^{1} + 2(\varphi/\rho^{2} - \varphi^{2})e^{2} \wedge e^{3}\,, \quad \text{cyclic.} \tag{16}$$

It can now be verified that both the torsion and the curvature are C^{∞} forms everywhere, even at the origin. Clearly the connections with torsion do not provide a solution to the vacuum Einstein equations. The appearance of nonvanishing torsion is a necessary consequence of the presence of a zero in the vierbein and the regularization of the connection at the origin.

A typical choice for φ such as

$$\varphi = \rho^{-2}(1 - e^{-\rho^{2}/\lambda}) \tag{17}$$

in fact gives finite action,

$$\int R\, g^{\frac{1}{2}} d^{4}x = \frac{1}{4}\int_{M} R^{ab} \wedge e^{c} \wedge e^{d}\, \epsilon_{abcd} = \lambda^{2}/16\,. \tag{18}$$

(If one chooses φ so that $\omega^{a}{}_{b}$ corresponds to the SU(2) Yang-Mills instanton connection, the action turns out to be infinite.) The topological invariants of our manifold and the bundle for which (13) is the connection are now drastically altered. So long as $\varphi \neq \rho^{-2}$, we find that the "Euler characteristic" volume term and surface corrections are [9]

$$\chi_{V} = -1\,, \qquad \chi_{S} = 1\,. \tag{19}$$

While our original flat $\mathbb{R}^{4}$ connection had $\chi = 1$, our new system has

$$\chi = \chi_{V} + \chi_{S} = 0\,. \tag{20}$$

Similarly, we find for the new Pontrjagin number

$$P_{1} = -\frac{1}{8\pi^{2}}\int_{M} \mathrm{Tr}\, R \wedge R = +2\,. \tag{21}$$

The flat connection had Hirzebruch signature $\tau = P_{1}/3 = 0$. The altered topological invariants prove that our new SO(4) principal bundle is not isomorphic to the bundle of orthonormal tangent-space frames of Euclidean space.

CONCLUSIONS

We remark that there may be two phases of the system we have described. Since the presence of too many magnetic field vortices will cause a superconductor to undergo a phase transition to the normal state with $\langle\varphi\rangle = 0$, we can also conceive of a dense phase of torsion vortices with $\langle e^a{}_\mu\rangle = 0$ everywhere. In the presence of this "torsion foam," we may no longer have a sensible metric; only the SO(4) affine connection would remain well-defined. Physically, the "torsion foam" phase might dominate in the early universe or at very short distances [10].

The problem now is to develop a dynamical theory which has a stable torsion vortex as a solution. Such theories could of course include mechanisms (e.g. localized nonmetricity) more general than those discussed here. We would expect a theory with torsion vortices in the Euclidean regime to have a profound effect on our understanding of quantum gravity and its relation to elementary particle physics.

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