TORSION HOMOLOGY GROWTH AND CYCLE COMPLEXITY OF ARITHMETIC MANIFOLDS

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ABSTRACT. Let M be an arithmetic hyperbolic 3-manifold, such as a Bianchi manifold. We conjecture that there is a basis for the second homology of M, where each basis element is represented by a surface of 'low' genus, and give evidence for this. We explain the relationship between this conjecture and the study of torsion homology growth.

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1. INTRODUCTION

In this paper we formulate and discuss a conjecture about topological complexity of arithmetic manifolds, i.e. locally symmetric spaces associated to arithmetic groups. This conjecture is closely related to studying growth of torsion in homology. Roughly speaking, the conjecture is that

> *homology classes on arithmetic manifolds are represented by cycles of low complexity.*

From a strictly arithmetic perspective, what may be most interesting is that our proofs suggest that the *topological* complexity of these cycles reflect the *arithmetic* complexity of the (Langlands-)associated varieties (i.e. the height of equations needed to define the varieties).

We will study this in detail in a simple interesting case, namely, that of arithmetic hyperbolic 3-manifolds. To simplify matters as far as possible, we study only sequences that are coverings of a fixed base manifold M_0.

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1.1. **Conjecture.** *There is a constant $C = C(M_0)$ such that, for any arithmetic congruence hyperbolic 3-manifold $M \to M_0$ of volume V, there exist immersed surfaces S_i of genus $\leq V^C$ such that the $[S_i]$ span $H_2(M, \mathbb{R})$.*

Thus the conjecture is related to understanding the Gromov-Thurston norm on H_2; it can also be phrased in terms of a 'harmonic' norm on H_2 whose definition uses the hyperbolic metric. See §4. It follows from Gabai's generalization [25, p. 3] of Dehn's lemma to higher genus that we may as well ask the S_i to be *embedded* in Conjecture 1.1.

It is plausible, although we are not sure, that this conjecture is really a special feature of arithmetic manifolds. For the purpose of this paper, "arithmetic manifold" means more properly "arithmetic congruence manifold." Firstly, our proofs certainly use number theory heavily. Secondly, it seems that any 'naive' analysis yields only an exponential bound on $[S_i]$ in terms of V or the topological complexity of M – indeed, work in progress of Jeff Brock and Nathan Dunfield indeed strongly suggests that this exponential bound cannot be improved. Finally, numerical data (see e.g. [12] or [52]), although far from conclusive, also appears to differ between nonarithmetic and arithmetic cases. See §1.4 for a little further discussion.

This conjecture is motivated by the study of torsion classes, and indeed in trying to understand the obstruction to extending previous results (see [7]) on 'strongly acyclic' coefficient systems to the case of the trivial local system. We will prove:

1.2. **Theorem.** *Let $(M_i \to M_0)_{i\in\mathbb{N}}$ be a sequence of arithmetic congruence hyperbolic 3-manifolds s.t. M_0 is compact and $V_i = \mathrm{vol}(M_i)$ goes to infinity. Assume the following two conditions are satisfied:*

(i) *'Few small eigenvalues': For every $\varepsilon > 0$ there exists some positive real number c such that*

$$\limsup_{i\to\infty} \frac{1}{V_i} \sum_{0<\lambda\leq c} |\log\lambda| \leq \varepsilon. \tag{1.2.1}$$

Here λ ranges over eigenvalues of the 1-form Laplacian Δ on M_i. Indeed we may even replace the condition by the condition that

$$\lim_{i\to\infty} \frac{1}{V_i} \sum_{0<\lambda\leq V_i^{-\delta}} |\log\lambda| = 0 \tag{1.2.2}$$

for every $\delta > 0$.

(ii) *'Small Betti numbers': $b_1(M_i, \mathbb{Q}) = o(\frac{V_i}{\log V_i})$.*

Then, if Conjecture 1.1 holds, as $i \to \infty$, we have:

$$\frac{\log \# H_1(M_i, \mathbb{Z})_{\mathrm{tors}}}{V_i} \longrightarrow \frac{1}{6\pi}. \tag{1.2.3}$$

For the proof see §2 (it also uses results from §3 and §4). Heuristically, we expect (i) to be valid with very few exceptions, and (ii) to be always valid; see [16, 6] for evidence, and also [39] in a somewhat different direction.

The proof of this Theorem also gives a partial converse. For instance, if we suppose (1.2.3) and a strengthening of (ii) – that the Betti numbers b_1 actually remain bounded – then (i) must be true, and also a weak form of the Conjecture, with "polynomial" replaced by "subexponential," must hold.

Now the central result of our paper:

1.3. **Theorem.** *Conjecture 1.1 is true in the two following cases:*

(i) *When M_0 arises from a division algebra $D \otimes F$ where D is a quaternion algebra over $\mathbb{Q}$ and F is an imaginary quadratic field, M is defined by a principal congruence subgroup*[1]*, and all the cohomology of M is of base-change type (§6);*

(ii) *When M_0 is a Bianchi manifold (for us: an adelic manifold whose components are of the form $\Gamma_0(\mathfrak{n})\backslash\mathbb{H}^3$), and the cuspidal cohomology of M is 1-dimensional, associated to a non-CM elliptic curve of conductor $\mathfrak{n}$, for which we assume the equivariant BSD conjecture (see (8.7.3)) and the Frey–Szpiro conjecture (see [29, F.3.2]).*

What the proof of (ii) really gives is a relationship between the complexity of H_2-cycles and the height of the elliptic curve (i.e., the minimal size of A, B so it can be expressed as $y^2 = x^3 + Ax + B$.) Thus, "the topological complexity of cycles in H_2 reflect the arithmetic height of E." This may be a general phenomenon (it was also suggested in [17]).

A few words on the conjectures which appear in (ii): The Frey–Szpiro conjecture is a conjecture in Diophantine analysis which follows from the ABC conjecture (and thus is very strongly expected from a heuristic viewpoint). It asserts that the height of an elliptic curve cannot be too large relative to its conductor. Moreover, for the purposes of establishing growth of torsion, as in Theorem 1.2, we do not need the full strength of Conjecture 1.1; a weaker version with sub-exponential bounds would suffice, and correspondingly a very weak "sub-exponential" version of Frey–Szpiro would do.

We note that both case (i) and case (ii) are quite common over imaginary quadratic fields! For (i), we present data in §9.1: e.g. for the first 40 rational primes p that are inert in $\mathbb{Q}(\sqrt{-7})$, the cohomology of $\Gamma_0(\mathfrak{p})$, where $\mathfrak{p} = (p)$, is entirely base change in all but 6 cases. For (ii) we refer to [51, p.17]; in the data there, at prime level, situation (ii) occurs in the majority of cases where $b_{1,!} > 0$, see also §9.2.

Also, (i) and (ii) illustrate two different extremes of the Theorem:

For (i) it's easy to think of candidate surfaces in H_2 — the challenge is, rather, that the dimension of H_2 is increasing rapidly and it is not clear that the candidate surfaces span 'enough' homology. In fact, our result applies to all M, but bounds only the regulator of the 'base-change part' of cohomology. One can see (i) as an effectivization of a result of Harder, Langlands and Rapoport [28], although they work with Hilbert modular surfaces rather than hyperbolic 3-manifolds. The main global ingredient is a (polynomially strong) quantitative form of the 'multiplicity one' theorem in the theory of automorphic forms but there is also (surprisingly) a nontrivial local ingredient: one needs good control on (e.g.) support of matrix coefficients of supercuspidal representations. In fact, one motivation to study example (i) is that our result shows that the regulator R_2 (see §2) grows subexponentially, whereas this was not at all clear by looking at numerical evidence! — see §9.3. (There is actually another setting where H_2 grows quickly for easily comprehensible reasons – the setting of "oldforms," whereby one pulls back forms from a surface of lower level. In that case, it is not difficult to see that the complexity of the cycles remains controlled.)

For (ii) the challenge is instead that there are no obvious cycles in H_2; we work with H_1 and modular symbols, and dualize; the main point is to replace a modular symbol by the sum of two well-chosen others to avoid unpleasant dominators. The equivariant BSD conjecture enters to compute cycle integrals over modular symbols. The Szpiro conjecture enters to give a lower bound on the period of an elliptic curve. We note that

[1] This is not an onerous restriction; is easy to reduce the conjecture for other standard subgroup structures, such as Γ_0-structure, to this case.

this result is closely related to prior work of Goldfeld [26], although the techniques of proof are necessarily different owing to the lack of an algebraic structure.

1.4. **The role of arithmeticity.** As we have mentioned, it seems plausible that Conjecture 1.1 is really specific to arithmetic. It would be desirable to have a specific counterexample in this direction, that is to say, exhibiting the behavior that Conjecture 1.1 disallows in the arithmetic case.

From the point of view of mirroring the situation of this paper, it would be ideal to have an answer to the following:

Question. *Can one produce a sequence of hyperbolic manifolds M_i with the following properties?*

- *the volumes of M_i go to infinity (or, even better, the sequence (M_i) BS-converges toward $\mathbb{H}^3$, see §2.3),*
- *The injectivity radii of M_i remain bounded below, and*
- *in any basis for $H_2(M_i,\mathbb{Z})$, at least one basis element* cannot *be represented by a surface of genus $\leq (\mathrm{vol} M_i)^i$?*

Jeff Brock and Nathan Dunfield have made progress in constructing such a sequence.

Here is some intuition as to why arithmeticity might play a role: In general, generators for $H_2(M,\mathbb{Z})$ might be of exponential complexity. This comes down to analyzing the kernel of a matrix M that expresses adjacency between 1-cells and 2-cells in a triangulation. Now, even given a matrix $A \in \mathrm{M}_n(\mathbb{Z})$ of zeroes and ones, generators for the kernel of A on $\mathbb{Z}^n$ could have exponentially large (in n) entries. However, in the arithmetic case, the existence of Hecke operators means that the 'adjacency matrix' A is (heuristically speaking) forced to commute with many other symmetries. One might expect this to reduce its effective size — a phenomenon that is perhaps parallel to the observed difference between eigenvalue statistics in the arithmetic and nonarithmetic case (see [31] for discussion).

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2. RELATIONSHIP TO TORSION AND THE PROOF OF THEOREM 1.2

In this section and the next, we will give the proof of Theorem 1.2. We first recall the definition of 'regulators' from a prior paper [7] by the first- and last- named author (N.B. and A.V.)

2.1. **Regulators.** Let M be a compact Riemannian manifold of dimension n. We define the H_j*-regulator* of M as the volume of $H_j(M,\mathbb{Z})$ with respect to the metric on $H_j(M,\mathbb{R})$ defined by harmonic forms — the 'harmonic metric.' That is,

$$R_j(M) = \frac{\det\left(\int_{\gamma_k}\omega_\ell\right)}{\sqrt{\det\langle\omega_k,\omega_\ell\rangle}} \tag{2.1.1}$$

where $\gamma_k \in H_j(M,\mathbb{Z})$ project to a basis for $H_j(M,\mathbb{Z})/H_j(M,\mathbb{Z})_{\mathrm{tors}}$ and ω_ℓ are a basis for the space of L^2 harmonic forms on M. Note that $R_0(M) = \frac{1}{\sqrt{\mathrm{vol}(M)}}$, $R_n(M) = \sqrt{\mathrm{vol}(M)}$ and, by Poincaré duality, we have:

$$R_j(M)\cdot R_{n-j}(M) = 1.$$

2.2. A celebrated theorem of Cheeger and Müller [19, 43] relates the torsion homology groups and the regulators to the analytic torsion of M. In the special case $n=3$, the theorem of Cheeger and Müller implies that

$$|H_1(M,\mathbb{Z})_{\mathrm{tors}}|\cdot\frac{R_0R_2}{R_1R_3}=T_{\mathrm{an}}(M)^{-1},$$

where $T_{\mathrm{an}}(M)$ is the analytic torsion of the manifold M. We furthermore note that

$$\frac{R_0R_2}{R_1R_3}=\frac{R_2^2}{\mathrm{vol}(M)}=\frac{1}{R_1^2\mathrm{vol}(M)}.$$

2.3. **Benjamini-Schramm convergence.** For a hyperbolic manifold M we define $M_{<R}$ to be the R-thin part of M, i.e. the part of M where the local injectivity radius is $<R$. Now let $(M_i\to M_0)_{i\in\mathbb{N}}$ be a sequence of finite covers of a fixed compact hyperbolic 3-manifolds. Following [1], we say that the sequence $(M_i)_{i\in\mathbb{N}}$ *BS-converges* to $\mathbb{H}^3$ if for all $R>0$ one has $\mathrm{vol}(M_i)_{<R}/\mathrm{vol}(M_i)\to 0$. It follows from [1, Theorem 1.12] that if $(M_i\to M_0)_{i\in\mathbb{N}}$ is a sequence of arithmetic congruence compact hyperbolic manifolds s.t. $V_i=\mathrm{vol}(M_i)$ goes to infinity then $(M_i)_{i\in\mathbb{N}}$ BS-converges to $\mathbb{H}^3$. The proof of Theorem 1.2 then follows from the following three ingredients:

2.4. **First ingredient.** We shall show in the next section (Proposition 3.1) that there exists some constant C s.t.

$$R_1(M_i)\ll \mathrm{vol}(M_i)^{Cb(M_i)}, \tag{2.4.1}$$

where $b(M)=b_1(M,\mathbb{Q})=b_2(M,\mathbb{Q})$ is the Betti number. In particular, so long as $b(M_i)$ grows as $o(\frac{V_i}{\log V_i})$, the subexponential growth of $R_1(M_i)$ follows. (Here and below, subexponential means subexponential in V_i).

2.5. **Second ingredient.** We will also show in §4.6 that, assuming Conjecture 1.1, there exists a constant C such that

$$R_2(M_i)\ll \mathrm{vol}(M_i)^{Cb(M_i)}. \tag{2.5.1}$$

So here again, as long as $b(M_i)$ grows as $o(\frac{V_i}{\log V_i})$, the subexponential growth of $R_2(M_i)$ *follows* from Conjecture 1.1.

2.6. **Third ingredient.** Finally, the condition 'few small eigenvalues' from Theorem 1.2 implies that

$$\frac{\log T_{\mathrm{an}}(M_i)}{V_i}\to\tau^{(2)}_{\mathbb{H}^3}=-\frac{1}{6\pi}. \tag{2.6.1}$$

It follows from the definition of analytic torsion and well known properties of the spectrum of the Laplace operators on Riemannian 3-manifolds (see e.g. in [7]) that it is enough to prove that

$$\frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_0^{+\infty}t^{s-1}\frac{1}{V_i}\int_{M_i}\Big(\mathrm{tr}e^{-t\Delta^{(2)}}(\tilde{x},\tilde{x})-(\mathrm{tr}e^{-t\Delta_i}(x,x)-b_1(M_i))\Big)\,dx\,dt\to 0. \tag{2.6.2}$$

Here Δ_i, resp. $\Delta^{(2)}$, is the Laplace operator on square-integrable 1-forms on M_i, resp. $\mathbb{H}^3$, and $\tilde{x}$ is an arbitrary lift of x to $\mathbb{H}^3$.

Since $b(M_i)$ grows as $o(\frac{V_i}{\log V_i})$ the proof of the limit (2.6.2) follows the same lines as [7, Theorem 4.5] under the assumptions that

(1) the injectivity radius of M_i goes to infinity; and

(2) there exists some positive c such that for all M_i the lowest eigenvalue of Δ_i is bigger than c.

The first assumption is used to handle the 'small t' contribution to the limit (2.6.2). In fact the proof only uses the fact that the local injectivity radius is 'almost everywhere' going to infinity, the condition is precisely that the sequence $(M_i)_{i\in\mathbb{N}}$ BS-converges to $\mathbb{H}^3$. We refer to [1, §8 and 9] for more details in particular on how to bound the size of the heat kernel at the bad points.

The second assumption is used to handle the 'large t' contribution; it more precisely implies that for sufficiently large t each individual term of the difference in (2.6.2) can be made arbitrary small. However this spectral gap assumption never holds for the trivial coefficient system; we replace that instead by assumption (i) of Theorem 1.2.

Let ε and c be as in assumption (i) of Theorem 1.2. Without loss of generality, $c < 1$. Spectral expansion on the compact manifold M_i and classical Sobolev estimates yield that for any $t \geq 1$ we have:

$$\begin{aligned}\int_{M_i} \mathrm{tr} e^{-t\Delta_i'}(x,x) &= \sum_{0<\lambda} e^{-\lambda t} \\ &\ll \sum_{0<\lambda\leq c} e^{-t\lambda} + e^{-c(t-1)} \sum_{\lambda>c} e^{-\lambda} \\ &\ll \sum_{0<\lambda\leq c} e^{-t\lambda} + e^{-c(t-1)} V_i,\end{aligned}$$

where we have denoted by Δ_i' the restriction of Δ_i to the orthogonal complement of its kernel and the implicit constant does not depend on i and x; we used the fact that the trace of $e^{-\Delta_i'}$ on M_i can be bounded by a multiple of V_i. To conclude the proof we just have to remark that for any $T \geq 1$ fixed

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_T^{+\infty} t^{s-1} \sum_{0<\lambda\leq c} e^{-t\lambda} dt &= \sum_{0<\lambda\leq c} \int_T^{\infty} e^{-t\lambda} \frac{dt}{t} \\ &= \sum_{0<\lambda\leq c} \int_T^{\infty} \frac{e^{-t}}{t} dt + \sum_{0<\lambda\leq c} \int_T^{\infty} \frac{e^{-t\lambda} - e^{-t}}{t} dt \\ &< \big(\text{number of eigenvalues in } (0,c]\,\big) e^{-T} + \sum_{0<\lambda\leq c} \log|\lambda|.\end{aligned}$$

There is a constant A so that the number of eigenvalues in $(0,c]$ is $\leq AV_i$, and we may then choose T sufficiently large so that $Ae^{-T} < \varepsilon$. Thus the integral above contributes at most 2ε to the limit. Using (1.2.1), this holds for every ε, so the proof of (2.6.2) follows as in [7].

We have now completed the proof of the Theorem, but assuming (i) in the stronger form (1.2.1). To see that (1.2.2) suffices:

2.7. **Lemma.** *Assume that* (1.2.2)*. Then, for every* $\varepsilon > 0$ *there exists some positive real number c such that*

$$\lim_{i\to\infty} \frac{1}{V_i} \sum_{0<\lambda\leq c} |\log \lambda| \leq \varepsilon.$$

Here λ *ranges over eigenvalues of the* 1*-form Laplacian* Δ_i *for* M_i*.*

Proof. In [1, Theorem 1.12] a quantitative version of BS-convergence is proven; in particular there exist positive constants c and δ such that for every i one has

$$\mathrm{vol}(M_i)_{<c\log V_i} \leq V_i^{1-\delta}.$$

Fix $M = M_i$ and $V = V_i$. Employing the trace formula — as in [47, 50] — with a test function supported in an interval of length $1/(c' \log V)$ for some positive constant c', and using the estimates of [1, Lemma 7.23], we can show that for every $k \in \mathbb{N}$ the number of eigenvalues of Δ between $\frac{k}{c' \log V}$ and $\frac{k+1}{c' \log V}$ is bounded by some uniform constant times $V/\log V$. It follows that

$$\prod_{\frac{k}{c' \log V} < \lambda \leq \frac{k+1}{c' \log V}} \lambda \gg \left(\frac{k}{\log V} \right)^{\frac{V}{\log V}}.$$

Taking a further product for $k = 1, \ldots, \alpha \log V$ for some positive α (and using Stirling's formula) we get that

$$\prod_{\frac{1}{c' \log V} < \lambda \leq \alpha} \lambda \gg \left(\frac{(\alpha \log V)!}{(\log V)^{\alpha \log V}} \right)^{\frac{V}{\log V}} \gg e^{-o(1)V} \tag{2.7.1}$$

as $\alpha \to 0$.

Now given a positive real number δ we similarly have:

$$\prod_{V^{-\delta} < \lambda \leq \frac{1}{\log V}} \lambda \gg \left(\frac{1}{V^{\delta}} \right)^{\frac{V}{\log V}} = e^{-\delta V}. \tag{2.7.2}$$

The lemma follows from (2.7.1), (2.7.2) and the 'few eigenvalues' assumption. □

3. Bounding $R_1(M)$

Here we prove (2.4.1) that was used in the proof of Theorem 1.2. Let M_0 be a complete Riemannian n-dimensional manifold of pinched nonpositive sectional curvature. We more generally prove the following:

3.1. **Proposition.** *If M varies through a sequence of finite coverings of a fixed compact manifold M_0, we have:*

$$|R_1(M)| \ll \mathrm{vol}(M)^{Cb(M)}.$$

Here the implicit constants only depend on M_0.

The following is a consequence of Sobolev estimates:

3.2. **Lemma.** *Let M be as in Proposition 3.1, let $S \subset M$ be a k-submanifold of (Riemannian) volume v and let ω be an L^2-normalized harmonic differential k-form on M. Then:*

$$\int_S \omega \ll v.$$

We now explain how to prove Proposition 3.1 using Lemma 3.2.

3.3. Fix M_0 and let Γ_0 be the fundamental group of M_0, let S be a set of generators of Γ_0 and let d_0 be the cardinality of S.

To any finite covering $M \to M_0$ — corresponding to a finite index subgroup $\Gamma < \Gamma_0$ — we associate the Schreier graph $\mathcal{G}(\Gamma_0/\Gamma, S)$; it is a finite cover of degree $[\Gamma_0 : \Gamma]$ of the wedge product of d_0 circles. Computing the Euler characteristic we conclude that $\mathcal{G}(\Gamma_0/\Gamma, S)$ has the homotopy type of the wedge product of d circles where:

$$(d-1) = [\Gamma_0 : \Gamma](d_0 - 1).$$

The group Γ is therefore generated by at most d elements; moreover each of these elements has length at most $[\Gamma_0 : \Gamma]$ in the S-word metric of Γ_0.

Since Γ_0 with the S-word metric is quasi-isometric to the universal cover $\widetilde{M}$ of M with its induced Riemannian metric we have the following:

3.4. **Lemma.** *There exists a constant $c = c(M_0)$ such that Γ is generated by at most $c[\Gamma_0 : \Gamma]$ elements which can be represented by closed geodesics of length $\leq c[\Gamma_0 : \Gamma]$.*

Note that up to a constant (depending only on M_0) $\mathrm{vol}(M)$ equals $[\Gamma_0 : \Gamma]$. Hadamard's inequality, Lemma 3.2 and Lemma 3.4 therefore imply Proposition 3.1. (Note that, in the definition (2.1.1) of the regulator, replacing the γ_j by elements γ'_j that generate a finite index sublattice of homology only increases the regulator.)

3.5. Assuming Conjecture 1.1 we can apply a similar scheme to bound $R_2(M)$, but we now need to compare two different norms on $H_2(M,\mathbb{R})$. This is the purpose of the next section.

4. Relationship of the harmonic norm and the Gromov–Thurston norm

In this section, M will denote a compact hyperbolic 3-manifold. The second homology group $H_2(M,\mathbb{R})$ is equipped with two natural norms: the Gromov–Thurston norm, which measures the number of simplices needed to present a cycle, and the harmonic norm, which arises from the identification of $H_2(M,\mathbb{R}) \simeq H^1(M,\mathbb{R})$ with harmonic 1-forms on M. We will relate the two norms and use it to prove (2.5.1), used in the proof of Theorem 1.2.

More precisely: if $\delta \in H_2(M,\mathbb{R})$ we set

$$\|\delta\|_{GT} = \inf\left\{\sum |n_k| \;\middle|\; [\sum n_k\sigma_k] = \delta \text{ where } \sum n_k\sigma_k \text{ is a singular chain}\right\}.$$

Note that Gabai [25, Corollary 6.18] shows that if $\delta \in H_2(M,\mathbb{Z})$ then

$$(4.0.1)\quad \|\delta\|_{GT} = 2\min\left\{ \sum_{i,\ \chi(S_i)<0} |\chi(S_i)| \;\middle|\; \begin{array}{l} S = \cup_i S_i, \text{ where } S_i \text{ is a properly embedded} \\ \text{connected surface in } M \text{ and } [S] = \delta \text{ in } H_2(M,\mathbb{Z}) \end{array} \right\}.$$

Note that, since M is compact hyperbolic, we may suppose that each S_i actually a surface of genus ≥ 2, since if S is either a sphere or a torus the image of $H_2(S,\mathbb{Z})$ in $H_2(M,\mathbb{Z})$ will be trivial. In particular, to prove the theorem, it is enough to exhibit a set in $H_2(M,\mathbb{Z})$ of full rank, and with polynomially bounded Gromov-Thurston norm.

We also define $\|\delta\|_{L^2} = \|\omega\|_{L^2}$ where ω is the L^2 harmonic 1-form on M which is dual to δ, i.e.

$$\int_M \omega \wedge \alpha = \int_\delta \alpha,$$

for every closed 2-form α on M. Note in particular that

$$(4.0.2)\qquad \|\delta\|_{L^2}^2 = \left|\int_\delta *\omega\right|.$$

In this section we compare $\|\cdot\|_{L^2}$ and $\|\cdot\|_{GT}$. In particular, we prove the following:

4.1. **Proposition.** *If M varies through a sequence of finite coverings of a fixed manifold M_0, we have:*

$$\frac{1}{\mathrm{vol}(M)}\|\cdot\|_{GT} \ll \|\cdot\|_{L^2} \ll \|\cdot\|_{GT}.$$

Proof. The proof occupies §4.2–§4.5 below.

4.2. Given a cycle $\delta \in H_2(M,\mathbb{R})$ with $\|\delta\|_{GT} \le 1$, we may write (see e.g. [46, Theorem 11.4.2 and the Remark following it])

$$\delta = \sum_k n_k \sigma_k$$

where each σ_k is a *straight* simplex (or triangle), i.e. the image of the convex hull of 3 points in $\mathbb{H}^3$, and $\sum_k |n_k| \le 1$. Now if α is a harmonic 2-form, then

$$\int_{\sigma_k} \alpha \ll ||\alpha||_\infty \mathrm{area}(\sigma_k) \le \pi ||\alpha||_\infty \ll ||\alpha||_2$$

is uniformly bounded so that $\int_\delta \alpha \ll \|\alpha\|_2$. Since we can compute the harmonic norm of δ as the operator norm of $\alpha \mapsto \int_\delta \alpha$, this has shown the second inequality of Proposition 4.1; we pass now to the first inequality.

4.3. In the reverse direction, suppose given an element $\delta \in H_2(M,\mathbb{R})$ of harmonic norm ≤ 1; equivalently, its image under $H_2(M,\mathbb{R}) \simeq H^1(M,\mathbb{R})$ is represented by a harmonic 1-form ω of L^2-norm ≤ 1.

Fix a triangulation K of M by lifting a triangulation K_0 of M_0. We can suppose that every edge has length ≤ 1 and every triangle has area ≤ 1. Let K' be the dual cell subdivision. We denote by

$$\langle , \rangle : C_i(K,\mathbb{Z}) \times C_{3-i}(K',\mathbb{Z}) \to \mathbb{Z}$$

the (integer) intersection number; it canonically identifies $C_{3-i}(K',\mathbb{Z})$ with the dual $C^i(K,\mathbb{Z}) = C_i(K,\mathbb{Z})^*$ of $C_i(K,\mathbb{Z})$. Furthermore, the boundary homomorphism

$$\partial : C_{3-i}(K',\mathbb{Z}) \to C_{3-i-1}(K',\mathbb{Z})$$

is (up to sign) dual to the corresponding boundary homomorphism $C_{i+1}(K,\mathbb{Z}) \to C_i(K,\mathbb{Z})$, in other words ∂ identifies (up to sign) with the coboundary homomorphism $C^i(K,\mathbb{Z}) \to C^{i+1}(K,\mathbb{Z})$. Both complexes $C_\bullet(K,\mathbb{Z})$ and $C_\bullet(K',\mathbb{Z})$ compute $H_\bullet(M,\mathbb{Z})$. Now the latter identifies with $C^{3-\bullet}(K,\mathbb{Z})$ and computes $H^{3-\bullet}(M,\mathbb{Z})$. This realizes the Poincaré duality.

4.4. Consider, then, the two-cycle

$$Z := \sum_e \left(\int_e \omega \right) e^* \in C^1(K,\mathbb{R}) = C_2(K',\mathbb{R}).$$

Since ω is closed, it follows from Stokes formula that

$$\partial Z = \pm \sum_t \left(\int_{\partial t} \omega \right) t^* = 0.$$

On the other hand, Z represents the image of the class of $[\omega]$ under the Poincaré duality pairing

$$H^1(M,\mathbb{R}) \xrightarrow{\sim} H_2(M,\mathbb{R}).$$

4.5. Subdivide K_0' to get a triangulation T_0 of M_0 and lift this triangulation to a triangulation T of M. There exists a constant c which only depends on M_0 (and T_0) such that the number of triangles of T_0 in each cell of K_0' dual to an edge of K_0 is bounded by c. Then the number of triangles of T in each cell of K' is $\le c[M : M_0]$, where $[M : M_0]$ is the degree of the cover $M \to M_0$. By definition of the Gromov–Thurston norm we conclude that

$$\|[Z]\|_{GT} \ll ||\omega||_\infty \mathrm{vol}(M) \ll \mathrm{vol}(M),$$

the last by the Sobolev inequality. Proposition 4.1 now follows. □

4.6. **Relation with $R_2(M)$.** Let us now assume Conjecture 1.1. Then each $[S_i]$ has Gromov–Thurston norm — and therefore, by Proposition 4.1, harmonic norm — which is bounded by a polynomial in $\mathrm{vol}(M)$. Thus here again Hadamard's inequality shows that

$$R_2(M) \ll \mathrm{vol}(M)^{Cb(M)},$$

where $b(M)$ is the Betti number.

We have now concluded the proof of Theorem 1.2.

5. Arithmetic manifolds

Let F be an imaginary quadratic field. We consider arithmetic manifolds associated to an algebraic group $\mathbf{G}$ over $\mathbb{Q}$ such that $\mathbf{G}(\mathbb{R}) = \mathrm{PGL}_2(\mathbb{C})$. In this paper we are interested in the two examples:

- $\mathbf{G}_1 = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$ mod center;
- $\mathbf{G}_2 = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_1(D')$ mod center, where D' is a division algebra over F of the form $D' = D \otimes F$, with D a quaternion algebra over $\mathbb{Q}$.

Let $\mathbb{A}$ and $\mathbb{A}_F$ be the ring of adèles of $\mathbb{Q}$ and F respectively. We denote by $\mathbb{A}_f$ and $\mathbb{A}_{F,f}$ the corresponding rings of finite adèles. We also write $F_\infty = F \otimes \mathbb{R} \simeq \mathbb{C}$. In remaining part of this paper $\mathbf{G}$ stands for either $\mathbf{G}_1$ or $\mathbf{G}_2$.

In the second case $\mathbf{G}_2$ admits a $\mathbb{Q}$-subgroup which will be of importance to us: Let $\mathbf{H} = \mathrm{GL}_1(D)$ modulo center, considered as a subgroup of $\mathbf{G}_2$. Thus, $\mathbf{H}(\mathbb{R}) = \mathrm{PGL}_2(\mathbb{R})$.

5.1. **The arithmetic manifold $X(\mathfrak{n})$.** Let $\mathfrak{n}$ be an ideal of the ring of integers $\mathcal{O}$ of F. We associate to $\mathfrak{n}$ a compact open subgroup

$$K(\mathfrak{n}) = \prod_v K_v(\mathfrak{n}) \subset \mathbf{G}(\mathbb{A}_f)$$

in the following way. If $\mathbf{G} = \mathbf{G}_1$ as usual we define $K(\mathfrak{n}) = K_1(\mathfrak{n})$ as the subgroup corresponding — after restriction of scalars and mod center — to

$$\{g \in \mathrm{GL}_2(\widehat{\mathcal{O}}) \,:\, g \equiv I_2 \ (\mathfrak{n}\widehat{\mathcal{O}})\}. \tag{5.1.1}$$

Here $\widehat{\mathcal{O}}$ is the closure of $\mathcal{O}$ in $\mathbb{A}_{F,f}$. In this case, we also define $K_0(\mathfrak{n})$ in the usual way

$$K_0(\mathfrak{n}) = \{g \in \mathrm{GL}_2(\widehat{\mathcal{O}}) \,:\, g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (\mathfrak{n}\widehat{\mathcal{O}})\}.$$

Now if $\mathbf{G} = \mathbf{G}_2$ we make a corresponding definition of $K(\mathfrak{n})$ as follows: Regard $\mathbf{G}_2(\mathbb{A}_f)$ as the $\mathbb{A}_{F,f}$-poitns of $\mathrm{GL}_1(D')$ mod center. In the paragraph that follows, products over places v will be over places of F. First make an arbitrary choice $K = \prod_v K_v \subset \mathbf{G}_2(\mathbb{A}_f)$ of a compact open subgroup such that K_v is hyperspecial at each unramified place v. At those places we may then fix isomorphisms $\phi_v : \mathbf{G}_2(\mathbb{Q}_v) \to \mathrm{PGL}_2(F_v)$ such that $\phi_v(K_v) = \mathrm{PGL}_2(\mathcal{O}_v)$, where $\mathcal{O}_v \subset F_v$ is the ring of integers. We finally define $K(\mathfrak{n}) = \prod_v K_v(\mathfrak{n})$ by setting $K_v(\mathfrak{n}) = K_v$ at each ramified place and $K_v(\mathfrak{n}) = \phi_v^{-1}(K_{1,v}(\mathfrak{n}))$ at each unramified place, where $K_{1,v}(\mathfrak{n})$ is the local analog of (5.1.1). We will also suppose, for at least one ramified place v, the subgroup K_v is sufficiently small so as to force any group $\mathbf{G}(\mathbb{Q}) \cap K(\mathfrak{n})$ to be torsion-free.

Given any compact open subgroup $K \subset \mathbf{G}(\mathbb{A}_f)$, we define the arithmetic manifold $X(K)$ by

$$X(K) = \mathbf{G}(F)\backslash(\mathbb{H}^3 \times \mathbf{G}(\mathbb{A}_f))/K.$$

We simply denote by $X(\mathfrak{n})$ the arithmetic manifold $X(K(\mathfrak{n}))$.

5.2. **Connected components of $X(\mathfrak{n})$.** The connected components of $X(\mathfrak{n})$ can be described as follows. Write $\mathbf{G}(\mathbb{A}_f) = \sqcup_j \mathbf{G}(\mathbb{Q}) g_j K(\mathfrak{n})$; then

$$X(\mathfrak{n}) = \sqcup_j \Gamma_j \backslash \mathbb{H}^3,$$

where Γ_j is the image in $\mathrm{PGL}_2(\mathbb{C})$ of $\Gamma'_j = \mathbf{G}(\mathbb{Q}) \cap g_j K(\mathfrak{n}) g_j^{-1}$. We let

$$Y(\mathfrak{n}) = \Gamma \backslash \mathbb{H}^3$$

denote the connected component of $X(\mathfrak{n})$ associated to the class $g_j = e$ of the identity element so that Γ is the image in $\mathrm{PGL}_2(\mathbb{C})$ of $\mathbf{G}(\mathbb{Q}) \cap K(\mathfrak{n})$.

Note that $X(K(\mathfrak{n}))$ is the quotient of

$$\mathbf{G}(\mathbb{Q}) \backslash (\mathrm{PGL}_2(\mathbb{C}) \times \mathbf{G}(\mathbb{A}_f)) / K(\mathfrak{n}),$$

by

$$K_\infty = \text{ image in } \mathrm{PGL}_2(\mathbb{C}) \text{ of } \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : |a|^2 + |b|^2 = 1 \right\}$$

the maximal compact subgroup at infinity. Here we have chosen an identification of $\mathbf{G}(\mathbb{C})$ with $\mathrm{PGL}_2(\mathbb{C})$; in case (ii), we require that this identification carry $(D')^\times$ into $\mathrm{PGL}_2(\mathbb{R})$, so that in particular K_∞ intersects $\mathbf{H}(\mathbb{R})$ in a maximal compact subgroup.

In the $\mathbf{G}_2$ case both $Y(\mathfrak{n})$ and $X(\mathfrak{n})$ are compact manifolds. In the $\mathbf{G}_1$ case both are noncompact of finite volume.

5.3. **Hecke operators.** Suppose $g \in \mathbf{G}(\mathbb{A}_f)$, that K' is another compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$ and $K' \subset gK(\mathfrak{n})g^{-1}$. The map $\mathbf{G}(\mathbb{A}) \to \mathbf{G}(\mathbb{A})$ given by $h \mapsto hg$ defines a continuous mapping

$$r(g) : X(K') \to X(\mathfrak{n}).$$

Taking $K' = K(\mathfrak{n}) \cap gK(\mathfrak{n})g^{-1}$ we define the *Hecke operator* $\mathbb{T}(g) : H^\bullet(X(\mathfrak{n})) \to H^\bullet(X(\mathfrak{n}))$ to be the composition

$$H^\bullet(X(\mathfrak{n})) \stackrel{r(g)^*}{\to} H^\bullet(X(K')) \stackrel{r(1)_*}{\to} H^\bullet(X(\mathfrak{n})).$$

For suitable choice of g this gives rise to the usual Hecke operators $T_\mathfrak{m}$, which are attached to any ideal $\mathfrak{m}$ of F which is "relatively prime to ramification," i.e. no prime divisor of $\mathfrak{m}$ lies above any place v of $\mathbb{Q}$ where K_v is non-maximal.

5.4. **The truncation in the Bianchi case.** Now assume that $\mathbf{G} = \mathbf{G}_1$, so that we are in the noncompact case. We denote by $X(\mathfrak{n})_{\mathrm{tr}}$ a 'truncation' of $X(\mathfrak{n})$, where we "chop off the cusps." Thus $X(\mathfrak{n})_{\mathrm{tr}}$ is a manifold with boundary, and up to homeomorphism it does not depend on the height at which the cusps were cut off.

Connected components of $Y(\mathfrak{n})_{\mathrm{tr}}$ are homeomorphic to the compact quotient $\Gamma \backslash \mathbb{H}^3_*$ where

$$\mathbb{H}^3_* := \mathbb{H}^3 \setminus \bigcup_{\sigma \in \mathbb{P}^1(F)} B(\sigma)$$

where the $B(\sigma)$ are a disjoint collection of horospheres in $\mathbb{H}^3$ tangent to the rational boundary point σ. In particular, $\Gamma \backslash \mathbb{H}^3_*$ looks like a thickening of the 2-skeleton of $\Gamma \backslash \mathbb{H}^3$.

6. Complexity of base-change cohomology classes

In this section $\mathbf{G} = \mathbf{G}_2$ and $M = Y(\mathfrak{n})$ is an associated congruence arithmetic manifold. We address Conjecture 1.1 for base-change cohomology classes of M. We recall below the definition of the base-change part $H^2_{\mathrm{bc}}(M)$ of the cohomology $H^2(M)$. Note that in this section, when we write $H^*(M)$ etc. without coefficients, we always mean complex cohomology. Here we prove:

6.1. **Theorem.** *There is a constant $C = C(F)$ such that, for any arithmetic hyperbolic manifold $M = Y(\mathfrak{n})$ of volume V, there exist compact immersed surfaces S_i of genus $\leq V^C$ such that the homology classes $[S_i] \in H_2(M)$ span $H_2^{\mathrm{bc}}(M)$.*

Note that §9 gives evidence that 'often' we actually have $H^2(M) = H^2_{\mathrm{bc}}(M)$.

It is enough to prove this theorem for the non-connected $X(\mathfrak{n})$ rather than for $M = Y(\mathfrak{n})$. Since $X(\mathfrak{n})$ is compact, we can compute $H^2(X(\mathfrak{n}))$ by means of L^2-cohomology, and indeed there is a Hecke-equivariant isomorphism (Matsushima's theorem, see [10]):

$$H^2(X(\mathfrak{n})) = H^2(\mathfrak{g}, K_\infty; L^2([\mathbf{G}]))^K$$

where $L^2([\mathbf{G}])$ is the Hilbert space of measurable functions f on $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$ such that $|f|$ is square-integrable on $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$ and we abridge by $K = \prod_v K_v$ the compact open subgroup $K(\mathfrak{n})$.

Since $\mathbf{G}$ is anisotropic the space $L^2([\mathbf{G}])$ decomposes as a direct sum of irreducible unitary representations of $\mathbf{G}(\mathbb{A})$ with finite multiplicities (in fact equal to 1). A representation σ which occurs in this way is called an *automorphic representation* of $\mathbf{G}$; it is factorizable as a restricted tensor product of admissible representations of $\mathbf{G}(\mathbb{Q}_p)$ (or more precisely, the unitary completions of these admissible representations). In particular,

$$\sigma = \sigma_\infty \otimes \sigma_f$$

where σ_∞ is a unitary representation of $\mathbf{G}(\mathbb{R})$ and σ_f is a representation of $\mathbf{G}(\mathbb{A}_f)$. In the following we let $\mathscr{A}$ be the set of all irreducible automorphic representations (σ, V_σ) of $\mathbf{G}(\mathbb{A})$, realized on the subspace $V_\sigma \subset L^2([\mathbf{G}])$.

6.2. **Representations with cohomology.** Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ be the Lie algebra of the real Lie group $\mathbf{G}(\mathbb{R}) = \mathrm{PGL}_2(\mathbb{C})$. There exists a unique non-trivial irreducible $(\mathfrak{g}, K_\infty)$-module (π, V_π) such that $H^\bullet(\mathfrak{g}, K_\infty; V_\pi) \neq \{0\}$. Furthermore:

$$H^q(\mathfrak{g}, K_\infty; V_\pi) = \begin{cases} 0 & \text{if } q \neq 1,\ 2, \\ \mathbb{C} & \text{if } q = 1,\ 2. \end{cases}$$

If we let $\mathfrak{p} = \mathfrak{sl}_2(\mathbb{C})/\mathfrak{su}_2$, the compact group K_∞ acts by conjugation on $\wedge^q\mathfrak{p}$; this yields an irreducible representation of K_∞. There is a natural isomorphism

$$H^q(\mathfrak{g}, K_\infty; V_\pi) \simeq \mathrm{Hom}_{K_\infty}(\wedge^q\mathfrak{p}, V_\pi).$$

We denote by $\mathscr{C}$ the subset of $\mathscr{A}$ which consists of automorphic representation $\sigma = \sigma_\infty \otimes \sigma_f$ of $\mathbf{G}(\mathbb{A})$ such that $\sigma_\infty \cong \pi$ (where, by a slight use of notation, we use π also to denote the unitary completion of the $(\mathfrak{g}, K_\infty)$-module described above).

6.3. Let $\mathscr{H}_K$ be the Hecke algebra of finite $\mathbb{Q}$-linear combinations of K-double cosets in $\mathbf{G}(\mathbb{A}_f)$. It is generated by the Hecke operators $\mathbb{T}(g) := KgK$. If σ is a representation of $\mathbf{G}(\mathbb{A})$, then $\mathscr{H}_K$ acts on the space of K-fixed vectors of σ. On the other hand, §5.3 gives an action of $\mathscr{H}_K$ on $H^*(X(K))$.

To summarize the prior discussion, then, we have a $\mathscr{H}_K$-isomorphism:

$$H^q(X(K)) = \bigoplus_{\sigma\in\mathscr{C}} \mathrm{Hom}_{K_\infty}(\wedge^q \mathfrak{p}, V_\sigma^K). \tag{6.3.1}$$

6.4. **Base-change classes.** Given an automorphic representation σ of $\mathbf{G}(\mathbb{A})$ we let $\mathrm{JL}(\sigma)$ be the automorphic representation of $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_{2|F})$ with trivial central character associated to σ by the Jacquet-Langlands correspondence.

We say that an automorphic representation σ of $\mathbf{G}$ *comes from base-change* if $\mathrm{JL}(\sigma)$ is isomorphic to $\mathrm{BC}(\sigma_0)\otimes\chi$, where σ_0 is a cuspidal automorphic representation of $\mathrm{GL}_{2|\mathbb{Q}}$, χ is an idele class character of F, and BC denotes base change.

We denote by $\mathscr{A}^{\mathrm{bc}}$ the set of all such representations (σ, V_σ) and define

$$H^2_{\mathrm{bc}}(X(K)) \subset H^2(X(K))$$

as the subspace corresponding, under (6.3.1), to those $\sigma\in\mathscr{C}$ that actually belong to $\mathscr{A}^{\mathrm{bc}}$.

A priori, this defines only a complex subspace, but it is actually defined over $\mathbb{Q}$, as one sees by consideration of Hecke operators. There is then a unique Hecke-invariant splitting

$$H^2(X(K),\mathbb{Q}) = H^2_{\mathrm{bc}}(X(K),\mathbb{Q}) \oplus H^2_{\mathrm{else}}(X(K)\mathbb{Q}),$$

and we define the base-change subspace H_2^{bc} of homology as the orthogonal complement of H^2_{else}. As it turns out, H_2^{bc} is spanned by some special cycles that we now describe.

6.5. **Special cycles.** Let $\mathbf{H}\subset\mathbf{G}_2$ be as in §5; recall that $\mathbf{H}(\mathbb{R})\simeq\mathrm{PGL}_2(\mathbb{R})$. Many notions we have defined for $\mathbf{G}$ make similar sense for $\mathbf{H}$, we won't recall definitions but just add H as a subscript to avoid confusion. For example, we write $\mathfrak{p}_H$ for the image inside $\mathfrak{p}$ of the Lie algebra of $\mathbf{H}(\mathbb{R})$ (recall that $\mathfrak{p}$ is defined as a quotient of the Lie algebra of $\mathbf{G}_2(\mathbb{R})$.)

Let $L = K\cap\mathbf{H}(\mathbb{A}_f)$, the quotient

$$Z(L) = \mathbf{H}(\mathbb{Q})\backslash(\mathbb{C}-\mathbb{R})\times\mathbf{H}(\mathbb{A}_f)/L = \mathbf{H}(\mathbb{Q})\backslash\mathbf{H}(\mathbb{A})/L_\infty^\circ L \quad (L_\infty = (\mathbf{H}(\mathbb{R})\cap K_\infty)),$$

is a union of (compact) Shimura curves. Here L_∞° denotes the connected component of L_∞, and $L_\infty/L_\infty^\circ\simeq\pm1$ acts on $Z(L)$.

The inclusion $\mathbf{H}\hookrightarrow\mathbf{G}$ defines a map $Z(L)\to X(K)$. Note that, since we are supposing K is sufficiently small (§5.1) both $Z(L)$ and $X(K)$ are genuine manifolds and not merely orbifolds. The submanifold $Z(L)$ defines a class $[Z(L)]$ in $H_2(X(K))$.

More generally, for every $g\in\mathbf{G}(\mathbb{A}_\mathrm{f})$ we set $L_g = gKg^{-1}\cap\mathbf{H}(\mathbb{A}_f)$; then right multiplication by g gives a map $Z(L_g)\to X(K)$, and by pushing forward the fundamental class from any component we obtain a class in $H_2(X(K))$. The components of $Z(L_g)$ are indexed by $\mathbb{A}^\times/(\det L_g)(\mathbb{A}^\times)^2$, where det denotes here the reduced norm. Accordingly, if $\mu:\mathbb{A}^\times/(\det L_g)(\mathbb{A}^\times)^2\to\mathbb{Z}$ is an integer-valued function we denote by $[Z(L)]_{g,\mu}$ the associated class in $H_2(X(K))$; in other words, $[Z(L)]_{g,\mu}$ is the image of

$$\mu\in H^0(Z(L_g),\mathbb{Z})\simeq H_2(Z(L_g),\mathbb{Z})\to H_2(X(K),\mathbb{Z}),$$

where the first map is Poincaré duality.

We let $\mathcal{Z}_K$ be the subspace of $H_2(X(K))$ spanned by all such $[Z(L)]_{g,\mu}$. Note that this subspace is spanned by classes of totally geodesic immersed surfaces that we call *special cycles.*

6.6. We will need a precise description of the dual pairing $\langle -, - \rangle : H_2 \times H^2 \to \mathbb{C}$: Choose a Haar measure dh on $\mathbf{H}(\mathbb{Q})\backslash\mathbf{H}(\mathbb{A})$ and fix a generator ν_H of the line $\left(\wedge^2 \mathfrak{p}_H\right)$.

Now let $T \in \mathrm{Hom}_{K_\infty}(\wedge^2\mathfrak{p}, V_\sigma^K)$, for some $\sigma \in \mathscr{C}$. By (6.3.1) we can identify T with an element of $H^2(X(K))$. We compute

$$\langle [Z(L)]_{g,\mu}, T\rangle = c\int_{\mathbf{H}(\mathbb{Q})\backslash\mathbf{H}(\mathbb{A})/L_\infty L_g} T(\nu_H)(hg)\,\mu(\det(h))dh, \tag{6.6.1}$$

where c is a nonzero constant of proportionality, depending on g, the choice of measure dh and the choice of ν_H.

6.7. **Distinguished representations.** Let $(\sigma, V_\sigma) \in \mathscr{A}$. A function $\varphi \in V_\sigma$ can then be seen as a function in $L^2([\mathbf{G}])$. Let χ be a quadratic idele class character of $\mathbb{A}^\times/\mathbb{Q}^\times$. We define the period integral

$$P_\chi(\varphi) = \int_{\mathbf{H}(\mathbb{Q})\backslash\mathbf{H}(\mathbb{A})} \varphi(h)\chi(\det\, h)dh \tag{6.7.1}$$

where $dh = \otimes_v dh_v$ is a Haar measure on $\mathbf{H}(\mathbb{Q})\backslash\mathbf{H}(\mathbb{A})$ as above. Let us say that σ is χ-*distinguished* if $P_\chi(\varphi) \neq 0$ for some $\varphi \in \sigma$. . We say simply that σ is *distinguished* if it is χ-distinguished for some χ. For the following, see [4, Theorem 4.1] (see also the discussion above that theorem, §3 of *loc. cit.*, and [3, Proposition 3.4]):

6.8. **Proposition.** *An automorphic representation $\sigma \in \mathscr{A}$ is distinguished if and only if σ comes from base change.*

6.9. **Proposition.** *Let $\sigma \in \mathscr{C}$ be such that $\sigma_f^K \neq \{0\}$. Then σ is not distinguished if and only if the subspace*

$$\mathrm{Hom}_{K_\infty}(\wedge^2\mathfrak{p}, V_\sigma^K) \subset H^2(X(K))$$

is orthogonal to the subspace $\mathscr{Z}_K$ spanned by the cycles $[Z(L)]_{g,\mu}$. Equivalently, cycles $[Z(L)]_{g,\mu}$ span H_2^{bc}.

Proof. The direct implication 'only if' follows immediately from (6.6.1), together with the fact that μ lies in the span of functions $h \mapsto \chi \circ \det$.

In the converse direction: Suppose that P_χ is not identically zero for some χ, but $\mathrm{Hom}_{K_\infty}(\wedge^2\mathfrak{p}, V_\sigma^K)$ is orthogonal to $\mathscr{Z}_K$. Then (6.6.1) says at least that $P_\chi(\varphi)$ vanishes for every φ of the form $\varphi_\infty \otimes g\varphi_f \in \sigma_\infty \otimes \sigma_f$ where φ_f is K-fixed, $g \in \mathbf{G}(\mathbb{A}_\mathrm{f})$ is arbitrary, and φ_∞ is the image of ν_H under a nontrivial element of $\mathrm{Hom}_{K_\infty}(\wedge^2\mathfrak{p}, \sigma_\infty)$. Since such vectors $g\varphi_f$ span all of σ_f, we see that P_χ vanishes on $\varphi_\infty \otimes \sigma_f$.

Now factor P_χ on $\sigma = \sigma_\infty \otimes \sigma_f$ as $P_\infty \otimes P_f$ (this can be done by multiplicity one, cf. §6.13). It remains to show that $P_\infty(\varphi_\infty) \neq 0$.

However, χ_∞ is the nontrivial quadratic character of $\mathbb{R}^*$. This is because, if $\sigma_\infty \simeq \pi$ were distinguished by the trivial character χ_∞, then σ – considered as a representation of $\mathrm{GL}_2(\mathbb{C})$ – would be distinguished by $\mathrm{GL}_2(\mathbb{R})$. It is known [24, Theorem 7] that such representations of $\mathrm{GL}_2(\mathbb{C})$ are the *unstable* base-changes of representations of $\mathrm{U}(1,1)$ but σ is not such a representation; it is the *stable* base-change. of the weight 2 discrete series representation.

Now, if $P_\infty(\varphi_\infty) = 0$, the above argument shows that $[Z(L)]_{g,\mu}$ would be zero for *every* choice of L, χ, g_f as above, and this is not so as follows e.g. from [41]. □

6.10. **Outline of the proof of Theorem 6.1.** Write $V = \mathrm{vol}(Y(K))$. Fix an embedding of $\iota : \mathbf{G} \hookrightarrow \mathrm{SL}_N$ over F. For $g \in \mathbf{G}(\mathbb{Q}_v)$, we denote by $\|g\|_v$ the largest v-adic valuation of any entry of $\iota(g)$. For $g = (g_v) \in \mathbf{G}(\mathbb{A})$ we put $\|g\| = \prod_v \|g_v\|_v$.

Let σ_j (for j in some index set J) be all the $\sigma \in \mathscr{C}$ such that $\sigma^K \neq 0$ and such that σ comes from base change. Let R be the set of ramified places, i.e. the set of places at which $K_v \subset \mathbf{G}(\mathbb{Q}_v)$ is not maximal or where $K_v \cap \mathbf{H}(\mathbb{Q}_v) \subset \mathbf{H}(\mathbb{Q}_v)$ is not maximal, and let $\mathbb{Q}_R = \prod_{v \in R} \mathbb{Q}_v$. We decompose accordingly each $\sigma = \sigma_j$ ($j \in J$) as

$$\sigma = \sigma_R \otimes \sigma^R$$

where σ_R is a representation of $\mathbf{G}(\mathbb{Q}_R)$ and σ^R is a representation of $\mathbf{G}(\mathbb{A}_F^{(R)})$, the group $\mathbf{G}$ over the "adeles omitting R."

The proof now proceeds in 4 steps. After giving the outline we discuss steps 1 and step 3 in more detail (§6.11 and §6.13).

Fix $j_0 \in J$ and let $\sigma_0 = \sigma_{j_0}$. Let χ_0 be so that σ_0 is χ_0-distinguished. Factor P_{χ_0} on $\sigma_0 = \sigma_{0,R} \otimes \sigma_0^R$:

$$P_{\chi_0} = P_R \otimes P^R.$$

(1) We will show, first of all, that there exist ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ of F relatively prime to R (that is, they do not lie above any place in R) whose norms $\mathrm{N}\mathfrak{p}_i$ are all bounded by aV^b (with a, b constants depending only on F) and constants $c_i \in \mathbb{C}$ such that the Hecke operator $\sum_i c_i \mathbb{T}_{\mathfrak{p}_i}$ is non-zero on V_{σ_0} and trivial on V_{σ_k} for every $k \in J$, $k \neq j_0$.

   In other words, if $\lambda_{\mathfrak{q}}(\sigma)$ is the eigenvalue by which $T_{\mathfrak{q}}$ acts on σ, we have $\sum c_i \lambda_{\mathfrak{p}_i}(\sigma_0) \neq 0$ whereas $\sum c_i \lambda_{\mathfrak{p}_i}(\sigma_k) = 0$ for $k \neq j_0$.

(2) If v_{sph}^R denotes the non trivial spherical vector in the space of σ_0^R, we have:

$$P^R(v_{\mathrm{sph}}^R) \neq 0.$$

   We will omit the proof; for discussion of this type of result, see [48, Corollary 8.0.4].

   (We have not verified that the auxiliary conditions of [48] apply here, although the method surely does. In any case, this can be verified here by direct computation: Because of multiplicity one, it is enough for each $v \in R$ to show that there exists a function on $\mathbf{G}(\mathbb{Q}_v)$ such that $f(hgk) = \chi(h)f(g)$ for $k \in K_v$ and $h \in \mathbf{H}(\mathbb{Q}_v)$, such that $f(1) \neq 0$, and such that the Hecke eigenvalue of f is the same as σ_v. Now the cosets $\mathbf{H}(\mathbb{Q}_v) \backslash \mathbf{G}(\mathbb{Q}_v)/K$ are parameterized by non-negative integers and one constructs the required f as a solution to a linear recurrence.)

(3) Now let $\varphi_1, \ldots, \varphi_s$ be a basis for $V_{\sigma_0}^K$. Write $\varphi_j = \varphi_{j,R} \otimes v_{\mathrm{sph}}^R$. We will show that there exist $g_1, \ldots, g_s \in \mathbf{G}(\mathbb{Q}_R)$ such that $\|g_i\| \leq cV^d$ for constants c, d depending only on F, and the matrix $(P_R(g_k \cdot \varphi_{j,R}))_{1 \leq j,k \leq s}$ is nonsingular.

(4) From the two first steps we conclude that for every $j = 1, \ldots, s$ we have:

$$\begin{aligned} P_{\chi_0}(\sum_i c_i \mathbb{T}_{\mathfrak{p}_i} \varphi_j) &= (\sum_i c_i \lambda_{\mathfrak{p}_i}(\sigma_0)) P_R(\varphi_{j,R}) P^R(v_{\mathrm{sph}}^R), \\ P_{\chi_0}(\sum_i c_i \mathbb{T}_{\mathfrak{p}_i} \psi) &= 0, \ \psi \in \sigma_j \neq \sigma_0. \end{aligned}$$

   where — according to steps 1 and 2 — the scalars $\mu_1 := \sum_i c_i \lambda_{\mathfrak{p}_i}(\sigma_0)$ and $\mu_2 := P^R(v_{\mathrm{sph}}^R)$ are both non zero. Since the g_k belong to $\mathbf{G}(\mathbb{Q}_R)$ and the ideals $\mathfrak{p}_i$ are relatively prime to R, Step 3 finally implies that the matrix

$$\left(P_{\chi_0}(g_k \cdot \sum_i c_i \mathbb{T}_{\mathfrak{p}_i} \cdot \varphi_j)\right)_{1\le k\le s, 1\le j\le s} = \mu_1 \cdot \mu_2 \cdot \left(P_R(g_k \cdot \varphi_{j,R})\right)_{j,k}$$

is non singular.

Repeating the same reasoning for each σ_j leads to the following refinement of Proposition 6.9:

H_2^{bc} is spanned by cycles of the form $\mathbb{T}_{\mathfrak{p}}[Z(L)]_{g,\mu}$, where both $\mathrm{N}\mathfrak{p}$ and $\|g\|$ are bounded by a polynomial in V.

Using trivial estimates, we see all the cycles appearing in this statement have volume bounded by a power of V. That will conclude the proof of Theorem 6.1.

In the following sections we provide details for steps 1 and 3.

6.11. **Step 1 of §6.10: a quantitative 'multiplicity one theorem'.** We first deal with automorphic representations of $\mathrm{GL}_{2|F}$. Recall the definition of the *analytic conductor* of Iwaniec-Sarnak:

Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $\mathrm{GL}_{2|F}$. For each finite place v we denote by $\mathrm{Cond}_v(\pi) = q_v^{m_v}$, where m_v is the smallest non-negative integer such that π_v possesses a fixed vector under the subgroup of $\mathrm{GL}_2(\mathfrak{o}_{F_v})$ consisting of matrices whose bottom row is congruent to $(0,0,\dots,0,1)$ modulo ϖ_v^m. Here $\varpi_v \in F_v$ is a uniformizer. For the infinite place v, let $\mu_{j,v} \in \mathbb{C}$ satisfy $L(s,\pi_v) = \prod (2\pi)^{-s-\mu_{j,v}}\Gamma(s+\mu_{j,v})$, and put $\mathrm{Cond}_v(\pi) = \prod(2+|\mu_{j,v}|)^2$. We then put $\mathrm{Cond}(\pi) = \prod_v \mathrm{Cond}_v(\pi)$ (this is within a constant factor of the Iwaniec-Sarnak definition).

6.12. **Lemma.** *(Linear independence of Hecke eigenvalues) Given automorphic representations $\pi_1,\dots,\pi_r$ of $\mathrm{GL}_{2|F}$, all of which have analytic conductor at most X; let $\mathcal{Q}$ be a set of prime ideals of F of cardinality $\le B\log X$ containing all ramified primes for the π_i; let $\{\mathfrak{q}_j : j = 1,\dots,s\}$ be the set of all ideals of F of norm $< Y$ that are relatively prime to $\mathcal{Q}$. Then the $r \times s$ matrix of Hecke eigenvalues*

$$M_{ij} = (\lambda_{\mathfrak{q}_j}(\pi_i))_{i,j} \quad (j = 1,\dots,s,\ i = 1,\dots,r)$$

has rank r so long as $Y \ge (rX)^A$, where A is a constant depending only on B and the field F.

Before the proof, we show how this gives Step 1: The Jacquet-Langlands correspondence associates to any automorphic representation σ_j as in §6.10, a cuspidal automorphic representation $\pi_j = \mathrm{JL}(\sigma_j)$ of $\mathrm{GL}_{2|F}$ with the same Hecke eigenvalues. Since $\sigma_f^{K(\mathfrak{n})} \neq \{0\}$ then $\mathrm{Cond}(\pi_j) \ll \mathrm{N}(\mathfrak{n})^4$. (We do not know a reference for this bound, but that such a polynomial bound exists can be readily derived by reducing to the supercuspidal case and using the relationship between depth and conductor; see [37] and references therein, especially [15]). In particular, the conductor is bounded by a polynomial in $\mathrm{vol}(Y(\mathfrak{n}))$.

To obtain Step 1, then, we apply the Lemma with $\mathcal{Q} = R$, the set of "bad" places – i.e. the places that are ramified for D together with primes dividing $\mathfrak{n}$. Note that the number of primes dividing $\mathfrak{n}$ is $\le \log_2(\mathrm{N}\,\mathfrak{n})$; the desired result follows, since (in the setting of Step 1) the integer r is bounded by $\dim H^2(Y(\mathfrak{n}))$, and thus by a linear function in $\mathrm{vol}\, Y(\mathfrak{n})$.

Proof. This is a certain strengthening of multiplicity one and will be deduced from the quantitative multiplicity one estimate of Brumley [13]. (See also [33, 42] for earlier results in the same vein.)

Consider, instead of the matrix M, the smoothed matrix N wherein we multiply the matrix entry M_{ij} by $h(\mathrm{Norm}(\mathfrak{q}_j/Y))$, where h is a smooth real-valued bump function on the positive reals such that $h(x) = 0$ when $x > 1$ and h is positive for $x < 1$. Clearly the rank of M and the rank of N are the same.

It is enough to show that the square $(r \times r)$ Hermitian matrix

$$N \cdot {}^t\overline{N}$$

is of full rank r. Its (i, j) entry is equal to

$$\sum N_{ik}\overline{N}_{jk} = \sum_{\mathfrak{q}} \lambda_{\mathfrak{q}}(\pi_i)\overline{\lambda_{\mathfrak{q}}(\pi_j)}h(\mathfrak{q}/Y)^2,$$

where the sum extends over the set of $\mathfrak{q}$ with norm $< Y$ and prime to $\mathcal{Q}$.

This is very close to [13, page 1471, equation (23)], with a minor wrinkle: *loc. cit.* discusses the corresponding sum but with $\lambda_{\mathfrak{q}}(\pi_i)\overline{\lambda_{\mathfrak{q}}(\pi_j)}$ replaced by $\lambda_{\mathfrak{q}}(\pi_i \times \overline{\pi_j})$. But the proof of [13] applies word for word here, using the equality

$$\sum_{\mathfrak{q}} \frac{\lambda_{\mathfrak{q}}(\pi_i)\overline{\lambda_{\mathfrak{q}}(\pi_j)}}{\mathrm{N}(\mathfrak{q})^s} = \frac{L^{\mathcal{Q}}(\pi_i \times \overline{\pi_j}, s)}{L^{\mathcal{Q}}(\omega_i\overline{\omega_j}, 2s)}. \tag{6.12.1}$$

where ω_i is the central character of π_i, and the superscript $\mathcal{Q}$ means we take the finite L-function and omit all factors at the set $\mathcal{Q}$. It leads to the corresponding bound:

$$\sum_{\mathfrak{q}} \lambda_{\mathfrak{q}}(\pi_i)\overline{\lambda_{\mathfrak{q}}(\pi_j)}h(\mathfrak{q}/Y)^2 = \delta_{ij}Y \cdot R_i + O(Y^{1-\theta}X^{B'}).$$

Here R_i is a residue of the L-function on the right of (6.12.1), θ is a positive real number (one can take $\theta = 1/2$) and B' is a constant that depends only on the constant B and the field F. It moreover follows from [13, equation (21)] that R_i is bounded below by X^{-C} for some absolute (positive) constant C.

Now the proof follows from 'diagonal dominance': Given a square hermitian matrix $S = (S_{ij})$ such that, for every α,

$$S_{\alpha\alpha} > \sum_{j \neq \alpha} |S_{\alpha j}| \tag{6.12.2}$$

then S is nonsingular, by an elementary argument.

Now one may choose A, depending only on B and F, so that (6.12.2) holds as long as $Y \geq (rX)^A$. □

6.13. **Step 3 of §6.10.** Let $(\sigma, V^\sigma) \in \mathscr{C}$ and χ be such that the functional P_χ is not identically vanishing on σ. For p a prime of $\mathbb{Q}$, let $H_p = \mathbf{H}(\mathbb{Q}_p)$ and $G_p = \mathbf{G}(\mathbb{Q}_p)$.

The multiplicity one theorem shows that the functional P_χ factorizes over places:

6.14. **Lemma.** *For any irreducible G_p-module σ_p we have:*

$$\dim \mathrm{Hom}_{(H_p,\chi_p)}(\sigma_p, \mathbb{C}) \leq 1.$$

Proof. If p is split in F the result is easy. If D_p is split this amounts to [23, Prop. 11] or [44, Theorem A] note that by twisting one reduces to the case of $\chi_p = 1$, at the cost of allowing σ_p to have a central character, so one can indeed apply Prop. 11. In the nonsplit case, there does not appear to be a convenient reference: one can also reduce to the results of [2] using an exceptional isomorphism, and see also [44, Theorem B] for a closely related result; □

For simplicity in what follows, we suppose that actually χ_p is trivial; the general case is a twisted case of what follows. So let P_p be a nonzero H_p-invariant functional on σ_p. Denote by V_p the index of K_p inside a maximal compact subgroup. We will now sketch a proof of the following result, which implies step (3):

> If $v_1, \ldots, v_r$ form a basis for $\sigma_p^{K_p}$, then there exist $g_i \in G_p$ with $\|g_i^{-1}\| \le cV_p^d$ – where c, d are constants, depending only on the embedding ι used in the definition of $\|g\|$ – such that the matrix $P_p(g_i^{-1} v_j)$ is nonsingular.

Consider the functions f_j on $X = G_p/H_p$ defined by the rule $g \mapsto P_p(g^{-1}v_j)$. We will show that, when restricted to the compact set

$$\Omega = \{gH : \|g^{-1}\| \le c \cdot V_p^d\}$$

the functions f_j are linearly independent.

Suppose to the contrary, i.e. there exists $a_1, \ldots, a_r$ not all 0 such that $\sum a_j f_j$ is zero on Ω. However, the asymptotics of $\sum a_j f_j$ can be computed by the theory of asymptotics on spherical varieties or even symmetric varieties (see [36, 35] or [49]); this theory of asymptotics shows that if $\sum a_j f_j$ vanish identically on a sufficiently large compact set, it must in fact identically vanish everywhere, contradiction. All that is needed is to give a sufficiently effective version of this asymptotic theory, which we sketch:

The wavefront lemma ([5, Proposition 3.2] or [49, Corollary 5.3.2]) shows that there is a set $F \subset G$ such that $HF = G$ and $P_p(gv_j)$ coincides for $g \in F$ with a *usual* matrix coefficient $\langle gu, v_j\rangle$, where u is a vector obtained by 'smoothing' P_p. The desired asymptotics for $\sum a_j f_j$ then follow from known asymptotics of matrix coefficients, see e.g. [18]; but what is needed is an explicit control on when matrix coefficients follow their asymptotic expansion. For supercuspidal representations of GL_n a sufficiently strong bound has been given by Finis, Lapid and Müller: [21, Corollary 2]. In our case of GL_2 the remaining possibilities of principal series (and their subrepresentations) can be verified by direct computation. (An alternate approach that treats the two together is to compute in the Kirillov model, using the local functional equation to control support near 0).

7. THE NONCOMPACT CASE

7.1. **The main result.** In this section $\mathbf{G} = \mathbf{G}_1$.

If M is a noncompact manifold we define, as usual, $H_!^i$ to be the image of compactly supported cohomology H_c^i inside cohomology H^i; and $H_{i,!}$ to be the image of usual homology H_i inside Borel–Moore homology $H_{i,\mathrm{BM}}$. All these definitions make sense with any coefficients, in particular, either integral or complex. If we do not specify the coefficients we will understand them to be $\mathbb{C}$.

We now suppose that

(i) $K = K_0(\mathfrak{n})$ where $\mathfrak{n}$ is a squarefree ideal, i.e. $K = \prod K_v$ where

$$K_v = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathcal{O}_v) : c \in \mathfrak{n}\mathcal{O}_v\}.$$

(ii) The corresponding (possibly disconnected) symmetric space $X_0(\mathfrak{n}) = X(K)$ satisfies $\dim H_!^1(X_0(\mathfrak{n}), \mathbb{C}) = 1$; let π be the associated automorphic representation (i.e. the unique representation whose Hecke eigenvalues coincide with those of a class in this $H_!^1$). As before we let $Y_0(\mathfrak{n})$ be the identity component of $X_0(\mathfrak{n})$.

(iii) π is associated to an elliptic curve E of conductor $\mathfrak{n}$ over F, which we moreover assume to not have complex multiplication. [2]

Under these assumptions our main result is:

7.2. **Theorem.** *There exists a L^2 harmonic 1-form ω representing a nonzero class in $H^1_!(Y_0(\mathfrak{n}),\mathbb{C})$, with integral periods (i.e. $\int_\gamma \omega \in \mathbb{Z}$ for every $\gamma \in H_1$) and moreover*

$$\langle \omega, \omega \rangle \ll A(\mathrm{Norm}\,\mathfrak{n})^B \tag{7.2.1}$$

for some constants A and B depending only on F.

By methods similar to §4 this proves Conjecture 1.1 in this case, i.e.

> If $Y_0(\mathfrak{n})$ is as above, there exist immersed compact surfaces S_i of genus $\ll \mathrm{vol}(Y_0(\mathfrak{n}))^C$ such that the images of $[S_i]$ span $H_2(Y_0(\mathfrak{n}),\mathbb{Z})$.

Passing ω through Poincaré-Lefschetz duality

$$H^1_!(Y_0(\mathfrak{n})) \simeq \mathrm{Im}(H_2(Y_0(\mathfrak{n})) \to H_{2,\mathrm{BM}}(Y_0(\mathfrak{n}))$$

one obtains a generator for the image of $H_2(Y_0(\mathfrak{n}))$ in $H_{2,\mathrm{BM}}(Y_0(\mathfrak{n})) \cong H_2(Y_0(\mathfrak{n})_{\mathrm{tr}}, \partial Y_0(\mathfrak{n})_{\mathrm{tr}})$. One can represent this generator as in §4, and we just outline the process:

Fix a triangulation of $Y_0(1)_{\mathrm{tr}}$ such that the boundary is a full subcomplex; it then follows that the boundary is a deformation retract of the subcomplex which consists of all simplices that intersect the boundary. We will also assume that all the edges of the dual cell subdivision have length ≤ 1. We finally lift this triangulation to a triangulation K of $Y_0(\mathfrak{n})_{\mathrm{tr}}$, denote by ∂K the full subcomplex corresponding to the (tori) boundary components and let K' denote the subcomplex of the first barycentric subdivision of K consisting of all simplices that are disjoint from ∂K. Then the 2-cycle

$$Z := \sum_e \left(\int_e \omega\right) e^* \in C_2(K, \partial K, \mathbb{R}). \tag{7.2.2}$$

represents the image of the class $[\omega]$ under the Poincaré-Lefschetz duality.

Thus, as in in §4, we have

(7.2.3)
$$\inf\{\sum |n_k| \mid [\sum n_k \sigma_k] = [Z] \text{ where } \sum n_k \sigma_k \text{ is a singular chain in } C_2(Y_0(\mathfrak{n})_{\mathrm{tr}}, \partial Y_0(\mathfrak{n})_{\mathrm{tr}})\} \ll \mathrm{N}\mathfrak{n}^{\mathrm{A}}.$$

Now Gabai's theorem — used in §4 — holds for $H_2(Y_0(\mathfrak{n})_{\mathrm{tr}}, \partial Y_0(\mathfrak{n})_{\mathrm{tr}})$: the 2-cycle Z is homologous into a (maybe disconnected) embedded surface

$$(S, \partial S) \subset (Y_0(\mathfrak{n})_{\mathrm{tr}}, \partial Y_0(\mathfrak{n})_{\mathrm{tr}})$$

such that the LHS of (7.2.3) is $\sum_{\chi(S_i)<0} -2\chi(S_i)$, the sum being taken over components S_i of S. Since $\partial Y_0(\mathfrak{n})_{\mathrm{tr}}$ is incompressible and $Y_0(\mathfrak{n})_{\mathrm{tr}}$ is atoroidal and aspherical we may furthermore assume that all components S_i have negative Euler characteristic.

Note that the surface S could a priori have boundary, but since $[S] = [Z]$ belongs to the image of $H_2(Y_0(\mathfrak{n})_{\mathrm{tr}})$ in $H_2(Y_0(\mathfrak{n})_{\mathrm{tr}}, \partial Y_0(\mathfrak{n})_{\mathrm{tr}})$, the image of $[S]$ in $H_1(\partial Y_0(\mathfrak{n})_{\mathrm{tr}})$ by the boundary operator in the long exact sequence associated to the pair $(Y_0(\mathfrak{n})_{\mathrm{tr}}, \partial Y_0(\mathfrak{n})_{\mathrm{tr}})$ is trivial.

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[2] Over $\mathbb{Q}$ condition (ii) automatically means that π must be associated to an elliptic curve. Over F, π is still conjecturally associated to a rank 2 motive over F with Hodge numbers $(0,1),(1,0)$ and coefficient field equal to $\mathbb{Q}$. Such a motive might arise from an abelian variety A/F admitting a quaternion algebra of endomorphisms. We anticipate that the same method would work in this case also.

We can close S using discs or annuli on the boundary tori, because ∂S intersects each boundary torus in a union of simple closed curves γ_j. One first closes each γ_j which is null-homotopic by a disc; and the remaining γ_j must be be parallel and all define, up to sign, the same primitive class in homology; we can close them in pairs by annuli.

Let f be the total number of discs adjoined when closing the boundary curves. The closing process has only increased the total Euler characteristic of S by f, so we arrive now at a closed surface S' with Euler characteristic

$$\chi(S') = \chi(S) + f = \sum_{\chi(S_i)<0} \chi(S_i) + f.$$

Finally, we may remove from S' all components that are either tori or spheres, because both cases must have trivial class inside $H_2(Y_0(\mathfrak{n})_{\mathrm{tr}}, \partial Y_0(\mathfrak{n})_{\mathrm{tr}})$. Removing the tori components does not change the Euler characteristic but removing the sphere components decreases it and therefore increase the complexity. This is the last issue we have to deal with.

Each component S'_i of S' that is a sphere meets S along spheres with ≥ 3 boundary components. So each such S'_i corresponds to a component of S^*_i of S with $\chi(S^*_i) \leq -1$, and distinct i's give rise to distinct components. So

$$\sum_{\chi(S'_i)=2} \chi(S'_i) \leq \sum_{\chi(S_i)<0} -2\chi(S_i).$$

Therefore, the total Euler characteristic of all sphere components of S' is at most $\sum_{\chi(S_i)<0} -2\chi(S_i)$, and removing these and tori gives a closed surface S'' with Euler characteristic

$$\chi(S') \geq \sum_{\chi(S_i)<0} 3\chi(S_i) + f \geq \sum_{\chi(S_i)<0} 3\chi(S_i)$$

where S'' still represents $Z \in H_2(Y_0(\mathfrak{n})_{\mathrm{tr}}, \partial Y_0(\mathfrak{n})_{\mathrm{tr}})$.

This bounds the complexity of the (1-dimensional) image of $H_2(Y_0(\mathfrak{n}))$ in $H_{2,\mathrm{BM}}(Y_0(\mathfrak{n}))$. Finally, since the homology classes of the cusps are represented by surfaces of genus 1, the Conjecture follows.

7.3. **Modular symbols.** We henceforth suppose we are in the situation of §5 with $\mathbf{G} = \mathbf{G}_1 (= \mathrm{Res}_{F/\mathbb{Q}} \mathrm{PGL}_2)$; in what follows, we will usually think of $\mathbf{G}$ as PGL_2 over F, rather than the scalar-restricted group to $\mathbb{Q}$.

Let $\alpha, \beta \in \mathbf{P}^1(F)$ and $g_f \in \mathbf{G}(\mathbb{A}_f)/K_f$. Then the geodesic from α to β (considered as elements of $\mathbf{P}^1(\mathbb{C})$, the boundary of $\mathbf{H}^3$), translated by g_f, defines a class in $H_{1,\mathrm{BM}}(X(K))$ that we denote by $\langle \alpha, \beta; g_f \rangle$. Evidently these satisfy the relation

$$\langle \alpha, \beta; g_f \rangle + \langle \beta, \gamma; g_f \rangle + \langle \gamma, \alpha; g_f \rangle = 0$$

the left-hand side being the (translate by g_f of the) boundary of the Borel–Moore chain defined by the ideal triangle with vertices at α, β, γ. Note that $\langle \alpha, \beta; g_f \rangle = \langle \gamma\alpha, \gamma\beta; \gamma g_f \rangle$ for $\gamma \in \mathrm{PGL}_2(F)$.

For a finite place v of F, let $F_v, \mathscr{O}_v, q_v$ denote the completion of F at v, the ring of integers of F_v and the cardinality of the residue field of F_v, respectively. By the *valuation* at v of the triple $\langle \alpha, \beta; g_f \rangle$ we shall mean the distance between:

- the geodesic from $\alpha_v, \beta_v \in \mathbf{P}^1(F_v)$ inside the Bruhat-Tits tree of $\mathbf{G}(F_v)$, and
- the point in that tree defined by $g_f \mathscr{O}_v^2$.

i.e., the minimum distance between a vertex on this geodesic and the vertex whose stabilizer is $\mathrm{Ad}(g_f)\mathrm{PGL}_2(\mathscr{O}_v)$.

Let n_v be the valuation of the symbol $\langle \alpha, \beta; g_f\rangle$ at v. We define the *conductor* of the symbol to be $\mathfrak{f} = \prod_v \mathfrak{q}_v^{n_v}$, where $\mathfrak{q}_v$ is the prime ideal associated to the place v; and the *denominator* of the symbol $\langle \alpha, \beta; g_f\rangle$ is then defined

$$\mathrm{denom}(\langle \alpha, \beta; g_f\rangle) = |(\mathscr{O}/\mathfrak{f})^\times| = \prod_{v: n_v \geq 1} \left(q_v^{n_v - 1}(q_v - 1)\right), \tag{7.3.1}$$

where q_v is the norm of $\mathfrak{q}_v$. We sometimes write this as the Euler φ function $\varphi(\mathfrak{f})$.

Let $\mathbf{T}$ be the stabilizer of α, β in PGL_2; it is isomorphic to the multiplicative group $\mathbf{T} \simeq \mathbb{G}_m$ and the isomorphism is unique up to sign. Then $\mathbf{T}(\mathscr{O}_v) \cap \mathrm{Ad}(g_f)\mathrm{PGL}_2(\mathscr{O}_v)$ corresponds to the subgroup $1 + \mathfrak{q}_v^{n_v} \subset \mathbb{G}_m(F_v) = F_v^\times$ if $n_v \geq 1$, and otherwise to the maximal compact subgroup of $F_v^\times$. (For example, to see the latter statement, note that $\mathbf{T}(\mathscr{O}_v)$ fixes exactly the geodesic from α to β inside the building of $\mathrm{PGL}_2(F_v)$.)

In particular, any finite order character ψ of $\mathbf{T}(\mathbb{A}_F)/\mathbf{T}(F) \simeq \mathbb{A}_F^\times/F^*$ that is trivial on $\mathbf{T}(\mathbb{A}_F) \cap \mathrm{Ad}(g_f)\mathrm{PGL}_2(\widehat{\mathscr{O}})$ has conductor dividing $\mathfrak{f}$ and order dividing $h_F\varphi(\mathfrak{f})$, where

$$h_F = \text{ order of narrow class group } C_F \text{ of } F.$$

More generally, if ψ is trivial on $\mathbf{T}(\mathbb{A}_F) \cap \mathrm{Ad}(g_f)K_0(\mathfrak{n})$, with $\mathfrak{n}$ a squarefree ideal, then – by a similar argument – the conductor of ψ divides $\mathfrak{n}\mathfrak{f}$ and its order divides $h_F\varphi(\mathfrak{n}\mathfrak{f})$, in particular, its order divides

$$h_F\varphi(\mathfrak{f}) \cdot \mathrm{Norm}(\mathfrak{n}) \cdot \varphi(\mathfrak{n}). \tag{7.3.2}$$

Note that another way to present our arguments would be to use a stronger version of "conductor" designed so that it takes account of level structure at $\mathfrak{n}$. This leads to a more elaborate version of §7.4 but simplifies other parts of the argument, because the factors of $\mathfrak{n}$ are no longer present in (7.3.2). See §7.5 for comments on that.

7.4. **Denominator avoidance and its proof.**

Lemma. *Fix any integer M. Let p be a prime number. If $p > 5$ (resp. $p \leq 5$) any class in $H_{1,\mathrm{BM}}(Y(K), \mathbb{Z})$ is represented as a sum of symbols $\langle \alpha, \beta, g_f\rangle$, each of which has conductor relatively prime to Mp and denominator indivisible by p (resp. divisible by at most p^A, for an absolute constant A).*

Proof. This is a slight sharpening of results in [17, §6.7.5]. In fact, there is a slight error in [17] which does not deal properly with the case when $g_f \notin \mathrm{PGL}_2(\mathscr{O}_v)$; the argument below in any case fixes that error.

As in [17] the Borel–Moore homology is generated by $\langle 0, \infty; g_f\rangle$ for varied g_f (in the classical case, this goes back to Manin, and the proof is the same here). Set $A_p = 1$ for $p > 5$ and $A_p = 3$ for $p \leq 5$.

One writes

$$\langle 0, \infty; g_f\rangle = \langle 0, x; g_f\rangle + \langle x, \infty; g_f\rangle$$

for a suitable $x \in \mathbf{P}^1(F)$.

First of all, if $g_v \in \mathrm{PGL}_2(\mathscr{O}_v)$, and the prime ideal $\mathfrak{q}_v$ associated to v divides the conductor of either $\langle 0, x; g_f\rangle$ or $\langle x, \infty; g_f\rangle$, then $v(x) \neq 0$.

Now suppose that v belongs to the set of finite set $\mathscr{B}$ of places such that $g_v \notin \mathrm{PGL}_2(\mathscr{O}_v)$. In the Bruhat-Tits tree of $\mathbf{G}(F_v)$ consider the subtree rooted at $[g_v\mathscr{O}_v^2]$ which consists of the half-geodesics that intersect the geodesic from 0 to ∞ at most in the vertex $g_v\mathscr{O}_v^2$. Its boundary at infinity defines an open subset $S_v \subset \mathbf{P}^1(F_v)$, and the conductors of both $\langle 0, x; g_f\rangle$ and $\langle x, \infty; g_f\rangle$ are prime to $\mathfrak{q}_v$ if x belongs to this subset.

Being open, S_v contains a subset S'_v of the form

$$S'_v = \varpi_v^{n_v}\beta_v(1+\varpi_v^{m_v}\mathcal{O}_v). \tag{7.4.1}$$

where n_v is an integer, m_v is an integer ≥ 1, ϖ_v is a uniformizer, and $\beta_v \in \mathcal{O}_v^\times$. Write $n_v^+ = \max(n_v, 0)$ and $n_v^- = \max(-n_v, 0)$ and set

$$\mathfrak{n}_0 = \prod_{v\in\mathscr{B}} \mathfrak{q}_v^{m_v}, \quad \mathfrak{a}_1 = \prod_{v\in\mathscr{B}} \mathfrak{q}_v^{n_v^+}, \quad \mathfrak{a}_2 = \prod_{v\in\mathscr{B}} \mathfrak{q}_v^{n_v^-}.$$

We say a prime ideal $\mathfrak{p}$ is *good* if it is prime to Mp, its norm is not congruent to 1 modulo p^{A_p}, and it does not lie in the set $\mathscr{B}$.

Now we claim that we may always find $x = \frac{a_1 b_1}{a_2 b_2}$ with the following properties:

(i) a_1, a_2 have the prime factorization

$$(a_i) = \mathfrak{a}_i \cdot \mathfrak{a}'_i,$$

where the $\mathfrak{a}'_i$ are good prime ideals. In particular, $v(\frac{a_1}{a_2}) = n_v$ for every $v \in \mathscr{B}$.

(ii)

$$\frac{b_1}{b_2} \in \left(\frac{a_2}{a_1}\varpi_v^{n_v}\right)\beta_v\left(1+\varpi_v^{m_v}\mathcal{O}_v\right) \tag{7.4.2}$$

for every $v \in \mathscr{B}$ (note that this forces $x \in S'_v$ for every $v \in \mathscr{B}$).

(iii) b_1, b_2 generate principal good prime ideals $\mathfrak{b}_1, \mathfrak{b}_2$;

Given such a_i, b_i we are done: Because of (7.4.2) and (7.4.1), the conductor of $\langle 0, x; g_f\rangle$ and $\langle x, \infty; g_f\rangle$ is not divisible by $\mathfrak{q}_v$ if $v \in \mathscr{B}$. Otherwise, if $v \notin \mathscr{B}$, then $g_v \in \mathrm{PGL}_2(\mathcal{O}_v)$. In that case, $\mathfrak{q}_v$ divides the conductor of either symbol only when $v(x) \neq 0$. In other words, the only primes dividing the conductor will be primes in the set $\{\mathfrak{a}'_1, \mathfrak{a}'_2, \mathfrak{b}_1, \mathfrak{b}_2\}$. Any prime $\mathfrak{q}$ in this set is prime to Mp, so that the conductor is prime to Mp. Also, for any prime $\mathfrak{q}$ in this set, $\mathrm{N}\mathfrak{q} - 1$ is not divisible by p^{A_p}. Thus the denominator of either symbol is divisible at most by $p^{2(A_p-1)}$.

We first find a_1, a_2 to satisfy (i). We then find b_1, b_2 to satisfy (ii), (iii).

For (i), we apply the Chebotarev density theorem to the homomorphism $\mathrm{Gal}(\bar{F}/F) \to C_F \times (\mathbb{Z}/p^{A_p}\mathbb{Z})$ arising from the Hilbert class field (for the $C_F =$ class group factor) and from the extension $F(\mu_{p^{A_p}}) \supset F$ (for the $(\mathbb{Z}/p^{A_p}\mathbb{Z})^\times$). Now the kernel of $\mathrm{Gal} \to C_F$ does not project trivially to the second factor; considering inertia shows that the image has size at least $\frac{p^{A_p-1}(p-1)}{2} > 1$. The Chebotarev density theorem now shows that there are infinitely many prime ideals $\mathfrak{p}$ whose image in C_F is the same class as $\mathfrak{a}_1^{-1}$ (or $\mathfrak{a}_2^{-1}$), and whose image in $(\mathbb{Z}/p^{A_p}\mathbb{Z})^\times$ is nontrivial. Now take a_1 to be a generator for the principal ideal $\mathfrak{p}\mathfrak{a}_1$, where the norm of $\mathfrak{p}$ is taken sufficiently large to guarantee that $\mathfrak{p}$ is prime to $Mp\mathscr{B}$. Similarly for a_2.

Now, once we have found a_1, a_2, then condition (ii) amounts to the following: for a certain class $\lambda \in (\mathcal{O}_v/\mathfrak{n}_0)^\times$ defined by the right-hand side of (7.4.2), we want to have

$$\frac{b_1}{b_2} \equiv \lambda \text{ modulo } \mathfrak{n}_0. \tag{7.4.3}$$

To get (7.4.3) and (iii) is another application of Chebotarev: Write $\mathfrak{n}_0 = \mathfrak{n}_1\mathfrak{n}_2$ where $\mathfrak{n}_1$ is prime-to-p and $\mathfrak{n}_2$ is divisible only by primes above p. Choose $\bar{b}_1, \bar{b}_2 \in (\mathcal{O}_F/p^{A_p}\mathfrak{n}_2)^\times$ such that $\bar{b}_1 \equiv \lambda\bar{b}_2 \bmod \mathfrak{n}_2$ and the norms of $\bar{b}_1, \bar{b}_2$ (under the map

$$\mathcal{O}_F/p^{A_p}\mathfrak{n}_2 \to \mathcal{O}_F/p^{A_p} \xrightarrow{\mathrm{N}} \mathbb{Z}/p^{A_p}\mathbb{Z}$$

are not congruent to 1. This can be done, for the image of the norm map $(\mathcal{O}_F/p^{A_p})^\times \to (\mathbb{Z}/p^{A_p})^\times$ has size strictly larger than 2. Now take for b_1 a lift of

$$(\lambda \bmod \mathfrak{n}_1) \times \bar{b}_1 \in (\mathcal{O}_F/\mathfrak{n}_1)^\times \times \left(\mathcal{O}_F/p^{A_p}\mathfrak{n}_2\right)^\times \simeq (\mathcal{O}_F/\mathfrak{n}_1\mathfrak{n}_2 p^{A_p})^\times$$

to a generator π of a principal prime ideal, and take b_2 similarly to be a lift π' of $1 \times \bar{b}_2$; these lifts can be done in infinitely many ways, so certainly the prime ideals can be taken prime to $Mp\mathcal{B}$. Moreover, the norm of (b_1) equals the norm of π (note this is automatically positive) and thus is not congruent to 1 modulo p^{A_p}. Similarly for (b_2). □

7.5. This section is not necessary for the proof. It is rather a commentary on how parts of the proof could be simplified at the cost of expanding the prior subsection.

A complication in the later proof arises at various points because of primes dividing $\mathfrak{n}$. For example, we have to explicitly evaluate some local integrals ((**??**)), we cannot assume that the conductors of E, ψ are relatively prime in Proposition 7.7, and so on. We outline here a refined version of the prior Lemma that would allow us to avoid these points.

Suppose for finitely many places V we specify a geodesic segment ℓ_v $(v \in V)$ of length 1 inside the Bruhat-Tits tree of $\mathrm{PGL}_2(F_v)$ containing $\mathcal{O}_v^2$ (i.e., $\mathcal{O}_v^2$ and one adjacent vertex). Now define the valuation at $v \in V$ of a triple $\langle \alpha, \beta; g_f\rangle$ to be the distance between the geodesic from α_v to β_v and the set of vertices of $g_v \ell_v$, i.e.

$$\text{valuation at } v = \max(\text{ distance between } P \text{ and } [\alpha_v, \beta_v]\text{ , for } P \in g_v\ell_v).$$

Thus the valuation is 0 if and only if the segment $g_v\ell_v$ is contained in the geodesic from α_v to β_v. At places outside V, the valuation is defined as before.

Then *with this refined notion the same statement as in the Lemma still holds.*

The proof, however, is slightly more involved: In the proof above, take $\mathcal{B}$ to consist of all places in V together with all places where $g_v \notin \mathrm{PGL}_2(\mathcal{O}_v)$. The problem is that the set of x such that $\langle 0, x; g_f\rangle$ and $\langle x, \infty; g_f\rangle$ both have conductor indivisible by v, for $v \in V$, need not contain an open subset of $\mathbf{P}^1(F_v)$. The problem arises when $g_v\ell_v \subset [0,\infty]$.

Call a modular symbol $\langle \alpha, \beta; g_f\rangle$ *good* if, for every $v \in V$, the segment $g_v\ell_v$ is not contained in $[\alpha, \beta]$ for every $v \in V$. Thus, what the proof still gives is:

(7.5.1)
A good modular symbol is the sum of two modular symbols with the desired properties

where "desired properties" refers to the relevant divisibility statements for conductor and denominator.

Now if a modular symbol – without loss $\langle 0, \infty; g_f\rangle$ is *not* good, then, for every $v \in V$, the set of $x \in \mathbf{P}^1(F_v)$ such that:

- $\langle 0, x; g_f\rangle$ has v-valuation 0, and
- $\langle x, \infty; g_f\rangle$ is good

is open and nonempty. The above argument then works to show that we can write

$$\langle 0, \infty; g_f\rangle = \langle 0, x; g_f\rangle + \langle x, \infty; g_f\rangle$$

where $\langle 0, x; g_f\rangle$ has the desired divisibility properties, and $\langle x, \infty; g_f\rangle$ is good. Then we are done by (7.5.1).

7.6. **The proof of (7.2.1) assuming equivariant BSD.** Fix in what follows a symbol $\langle \alpha, \beta; g_f \rangle$ with conductor $\mathfrak{f}$ and denominator $D = \varphi(\mathfrak{f})$; without loss of generality we can suppose $\alpha = 0, \beta = \infty$. We write N for the norm of $\mathfrak{f}$. Also we can factorize $D = \prod_v D_v$ over places v of F. Finally we write $N_E = \mathrm{Norm}(\mathfrak{n})$ for the absolute conductor of the elliptic curve E.

7.6.1. *Normalizations.* Fix an additive character θ of $\mathbb{A}_F/F$: for definiteness we take the composition of the standard character of $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$ with the trace. Fix the measure on $\mathbb{A}_F$ that is self-dual with respect to θ, and similarly on each F_v.

For a function φ on $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}) \simeq \mathrm{PGL}_2(F)\backslash\mathrm{PGL}_2(\mathbb{A}_F)$ we define the Whittaker function W_φ by the rule

$$W_\varphi(g) = \int_{x\in\mathbb{A}_F} \theta(x)\varphi(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g)\,dx.$$

Let $X = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$ in the Lie algebra of the diagonal torus $\mathbf{A}$ of $\mathfrak{pgl}_2$; we will also think of it as an element in the Lie algebra of $\mathfrak{pgl}_2$.

For $y \in F$ or F_v, we set $a(y) = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$.

Write $U_\infty = \mathbf{A}(F_\infty) \cap K_\infty$. It is a maximal compact subgroup of $\mathbf{A}(F_\infty)$.

On every $\mathbf{A}(F_v)$, for v finite, choose the measure μ_v which assigns the maximal compact subgroup mass 1. On the 1-dimensional Lie group $\mathbf{A}(F_\infty)/U_\infty$ we put the measure that is dual to the vector field $\underline{X}$ defined by $X \in \mathrm{Lie}(\mathbf{A})$, in other words, induced by a differential form dual to $\underline{X}$. Finally, on $\mathbf{A}(F_\infty)$ itself, take the Haar measure which projects to the measure just defined on $\mathbf{A}(F_\infty)/U_\infty$.

The product measure $\mu = \prod_v \mu_v$ has been chosen to have the following property: if ν is a 1-form on the quotient $\mathbf{A}(F)\backslash\mathbf{A}(\mathbb{A}_F)/K_\infty U$ for some open compact $U \subset \mathbf{A}(\mathbb{A}_{F,f})$, we have

$$\int_{\mathbf{A}(F)\backslash\mathbf{A}(\mathbb{A}_F)/U_\infty U} \nu = \frac{1}{\mathrm{vol}(U)} \int_{\mathbf{A}(F)\backslash\mathbf{A}(\mathbb{A}_F)} \langle \underline{X}, \nu\rangle\, d\mu. \tag{7.6.1}$$

Here is how to interpret the right-hand side: X defines a vector field $\underline{X}$ on $\mathbf{A}(F)\backslash\mathbf{A}(\mathbb{A}_F)/U_\infty U$; pairing with ν gives a function, which we then pull back to $\mathbf{A}(F)\backslash\mathbf{A}(\mathbb{A}_F)$ and integrate against the measure we have just described. The volume $\mathrm{vol}(U)$ is measured with respect to the measure $\prod \mu_v$ over finite v. Finally, the left-hand side requires an orientation to make sense; we orient so that $\underline{X}$ is positive.

To prove (7.6.1), note the ν-integral is a sum of integrals over components. Each component is a quotient of $\mathbf{A}(F_\infty)/U_\infty$. On each such components, the integral is (by definition) obtained by pushing forward the measure $\langle \underline{X}, \nu\rangle \mu_\infty$ to this quotient, and integrating. One also computes the right-hand side to induce the same measure on each component.

7.6.2. *Normalization of* $T(X)$. Let $T \in \mathrm{Hom}_{K_\infty}(\mathfrak{g}/\mathfrak{k}, \pi)^K$; here π is the unique cohomological representation of level $\mathfrak{n}$ as per our assumptions (§7.1) and, as in the previous section, $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebra of the groups $\mathbf{G}(F_\infty)$ and its maximal compact subgroup. Now T defines a differential form on $Y_0(\mathfrak{n})$, which we call simply ω. Put $T_X := T(X) \in \pi$; in our case it will be a factorizable vector $\bigotimes f_v$.

We normalize T by requiring that the

$$\text{Whittaker function } W_{T(X)} \text{ of } T(X) = \prod_v W_v, \tag{7.6.2}$$

where W_v is the new vector of [32] (in particular, $W_v(e) = 1$ when θ_v is unramified) and at ∞ we normalize by the requirement

$$\int_{F_\infty^*} W_\infty\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} dy = 1$$

where dy is chosen to correspond to the measure on $\mathbf{A}(F_\infty)$ fixed above (a simple computation is necessary to check this is possible, since the integral might, *a priori*, always equal 0). By Rankin-Selberg and standard estimates, we check that

$$\mathrm{N}(\mathfrak{n})^{-\varepsilon} \ll \frac{\langle \omega, \omega \rangle_{L^2(Y_0(\mathfrak{n}))}}{\mathrm{vol}(Y_0(\mathfrak{n}))} \ll \mathrm{N}(\mathfrak{n})^{\varepsilon}. \tag{7.6.3}$$

Indeed, all we need is polynomial bounds of this form, with lower bound $\mathrm{N}(\mathfrak{n})^{-A}$ and upper bound $\mathrm{N}(\mathfrak{n})^{A}$ for a constant A depending only on F; such bounds are given in [13, eq. (10) and Theorem 5]; for the case of $F = \mathbb{Q}$ the sharper lower bound is due to Hoffstein and Lockhart [30], and that also contains references for the sharper upper bound.

7.6.3. *Adelic torus orbits versus modular symbols.* We want to express the integral of ω over a modular symbol $\langle 0, \infty; g_f \rangle$ in terms of an adelic integral, similar to what was done in (6.6.1). *We will assume that the conductor of* $\langle 0, \infty; g_f \rangle$ *is relatively prime to* $\mathfrak{n}$.

Let $U = \mathbf{A}(\mathbb{A}_{F,f}) \cap g_f K g_f^{-1}$. Consider now the map

$$\mathbf{A}(F)\backslash \mathbf{A}(\mathbb{A}_F)/U_\infty U \to Y(K)$$

defined by $t \mapsto t g_f$. Its image can be regarded as a finite union of modular symbols $\langle 0, \infty; t g_f \rangle$ where t varies through representatives in $\mathbf{A}(\mathbb{A}_{F,f})$ for the group $Q = \mathbf{A}(\mathbb{A}_{F,f})/\mathbf{A}(F)U$. There is an exact sequence:

$$\mu_F/\mu_F \cap g_f K g_f^{-1} \to \underbrace{\mathbf{A}(\widehat{\mathcal{O}})/U}_{\text{size } = \mathrm{vol}(U)^{-1}} \to Q \to \text{class group}.$$

where μ_F is the group of roots of unity and we regard it as a subgroup of $\mathbf{A}(F) \simeq F^\times$ via $\mu_F \subset F^\times$. Call w'_F the size of the group on the far left-hand side. So $|Q| w'_F = h_F \mathrm{vol}(U)^{-1}$. Any character ψ of Q extends to a character of $[\mathbf{A}] = \mathbf{A}(F)\backslash \mathbf{A}(\mathbb{A}_F)$ which is trivial at infinity and it follows from (7.6.1) that

$$\sum_{q \in Q} \psi(q) \int_{\langle 0, \infty; q g_f \rangle} \omega = \frac{1}{\mathrm{vol}(U)} \cdot \int_{\mathbf{A}(F)\backslash \mathbf{A}(\mathbb{A}_F)} \psi(t) \langle (g_f)^* \omega, \underline{X} \rangle (t)\, d\mu(t)$$

where the right-hand side is intepreted in the same way as in (7.6.1); recall that ω has been defined in §7.6.2. 'Fourier analysis' on the finite group Q then gives:

$$\begin{aligned}
\int_{\langle 0, \infty; g_f \rangle} \omega &= \frac{1}{|Q|} \sum_{\psi \in \widehat{Q}} \frac{1}{\mathrm{vol}(U)} \cdot \int_{[\mathbf{A}]} \psi(t)\ \langle (g_f)^* \omega, \underline{X} \rangle (t)\, d\mu(t) \\
&= \frac{w'_F}{h_F} \sum_{\psi \in \widehat{Q}} \int_{[\mathbf{A}]} \psi(t)\ \langle (g_f)^* \omega, \underline{X} \rangle (t)\, d\mu(t) \\
&= \frac{w'_F}{h_F} \sum_{\psi \in \widehat{Q}} \int_{[\mathbf{A}]} T_X(t g_f) \psi(t)\, d\mu \qquad (7.6.4) \\
&= \frac{w'_F}{h_F} \sum_{\psi} L(\frac{1}{2}, \pi \times \psi) \cdot \prod_v \frac{I_v}{L_v(\frac{1}{2}, \pi \times \psi)}. \qquad (7.6.5)
\end{aligned}$$

Here

$$I_v := \int_{y \in F_v^\times} W_v(a(y)g_v)\psi_v(y)dy,$$

where g_v is the component at v of g_f; W_v is in (7.6.2); and measures are as normalized earlier. We have used at step (7.6.5) unfolding, as in the theory of Hecke integrals [14, §3.5].

Let S be the set of archimedean places, together with all places where the conductor of the symbol $\langle 0,\infty; g_f\rangle$ is not 1. Let S' be the set of finite places dividing $\mathfrak{n}$. Because of our assumption, S and S' are disjoint.

Note that if $v \notin S \cup S'$ we have $g_v \in \mathbf{A}(F_v)\cdot \mathrm{PGL}_2(\mathscr{O}_v)$. So ψ_v must be unramified for I_v to be nonzero. By choice of W_v we have $I_v = u_v \cdot L_v(\frac{1}{2}, \pi_v \times \psi_v)$ whenever $v \notin S \cup S'$, where u_v is an algebraic unit.

For finite $v \in S$, the values of $W_v^{g_v}$ at least lie in $\overline{\mathbb{Z}}[\frac{1}{q_v}]$, as one verifies by explicit computation. On the other hand, the function $y \mapsto W_v(a(y)g_v)$ is now constant on each coset of $1+\mathfrak{q}_v^{n_v}$, where n_v is the local conductor – see discussion just before §7.4; for the integral to be nonzero, then $\psi_v(y)$ must be identically 1 on $1+\mathfrak{q}_v^{n_v}$ and constant on each of its cosets. Each of these cosets has measure D_v^{-1}, where D_v is the local denominator.

Now $J_v(s) := \frac{\int_{F_v^*} W_v(a(y)g_v)\psi_v(y)|y|^s dy}{L(s+1/2,\pi_v\times\psi_v)}$ can be rewritten as $\int_{F_v^*} f(y)\psi_v(y)|y|^s dy$ where f is a certain sum of translates of W_v. [3] But $J_v(s)$ is a polynomial in q_v^{-s} (this again by the theory of Hecke integrals) and so $f_v(y)$ is compactly supported. Also, $\alpha_v, \beta_v \in \overline{\mathbb{Z}}[1/q_v]$. So $J_v(0)$ is actually a finite sum of elements, each lying in $\overline{\mathbb{Z}}[\frac{1}{q_v}]\cdot D_v^{-1}$. This shows that

$$D_v \cdot I_v \in L_v(\frac{1}{2}, \pi_v \times \psi_v)\overline{\mathbb{Z}}[\frac{1}{N}] \; (v \in S) \tag{7.6.6}$$

where $\overline{\mathbb{Z}}$ is the ring of algebraic integers, N is the norm of the conductor of $\langle 0,\infty; g_f\rangle$, and D_v the contribution of v to the denominator of $\langle 0,\infty; g_f\rangle$.

Now for $v \in S'$. Although in fact exactly the same reasoning that was just applied to $v \in S$ also applies to $v \in S'$, we will argue separately because we actually want a slightly more precise result for $v \in S'$, i.e. the set of primes dividing $\mathfrak{n}$, with better denominator control. Because $S \cap S' = \emptyset$ we have $g_v \in \mathbf{A}(F_v)\cdot \mathrm{PGL}_2(\mathscr{O}_v)$ for each $v \in S'$. In particular, we may suppose that $g_v \in \mathrm{PGL}_2(\mathscr{O}_v)$ while only modifying the value of I_v by an algebraic unit. By a direct computation with Steinberg representations we find that in fact , for $k_v \in \mathrm{PGL}_2(\mathscr{O}_v)$ and W_v the new vector for a Steinberg representation π_v, we have

$$\int W_v(a(y)k_v)\psi_v(y)d^*y \in \frac{1}{q_v(q_v-1)} L(\frac{1}{2}, \pi_v \times \psi_v)\cdot \overline{\mathbb{Z}} \tag{7.6.7}$$

This is a matter of explicit computation, as we now detail:

(i) If k_v belongs to $K_0(\mathfrak{n})$ this is clear.

(ii) Otherwise we can write $k_v = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot w \cdot k_v'$ where $x \in \mathscr{O}_v$, $k_v' \in K_0(\mathfrak{n})$ and $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In that case we can rewrite the integral as $\int W_v^{w_v}(a(y))\theta_v(xy)\psi_v(y)dy$. The function $W_v^{w_v}(a(y))$ is supported on $v(y) \geq -1$ and its values are algebraic units when $v(y) = -1$ (see [56, (11.14)]). Moreover, it is invariant on each coset $y\mathscr{O}_v^*$.

[3] Namely, write $L(s+1/2, \pi_v \times \psi_v)^{-1}$ as $(1-\alpha_v\psi_v(\varpi_v)q_v^{-s})(1-\beta_v\psi_v(\varpi_v)q_v^{-s})$, where ϖ_v is a uniformizer and α_v, β_v could be 0, and then take $f(y) = (1-\alpha_v T)(1-\beta_v T)W_v(a(y)g_v)$, where T is the operation which translates a function by ϖ_v.

If ψ_v is ramified the only contribution comes from $v(y) = -1$, since on any other coset $y\mathcal{O}_v^*$ with $v(y) \geq 0$ both $W_v^{w_v}(a(y))$ and $\theta_v(xy)$ remain constant on that coset. The integral over $v(y) = -1$ then amounts to a Gauss sum; it belongs to $\frac{1}{q_v - 1}\overline{\mathbb{Z}}$.

On the other hand, if ψ is unramified, the value of the integral is – by explicit computation – $\pm q_v^{-1} \cdot L_v(1/2, \pi_v \times \psi_v) + \frac{u}{q_v - 1}$ where u is an algebraic unit. In fact, the term $\frac{u}{q_v-1}$ comes from $v(y) = -1$, and the remaining term $\pm q_v^{-1} \cdot L_v(1/2, \pi_v \times \psi_v)$ comes from the contribution of $v(y) \geq 0$.

We deduce that, *if the conductor of* $\langle 0, \infty; g_f\rangle$ *is relatively prime to* $\mathfrak{n}$,

$$\int_{\langle 0,\infty; g_f\rangle} \omega = \frac{1}{h_F D \mathrm{N}(\mathfrak{n})\varphi(\mathfrak{n})} \sum_{\psi} L(\frac{1}{2}, \pi \times \psi) \cdot a_\psi, \quad a_\psi \in \overline{\mathbb{Z}}[\frac{1}{N}]. \tag{7.6.8}$$

where $D = \prod D_v$ is the denominator of $\langle 0, \infty; g_f\rangle$, every character ψ that occurs on the right-hand side has conductor dividing $\mathfrak{n}\mathfrak{f}$, with $\mathfrak{f}$ the conductor of $\langle 0, \infty; g_f\rangle$; and $N = \mathrm{Norm}(\mathfrak{f})$. Note also that the order of ψ is bounded, as in as in (7.3.2).

We will now apply equivariant BSD. We first normalize a period Ω_E. Let $\Omega^1_{\mathscr{E}}$ be the $\mathcal{O}_F$-submodule of differential 1-forms on E which extend to a Néron model; it's an $\mathcal{O}_F$-module of rank 1. It will be slightly more convenient for us to deal with $\Omega^1_{\mathscr{E}}\mathfrak{d}_F^{-1} := \Omega^1_{\mathscr{E}} \otimes \mathfrak{d}_F^{-1}$, where $\mathfrak{d}_F$ is the different of $F/\mathbb{Z}$. (The reason why this is more convenient will become clear before (8.3.3).) We regard it as a submodule of the F-vector space Ω^1 of all differential 1-forms. Pick a $\mathbb{Z}$-basis ξ_1, ξ_2 for $\Omega^1_{\mathscr{E}}\mathfrak{d}_F^{-1}$ (we'll only use ξ_2 later). Now put

$$\Omega_E = \left| \frac{1}{[\Omega^1_{\mathscr{E}}\mathfrak{d}_F^{-1} : \mathcal{O}_F \xi_1]} \int_{E(\mathbb{C})} \xi_1 \wedge \overline{\xi_1} \right|. \tag{7.6.9}$$

This is independent of the choice of ξ_1. For later usage, note the following: If $a = [\Omega^1_{\mathscr{E}}\mathfrak{d}_F^{-1} : \mathcal{O}_F \xi_1]$, then we have $\frac{\sqrt{-D_F/4}}{a} = \mathrm{Im}(\xi_2/\xi_1)$, by an area computation.

7.7. **Proposition.** *(Equivariant BSD conjecture; see §8 for full discussion.) Assume the equivariant BSD conjecture, in the formulation* (8.7.3)*. Let E be a non-CM elliptic curve over the imaginary quadratic field F of conductor $\mathfrak{n}_E$. Let Ω_E be as in* (7.6.9)*. Let ψ be a character of $\mathbb{A}_F^\times / F^\times F_\infty^\times$ of finite order d and conductor $\mathfrak{n}_\psi$. Suppose that E has semistable reduction at all primes dividing $(\mathfrak{n}_E, \mathfrak{n}_\psi)$. Then we have*

$$(d\mathrm{N}(\mathfrak{n}_\psi))^2 \cdot L(\frac{1}{2}, \pi \times \psi) \in \overline{\mathbb{Z}}[\frac{1}{N'_\psi}] \cdot \frac{\Omega_E}{|E(F_\psi)_{\mathrm{tors}}|^2}, \tag{7.7.1}$$

where $E(F_\psi)_{\mathrm{tors}}$ denotes the torsion subgroup of the points of F over the abelian extension F_ψ corresponding to ψ, and $N'_\psi = \prod_{\mathfrak{p}^2 | \mathfrak{n}_\psi} \mathrm{N}(\mathfrak{p})$.

Note that the elliptic curve E in our context does have semistable reduction at every place, because its conductor is squarefree, so we can freely apply this result.

7.8. **Proof of Theorem 7.2.** We now collect together what we have shown, in order to complete the proof. As above, ω is a differential 1-form of level $\mathfrak{n}$ belonging to the automorphic representation π.

Fix a prime $\mathfrak{l}$ of $\overline{\mathbb{Z}}$ above a prime ℓ of $\mathbb{Z}$. Let $\overline{\mathbb{Z}}_{\mathfrak{l}}$ consist of algebraic integers with valuation ≥ 0 at $\mathfrak{l}$. Also, let F_ℓ denote the largest abelian extension of F that is unramified at all primes above ℓ if ℓ is relatively prime to $\mathfrak{n}$. Otherwise, let F_ℓ be the largest abelian extension of F that is at worst tamely ramified at primes of F above ℓ. Begin with $\int_\gamma \omega$ for arbitrary $\gamma \in H_{1,BM}$, and use the Lemma of §7.4 to write γ as a sum of symbols $\langle 0, \infty; g_f\rangle$, where the conductor of each symbol is relatively prime to $N_E\ell$, and the denominator is prime to ℓ (or divisible by at most ℓ^A if $\ell \le 5$).

Now (7.6.8) writes $\int_\gamma \omega$ as a sum of L-values $L(\frac{1}{2}, E \times \psi)$, where the ψs which occur have conductor dividing $\mathfrak{n}\cdot\mathfrak{f}$, where $\mathfrak{f}$ is prime to $N_E\ell$. In particular, for any prime $\mathfrak{l}$ above l, the square $\mathfrak{l}^2$ doesn't divide the conductor of ψ. [4]

Combine (7.3.2), (7.7.1) and (7.6.8) to arrive at:

$$\int_\gamma \omega \in \frac{1}{(30 h_F \mathrm{N}(\mathfrak{n})\varphi(\mathfrak{n}))^B} \cdot \frac{\Omega_E}{|E(F_\ell)_{\mathrm{tors}}|^2} \cdot \overline{\mathbb{Z}}_{\mathfrak{l}},$$

for some absolute constant B.

Set

$$M = \prod_\ell \# E(F_\ell)[\ell^\infty] \text{ and } M' = (30 h_F \mathrm{N}(\mathfrak{n})\varphi(\mathfrak{n}))^B;$$

M is finite because $E(F^{\mathrm{ab}})_{\mathrm{tors}}$ is finite – that is a simple consequence of Serre's open image theorem (see e.g. [54]), using the fact that E does not have CM. Then

$$\int_\gamma \omega \in \frac{1}{M'} \cdot \frac{\Omega_E}{M^2} \cdot \overline{\mathbb{Z}}_{\mathfrak{l}},$$

and beause this is true for all $\mathfrak{l}$ we get

$$\int_\gamma \omega \in \frac{1}{M'} \cdot \frac{\Omega_E}{M^2} \cdot \overline{\mathbb{Z}}.$$

Thus, if we set

$$\omega' = \Omega_E^{-1} M' M^2 \cdot \omega$$

the form ω' has integral periods, i.e. $\int_\gamma \omega' \in \mathbb{Z}$ for all γ. Our desired result (Theorem 7.2) follows from (7.6.3) together with the bounds:

$$\Omega_E^{-1} \text{ and } M \ll_F A(\mathrm{N}\mathfrak{n})^B \tag{7.8.1}$$

for absolute constants A, B.

We now explain how to check (7.8.1).

For Ω_E, one uses the relationship with the Faltings height, together with Szpiro's conjecture and Frey's conjecture, see [29, F.3.2]. As commented in that reference, these conjectures are, up to the exact value of the constants involved, equivalent to the ABC conjecture. Then, up to constant factors, $\Omega_E^{-1/2}$ coincides with the exponential of the Faltings height; conjecture [29, F.3.2] now yields (7.8.1), using also the result stated in [29, Exercise F.5(c)].

Now let us examine M.

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[4] Note that, using §7.5, the situation can be simplified in the following way: Take ℓ_v of §7.5 to be the set of vertices fixed by $K_0(\mathfrak{n})$. Then §7.5 allows us to write γ as a sum of symbols $\langle 0, \infty; g_f\rangle$, where the "refined" conductor of each symbol is relatively prime to $N_E\ell$, and with controlled denominator as above. Now, (7.6.8) writes $\int_\gamma \omega$ as a sum of L-values $L(\frac{1}{2}, E \times \psi)$, where the ψs which occur have conductor relatively prime to $N_E\ell$ also. In particular, we can assume that ψ and E have relatively prime conductor – simplifying our later discussion.

The reason is the following: For any symbol $\langle 0, \infty; g_f\rangle$, refined valuation 0 actually means that $g_v K_0(\mathfrak{n})_v g_v^{-1}$ contains the maximal compact subgroup of $\mathbf{A}(F_v)$. In particular – looking above (7.6.5) – if v is any place of "refined" valuation 0, the vector $W(a(y)g_v)$ is actually invariant by $y \in \mathcal{O}_v^*$, and then ψ_v must actually be unramified for the local integral I_v to be nonzero. So, in the above reasoning, the only ψs that occur have conductor relatively prime to $N_E\ell$, because the modular symbols which occur had "refined" conductor relatively prime to $N_E\ell$.

- For $\ell > 3$ and ℓ relatively prime to $\mathfrak{n}$, $E(F_\ell)[\ell]$ must be either trivial or cyclic, because if $E(F_\ell)[\ell]$ were all of $E[\ell]$, that means that inertia groups $I_{\mathfrak{l}} \subset \mathrm{Gal}(\bar{F}/F)$ for any prime $\mathfrak{l}$ of F above ℓ would act trivially on $E[\ell]$. But this is never true, because the determinant of this action is the mod ℓ cyclotomic character. which is nontrivial.
- For $\ell = 2, 3$ or ℓ dividing $\mathfrak{n}$, we see similarly that $E(F_\ell)[\ell^3]$ cannot be of $E[\ell^3]$.

Write $Q = (6\mathrm{Norm}(\mathfrak{n}))^2$.

So $Q \cdot E(F_\ell)[\ell^\infty]$ is cyclic for every ℓ. Write K for the subgroup of $E(F^{\mathrm{ab}})_{\mathrm{tors}}$ generated by all the $Q \cdot E(F_\ell)[\ell^\infty]$. As we have just seen, K is a cyclic subgroup of order $\geq M/Q^2$, and it is stable by the Galois group of $\bar{F}/F$. Consider the isogeny $\varphi : E \to E' := E/K$. Masser-Wüstholz (see e.g. the main theorem of [40]) give an isogeny $\varphi' : E' \to E$ in the reverse direction, whose degree is bounded by a polynomial in the Faltings height of E. The composite isogeny $\theta = \varphi' \circ \varphi : E \to E$ must be multiplication by an integer r, because E does not have CM, and also $\#K$ divides r because $\theta(K) = 0$ and K is cyclic. From

$$(\#K) \cdot \deg \varphi' = r^2$$

we get the desired bound: $M \leq Q^2 \deg \varphi'$.

8. The equivariant conjecture of Birch and Swinnerton–Dyer

In the previous section we used Proposition 7.7 which says (see that section for notation):

> Assume equivariant BSD. Let E be an elliptic curve over the imaginary quadratic field F of conductor $\mathfrak{n}_E$. Let ψ be a character of $\mathbb{A}_F^\times / F^\times F_\infty^\times$ of finite order d and conductor $\mathfrak{n}_\psi$. We assume that E has semistable reduction at every prime dividing $(\mathfrak{n}_\psi, \mathfrak{n}_E)$. Put $N'_\psi = \prod_{\mathfrak{p}^2 | \mathfrak{n}_\psi} \mathrm{N}(\mathfrak{p})$.

(8.0.2) $$(d\mathrm{Norm}(\mathfrak{n}_\psi))^2 \cdot L(\frac{1}{2}, \pi \times \psi) \in \overline{\mathbb{Z}}[\frac{1}{N'_\psi}] \cdot \frac{\Omega_E}{|\#E(F_\psi)_{\mathrm{tors}}|^2},$$

> where N_E, N_ψ are the respective norms of $\mathfrak{n}_E, \mathfrak{n}_\psi$; and F_ψ is the abelian extension determined by F, and Ω_E is the period normalized as in (7.6.9).

This was quoted as a consequence of the "equivariant Birch/Swinnerton-Dyer conjecture." Unfortunately, there is no standardized form of such a conjecture in the literature, to our knowledge, in the generality we need it. That is why we have written the current section §8, to spell out exactly what we mean and how it gives rise to (8.0.2). We have chosen to directly formulate an equivariant BSD conjecture in (8.7.3) in a way that directly mirrors the formulation given by Gross [27] for CM elliptic curves. In principle, this should be routinely verifiable to be equivalent to the equivariant Tamagawa number conjecture of [22, §4], although we did not attempt to verify the details. In summary, when we say "equivariant BSD" in this paper, we *mean* the conjecture that is formulated in (8.7.3) below; and we anticipate, but have not verified, that this can be verified to be compatible with [22] in a routine fashion.

Here is the basic idea. To understand $L(\frac{1}{2}, E \times \psi)$ as below one needs to understand the L-function of E over a certain abelian extension F_ψ, i.e. the L-function of abelian variety $\mathrm{Res}_{F_\psi/F} E$, but *equivariantly* for the action of the Galois group G of F_ψ over F. One difficulty encountered is that $\mathbb{Z}[G]$ is not a Dedekind ring. This issue comes up in other work on the subject [9]. Of course, our goal is much less precise, since we may lose arbitrary denominators at N'_ψ and also some denominator at d. In any case we deal with this by instead passing to an abelian subvariety of A which admits an action of a

Dedekind quotient of $\mathbb{Z}[G]$. As a simple example, if ψ is a quadratic character, one can analyze $L(\frac{1}{2}, E\times\psi)$ as the L-function of a quadratic twist of E, rather than working with E over the quadratic extension defined by ψ.

In the actual derivation we will try to write formulas that are as explicit as possible. We write for short $\mathbb{Z}' = \mathbb{Z}[\frac{1}{N'_\psi}]$ and if M is a $\mathbb{Z}$-module we sometimes write M' for $M\otimes_{\mathbb{Z}}\mathbb{Z}'$.

8.1. **Basic setup.** Choose a prime ℓ that doesn't divide N'_ψ and extend the valuation at ℓ to a valuation of $\overline{\mathbb{Q}}$. We will prove (8.0.2) "at ℓ," i.e. verify that the ℓ-adic valuation of the ratio $\frac{\mathrm{LHS}}{\mathrm{RHS}}$ behaves as predicted.

We regard F as a subfield of $\mathbb{C}$, i.e. we choose a fixed embedding ι of F into $\mathbb{C}$. When we write $E(\mathbb{C})$ we understand it as the complex points of E considered as a complex variety via ι.

Let F_ψ be the abelian field extension of F determined, according to class field theory, by the kernel of ψ. Note that F_ψ is tamely ramified above F at all primes above ℓ, because any prime $\mathfrak{l}$ above ℓ divides $\mathfrak{n}_\psi$ with multiplicity ≤ 1.

Let μ be the cokernel of ψ (so that ψ gives an isomorphism of μ with a cyclic subgroup of $\mathbb{C}^\times$); thus $\mathrm{Gal}(F_\psi/F)\simeq\mu$. We fix an extension of ι to an embedding $\sigma_1 : F_\psi\to\mathbb{C}$; for $\alpha\in\mu$ we put $\sigma_\alpha = \sigma_1\circ\alpha$.

8.2. **Background on cyclotomic rings.** The size of μ is d, i.e. the order of ψ; and let $R = \mathbb{Z}[\mu]$ be the group algebra of μ, so that ψ gives an algebra homomorphism $\psi : R\to\mathbb{C}$. For $a\in R$ we will sometimes write a^ψ instead of $\psi(a)$; we will also use this notation for $a\in R_{\mathbb{R}} := R\otimes\mathbb{R}$ (i.e. a^ψ is the value, in $\mathbb{C}$, of the real-linear extension $\psi : R_{\mathbb{R}}\to\mathbb{C}$).

Choose a generator ζ for μ. Let $\phi_d\in\mathbb{Z}[x]$ be the dth cyclotomc polynomial and $\theta_d = \frac{x^d-1}{\phi_d}\in\mathbb{Z}[x]$. Let $\Phi_d = \phi_d(\zeta)$ and $\Theta_d = \theta_d(\zeta)$ be the elements of R obtained by evaluating these at ζ. Note that $\Theta_d\Phi_d = 0$ in R.

Set

$$S = R/(\Phi_d).$$

Then S is a Dedekind ring, isomorphic to the ring of integers in the dth cyclotomic field, and the homomorphism $\psi : R\to\mathbb{C}$ then factors through S. Note that by differentiating

$$\text{image in } S \text{ of } (\Theta_d\cdot\phi'_d(\zeta)) = \frac{d}{dx}(x^d-1)|_{x=\zeta} = d\zeta^{-1}, \tag{8.2.1}$$

where this equality is in S. This shows that d is divisible, in S, by the product of Θ_d and $\phi'_d(\zeta)$. Note that the image of $\phi'_d(\zeta)$ in S is exactly the different of S over $\mathbf{Z}$.

Consider the abelian category $S-\mathrm{modf}$ of finite S-modules: modules that are finite as abelian groups. Then the rule $S/\mathfrak{n}\mapsto\mathfrak{n}$ gives an isomorphism

$$K_0(S-\mathrm{modf})\simeq\{\text{fractional ideals of } S.\}. \tag{8.2.2}$$

We use $[X]$ to denote the (fractional) ideal corresponding to a torsion S-module X, and write $[X]\ge[Y]$ if the ideal for X is divisible by the ideal for Y. We write $[X]\ge_\ell[Y]$ if this holds "at ℓ," i.e. the valuation at any prime $\mathfrak{l}$ above ℓ for $[X]$ is $\ge$ the same for $[Y]$.

If $[X]$ corresponds to a *principal* ideal, we will say that X is "virtually principal." By an abuse of notation, we may regard then $[X]$ as an element of $S^*_{\mathbb{Q}}/S^*$, namely, a generator for that principal ideal. Here we have written $S_{\mathbb{Q}}$ as an abbreviation for $(S\otimes\mathbb{Q})$.

Involutions: We denote by $x\mapsto x^*$ the involution of R that is induced by inversion on μ. This descends to the canonical complex conjugation on the CM-field S. Later we also consider the "complex conjugation" $x\mapsto\bar{x}$ on $R_{\mathbb{C}} := R\otimes_{\mathbb{Z}}\mathbb{C}$ arising from the conjugation of $\mathbb{C}/\mathbb{R}$.

Given $x \in F_\psi$ we define $[x] \in R_{\mathbb{C}}$ by

$$[x] = \sum_\alpha \sigma_\alpha(x)\alpha^{-1}.$$

The morphism $x \mapsto [x]$ is actually equivariant for the action of $\mathcal{O}_F[\mu]$, which acts on F_ψ by linear extension of the μ-action and acts on $R_{\mathbb{C}}$ via the map $\mathcal{O}_F[\mu] \to R_{\mathbb{C}}$ induced from the natural embedding $\mu \to R$ and the inclusion $\iota : \mathcal{O}_F \hookrightarrow \mathbb{C}$.

8.3. **The abelian varieties and their Néron models.** We put

$$A = \mathrm{Res}_{F_\psi/\mathbf{Q}} E.$$

Then A is a $2d$-dimensional abelian variety which admits an action of $\mu \simeq \mathrm{Gal}(F_\psi/F)$ and so also of the algebra R. Consider the $2\varphi(d)$-dimensional abelian variety B which is given by the connected component of the kernel of Φ_d acting on A:

$$B = \ker(\Phi_d : A \to A)^0.$$

Then the action of R on B factors through S. Also Θ_d gives a surjection of abelian varieties $A \to B$.

Denote by $\mathscr{E}$ the Néron model of E over $\mathcal{O}_F$, and by $\mathscr{B}$ the Néron model of B (now over $\mathbb{Z}$) and finally $\mathscr{A}$ that of A (also over $\mathbb{Z}$).

Denote by $\mathrm{Lie}(\mathscr{E})$ the tangent space to $\mathscr{E}$ above the identity section, and $\Omega^1_{\mathscr{E}}$ its $\mathcal{O}_F$-linear dual; these are both locally free $\mathcal{O}_F$-modules of rank one. We use similar notation for $\mathscr{A}$ and $\mathscr{B}$; in that case, they are free $\mathbb{Z}$-modules of rank $2d$ and $2\varphi(d)$ respectively. Thus, e.g. $\mathrm{Lie}(\mathscr{E})$ is the set of $\mathrm{Spec}\left(\mathcal{O}_F[\varepsilon]/\varepsilon^2\right)$-valued points of $\mathscr{E}$ that extend the identity section $\mathrm{Spec}\mathcal{O}_F \to \mathscr{E}$.

The connected Néron model of $E \otimes_F F_\psi$ over $\mathcal{O}_{F_\psi}$ coincides with the base-change of the connected Néron model for $\mathscr{E}$. Indeed, the universal property gives a map from $\mathscr{E} \otimes_{\mathbb{Z}} \mathcal{O}_{F_\psi}$ *to* this Néron model, and this is an open immersion. We check this after localizing at each prime $\mathfrak{p}$:

- If a prime $\mathfrak{p}$ of F doesn't divide $\mathfrak{n}_\psi$, this is the commutation of Néron models with unramified base change [11, Theorem 1, Chapter 7].
- If a prime $\mathfrak{p}$ of F *does* divide $\mathfrak{n}_\psi$ then by assumption E has semistable reduction at $\mathfrak{p}$, and the result is known [11, Prop 3, Chapter 7].

Now it is known ([20, Proposition 4.1]) that $\mathscr{A}$ is the restriction of scalars $\mathrm{Res}_{\mathcal{O}_{F_\psi}/\mathbb{Z}}$ for the Néron model of E over $\mathcal{O}_{F_\psi}$. Using the description of Lie algebra noted above and the defining property of restriction of scalars we see that

$$\mathrm{Lie}(\mathscr{A}) = \left(\mathrm{Lie}(\mathscr{E}) \otimes_{\mathcal{O}_F} \mathcal{O}_{F,\psi}\right)$$

We obtain the respective Ω^1 spaces by dualizing. Now the $\mathbb{Z}$-dual of a locally-free $\mathcal{O}_F$-module M is isomorphic to $\mathrm{Hom}_{\mathcal{O}_F}(M, \mathcal{O}_F) \otimes_{\mathcal{O}_F} \mathfrak{d}_F^{-1}$, where $\mathfrak{d}_F$ denotes the different. The pairing $x \in M, y \otimes \delta \in \mathrm{Hom}_{\mathcal{O}_F}(M, \mathcal{O}_F) \otimes_{\mathcal{O}_F} \mathfrak{d}_F^{-1} \mapsto \mathrm{trace}_{F/\mathbb{Z}}(\langle x, y\rangle \delta)$ induces this isomorphism. Thus –

$$\begin{aligned}
\Omega^1_{\mathscr{A}/\mathbb{Z}} &= \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Lie}(\mathscr{A}), \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Lie}(\mathscr{E}) \otimes_{\mathcal{O}_F} \mathcal{O}_{F_\psi}, \mathbb{Z}) \\
&= \left(\mathrm{Hom}_{\mathcal{O}_{F_\psi}}(\mathrm{Lie}(\mathscr{E}) \otimes_{\mathcal{O}_F} \mathcal{O}_{F_\psi}, \mathcal{O}_{F_\psi}) \otimes_{\mathcal{O}_{F_\psi}} \mathfrak{d}_{F_\psi}^{-1}\right)
\end{aligned}$$

$$= \left(\mathrm{Hom}_{\mathcal{O}_F}(\mathrm{Lie}(\mathscr{E}), \mathcal{O}_{F_\psi}) \otimes_{\mathcal{O}_{F_\psi}} \mathfrak{d}_{F_\psi}^{-1}\right) \tag{8.3.1}$$

$$= \left(\Omega^1_{\mathscr{E}} \otimes_{\mathcal{O}_F} \mathfrak{d}_{F_\psi}^{-1}\right) \tag{8.3.2}$$

$$= \left(\Omega^1_{\mathscr{E}} \mathfrak{d}_F^{-1} \otimes_{\mathcal{O}_F} \mathfrak{d}_{F_\psi/F}^{-1}\right) \tag{8.3.3}$$

To be more precise, there is a natural map $\Omega^1_{\mathscr{E}} \otimes_{\mathscr{O}_F} \mathfrak{d}^{-1}_{F_\psi} \to \Omega^1_A$, and the assertion is that the image consists of 1-forms on A which extend to the Néron model. In the last equations, $\mathfrak{d}_{F_\psi/F}$ denotes the relative different and we used transitivity of the different, which means $\mathfrak{d}_{F_\psi} \simeq \mathfrak{d}_{F_\psi/F} \otimes_{\mathscr{O}_F} \mathfrak{d}_F$.

Our next order of business is to get some understanding of $\Omega^1_{\mathscr{B}}$. There's a natural morphism $\Omega^1_{\mathscr{A}} \to \Omega^1_{\mathscr{B}}$ induced by $B \hookrightarrow A$, and we want to put an upper bound on the size of the cokernel. To do so we examine the morphism $A \to B$ given by "multiplication by Θ_d." The composite $B \to A \to B$ is given by multiplication by Θ_d on B; that shows us that

$$\Theta_d \Omega^1_{\mathscr{B}} \subset \text{image of } \Omega^1_{\mathscr{A}}.$$

Note that $\Omega^1_{\mathscr{B}}$ is a locally free S-module of rank 2. (In fact, it is a free $\mathbb{Z}$-module of rank $2\varphi(d)$, and if we tensor $\otimes_{\mathbb{Z}}\mathbb{C}$ we get a free $S \otimes \mathbb{C}$ module of rank 2. From there we see that $\Omega^1_{\mathscr{B}}$ is contained with finite index in a free S-module, which easily implies it is locally free.)

So in the exact sequence

$$\Omega^1_{\mathscr{A}} \to \Omega^1_{\mathscr{B}} \to C, \tag{8.3.4}$$

the cokernel C has the property that $[C] \le 2[S/\Theta_d]$; in particular (see (8.2.1)) $[C] \le [S/d^2]$.

8.4. **The homology of $A(\mathbb{C})$.** Note that

$$A(\mathbb{C}) = E(F_\psi \otimes_{\mathbb{Q}} \mathbb{C}) = \bigoplus_{\sigma: F_\psi \to \mathbb{C}} E^\sigma(\mathbb{C}).$$

The set of σs which occur is the set σ_α $(\alpha \in \mu)$ defined earlier, together with their conjugates $\overline{\sigma_\alpha}$ $(\alpha \in \mu)$.

Choose generators γ_1, γ_2 for $H_1(E^{\sigma_1}(\mathbb{C}))$ and let $\overline{\gamma_1}, \overline{\gamma_2}$ be their images under the antiholomorphc map $E^{\sigma_1} \to E^{\overline{\sigma_1}}$. A free R-basis for $H_1(A(\mathbb{C}), \mathbb{Z})$ is given by $\gamma_1, \gamma_2, \overline{\gamma_1}, \overline{\gamma_2}$. The complex conjugation of $\mathbb{C}/\mathbb{R}$ induces an antiholomorphic involution of $A(\mathbb{C})$; that involution switches γ_i and $\overline{\gamma_i}$. Later we will also set $\delta_i = \gamma_i + \overline{\gamma_i}$.

Now $H_1(B(\mathbb{C}), \mathbb{Z})$ is given by the kernel of Φ_d acting on $H_1(A(\mathbb{C}), \mathbb{Z})$. Since the latter is free, as R-module on $\gamma_1, \gamma_2, \overline{\gamma_1}, \overline{\gamma_2}$, it follows that $H_1(B(\mathbb{C}), \mathbb{Z})$ is free as S-module on the same generators multiplied by Θ_d.

(In fact, the kernel of $\Phi_d : R \to R$ is just $R\Theta_d$, and is free of rank 1 as an S-module: regard $R = \mathbb{Z}[x]/(x^d - 1)$; if the class of $f(x)$ is killed by ϕ_d, then $(x^d - 1)$ divides $f(x) \cdot \phi_d$, so that θ_d divides f.)

8.5. **Torsion subgroups.** Later we will need to understand the torsion subgroups of both $B(\mathbb{Q})$ and $\hat{B}(\mathbb{Q})$ where $\hat{B}$ is the dual abelian variety.

Clearly $B(\mathbb{Q})_{\text{tors}} \subset A(\mathbb{Q})_{\text{tors}} = E(F_\psi)_{\text{tors}}$. To bound torsion in $\hat{B}$ note that we have a map $\Theta_d : A \to B$ and thus also a dual map $\widehat{\Theta_d} : \hat{B} \to \hat{A}$. We compute the kernel of $\widehat{\Theta_d}$ over $\mathbb{C}$: it is dual, as an abelian group, to the cokernel of

$$\Theta_d : H_1(A(\mathbb{C}), \mathbb{Z}) \to H_1(B(\mathbb{C}), \mathbb{Z}),$$

but this is trivial, as we have seen. Thus also $\hat{B}(\mathbb{Q})_{\text{tors}}$ is isomorphic to a subgroup of $\hat{A}(\mathbb{Q})_{\text{tors}} = E(F_\psi)_{\text{tors}}$ (A carries a principal polarization and so is isomorphic to $\hat{A}$).

8.6. **Integration.** By integration we get a mapping

$$f : \Omega^1_{\mathscr{B}} \otimes \mathbb{R} \xrightarrow{f} \left(H^1(B(\mathbb{C}), \mathbb{R})_+\right), \tag{8.6.1}$$

where on the right hand side the subscript + denotes *coinvariants* of complex conjugation considered as an antiholomorphic involution of $B(\mathbb{C})$. We can regard the right-hand side as the $\mathbb{R}$-dual of $H_1(B(\mathbb{C}), \mathbb{R})^+$, the conjugation-invariants on homology, and then the map is $\omega \mapsto \int_\gamma \omega$ for $\gamma \in H_1(B(\mathbb{C}), \mathbb{R})^+$. This is an isomorphism of free $\mathbb{R}$-modules, both of rank $r = 2\varphi(d)$.

Both sides have integral structures: On the left-hand side $\Omega^1_{\mathscr{B}}$. On the right-hand side we put the integral structure that is the image of $H^1(B(\mathbb{C}), \mathbb{Z})$. Thus we can compute the "period determinant" of (8.6.1), well-defined up to sign. That determinant is given by the volume

$$\Lambda = \pm \int_{B(\mathbb{R})^\circ} |\omega_1 \wedge \ldots \omega_r|, \tag{8.6.2}$$

where ω_i is an integral basis for $\Omega^1_{\mathscr{B}}$.

The usual BSD conjecture [8] for B says

$$L(\frac{1}{2}, B) = \pm \frac{Ш_B \cdot \prod_v c_v(B)}{B(\mathbb{Q}) \cdot \hat{B}(\mathbb{Q})} \cdot \Lambda \tag{8.6.3}$$

where we *allow ourselves to write a finite group in place of its order*, and if $B(\mathbb{Q})$ is infinite we understand the right-hand side as 0. Note that every term on the right is a finite S-module (e.g. $c_v(B)$ is the local component group of the Néron model, and S acts on it too.) Also note that (the way we have set things up) the archimedean component groups $c_\infty(B) \simeq B(\mathbb{R})/B(\mathbb{R})^\circ$ also counts.

As preparation for the equivariant version, we phrase this a little differently. Suppose that we give ourselves finite index subgroups $\mathscr{H} \subset H^1_+$ and $\mathscr{W} \subset \Omega^1_{\mathscr{B}}$ of the respective integral structures. We can form the period determinant Λ' with respect to $\mathscr{H}$ and $\mathscr{W}$, i.e. $f_*(\det \mathscr{W}) = \Lambda' \cdot \det \mathscr{H}$, where e.g. $\det \mathscr{H}$ denotes the element of the top exterior power of $\mathscr{H} \otimes \mathbb{R}$ determined by the lattice $\mathscr{H}$. Since $[H^1_+ : \mathscr{H}] \cdot \det(H^1_+) = \det(\mathscr{H})$ and similarly for $\mathscr{W}$, we deduce the following variant form of BSD:

$$L(\frac{1}{2}, B) = \pm \frac{Ш_B \cdot \prod_v c_v(B)}{B(\mathbb{Q}) \cdot \hat{B}(\mathbb{Q})} \cdot \frac{[H^1_+ : \mathscr{H}]}{[\Omega^1_{\mathscr{B}} : \mathscr{W}]} \cdot \Lambda' \tag{8.6.4}$$

8.7. **Statement of the conjecture.** In order to make the equivariant conjecture we need to break up the right hand side of (8.6.3) in a way that corresponds to the factorization $L(\frac{1}{2}, B) = \prod_\chi L(\frac{1}{2}, E \times \chi)$, where the product is taken over all powers $\chi = \psi^i$ with $i \in (\mathbb{Z}/d)^*$.

First of all choose integral elements $e_1, e_2 \subset H^1(B(\mathbb{C}), \mathbb{R})_+$ so that the $S_{\mathbb{R}}$-module generated by e_1, e_2 is free, and similarly choose $v_1, v_2 \in \Omega^1_{\mathscr{B}}$. Then (8.6.4) says

$$L(\frac{1}{2}, B) = \left(\frac{Ш_B \cdot \prod_v c_v(B)}{B(\mathbb{Q}) \hat{B}(\mathbb{Q})} \frac{H^1_+ / Se_1 + Se_2}{\Omega^1_{\mathscr{B}} / Sv_1 + Sv_2} \right) \cdot \Lambda' \tag{8.7.1}$$

where Λ' is the period determinant taken relative to the integral lattices $Se_1 + Se_2$ and $Sv_1 + Sv_2$. Note that all the finite groups inside the brackets on the right-hand side are actually S-modules. We will next examine how to refine each term on the right to an element of $S^\times_{\mathbb{R}}$, so that we recover (8.6.4) by taking norms.

Firstly let us examine Λ'. The map (8.6.1) is an *isomorphism* of free $S_{\mathbb{R}} = (S\otimes_{\mathbb{Q}}\mathbb{R})$ modules of rank 2. We will obtain an element of $S_{\mathbb{R}}^{\times}$ by comparing generators for $\wedge^2_{S_{\mathbb{R}}} LHS$ and $\wedge^2_{S_{\mathbb{R}}} RHS$: We have

$$\text{image of } v_1 \wedge_{S_{\mathbb{R}}} v_2 = \lambda(e_1 \wedge_{S_{\mathbb{R}}} e_2) \quad \text{some } \lambda \in S_{\mathbb{R}} \tag{8.7.2}$$

and $\lambda \in S_{\mathbb{R}}$ is the desired element; its norm is equal to Λ'. A more explicit way to think about this is the following: There are elements $\alpha,\beta,\gamma,\delta \in S_{\mathbb{R}}$ so that the period map (8.6.1) is given by

$$f\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta. \end{pmatrix}\cdot\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

Then simply $\lambda = \alpha\delta - \beta\gamma \in S_{\mathbb{R}}$; also the norm of λ is Λ' as before.

We can now state the equivariant BSD conjecture. The $\mathbb{R}$-linear extension of $\psi : S \to \mathbb{C}$ gives $\psi : S_{\mathbb{R}} \to \mathbb{C}$. We then allow ourselves to denote $\psi(a)$ also by a^{ψ}. Then:

$$L(\frac{1}{2}, E\times\psi) = \left(\left[\frac{Ш_B \cdot \prod_v c_v(B)}{B(\mathbb{Q})\hat{B}(\mathbb{Q})} \cdot \frac{(H^1_+/Se_1+Se_2)}{(\Omega^1_{\mathscr{B}}/Sv_1+Sv_2)}\right].\lambda.\right)^{\psi} \text{ modulo } \psi(S^{\times}). \tag{8.7.3}$$

part of the conjecture is that the square-bracketed term is a virtually principal S-module (see §8.2) so that it gives an element of $S_{\mathbb{Q}}^{\times}/S^{\times}$ according to our conventions. As before, we regard the right-hand side as 0 if $B(\mathbb{Q})$ is infinite. Also, "modulo $\psi(S^{\times})$" means that the ratio of the two sides belongs to $\psi(S^{\times})$.

As remarked previously, it is likely one can derive this from the equivariant Tamagawa number conjecture [22], although we did not verify the details of this process. For the purpose of this paper, the phrase "equivariant BSD" refers to the formulation (8.7.3) above.

Taking the corresponding conjecture with ψ replaced by ψ^i, and taking product over $i \in (\mathbb{Z}/d)^*$, recovers the original BSD conjecture for B – at least up to algebraic units.

8.8. **Explication.** Assume equivariant BSD. Now since we are interested only in proving Proposition 7.7 we may suppose that $L(\frac{1}{2}, E\times\psi) \neq 0$; then also (by (8.7.3)) we have that $B(\mathbb{Q})$ is finite and so $L(\frac{1}{2}, B) \neq 0$. We assume these in what follows.

Next let us explicitly choose v_1, v_2, e_1, e_2 as in the discussion above (8.7.1). The inclusion $B \hookrightarrow A$ induces $H^1(A(\mathbb{C})) \to H^1(B(\mathbb{C}))$. It's enough to produce forms v_1, v_2, e_1, e_2 on A so that the R-modules spanned by v_1, v_2, e_1, e_2 are free, and then we pull them back to B. Then the S-modules spanned by v_1, v_2 and by e_1, e_2 are also free. In order to compute the number λ as above, it will be enough to do the corresponding computation on A and then pull back to B.

- Choice of e_i:

  Recall that $H_1(A(\mathbb{C}),\mathbb{Z})$ is free as R-module on basis $\gamma_1, \gamma_2, \overline{\gamma_1}, \overline{\gamma_2}$.

  We let $x_1, x_2, y_1, y_2 \in H^1(A(\mathbb{C}),\mathbb{Z})$ be the dual basis (to the basis for $H_1(A(\mathbb{C}),\mathbb{Z})$ as $\mathbb{Z}$-module obtained by applying μ to $\gamma_1, \gamma_2, \overline{\gamma_1}, \overline{\gamma_2}$). In other words, for any $\alpha \in \mu$,

  $$\langle x_1, \alpha(\gamma_i)\rangle = \begin{cases} 1, i=1, \alpha = \text{id} \\ 0, else \end{cases} \quad ; \langle x_1, \overline{\alpha(\gamma_i)}\rangle = 0,$$

  and x_2 is similarly dual to γ_2, y_1 to $\overline{\gamma_1}$, y_2 to $\overline{\gamma_2}$.

  Then $H^1(A(\mathbb{C}),\mathbb{Z})$ is a free R-module on x_1, x_2, y_1, y_2. Also, the images of x_1, y_1 in H^1_+ coincide, as do x_2, y_2; and

  $$H^1(A(\mathbb{C}),\mathbb{Z})_+ \text{ is a free } R\text{-module on } x_1, x_2,$$

where we abuse notation by writing x_1 also for its image in H^1_+.

- Choice of v_i:

  We have an isomorphism, from (8.3.3),

$$\left(\Omega^1_{\mathscr{E}}\mathfrak{d}_F^{-1} \otimes_{\mathscr{O}_F} \mathfrak{d}^{-1}_{F_\psi/F}\right) \xrightarrow{\sim} \Omega^1_{\mathscr{A}/\mathbb{Z}}.$$

  Now choose a $\mathbb{Z}$-basis ξ_1, ξ_2 for $\Omega^1_{\mathscr{E}}\mathfrak{d}_F^{-1}$ and take v_i to be (the image of) $\xi_i \otimes x$, where $x \in \mathfrak{d}^{-1}_{F_\psi/F}$ is chosen to have the property that

$$[\mathfrak{d}^{-1}_{F_\psi/F} : \sum_{\alpha\in\mu} \mathscr{O}_F x^\alpha]$$

  is prime-to-ℓ (see next paragraph for why this is possible.) In particular,

$$\Omega^1_{\mathscr{A}}/(Rv_1 + Rv_2) \text{ is prime to } \ell. \tag{8.8.1}$$

  As for why we can choose such an x: We want to show that $\mathfrak{d}^{-1}_{F_\psi/F} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ has a "normal basis" over $\mathscr{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$, i.e. there is $x \in \mathfrak{d}^{-1}_{F_\psi/F} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ so that αx ($\alpha \in \mu$) spans as $\mathscr{O}_F \otimes \mathbb{Z}_\ell$-module. This comes down to the fact that F_ψ/F is tamely ramified at primes above ℓ, and Galois-stable ideals in tamely ramified extensions have (locally) normal bases:

  In other words, let $\mathfrak{l}_i$ be the primes of F above ℓ, and let λ_{ij} be the primes of F_ψ above $\mathfrak{l}_i$. Let m_{ij} be the valuation of $\mathfrak{d}^{-1}_{F_\psi/F}$ at λ_{ij}. We are asking that

$$\prod_{i,j} \lambda_{ij}^{m_{ij}} \mathscr{O}_{F_\psi,\lambda_{ij}} \text{ have a normal basis over } \prod \mathscr{O}_{F,\mathfrak{l}_i}.$$

  It is enough that $\prod_j \lambda_{ij}^{m_{ij}} \mathscr{O}_{F_\psi,\lambda_{ij}}$ have a normal basis over $\mathscr{O}_{F,\mathfrak{l}_i}$ for each i separately; say $i = 1$. Next, because the Galois group permutes the various λ_{1j}, and m_{1j} doesn't depend on j, it is enough to show that $\lambda_{11}^{m_{11}} \mathscr{O}_{F_\psi,\lambda_{11}}$ has a normal basis over over $\mathscr{O}_{F,\mathfrak{l}_1}$. But that is a theorem of S. Ullom [55, Theorem 1] because F_ψ/F is tamely ramified.

  Finally, we note for later use that in fact

$$\mathrm{Norm}(\mathfrak{n}_\psi) x \in \mathscr{O}_{F_\psi} \otimes \mathbb{Z}_\ell \tag{8.8.2}$$

  i.e. it is an algebraic integer above ℓ. Here $\mathrm{N}(\mathfrak{n}_\psi) \in \mathbb{Z}$ is the absolute ideal norm from ideals of F.

  In fact, it's enough to see that $\mathfrak{n}_\psi x \subset \mathscr{O}_{F_\psi} \otimes \mathbb{Z}_\ell$, i.e. $\mathfrak{n}_\psi \mathfrak{d}^{-1}_{F_\psi/F} \subset \mathscr{O}_{F_\psi} \otimes \mathbb{Z}_\ell$. But, if L/K is a tamely ramified extension of global fields then $\prod_{\mathfrak{q}} \mathfrak{q} \cdot \mathfrak{d}^{-1}_{L/K} \subset \mathscr{O}_L$, where the product is over ramified primes $\mathfrak{q}$. (We want just the "version above ℓ" of this.) One reduces immediately to the case of a tamely ramified extension of local fields, say with ramification index e and residue field degree f. In that case, we can check the inclusion by taking norms of both sides; the valuation of the norm of $\mathfrak{q}$ is ef and the valuation of the norm of $\mathfrak{d}_{L/K}$ is $e-1$; clearly $ef \geq e-1$.

Recall that we may choose ξ_1, ξ_2 in such a way that

$$\mathrm{Im}\left(\frac{\xi_2}{\xi_1}\right) = \frac{\sqrt{-\Delta_F/4}}{a} \tag{8.8.3}$$

where $a = [\Omega^1_{\mathscr{E}}\mathfrak{d}_F^{-1} : \mathscr{O}_F \xi_1]$ is as in (7.6.9).

8.9. **Exterior product computations.** We compute $\lambda \in S_{\mathbb{R}}$ as in (8.7.2) – with respect to the images of $e_1, e_2, v_, v_2$ under the natural maps induced by $B \hookrightarrow A$ – by computing its analog $\widetilde{\lambda} \in R_{\mathbb{R}}$ computed "on A": There exists $\widetilde{\lambda} \in R_{\mathbb{R}}$ such that

(8.9.1) $$\text{image of } v_1 \wedge_{R_{\mathbb{R}}} v_2 = \widetilde{\lambda}(e_1 \wedge_{R_{\mathbb{R}}} e_2) \quad \text{some } \widetilde{\lambda} \in R_{\mathbb{R}}$$

(where everything is computed on the abelian variety A). Then the desired $\lambda \in S_{\mathbb{R}}$ is simply the image of $\widetilde{\lambda} \in R_{\mathbb{R}}$ under the natural map.

In what follows, we write simply $v_1 \wedge v_2$ instead of $v_1 \wedge_{R_{\mathbb{R}}} v_2$.

Recall that we take $e_1 = x_1, e_2 = x_2$ (see above).

Put $\delta_i = \gamma_i + \overline{\gamma_i}$. Regard integration on δ_i as being functionals $H^1(A(\mathbb{C}), \mathbb{R}) \to \mathbb{R}$; they factor through H^1_+. By averaging them over μ we get R-linear functionals: we define $\Delta_i : H^1(A(\mathbb{C}), \mathbb{R})_+ \to R_{\mathbb{R}}$ by the rule

$$\Delta_i(\omega) = \sum \alpha \int_{\alpha(\delta_i)} \omega, \ \ (i = 1, 2; \omega \in H^1(A(\mathbb{C}), \mathbb{R}).),$$

which is now R-linear. We now pair both sides of (8.9.1) with $\Delta_1 \wedge \Delta_2$ in order to compute λ.

Firstly,

$$\langle x_1, \Delta_1 \rangle = \sum_{\alpha} \alpha \int_{\alpha(\delta_1)} x_1 = 1 \in R,$$

and similarly

$$\langle x_2, \Delta_2 \rangle = 1; \ \langle x_1, \Delta_2 \rangle = \langle x_2, \Delta_1 \rangle = 0.$$

Therefore,

(8.9.2) $$\langle x_1 \wedge x_2, \Delta_1 \wedge \Delta_2 \rangle = x_1(\Delta_1) x_2(\Delta_2) - x_2(\Delta_1) x_1(\Delta_2) = 1.$$

Next, compute $\langle v_1 \wedge v_2, \Delta_1 \wedge \Delta_2 \rangle$; it equals

$$v_1(\Delta_1) v_2(\Delta_2) - v_2(\Delta_1) v_1(\Delta_2) = \sum_{\alpha,\beta} \alpha^{-1}\beta^{-1} \left(\langle v_1, \alpha^{-1}(\delta_1) \rangle \langle v_2, \beta^{-1}(\delta_2) \rangle - \langle v_2, \alpha^{-1}(\delta_1) \rangle \langle v_1, \beta^{-1}(\delta_2) \rangle \right),$$

wher $\int_{\alpha^{-1}(\delta_1)} v_1$ has been abbreviated $\langle v_1, \alpha_1^{-1}(\delta_1) \rangle$ and so on. In turn [5]

(8.9.3) $$\frac{1}{4} \left(\langle v_1, \alpha^{-1}(\delta_1) \rangle \langle v_2, \beta^{-1}(\delta_2) \rangle - \langle v_2, \alpha_1^{-1}(\delta_1) \rangle \langle v_1, \beta^{-1}(\delta_2) \rangle \right)$$

$$= \left(\mathrm{Re}(\int_{\gamma_1} \sigma_\alpha(x)\xi_1) \mathrm{Re}(\int_{\gamma_2} \sigma_\beta(x)\xi_2) - \mathrm{Re}(\int_{\gamma_2} \sigma_\alpha(x)\xi_1) \mathrm{Re}(\int_{\gamma_1} \sigma_\beta(x)\xi_2) \right)$$

(8.9.4) $$= \mathrm{Im}\left(\frac{\sigma_\beta(x)}{\sigma_\alpha(x)} \frac{\xi_2}{\xi_1} \right) \cdot (\text{ area of lattice of } E \text{ with respect to 1-form } \sigma_\alpha(x)\xi_1)$$

(8.9.5) $$= \frac{i}{2} \left(\int_{E(\mathbb{C})} \xi_1 \wedge \overline{\xi_1} \right) \cdot \mathrm{Im}\left(\sigma_\beta(x) \overline{\sigma_\alpha(x)} \frac{\xi_2}{\xi_1} \right)$$

and $\langle v_1 \wedge v_2, \Delta_1 \wedge \Delta_2 \rangle$ is obtained by summing this expression multiplied by $4\alpha^{-1}\beta^{-1}$, over α and β.

---

[5] Write ξ_2/ξ_1 for the element $t \in F$ with $t\xi_1 = \xi_2$. We used the following simple fact at (8.9.4), with $q = \frac{\sigma_\beta(x)\xi_2}{\sigma_\alpha(x)\xi_1}$,

$$\Re(z_1)\Re(qz_2) - \Re(z_2)\Re(qz_1) = \mathrm{Im}(q) \cdot \Re(z_1 \overline{z_2}).$$

Note now that if we modify ξ_2 by a real multiple of ξ_1 the answer is unchanged (the contribution of α, β and of β, α cancel in the summation). Thus we may take $\xi_2 = \frac{1}{a}\sqrt{\Delta_F/4}\cdot\xi_1$, where a as in (8.8.3), and then from the definition of Ω_E we see that:

$$\begin{aligned}\langle v_1 \wedge v_2, \Delta_1 \wedge \Delta_2\rangle &= \sum_{\alpha,\beta} \sqrt{-\Delta_F}\Omega_E \cdot \mathrm{Re}\left(\sigma_\beta(x)\overline{\sigma_\alpha(x)}\right)\alpha^{-1}\beta^{-1} \\ &= \sqrt{-\Delta_F}[x][\overline{x}]\Omega_E \in R_{\mathbb{R}}.\end{aligned}$$

(Recall that $[x] = \sum_\alpha \sigma_\alpha(x)\alpha^{-1}$ and $\overline{[x]} = \sum_\alpha \overline{\sigma_\alpha(x)}\cdot\alpha^{-1}$; these belong to $R_{\mathbb{C}}$ but their product $[x]\overline{[x]}$ belongs to $R_{\mathbb{R}}$.) Comparing with (8.9.2) and (8.9.1) we see that

$$\widetilde{\lambda} = \sqrt{-\Delta_F}[x][\overline{x}]\Omega_E \in R_{\mathbb{R}}.$$

We can now rewrite equivariant BSD from the form (8.7.3). We have seen that H^1_+ is free on e_1, e_2, so

$$\frac{L(\frac{1}{2}, E\times\psi)}{\Omega_E\sqrt{-\Delta_F}} = \left([x]\overline{[x]}\cdot\left[\frac{Ш_B \cdot \prod_v c_v(B)}{B(\mathbb{Q})\hat{B}(\mathbb{Q})}\cdot\frac{1}{\left(\Omega^1_{\mathscr{B}}/Sv_1+Sv_2\right)}\right]\right)^\psi \bmod \psi(S^\times). \tag{8.9.6}$$

8.10. **Conclusion.** We are almost finished. First note that for M a finite S-module, we always have $[M] \le [S/\#M]$; this is simply the fact that an ideal of S divides its norm. As we have mentioned, we can suppose that $B(\mathbb{Q})$ and $\hat{B}(\mathbb{Q})$ are finite. So, examining the denominator of (8.9.6),

$$\begin{aligned}[\hat{B}(\mathbb{Q})_{\mathrm{tors}}] + [B(\mathbb{Q})_{\mathrm{tors}}] + [\Omega^1_{\mathscr{B}}/Sv_1+Sv_2] &\le 2[E(F_\psi)_{\mathrm{tors}}] + \\ &\quad +[\Omega^1_{\mathscr{B}}/\text{image of }\Omega^1_{\mathscr{A}}] + [\text{image of }\Omega^1_{\mathscr{A}}/Rv_1+Rv_2] \\ &\le_\ell [S/K]\end{aligned}$$

Here $K = \#E(F_\psi)^2_{\mathrm{tors}}\cdot d^2\cdot\left(\#\Omega^1_{\mathscr{A}}/Rv_1+Rv_2\right)$ and $\le_\ell$ means the equality holds "at ℓ," as mentioned before. This last inequality follows from (8.3.4). Since by (8.8.2) we have that $\mathrm{N}(\mathfrak{n}_\psi)^2\cdot\psi([x]\overline{[x]})$ is an algebraic integer, we have proved

$$\mathrm{N}(\mathfrak{n}_\psi)^2 K\cdot\frac{L(\frac{1}{2}, E\times\psi)}{\Omega_E\sqrt{-\Delta_F}} \text{ has valuation} \ge 0 \text{ at } \ell,$$

which finishes the proof, because $\#\Omega^1_{\mathscr{A}}/Rv_1+Rv_2$ is prime-to-ℓ by (8.8.1).

9. Numerical computations

9.1. **How much of the cohomology is base-change at higher levels ?** Let F be an imaginary quadratic field with its ring of integers $\mathscr{O}_F$. Given a positive integer N, consider the congruence subgroup $\Gamma_0((N))$ of level (N) inside the associated Bianchi group $\mathrm{SL}_2(\mathscr{O}_F)$. Let $H^2_{bc}(\Gamma_0((N)),\mathbb{C})^{new}$ denote the subspace of $H^2(\Gamma_0((N)),\mathbb{C})^{new}$ which corresponds to Bianchi modular forms that are base-change of classical elliptic newforms and their twists (see Section 6.4). We are interested in the following question:

How much of $H^2(\Gamma_0((N)),\mathbb{C})^{new}$ is exhausted by $H^2_{bc}(\Gamma_0((N)),\mathbb{C})^{new}$?

Note that this question was investigated in [45] for $N = 1$ and more general coefficient modules. As part of an ongoing project [53], Panagiotis Tsaknias and the second author (M.H.Ş.) computed the dimension of $H^2_{bc}(\Gamma_0((N)),\mathbb{C})^{new}$ for the following special case:

F is ramified at a unique prime $p > 2$, N is square-free and prime to p.

Over the fields $F = \mathbb{Q}(\sqrt{-d})$ with $d = 3, 7, 11$, we have collected data to investigate the above question. For efficiency reasons, we computed $H_1(\Gamma_0((N)), \mathbb{F}_\ell)$ for six primes ℓ lying between 50 and 100 and took the minimum of the dimensions we got from these six mod ℓ computations. By the Universal Coefficients Theorem, this minimum is an upper-bound on the dimension of $H_1(\Gamma_0((N)), \mathbb{C})$. However in practice, this upper-bound is very likely to give the actual dimension.

We focused on three classes of ideals $(N) \lhd \mathcal{O}_F$:

- $N = p$ with p rational prime that is *inert* in F (Table 1).

  Here there are no oldforms. Thus the base-change dimension formula, together with the number cusps, provides a lower bound for the dimension of $H_1(\Gamma_0((N)), \mathbb{C})$. If this lower-bound agrees with our upper-bound coming from the mod ℓ computations, then we know for sure that the whole (co)homology is exhausted by base-change. As a result, the zero entries in "non-BC" column of Table 1 are provenly correct.

  We also directly computed the char. 0 dimensions (the scope was smaller of course). The nonzero entries in the "non-BC" columns of Table 1 which are in bold are proven to be correct as a result of these char. 0 computations.

- $N = p$ with p rational prime that is *split* in F (Table 2).

  Here they may be oldforms and we can compute size of the oldforms using the data computed [51]. Now the base-change dimension formula, the dimension of the old part, together with the number cusps, provide a lower bound for the dimension of $H_1(\Gamma_0((N)), \mathbb{C})$. As above, if this lower-bound agrees with our upper-bound coming from the mod ℓ computations, then we know for sure that the whole (co)homology is exhausted by base-change. As a result, the zero entries in the "non-BC" columns of Table 2 are provenly correct.

  We also directly computed the char. 0 dimensions for Table 2. The nonzero entries in the "non-BC" columns of Table 2 which are in bold are proven to be correct as a result of these char. 0 computations.

- $N = pq$ with p, q rational primes that are *inert* in F (Table 3).

  To compute the size of the oldforms, one can use the data computed for Table 1 (note that we only have to refer to entries of Table 1 which are provenly correct). As before, the zero entries in the "non-BC" columns of Table 3 are provenly correct.

  We also directly computed the char. 0 dimensions for Table 3. The nonzero entries in the "non-BC" columns of Table 3 which are in bold are proven to be correct as a result of these char. 0 computations.

Of course, there can be non-base change classes in the oldforms part, but this is not common. As we mentioned before, in Case (1), there are no oldforms. In Case (2), extensive computations in [51] show that %90 of the time the cuspidal cohomology of $\Gamma_0(\mathfrak{p})$ (with trivial coefficients) vanishes, where $(p) = \mathfrak{p}\bar{\mathfrak{p}}$. So usually, we do not have oldforms in Case (2). But when we do, they are completely non-base change. For Case (3), there will be lots of old forms, however with little non-base change classes amongst them (which can detected via Table 1).

In the Tables 1,2,3, the columns labeled “new" denote the dimension of the new subspace and the columns labeled “non-BC" denote the dimension of the dimension of the complement of the base-change subspace inside the new subspace.

$d=3$			$d=7$			$d=11$		
p	new	non-BC	p	new	non-BC	p	new	non-BC
5	0	0	3	0	0	7	3	0
11	2	0	5	1	0	13	5	0
17	2	0	13	5	**2**	17	9	**2**
23	4	0	17	5	0	19	9	0
29	4	0	19	5	0	29	13	0
41	6	0	31	9	0	41	21	**2**
47	10	**2**	41	13	0	43	23	**2**
53	8	0	47	15	0	61	29	0
59	12	**2**	59	19	0	73	35	0
71	12	0	61	19	0	79	39	0
83	14	0	73	23	0	83	41	0
89	16	**2**	83	29	**2**	101	49	0
101	16	0	89	31	**2**	107	55	**2**
107	18	0	97	33	**2**	109	53	0
113	22	**4**	101	35	**2**	127	63	0
131	22	0	103	33	0	131	65	0
137	22	0	131	43	0	139	69	0
149	26	**2**	139	45	0	149	73	0
167	28	0	157	51	0	151	75	0
173	28	0	167	55	0	167	83	0
179	34	**4**	173	57	0	173	85	0
191	34	**2**	181	59	0	193	99	**4**
197	32	0	199	65	0	197	97	0
227	46	**8**	223	73	0	211	105	0
233	38	0	227	75	0	227	113	0
239	40	0	229	75	0	233	121	6
251	42	0	241	79	0	239	119	0
257	42	0	251	87	**4**	241	119	0
263	44	0	257	85	0	263	131	0
269	44	0	269	89	0	271	135	0
281	48	**2**	271	89	0	277	137	0
293	48	0	283	93	0	281	139	0
311	54	**2**	293	97	0	283	141	0
317	52	0	307	101	0	293	145	0
347	58	0	311	103	0	307	153	0
353	60	**2**	313	103	0	337	167	0
359	62	**2**	349	115	0	347	173	0
383	64	0	353	117	0	349	173	0
389	64	0	367	121	0	359	181	2
401	66	0	383	127	0	373	185	0
419	70	0	397	131	0	409	203	0

431	72	0	409	135	0	431	215	0
443	80	6	419	139	0	439	219	0
449	74	0	433	147	4	457	227	0
461	76	0	439	145	0	461	229	0
			461	153	0	479	239	0
			467	155	0	491	245	0
			479	159	0	503	251	0
			503	167	0	523	261	0
			509	169	0	541	269	0
			521	173	0	547	273	0
			523	173	0	557	277	0
			563	187	0	563	281	0
			577	191	0	569	283	0
			587	195	0	571	285	0
			593	197	0	593	295	0
			601	199	0	601	301	2
			607	201	0	607	303	0
			619	205	0	613	305	0

Table 1: Level is (p) with p rational prime, *inert* in F

$d = 3$			$d = 7$			$d = 11$		
p	new	non-BC	p	new	non-BC	p	new	non-BC
7	1	0	11	3	0	3	1	0
13	1	0	23	7	0	5	1	0
19	3	0	29	9	0	23	13	**2**
31	5	0	37	15	**4**	31	17	**2**
37	5	0	43	13	0	37	17	0
43	7	0	53	17	0	47	35	**12**
61	11	**2**	67	21	0	53	25	0
67	11	0	71	23	0	59	29	0
73	11	0	79	25	0	67	33	0
79	15	**2**	107	37	**2**	71	35	0
97	15	0	109	35	0	89	43	0
103	19	**2**	113	37	0	97	47	0
109	17	0	127	41	0	103	51	0
127	29	**8**	137	45	0	113	55	0
139	25	**2**	149	49	0	137	67	0
151	27	**2**	151	49	0	157	77	0
157	27	**2**	163	53	0	163	81	0
163	29	**2**	179	59	0	179	89	0
181	31	**2**	191	65	**2**	181	89	0
193	31	0	193	65	**2**	191	97	2
199	35	**2**	197	67	**2**	199	99	0
211	37	**2**	211	69	0	223	111	0
223	37	0	233	77	0	229	115	2
229	37	0	239	79	0	251	125	0

241	47	**8**	263	87	0	257	129	2
271	47	**2**	277	91	0	269	133	0
277	45	0	281	97	4	311	157	2
283	47	0	317	105	0	313	155	0
307	53	**2**	331	111	2	317	157	0
313	51	0	337	111	0	331	165	0
331	57	**2**	347	115	0	353	175	0
337	57	**2**	359	119	0	367	183	0
349	59	**2**	373	123	0	379	189	0
367	61	0	379	125	0	383	191	0
373	61	0	389	129	0	389	193	0
379	63	0	401	133	0	397	201	4
397	67	2	421	139	0	401	199	0
409	69	2	431	145	2	419	209	0
421	69	0	443	147	0	421	209	0
433	71	0	449	149	0	433	215	0
			457	151	0	443	221	0
			463	153	0	449	223	0
			487	161	0	463	231	0
			491	163	0	467	233	0
			499	167	2	487	243	0
						499	251	2
						509	255	2
						521	259	0
						577	289	2
						587	293	0

Table 2: Here the level is (p) with p rational prime, *split* in F

$d=3$			$d=7$			$d=11$		
pq	new	non-BC	pq	new	non-BC	pq	new	non-BC
55	7	0	15	3	0	91	41	**4**
85	11	0	39	11	**2**	119	59	**10**
115	15	0	51	11	0	133	53	0
145	19	0	57	13	0	203	85	0
187	27	0	65	17	0	221	97	0
205	31	**4**	85	21	0	247	109	0
235	33	**2**	93	25	**4**	287	121	0
253	35	0	95	25	0	301	127	2
265	35	0	123	27	0	323	145	0
295	39	0	141	31	0	377	171	2
319	47	0	155	43	**2**	427	181	0
355	47	0	177	39	0	493	225	0
391	59	0	183	43	**2**	511	219	2
415	55	0	205	55	**2**	533	243	2
445	61	**2**	219	49	0	551	253	0
451	67	0	221	65	0	553	233	0

493	75	0	235	61	0	559	253	0
505	67	0	247	73	0	581	245	0
517	75	0	249	55	0			
			267	59	0			
			291	65	0			
			295	81	**4**			
			303	67	0			
			305	83	**2**			
			309	69	0			
			323	97	0			
			365	97	0			
			393	87	0			
			403	121	0			
			415	111	2			
			417	93	0			
			445	117	0			
			471	105	0			
			485	131	2			
			501	113	2			
			505	133	0			
			515	139	2			
			519	117	2			
			527	161	0			
			533	161	0			
			543	121	0			
			589	181	0			
			597	133	0			
			611	185	0			
			655	173	0			

Table 3: Here the level is (pq) with p,q rational primes both *inert* in F

9.2. **Cases with one-dimensional cuspidal cohomology.** As mentioned in the Introduction, experiments in [51] show that for the five Euclidean imaginary quadratic fields F, the cuspidal part of $H_1(\Gamma_0(\mathfrak{p}),\mathbb{C})$, for $\Gamma_0(\mathfrak{p}) \le \mathrm{PSL}_2(\mathcal{O}_F)$ with residue degree one prime level $\mathfrak{p}$ of norm ≤ 45000, vanishes roughly %90 of the time. In the remaining non-vanishing cases, the dimension is observed to be one in the majority of cases (see Table 16 of [51] for details).

In a new experiment, we computed the dimension of the cuspidal part of $H_1(\Gamma_0(\mathfrak{n}),\mathbb{C})$, for $\Gamma_0(\mathfrak{n}) \le \mathrm{PGL}_2(\mathcal{O}_F)$ with *all* levels $\mathfrak{n}$ of norm ≤ 10000 for the fields $F = \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$. Again we see that a significant proportion of the non-vanishing cases have dimension exactly one. The distribution of the levels according to the dimension is given in Table 4.

dim	$\mathbb{Q}(\sqrt{-1})$	$\mathbb{Q}(\sqrt{-3})$
0	4170	3516
1	614	526

2	734	642
3	341	266
4	402	327
5	183	168
6 ≤	1409	1104
total:	7853	6048

Table 4: distribution of subgroups $\Gamma_0(\mathfrak{n})$ with $\mathrm{Norm}(\mathfrak{n}) \leq 10000$.

9.3. **Growth of Regulators of Hyperbolic Tetrahedral Groups.** In this section, we report on our numerical experiments related to the growth of regulators in the case of hyperbolic tetrahedral groups. Here we deal with combinatorial regulators rather than analytic ones. One may however prove that they both have either subexponential or exponential growth with respect to the index [38].

9.3.1. *Tetrahedral Groups.* A hyperbolic tetrahedral group is the index two subgroup consisting of orientation-preserving isometries in the discrete group generated by reflections in the faces of a hyperbolic tetrahedron whose dihedral angles are submultiples of π. It is well-known that there are 32 hyperbolic tetrahedral groups; 9 of them are cocompact. Among the 9 cocompact ones, only one is non-arithmetic (see [52]).

Let Δ be one of the 9 compact tetrahedra mentioned above, sitting in hyperbolic 3-space $\mathbb{H}^3$ and Γ be the associated hyperbolic tetrahedral group. Let Σ be a fundamental domain for Γ, viewed as a 3-dimensional simplicial complex. We can take Σ to be the union $\Delta \cup \Delta^*$ where Δ^* is the copy of Δ obtained by reflecting Δ along one of its faces. Let $\mathbf{T}$ denote the triangulation of $\mathbb{H}^3$ obtained from Δ, viewed as an infinite, locally finite simplicial complex with a cocompact cellular action of Γ so that $\Gamma \backslash \mathbf{T} = \Sigma$.

Let H be a finite index subgroup of Γ and consider the vector space $M = \mathbb{R}[H\backslash\Gamma]$ with the natural Γ-action. It is well-known that the Γ-equivarient cohomology of $\mathbf{T}$ equals to the usual cohomology of Γ:

$$H^*(\Gamma, M) \simeq H^*_\Gamma(\mathbf{T}, M).$$

Put an inner product on M by declaring the basis $H\backslash\Gamma$ to be orthonormal. We define the combinatorial Laplacians $\{\Delta_i\}_i$ on the Γ-equivarient cochain complex $\{C^i(\mathbf{T}, M)^\Gamma\}_i$ (which computes the RHS) using the inner product

$$\langle f, g\rangle_i^\Gamma := \sum_{\sigma\in\Sigma_i} \frac{1}{|\Gamma(\sigma)|} \langle f(\tilde{\sigma}), g(\tilde{\sigma})\rangle$$

where $\tilde{\sigma}$ is a lift of σ in $\mathbf{T}_i$. See, for example, [34, Section 2.] for details.

For $i = 1, 2$, let r_i denote volume of $H^i(\Gamma, \mathbb{Z}[H\backslash\Gamma])$ inside $H^i(\Gamma, \mathbb{R}[H\backslash\Gamma])$ with respect to the above inner product. According to our Proposition 4.1, asymptotically r_i behaves like the regulator R_i. For computational efficiency, we will compute another quantity, which, asymptotically speaking, gives us the desired information. Let $\tilde{r}_i$ denote the volume (w.r.t. to the same inner product) of the subspace of harmonic i-cochains, that is, the kernel of Δ_i. Then it is not hard to see that

$$\tilde{r}_i \geq r_i \geq \frac{1}{\tilde{r}_i}.$$

$[T6:H]$	rank	$\mathrm{Log}(\tilde{r}_1)$	$\mathrm{Log}(\tilde{r}_1)$ / $[T6:H]$	$\mathrm{Log}(\tilde{r}_2)$	$\mathrm{Log}(\tilde{r}_2)$ / $[T6:H]$
122	1	5.4161004	0.04439426559	4.4755991	0.03668523930
170	5	28.1536040	0.1656094354	44.7684568	0.2633438638
290	5	36.2058878	0.1248478891	30.5155222	0.1052259389
362	7	45.4762539	0.1256250109	44.4415985	0.1227668467
458	1	9.26712597	0.02023389951	6.4856782	0.01416086959
674	1	7.6487642	0.01134831485	8.3297444	0.01235867125
962	11	78.0538394	0.08113704729	79.9289185	0.08308619394
1034	2	17.8191345	0.01723320555	21.3528238	0.02065070001
1370	1	7.0476963	0.00514430392	7.52250931	0.005490882711
1682	15	105.2828487	0.06259384583	116.4427193	0.06922872726
1850	2	20.1698091	0.01090259953	20.7570214	0.01122001160
2210	15	109.5835840	0.04958533211		
2522	2	19.5918702	0.00776838630		

TABLE 5. Data for projective subgroups H of $T6$

$[T8:H]$	rank	$\mathrm{Log}(\tilde{r}_1)$	$\mathrm{Log}(\tilde{r}_1)$ / $[T8:H]$	$\mathrm{Log}(\tilde{r}_2)$	$\mathrm{Log}(\tilde{r}_2)$ / $[T8:H]$
42	1	5.67652517	0.1351553613	8.25960528	0.1966572686
82	4	15.39756182	0.1877751441	23.67090665	0.2886695933
962	9	60.87067153	0.06327512633	79.69735698	0.08284548543
1682	13	90.93035031	0.05406085036		
2402	4	26.74244163	0.01113340617		

TABLE 6. Data for projective subgroups H of $T8$

We computed $\tilde{r}_i$ for prime level Γ_0-type subgroups H (see [52]) of two cocompact hyperbolic tetrahedral groups $T6$ and $T8$, which, in the notation of [52], can be identified as $T(4,3,2;4,3,2)$ and $T(5,3,2;4,3,2)$. While $T6$ is arithmetic, $T8$ is non-arithmetic. The data we collected are depicted in Tables 5 and 6 respectively. In the tables, the first column shows the index of the subgroup H inside the tetrahedral group, the second column shows the dimension of the cohomology $H_1(H,\mathbb{R})$. For the cases where this dimension is zero, the space of harmonic cochains is trivial and thus these cases are not included in the tables.

9.3.2. *Arithmetic versus non-arithmetic.* As discussed at the end of the introduction (also see [7, Chapter 9]), subexponential growth of the regulator with respect to the volume might be related to *arithmeticity*. Unfortunately, the scope of the data we collected here on the growth of the regulator is too limited to infer anything on this speculation. However, the experiments in [12, 52], which inspect the growth of torsion, all suggest that if M_0 is non-arithmetic, then for a sequence $(M_i \to M_0)_{i\in\mathbb{N}}$ of finite covers of M_0 which is BS-converging to $\mathbb{H}^3$, the sequence

$$\frac{\log \# H_1(M_i,\mathbb{Z})_{\mathrm{tors}}}{V_i}$$

does *not* necessarily converge to $1/(6\pi)$ (the convergence is broken at covers with positive Betti numbers). If we believe that analytic torsion converges in this general setting then it must be that the regulator does not disappear in the limit, giving support to the above speculation.

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