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Corrigendum

Torsion in cocompact lattices in coverings of spin (2, *n*)

M.S. Raghunathan

Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

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A.S. Rapinchuk has drawn my attention to an error in the proof of Assertion 5.8 of the paper referred to in the title [1] (Math. Ann. 206 (1984), 403–419). The assertion itself and other results stated in the paper are correct. In this note we give a correct proof of Assertion 5.8. We will make free use of the notations of the earlier paper. Briefly they are as follows:

k is a numberfield and we fix a completion $k_v \simeq \mathbf{R}$ of k and treat k as a subfield of $\mathbf{R}$ through this identification. G is the algebraic group of norm 1 elements in a quaternion division algebra D over k which splits over k_v. We set $\mathscr{G} = SL(2,\mathbf{R}) = G(\mathbf{R}), \hat{\mathscr{G}}$ a central extension of $\mathscr{G}$ by $T = \mathbf{R}/\mathbf{Z}$ split over $G(k)$. For $x, y \in \mathscr{G}$, the commutator $\tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}$ of lifts $\tilde{x}, \tilde{y}$ of x, y to $\hat{\mathscr{G}}$ is independent of the choice of these lifts and defines thus a continuous map $\hat{C} : \mathscr{G} \times \mathscr{G} \to \hat{\mathscr{G}}$. If x, y, x', y' are in $G(k)$ with $xyx^{-1}y^{-1} = x'y'x'^{-1}y'^{-1}$, the $\hat{C}(x,y) = \hat{C}(x',y')$. (The main result of Sect. 5 in the paper is that $\hat{C}$ factors through $C : \mathscr{G} \times \mathscr{G} \to \mathscr{G}$ given by $(x,y) \mapsto xyx^{-1}y^{-1}$). One defines $f : \mathscr{G} \times \mathscr{G} \to \mathscr{G} \times \mathscr{G}$ by setting $f(x,y) = (xyx^{-1}, y^{-1})$ for $x, y \in \mathscr{G}$ and $E = \mathscr{G} \times \mathscr{G} \backslash C^{-1}(\pm 1)$, $P = f(E)$. Also $m : \mathscr{G} \times \mathscr{G} \to \mathscr{G}$ is the multiplication map so that $C = m \circ f$. It is shown in the earlier paper [1] that $\hat{C}$ on E factors through $E \xrightarrow{f} P : \hat{C} = \hat{m} \circ f$. With these notations we will establish the following (claim 5.8 of [1]).

Claim *$\hat{m}$ is constant along the fibres of $m : P \to \mathscr{G} \backslash \{\pm 1\}$.*

We begin by examining the fibres of $m : P \to \mathscr{G} \backslash \{\pm 1\}$. Let $\alpha \in \mathscr{G} \backslash \{\pm 1\}$ be a semisimple element. The fibre $m^{-1}(\alpha)$ then can be described as pairs $\{(\alpha y, y^{-1}) \mid y \in \mathscr{G}, \alpha y \text{ is conjugate to } y \text{ in } \mathscr{G}\}$. The map $m : \mathscr{G} \times \mathscr{G} \to \mathscr{G}$ given by $(\alpha y, y^{-1}) \mapsto y$ imbeds $m^{-1}(\alpha)$ as a subset of $\mathscr{G}$. The condition that αy and y be conjugates in $\mathscr{G}$ implies that trace $\alpha y =$ trace y (but the equality of the traces only ensures that αy and y are conjugates in $GL(2, \mathbf{R})$).

Case i. α is an elliptic element. In this case after a conjugation in $\mathscr{G}$ we may assume that $\alpha \in SO(2)$: for $-\pi < \theta \leqq \pi$, set $u(\theta) \begin{pmatrix} Cos\,\theta & Sin\,\theta \\ -Sin\,\theta & Cos\,\theta \end{pmatrix}$ and let $\alpha = u(\chi), -\pi < \chi \leqq \pi$. Let $\mathscr{P}$ be the set of positive definite matrices of determinant 1. Then for any $g \in \mathscr{G}$ we have the unique polar decomposition $g = u(\theta(g)) \cdot p(g)$ where $p(g) \in \mathscr{P}$. Let $y = \beta p, \alpha y = \gamma q, \beta, \gamma \in SO(2), p, q \in \mathscr{P}$. Then $\alpha\beta p = \gamma q$ so that $\gamma = \alpha\beta$ and $p = q$. If y and αy are conjugates, their traces are equal so that trace $\beta p =$ trace γq. Now let $\beta = u(\varphi)$ and $\gamma = u(\psi)$; this leads to the equation (note that p, q are symmetric) trace $p \operatorname{Cos} \varphi =$ trace $q \operatorname{Cos} \psi$. Thus $\varphi = \pm\psi \pmod{2\pi}$ and since $\varphi - \psi = \chi \pmod{2\pi}$, one has $\varphi = -\psi$ so that $\beta = \gamma^{-1}$ and $2\varphi = \theta \pmod{2\pi}$. It follows that $\varphi = \chi/2$ or $\chi/2 + \pi$. Thus $y = \beta p$ and $\alpha y = \beta^{-1} p$. Clearly the element $w_o = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ conjugates y into αy. Now if there is an $x \in \mathscr{G}$ such that $xyx^{-1} = \alpha y$ then $x^{-1}w_o(\in GL(2, \mathbf{R}))$ centralises y and $\det x^{-1}w_o = -1$. Conversely if there is a $z \in$ centraliser of y in $GL(2, \mathbf{R})$ such that $\det z = -1$, then $w_o z^{-1} \in SL(2, \mathbf{R})$ conjugates y into αy. Now the centraliser of y in $GL(2, \mathbf{R})$ has an element of determinant -1, if and only if y is hyperbolic i.e., if and only if $|\operatorname{trace} y| > 2$ and $|\operatorname{trace} y| = \operatorname{trace} p \operatorname{Cos} \chi/2$. Thus the fibre $m^{-1}(\alpha)$ can be identified with the set

$$\{\pm u(\chi/2)p \mid \operatorname{trace} p \cdot \operatorname{Cos} \chi/2 > 2\} .$$

This is evidently a disjoint union of two connected sets and hence $m^{-1}(\alpha)$ has two connected components. Further $m^{-1}(\alpha)$ is stable under the map $(\alpha y, y^{-1}) \mapsto (-\alpha y, y^{-1})$ which interchanges its two connected components.

Case ii. α hyperbolic $\alpha = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ after a conjugation with $\lambda > 0$. Suppose now that $(\alpha y, y^{-1}) \in m^{-1}(\alpha)$ with $y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The equation trace $\alpha y =$ trace y which holds in $m^{-1}(\alpha)$ leads us to conclude that $d = \lambda a : y = \begin{pmatrix} a & b \\ c & \lambda a \end{pmatrix}$. Now $\alpha y = \begin{pmatrix} \lambda a & \lambda b \\ \lambda^{-1} c & a \end{pmatrix}$. Suppose that $b/c < 0$ (so that $bc < 0$); then $-\lambda b/c = \rho > 0$ and if we set $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\tau = \begin{pmatrix} \rho^{1/2} & 0 \\ 0 & \rho^{-1/2} \end{pmatrix}$, τw conjugates y into αy and belongs to $\mathscr{G}$. Thus $(\alpha y, y^{-1})$ belongs to $m^{-1}(\alpha)$. Next if $bc = 0$ either $b = 0$ or $c = 0$. Suppose that $c = 0$; then $y = \begin{pmatrix} a & b \\ 0 & \lambda a \end{pmatrix}$ while $\alpha y = \begin{pmatrix} \lambda a & \lambda b \\ 0 & a \end{pmatrix}$ and $1 = \det y = \lambda a^2$. It follows that $y = \begin{pmatrix} \lambda^{-1/2} & b \\ 0 & \lambda^{1/2} \end{pmatrix}$ while $\alpha y = \begin{pmatrix} \lambda^{1/2} & \lambda b \\ 0 & \lambda^{-1/2} \end{pmatrix}$. Now y is conjugate to $\begin{pmatrix} \lambda^{-1/2} & 0 \\ 0 & \lambda^{+1/2} \end{pmatrix}$ in $\mathscr{G}$ – infact by an upper triangular unipotent. Similarly αy is conjugate in $\mathscr{G}$ to $\begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}$ and these two diagonal elements are conjugate under

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus αy and y are conjugates in $\mathcal{G}$ if $c = 0$; an entirely analogous argument holds in the case $b = 0$. Finally suppose now that $bc > 0$. This means $\lambda a^2 - 1 > 0$ so that $\lambda a^2 > 1$ leading to $|\lambda a| > |1/a|$. This means that $|\operatorname{trace} y| = |a| + |\lambda a| = |a| + 1/|a| > 2$ so that y is hyperbolic. Now $y = \begin{pmatrix} a & b \\ c & \lambda a \end{pmatrix}$ is conjugate under $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to $\begin{pmatrix} \lambda a & c \\ b & a \end{pmatrix}$ and this last element is conjugated into αy by $\begin{pmatrix} (\lambda b/c)^{1/2} & 0 \\ 0 & (\lambda b/c)^{-1/2} \end{pmatrix}$. Thus we find y and αy are conjugates by an element σ of determinant -1. Since y is hyperbolic, one can find $\tau \in GL(2, \mathbf{R})$ centralising y and with $\det \tau = -1$ evidently $\sigma\tau$ has determinant 1 and conjugates y into αy. We see thus that $m^{-1}(\alpha)$ can be identified with $\left\{ \begin{pmatrix} a & b \\ c & \lambda a \end{pmatrix} \middle| \lambda a^2 - bc = 1 \right\}$. Since $\lambda > 0$, this is a *connected* set. Thus $m^{-1}(\alpha)$ is connected for α hyperbolic and trace $\alpha > 0$.

Case iii. α hyperbolic, $\alpha = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ with $\lambda < 0$. Once again one has $y = \begin{pmatrix} a & b \\ c & \lambda a \end{pmatrix}$ and we need to examine when y and $\alpha y = \begin{pmatrix} \lambda a & \lambda b \\ \lambda^{-1} & a \end{pmatrix}$ are conjugates in $SL(2, \mathbf{R})$. Since $\lambda a^2 - bc = 1$, one has $bc = \lambda a^2 - 1 < 0$. Thus $b/c < 0$. Now the element $w_o = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ conjugates y into $\begin{pmatrix} \lambda a & -c \\ -b & a \end{pmatrix}$. If we set $\tau = \begin{pmatrix} \rho^{1/2} & 0 \\ 0 & -\rho^{-1/2} \end{pmatrix}$ where $\rho^{1/2}$ is the positive square root of $\lambda c/b, \tau w_o$ conjugates y into αy; and let $\tau w_o = -1$. Thus y and αy are conjugates if and only if the centraliser y contains an element of determinant -1 i.e., if and only if y is hyperbolic; and y is hyperbolic if and only if $|a + \lambda a| > 2$ i.e., if and only if $|a| > 2/|1 + \lambda|$. Thus we see that $m^{-1}(\alpha)$ can be identified with the set

$$\left\{ \begin{pmatrix} a & b \\ c & \lambda a \end{pmatrix} \middle| a\lambda^2 - bc = 1 \text{ and } |a| > 2/|(1 + \lambda)| \right\} .$$

It is now easily seen that $m^{-1}(\alpha)$ has exactly two connected components; the subsets corresponding to $b > 0$ or $b < 0$. It is also clear that the map $(\alpha y, y^{-1}) \mapsto (-\alpha y, -y^{-1})$ maps $m(\alpha)$ into itself interchanging the components.

Suppose now $\alpha \in \mathcal{G}\backslash\{\pm 1\}$ is semisimple, $\alpha \in G(k)$ and further $\alpha = xyx^{-1}y^{-1}$ with $x, y \in G(k)$. If α is elliptic or hyperbolic with trace $\alpha < 0$, then (xyx^{-1}, y^{-1}) and $(-xyx^{-1}, y^{-1})$ belong to different connected components $m^{-1}(\alpha)$. Thus we see that if $\alpha = xyx^{-1}y^{-1}$ with $\alpha \in G(k)$, α semisimple and not equal to ± 1, then $G(k) \times G(k)$ meets every connected component of $m^{-1}(\alpha)$. Suppose now $x, y \in G(k)$ with $xyx^{-1}y^{-1} = \alpha$. Let $Z(x)$ (resp. $(Z(\alpha))$ denote the centraliser of x (resp. α) in $\underline{G}$; then $Z(x)(k)$ (resp. $Z(x)(k)$) is dense in $Z(x)(\mathbf{R})$ (resp. $Z(\alpha)(\mathbf{R})$) weak approximation). Now the map $Z(x)(\mathbf{R}) \times Z(\alpha)(\mathbf{R}) \to m^{-1}(\alpha)$ given by $(z, \gamma) \mapsto (xyzx^{-1}, z^{-1}y^{-1})$ is easily checked to be of maximal rank at $(1, 1)$ (see proof of Assertion 5.9 of [1]).

It is now evident that the closure of $m^{-1}(\alpha) \cap (G(k) \times G(k))$ in $m^{-1}(\alpha)$ contains non-empty open subsets in every connected component. Now the continuous map $m^{-1}(\alpha) \times m^{-1}(\alpha) \to \hat{\mathscr{G}}$ given by $(\xi, \eta) \mapsto \hat{C}(\xi)\hat{C}(\eta)^{-1}, \xi, \eta \in m^{-1}(\alpha)$ is identically equal to 1 on $G(k) \times G(k)) \cap m^{-1}(\alpha)$ and is thus identically 1 on non-empty open subsets in *every* connected component $m^{-1}(\alpha)$. Thus $\hat{C}(\xi) = \hat{C}(\eta)$ for all $\xi, \eta \in m^{-1}(\alpha), \alpha = xyx^{-1}y^{-1}$, $x, y \in G(k)$. Now the set $\mathscr{E} = \{xyx^{-1}y^{-1} \mid x \in G(k), y \in G(k)\}$ is dense in $\mathscr{G}\backslash\{\pm 1\}$ and the map of $J = P \times_m P \to \mathscr{G}\backslash\{\pm 1\}$ of the fibre product of P with itself over $\mathscr{G}\backslash\{\pm 1\}$ on $\mathscr{G}\backslash\{\pm 1\}$ being of maximal rank, the union $\Omega = \bigcup_{\alpha \in \mathscr{E}} (m^{-1}(\alpha) \times m^{-1}(\alpha))$ is dense in J. Since the continuous map $(\xi, \eta)(\in P \times P) \mapsto \hat{m}(\xi)\hat{m}(y)^{-1}$ is identically 1 on Ω, it is identically 1 on J; in other words $\hat{m}$ is constant along the fibres of m proving the assertion.