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Torsional oscillations of neutron stars*

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Summary. Motivated by the possibility that torsional oscillations of neutron stars may be observable in the timing of pulsar subpulses and/or in future gravitational-wave detectors, this paper develops the detailed mathematical theory of such torsional oscillations and of the gravitational waves they emit. The oscillations are analysed using the formulation of first-order perturbations of a fully general relativistic spherical stellar model. All sources of damping are ignored except gravitational radiation reaction. The perturbations are resolved into spherical harmonics, which decouple from each other. For each harmonic this paper presents equations of motion, an action principle, an energy conservation law and a Liapunov-type proof that the oscillations are always stable. Each harmonic is then resolved into normal modes with outgoing gravitational waves (time dependence $e^{i\omega t}$ with ω complex) and an eigenvalue problem is posed for the eigenfunctions and the eigenfrequencies ω. Five methods of solving the eigenvalue problem are presented; three methods are valid in general (the method of resonances, the variational method and the method of energy conservation); one is valid in the slow-motion approximation (wavelength of waves large compared to star) and one is valid in the weak-gravity approximation. For stellar models with weak gravity and with radially constant density and shear modulus the eigenvalue problem is solved analytically.

An appendix develops a general theory of action principles for systems with radiative boundary conditions – a theory which is then used to derive the action principles in the body of the paper and which could be useful for a variety of other problems involving physical systems coupled to radiation.

1 Introduction

If torsional oscillations of neutron stars could be observed, then comparisons of their measured periods and Qs with theoretical models would give valuable information not only

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about neutron star structure, but also about the physics of matter at subnuclear and supranuclear densities. There are two hopes for such observations: pulsar timing data and gravitational radiation. Van Horn (1980) has pointed out that the 'marching subpulses' observed in some pulsars have the same range of periods, 10–50 ms, as low-order torsional oscillations of neutron star crusts; and on this basis he has argued that such oscillations may be the clock which regulates the marching subpulses. And Dyson (1972) has pointed out that, if neutron stars have solid cores, then quakes in those cores should generate torsional oscillations which might produce gravitational waves strong enough to detect on Earth.

With these two applications in mind, and with hope that they or others will materialize, we construct in this paper the detailed mathematical theory of torsional oscillations of non-rotating, general relativistic stellar models with isotropic shear moduli μ.

The analogous general relativistic theory of non-spherical compressional oscillations of non-rotating perfect-fluid stars was laid out a number of years ago by Thorne & Campolattaro (1967), Price & Thorne (1969), Thorne (1969a, b), Campolattaro & Thorne (1970), Ipser & Thorne (1973), Detweiler & Ipser (1973), Thorne (1983, in preparation). Those eight papers developed many facets of the theory. This paper is rather long because it attempts to develop, all at once, all of those same facets for the theory of torsional oscillations, and several more facets besides.

To set the stage for our analysis, we shall review briefly the structures of neutron stars and the characteristic magnitudes of various quantities associated with them; for further detail see, e.g., Baym & Pethick (1975, 1979) and references therein.

Observation and theory agree that typical neutron stars have masses $M \sim 1 M_\odot$ and radii $R \sim 10$ km. Theory predicts with great confidence that within minutes after the star is born, its crust will cool enough to solidify into a crystal governed by Coulomb forces between atomic nuclei. This crystalline crust should extend from the star's atmosphere inward to a depth of order 1 km, where the density is within a factor 2 of nuclear, $\rho \simeq (1.5\text{–}3) \times 10^{14}$ g cm^{-3}. Throughout the crust the shear modulus μ is computed to be nearly proportional to density ρ, with

$$(\mu/\rho)^{1/2} = v_s \simeq 1 \times 10^8 \text{ cm s}^{-1}. \tag{1}$$

Here v_s is the speed of non-relativistic shear waves (Ruderman 1968; Pandharipande, Pines & Smith 1976; Hansen & Cioffe 1980) (see equation 20 for a relativistic correction).

It is now widely believed that below the solid crust resides a superfluid mantle, which extends inward through a thickness of roughly 5 km and through a density range of $(1.5\text{–}3) \times 10^{14}$ to $(5\text{–}10) \times 10^{14}$ g cm^{-3}, until it meets the star's ~ 4 km core. The physical state of the core is highly uncertain. Possibilities include a pion-condensed state, which might or might not be a solid governed by nuclear forces; an 'abnormal state' in which the nucleons become practically massless; a degenerate Fermi liquid of quarks, etc. The possibility of a solid core was viewed with much favour between 1971 and 1974, both on grounds of nuclear many-body calculations and on grounds of a reasonable fit between the theory of core quakes and observations of glitches in the timing of the Vela pulsar (Pines, Shaham & Ruderman 1974, see Hansen 1974, p. 189). However, by 1975 improved many-body calculations had cast doubt on the likelihood that supranuclear matter will solidify. The doubt remains today, but the calculations are far from convincing either way; see Baym & Pethick (1975, 1979) for details and references. If the core is a solid, then its shear modulus μ could be as large as its pressure P, or it might be somewhat smaller:

$$(\mu/\rho)^{1/2} = v_s \lesssim (P/\rho)^{1/2} \simeq 1 \times 10^{10} \text{ cm s}^{-1}. \tag{2}$$

Hansen & Cioffe (1980) have used Newtonian theory to compute the torsional oscillation periods of neutron star crusts. As one might expect, they obtain for modes with no radial nodes (so transverse wavenumber dominates)

$$\text{Period} = 2\pi/\omega \simeq 2\pi\,[l(l+1)]^{-1/2} R/v_s \sim 20\,\text{ms} \qquad \text{for} \qquad l = 2, \tag{3a}$$

where $l = 1, 2, 3, \ldots$ is the spherical–harmonic index. Relativistic effects (especially gravitational redshifts and the dragging of inertial frames) are likely to change these periods by ~ 10–50 per cent. These periods are a factor ~ 10 longer than would be compressional–oscillation periods for the crust, because the electrostatic forces which govern the crystal and its torsional oscillations are ~ 100 times weaker than the degeneracy forces and nucleon–nucleon forces which govern compressional oscillations. Because the crust's torsional oscillations are so slow, $v_s/c \ll 1$, they can be described very accurately by the 'slow-motion approximation' to general relativity (Thorne 1980; Section 4.5 of this paper) which predicts gravitational waves so weak that it is hopeless to ever detect them:

$$h \sim 6\left(\frac{GM_{\text{cr}}}{rc^2}\right)\left(\frac{v_s}{c}\right)^3 \beta \sim 10^{-28}\left(\frac{10\,\text{kpc}}{r}\right)\left(\frac{\beta}{10^{-3}}\right) \qquad \text{for} \qquad l = 2. \tag{3b}$$

Here h is the dimensionless gravity-wave amplitude, r is the distance from the Earth to the star, β is the dimensionless amplitude of the star's shearing oscillations, $M_{\text{cr}} \simeq 0.1 M_\odot$ is the mass of the crust, and we have specialized to quadrupole modes which are the strongest emitters. Gravitational radiation reaction will damp the crustal oscillations with an e-folding time

$$\tau \sim 0.3\,(GM_{\text{cr}}/Rc^2)^{-1}(v_s/c)^{-5}\,\omega^{-1} \sim 10^4\,\text{yr}; \tag{3c}$$

cf. equations (76).

If the core is solid and has $\mu \sim P$ (as was widely believed in the early 1970s), then the periods of its torsional oscillations would be roughly the same as those of its compressional oscillations:

$$\text{Period} = 2\pi/\omega \simeq 2\pi R_{\text{co}}/v_s \sim 0.3\,\text{ms} \qquad \text{for} \qquad l = 2. \tag{4a}$$

where $R_{\text{co}} \simeq 4$ km is the core radius. Because the torsional oscillations emit 'current quadrupole' gravitational waves (gravitational analogue of magnetic quadrupole), whereas the compressional oscillations emit 'mass quadrupole' waves (analogue of electric quadrupole), the waves from torsional oscillations will be weaker by $(v_s/c) \sim 1/3$ and will be damped more slowly by $(v_s/c)^{-2} \sim 10$ than those from compressional oscillations:

$$h \sim 0.3\left(\frac{GM_{\text{co}}}{rc^2}\right)\left(\frac{v_s}{c}\right)^3 \beta \sim 3 \times 10^{-23}\left(\frac{10\,\text{kpc}}{r}\right)\left(\frac{\beta}{10^{-3}}\right), \tag{4b}$$

$$\tau \sim 30\left(\frac{GM_{\text{co}}}{R_{\text{co}}c^2}\right)^{-1}\left(\frac{v_s}{c}\right)^{-5}\omega^{-1} \sim 1\,\text{s}. \tag{4c}$$

(The coefficients used here are extrapolated from strong-gravity, fast-motion calculations of compressional oscillations by Thorne (1969a); the coefficients used in equations (3) for crustal oscillations are based on the weak-gravity, slow-motion calculations of equations (76) of this paper.) Assuming that the Vela pulsar has a solid core, and that the glitches observed every 2 or 4 yr in the Vela pulse arrival times are due to core quakes, Pines *et al.* (1974) have estimated that the total strain energy released in each quake is ~ 10^{45} erg corresponding to $\beta \sim 10^{-4}$, which at a distance $r \sim 500$ pc would produce $h \sim 6 \times 10^{-23}$. Other, younger neutron stars might be stronger emitters. For comparison, the best currently operating gravitational-wave detector (Stanford's bar; Boughn *et al.* 1982) has a burst sensitivity $h \sim 5 \times 10^{-18}$ (rms noise $h \sim 1 \times 10^{-18}$) at a period $P \sim 10^{-3}$ s; the design sensitivity of a multikilometre laser-interferometer gravity-wave detector being planned for the late 1980s (Drever *et al.* 1982) for 10 kHz waves that last 1 s, would be $h \sim 3 \times 10^{-23}$. Thus, it is not inconceivable that corequakes in neutron stars could be detected and studied routinely in the 1990s.

We turn now to the detailed analysis of torsional oscillations of spherical, non-rotating, relativistic stellar models. The spherical symmetry of the unperturbed star guarantees that the oscillations can be decoupled into modes of definite spherical–harmonic indices (l, m) and definite parity. In the language of previous analyses (e.g. Regge & Wheeler 1957) pure torsional oscillations are the normal modes of odd-type or magnetic-type parity, $\pi = (-1)^{l+1}$. Such modes do not exist for $l = 0$ (monopole). They exist for $l = 1$ (dipole) but cannot generate gravitational waves. For $l \geqslant 2$ they do generate waves. The differences between $l = 1$ and $l \geqslant 2$ are so fundamental that they are best analysed in different gauges and with different mathematical techniques. Sections 2–4 of this paper are devoted to modes with $l \geqslant 2$; Section 5 treats $l = 1$. Section 2 lays the foundation for the analysis with $l \geqslant 2$, including the description of the unperturbed star (Section 2.1), the coordinates, metric and Ricci tensor for the perturbed star (Section 2.2) and the description of the material motion – i.e. the displacement function, four-velocity and stress-energy tensor (Section 2.3). Section 3 presents the details of the analysis, including the equations of motion for the matter and the gravitational field (Section 3.1), the boundary conditions on the matter and field variables (Section 3.2), the form of the gravitational waves emitted and their energy loss rate (Section 3.3), an action principle and local law of energy conservation for the pulsations and their waves (Section 3.4), and a Liapunov-type proof that so long as the shear modulus is positive the star is stable against arbitrary (but first-order) torsional perturbations (Section 3.5). Section 4 analyses the star's outgoing wave modes (pulsations with sinusoidal time dependences and complex frequencies), including a formulation of the eigenvalue problem for the normal modes (Section 4.1) and various methods of solving the eigenvalue problem: the method of resonances (Section 4.2), a variational principle method (Section 4.3), an energy conservation method (Section 4.4), a method valid in the slow-motion approximation (Section 4.5), and a method for stars with weak internal gravity (Section 4.6, which also includes an analytic solution of the eigenvalue problem for weak-gravity stars with constant density and shear modulus). The analysis of dipole oscillations in Section 5 follows a similar outline – but with all issues of gravitational radiation absent. Some mathematical details are relegated to appendices. Of special interest may be Appendix B which elaborates and extends an elegant formulation (by Friedman & Schutz 1975) of the general theory of action principles for systems that can radiate waves – any kind of waves – to infinity.

Throughout this paper we use the mathematical conventions of Misner, Thorne & Wheeler (1973, cited henceforth as MTW), including setting the speed of light and Newton's gravitation constant to unity and denoting covariant derivatives by semicolons and partial derivatives by commas.

2 Foundations for the analysis: $l \geqslant 2$

2.1 THE UNPERTURBED STAR

The unperturbed spherical star is described in the standard manner (see, e.g. MTW). The metric, in Schwarzschild coordinates, is

$$ds^2 = (ds^2)_0 \equiv -e^{2\Phi} dt^2 + e^{2\Lambda} dr^2 + r^2 (d\vartheta^2 + \sin^2\vartheta \, d\phi^2)$$
$$\equiv \gamma_{\mu\nu} \, dx^\mu \, dx^\nu, \tag{5a}$$

where $\Phi(r)$ and $\Lambda(r)$ are functions of the radial coordinate, r. The 'mass inside radius r', $m(r)$, is defined by

$$e^{-2\Lambda} \equiv 1 - 2m/r. \tag{5b}$$

The density of total mass-energy and the (isotropic) pressure are denoted by ρ and P, respectively. The standard equations of structure for the equilibrium star are (i) the mass equation

$$m \equiv \int_0^r 4\pi r^2 \rho(r)\, dr. \tag{5c}$$

(ii) The Oppenheimer–Volkov equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{(\rho + P)(m + 4\pi r^3 P)}{r^2(1 - 2m/r)}, \qquad P(R) = 0 \tag{5d}$$

where R = (value of r at surface of star), and (iii) the source equation for $\Phi(r)$

$$\frac{d\Phi}{dr} = \frac{m + 4\pi r^3 P}{r^2(1 - 2m/r)}, \qquad \Phi(\infty) = 0. \tag{5e}$$

From equations (5b) to (5e) the following useful relations are easily derived:

$$\Lambda' = 1/2\, r^{-1}(1 - e^{2\Lambda}) + 4\pi r \rho e^{2\Lambda}, \tag{5f}$$

$$\Phi' = -\,1/2\, r^{-1}(1 - e^{2\Lambda}) + 4\pi r P e^{2\Lambda}, \tag{5g}$$

$$\Phi' + \Lambda' = 4\pi r(\rho + P)\, e^{2\Lambda} \quad \text{or} \quad (\rho + P) = -(4\pi r)^{-1}(e^{-\Phi-\Lambda})'\, e^{\Phi-\Lambda}, \tag{5h}$$

$$e^{2\Phi} = e^{-2\Lambda} = 1 - 2M/r \quad \text{outside the star,} \tag{5i}$$

where primes denote radial derivatives, $\partial/\partial r$, and where $M \equiv m(R)$ is the star's total mass and R is its radius. The complete unperturbed model is specified by giving the radial distributions of ρ, P, Φ, Λ (or m) and the shear modulus $\mu(r)$. We assume in this paper that μ is isotropic ('scalar field').

2.2 COORDINATES, METRIC AND RICCI TENSOR FOR PERTURBED STAR

For the perturbed star we introduce coordinates (t, r, ϑ, ϕ) which reduce to those of the unperturbed star when the oscillations vanish. We linearize our entire analysis about the unperturbed configuration and resolve the oscillations into spherical harmonics of definite indices l, m and parity π. The spherical symmetry of the unperturbed configuration guarantees that modes of different l, m, π superpose linearly (i.e. no mixing). Therefore, we can restrict attention to modes with fixed l, m, π (pure modes). In this paper we do not consider 'even-parity' modes $[\pi = (-1)^l]$ because they represent compressional oscillations rather than pure torsional oscillations; see Thorne & Campolattaro (1967) for discussion. The odd-parity torsional modes with fixed l but different m can be obtained from each other by linear combinations of rotations about the star's centre. Thus, without loss of generality, we can specialize to an odd-parity mode with definite l and with $m = 0$; and we henceforth use m exclusively to denote the mass inside radius r (equation 5b) and not a spherical harmonic index.

The metric $g_{\mu\nu}$ for our oscillating star consists of the unperturbed metric $\gamma_{\mu\nu}$ plus components $h_{\mu\nu}$ which describe our odd-parity perturbation:

$$ds^2 = (ds^2)_0 + h_{\mu\nu}\, dx^\mu\, dx^\nu. \tag{6a}$$

Clearly, h_{tt}, h_{tr} and h_{rr} are scalars under rotation and thus have even parity, which means they must vanish. Further (*cf.* appendix A of Thorne & Campolattaro 1967) we are free to specialize our coordinates (choose our gauge) so as to make all other components of $h_{\mu\nu}$ vanish except the following:

$$h_{t\phi} = h_{\phi t} \equiv - r^2 \dot{y}(t, r)\, b_\phi \equiv - r^2 \dot{y} \sin\vartheta\, \partial_\vartheta P_l(\cos\vartheta),$$

$$h_{r\phi} = h_{\phi r} \equiv - r e^{\Lambda - \Phi} Q(t, r)\, b_\phi. \tag{6b}$$

Here b_ϕ is equal to $[4\pi/(2l+1)]^{1/2}$ times the Regge–Wheeler (1957) odd-parity vector spherical harmonic Φ_ϕ^{l0}; in future equations we shall raise and lower the index on b_ϕ with the metric of the unit sphere:

$$b_\phi \equiv \sin^2\vartheta\, b^\phi = \sin\vartheta\, \partial_\vartheta P_l(\cos\vartheta); \tag{6c}$$

the indices of the metric perturbation functions $h_{\mu\nu}$ are raised and lowered with the unperturbed metric, $\gamma_{\mu\nu}$ (equation 5a). In equations (6b, c), $P_l(\cos\vartheta)$ is the Legendre polynomial of order l, and the dot over y denotes a time-derivative $\partial/\partial t \equiv \partial_t$. The perturbation function $\dot{y} b^\phi$ is equal to the angular velocity of a zero-angular momentum observer (ZAMO; *cf.*, e.g., Bardeen, Press & Teukolsky 1972); thus $y b^\phi$ is the angular displacement of a ZAMO and is dimensionless. Outside the star the perturbation function Q is equal to the Regge–Wheeler (1957) gravitational-wave variable, aside from a multiplicative constant.

The metric perturbation (6b) produces a perturbation of the Ricci tensor with the following non-vanishing components (Thorne & Campolattaro 1967, equation B3 as corrected in the erratum – but note the different notation and signature used there):

$$\delta R_{t\phi} = \delta R_{\phi t} = \left\{ \frac{e^{\Phi-\Lambda}}{2r^2} [r^4 e^{-\Phi-\Lambda}(\dot{y}' - e^{\Lambda-\Phi}\dot{Q}/r)]' + e^{-\Phi-\Lambda}(re^{\Phi-\Lambda})'\dot{y} - \frac{l(l+1)}{2}\dot{y} \right\} b_\phi, \tag{7a}$$

$$\delta R_{r\phi} = \delta R_{\phi r} = \left\{ 1/2\, r^2 e^{-2\Phi}(\ddot{y}' - e^{\Lambda-\Phi}\ddot{Q}/r) + r^{-1} e^{-2\Phi}(re^{\Phi-\Lambda})' Q - \frac{l(l+1)}{2r} e^{\Lambda-\Phi} Q \right\} b_\phi, \tag{7b}$$

$$\delta R_{\vartheta\phi} = \delta R_{\phi\vartheta} = \{1/2\, r^2 e^{-2\Phi}\ddot{y} - 1/2\, e^{-\Lambda-\Phi}(rQ)'\} \sin^2\vartheta\, b^\phi{}_{,\vartheta}. \tag{7c}$$

Here and below primes denote radial derivatives and dots denote time derivatives, $Q' \equiv \partial Q/\partial r \equiv Q_{,r}$ and $\dot{Q} \equiv \partial Q/\partial t \equiv Q_{,t}$. Note that $\sin^2\vartheta\, b^\phi{}_{,\vartheta} \equiv \sin^2\vartheta\, \partial_\vartheta b^\phi$ is equal to $[16\pi/(2l+1)]^{1/2}$ times the Regge–Wheeler odd-parity tensor spherical harmonic $\chi^{l0}_{\vartheta\phi}$.

We now go on to consider the motion of the star and the interaction of its matter with the surrounding spacetime geometry.

2.3 DISPLACEMENT FUNCTION, FOUR–VELOCITY AND STRESS–ENERGY TENSOR FOR PERTURBED STAR

In the perturbed star, the coordinate location of a specific particle of stellar matter oscillates. We describe its oscillating location by a displacement vector $\boldsymbol{\xi}$ whose components ξ^r, ξ^ϑ and ξ^ϕ are functions of the particle's original location (r, ϑ, ϕ) and of time t:

$$r_{\text{pert}} = r + \xi^r(t, r, \vartheta, \phi); \quad \vartheta_{\text{pert}} = \vartheta + \xi^\vartheta(t, r, \vartheta, \phi); \quad \phi_{\text{pert}} = \phi + \xi^\phi(t, r, \vartheta, \phi). \tag{8a}$$

Because ξ^r is a scalar under rotations about the centre of the star and thus has even parity, it must vanish. The angular displacements form a vector field on the unit sphere, $\boldsymbol{\xi} = \xi^\vartheta \partial_\vartheta +$

$\xi^\phi \partial_\phi$, and must therefore have the angular dependence of a vector spherical harmonic of definite l, of $m = 0$ and of parity $\pi = (-1)^{l+1}$:

$$\xi^r = 0; \quad \xi^\vartheta = 0; \quad \xi^\phi \equiv Y(t, r)(\sin\vartheta)^{-1}\partial_\vartheta P_l(\cos\vartheta) = Y(t, r) b^\phi. \tag{8b}$$

Note that just as yb^ϕ is the angular displacement of a ZAMO, so Yb^ϕ is the angular displacement of the stellar matter.

The four-velocity u^μ of a particle with world line (8a, b) is obtained from the relations $u^j/u^t = dx^j_{\text{pert}}/dt = \xi^j{}_{,t}$; $u^\mu u^\nu g_{\mu\nu} = -1$. The result, linearized in the perturbation functions y, Q, Y, is

$$u^t = e^{-\Phi}, \quad u^r = 0, \quad u^\vartheta = 0, \quad u^\phi = e^{-\Phi}\dot{Y} b^\phi. \tag{9}$$

The radial and angular variations of the azimuthal displacement ξ^ϕ produce deformations (shears) of the star's crystal lattice. These deformations are described by a shear tensor $S_{\alpha\beta}$. When viewed in the orthonormal comoving frame of a particle of the stellar material $S_{\alpha\beta}$ is purely spatial ($S_{00} = S_{0j} = S_{j0} = 0$), and its spatial components $S_{jk} = S_{kj}$ are precisely those of the non-relativistic theory of a stressed medium (see, e.g. Landau & Lifshitz 1970). Hence, in this proper reference frame of the particle, the shearing motion produces a restoring stress given by the standard non-relativistic formula $T^{\text{shear}}_{jk} = -2\mu S_{jk}$, where μ is the shear modulus. By general covariance (*cf.* MTW, chapter 16) this equation can be rewritten in the coordinate-independent form

$$T^{\text{shear}}_{\alpha\beta} = -2\mu S_{\alpha\beta}. \tag{10}$$

To calculate the components $S_{\alpha\beta}$ of the shear tensor in our Regge–Wheeler coordinate system we proceed as follows: First, we calculate the rate of shear $\sigma_{\alpha\beta}$ from standard formulae (see, e.g., MTW, exercise 22.6):

$$\sigma_{\alpha\beta} = 1/2\,(u_{\alpha;\mu} P^\mu{}_\beta + u_{\beta;\mu} P^\mu{}_\alpha) - 1/3\, P_{\alpha\beta} u^\mu{}_{;\mu}, \tag{11}$$

where

$$P_{\alpha\beta} \equiv g_{\alpha\beta} + u_\alpha u_\beta.$$

The result is

$$\sigma_{r\phi} = \sigma_{\phi r} = 1/2\, r^2 e^{-\Phi}(\dot{Y}' - e^{\Lambda - \Phi}\dot{Q}/r) b_\phi; \tag{12a}$$

$$\sigma_{\vartheta\phi} = \sigma_{\phi\vartheta} = 1/2\, r^2 e^{-\Phi}\dot{Y}\sin^2\vartheta\, b^\phi{}_{,\vartheta}; \tag{12b}$$

all other components vanish. (12c)

(Notice that even for a fluid at rest in the (r, ϑ, ϕ) coordinate system ($\dot{Y} = 0$) there is a changing radial shear $\sigma_{r\phi}$ associated with the changing metric ($\dot{Q} \neq 0$; 'deformation of coordinates'). It is only because we are in the Regge–Wheeler gauge where $h_{\vartheta\phi} \equiv 0$ that the non-radial shear $\sigma_{\vartheta\phi}$ vanishes when $\dot{Y} = 0$.) Next, we write in explicit form the relationship

$$\boldsymbol{\sigma} = \mathscr{L}_{\mathbf{u}}\, \mathbf{S}, \tag{13}$$

that the rate of shear is the Lie derivative of the shear along the world lines – a relationship which is best derived in the proper reference frame of a fiducial material particle; *cf.* Carter & Quintana (1972). The result, to first order in the oscillations, is

$$\sigma_{\alpha\beta} = e^{-\Phi} S_{\alpha\beta,t}. \tag{14}$$

Finally, we combine equations (12) and (14) and integrate with respect to time, using the initial condition that the shear $S_{\alpha\beta}$ is equal to zero in the unperturbed star (i.e. when Y and $h_{\mu\nu}$ vanish). The result is

$$S_{r\phi} = S_{\phi r} = 1/2\, r^2 (Y' - e^{\Lambda-\Phi} Q/r)\, b_\phi = \xi_{(r;\phi)} + 1/2\, h_{r\phi}; \tag{15a}$$

$$S_{\vartheta\phi} = S_{\phi\vartheta} = 1/2\, r^2\, Y \sin^2\vartheta\, b^\phi{}_{,\vartheta} = \xi_{(\theta;\phi)}. \tag{15b}$$

Note that the shear $S_{\alpha\beta}$ is generated both by the deformation of the crystal relative to the coordinate system (non-zero Y) and by the deformation of the coordinate system itself (non-zero $h_{r\phi}$).

The shear stress of equations (10) and (15) is only one contributor to the stress-energy tensor of the stellar material. The other contributors are the total density of mass-energy ρ and the isotropic pressure P, both of which maintain their unperturbed values because they are scalar fields and therefore cannot undergo odd-parity perturbations. The stress-energy tensor associated with ρ and P (the bulk part of the stress-energy tensor) has the standard perfect fluid form

$$T^{\text{bulk}}_{\alpha\beta} = (\rho + P)\, u_\alpha u_\beta + P g_{\alpha\beta}. \tag{16}$$

Using equations (5), (6), (9), (10), (15) and (16), we obtain for the total stress-energy tensor $T_{\alpha\beta} = T^{\text{bulk}}_{\alpha\beta} + T^{\text{shear}}_{\alpha\beta}$ of the oscillating star

$$T_{tt} = \rho\, e^{2\Phi}, \quad T_{rr} = P e^{2\Lambda}, \quad T_{\vartheta\vartheta} = Pr^2, \quad T_{\phi\phi} = Pr^2 \sin^2\vartheta; \tag{17a}$$

$$T_{t\phi} = T_{\phi t} = -r^2 [(\rho + P)\, \dot{Y} - \rho \dot{y}]\, b_\phi; \tag{17b}$$

$$T_{r\phi} = T_{\phi r} = -r\, [\mu r Y' - (\mu - P)\, e^{\Lambda-\Phi} Q]\, b_\phi; \tag{17c}$$

$$T_{\vartheta\phi} = T_{\phi\vartheta} = -r^2 \mu\, Y \sin^2\vartheta\, b^\phi{}_{,\vartheta}; \tag{17d}$$

all other components vanish. (17e)

For evaluation of the Einstein field equations, $R_{\mu\nu} = 8\pi\,(T_{\mu\nu} - 1/2\, T g_{\mu\nu})$, we shall need the first-order perturbations of $(T_{\mu\nu} - 1/2\, T g_{\mu\nu})$. These are easily found from equations (5), (6) and (17):

$$\delta\,(T_{t\phi} - 1/2\, T g_{t\phi}) = r^2\, [1/2\,(\rho + 3P)\, \dot{y} - (\rho + P)\, \dot{Y}]\, b_\phi; \tag{18a}$$

$$\delta\,(T_{r\phi} - 1/2\, T g_{r\phi}) = r\, [-\mu r Y' + (\mu - 1/2\,\rho + 1/2 P)\, e^{\Lambda-\Phi} Q]\, b_\phi; \tag{18b}$$

$$\delta\,(T_{\vartheta\phi} - 1/2\, T g_{\vartheta\phi}) = -\mu r^2\, Y \sin^2\vartheta\, b^\phi{}_{,\vartheta} \tag{18c}$$

where we have used the fact that $T \equiv T^\alpha_\alpha = 3P - \rho$.

3 Details of the analysis: $l \geqslant 2$

3.1 Equations of motion

Because our stellar oscillations are described by three functions of t and r – Y, y, Q – our analysis will require three equations of motion. Our chosen versions of these equations are obtained from the perturbed Einstein field equations $\delta R_{\mu\nu} = 8\pi\delta\,(T_{\mu\nu} - 1/2\, T g_{\mu\nu})$ by manipulations described in Appendix A.

Our first equation is an initial-value equation for the ZAMO angular displacement function y:

$$-\frac{e^{\Phi+\Lambda}}{r^4}(r^4 e^{-\Phi-\Lambda}y')' + e^{2\Lambda}\left[16\pi(\rho+P)+\frac{(l+2)(l-1)}{r^2}\right]y$$
$$-16\pi(\rho+P)e^{2\Lambda}Y+\frac{e^{\Phi+\Lambda}}{r^4}(r^3 e^{-2\Phi}Q)'=0. \quad (19a)$$

This equation can be solved at any moment of time to give y in terms of Y and Q.

Our second equation is a wave equation for the angular displacement Y of the stellar material:

$$(\rho+P)e^{-2\Phi}\ddot{Y}-\frac{e^{-\Phi-\Lambda}}{r^4}(\mu r^4 e^{\Phi-\Lambda}Y')'+\left[16\pi(\rho+P)+\frac{(l+2)(l-1)}{r^2}\right]\mu Y$$
$$-(\rho+P)\frac{e^{-\Phi-\Lambda}}{r^2}(rQ)'+\frac{e^{-\Phi-\Lambda}}{r^4}(\mu r^3 Q)'=0. \quad (19b)$$

The characteristics of this equation (the world lines of high-frequency, radially propagating wave packets) have a propagation speed, as measured by an observer at rest in the star, given by

$$v_s=\frac{e^{\Lambda}dr}{e^{\Phi}dt}=\left(\frac{\mu}{\rho+P}\right)^{1/2}. \quad (20)$$

When one recalls that $(\rho+P)$ is inertial mass per unit volume in relativity (see, e.g., exercise 5.4 of MTW), one recognizes this as the standard expression for the speed of propagation of shear waves in an isotropic solid; *cf.* Carter (1973a).

Our third equation is a wave equation for the Regge–Wheeler gravitational-wave-function, Q:

$$e^{-2\Phi}\ddot{Q}-e^{-\Phi-\Lambda}(e^{\Phi-\Lambda}Q')'+\left[16\pi\mu+\frac{(l+2)(l-1)}{r^2}-re^{-\Phi-\Lambda}\left(\frac{e^{\Phi-\Lambda}}{r^2}\right)'\right]Q$$
$$+16\pi r e^{-\Phi-\Lambda}(\mu e^{2\Phi})'Y=0. \quad (19c)$$

In the vacuum outside the star this reduces to the Regge–Wheeler (1957) equation for gravitational waves propagating in Schwarzschild spacetime. Both inside the star and out the characteristics of this equation are radial null lines (propagation speed equal to speed of light). Note that the ZAMO displacement function y has been completely decoupled from the wave equations (19b, c); they are coupled wave equations for Y and Q alone.

One can show (see Appendix A) that our equations of motion (19) are 'complete' in the sense that the set of all physically acceptable solutions of (19) is identical to the set of all physically acceptable solutions of the perturbed Einstein field equations – physical acceptability being defined as satisfaction of the boundary conditions as given in the next section of this paper.

One can also show from our equations of motion (19) plus boundary conditions (or, more easily, from equations 19c, 19b and $\epsilon_{\vartheta\phi}=0$ in A.3) that in any region of the star where the shear modulus vanishes, $\mu=0$, the perturbed gravitational field is decoupled from

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the stellar matter, and the matter cannot support torsional oscillations. More specifically, equation (19c) then becomes a homogeneous wave equation for the decoupled gravitational-wave variable Q; equation (19a) or (A.3) determines the ZAMO displacement y in terms of Q; and (A.3) together with (19b) guarantees that the fluid is at rest relative to the ZAMOs, $Y = y$. This decoupling has been noted previously by Thorne & Campolattaro (1967).

3.2 BOUNDARY CONDITIONS

The equations of motion (19) must be solved subject to suitable boundary conditions at the star's centre and surface, and at infinity.

At the star's centre the fluid motions and the spacetime geometry must be suitably smooth. Roughly speaking, smoothness means that the more rapid are the angular variations of Y, y, Q – i.e. the larger the value of l – the more rapidly must Y, y and Q approach zero at $r = 0$. To make this quantitative we introduce local Cartesian coordinates $\{x^a\}$ near $r = 0$:

$$x^1 \equiv r \sin\vartheta \cos\phi, \qquad x^2 \equiv r \sin\vartheta \sin\phi, \qquad x^3 \equiv r\cos\vartheta. \tag{21}$$

Because $\Lambda \sim \rho r^2$ near $r = 0$ the components of the unperturbed spatial metric (5a) are Cartesian at $r = 0$ in this coordinate system: $\gamma_{ab} = \delta_{ab} + O(r^2)$.

Consider the three-dimensional vector and tensor fields

$$\boldsymbol{\xi} = \xi^\phi \partial_\phi, \quad \boldsymbol{\alpha} = h_t^\phi \partial_\phi, \quad \boldsymbol{\beta} = h^{r\phi}(\partial_r \otimes \partial_\phi + \partial_\phi \otimes \partial_r); \tag{22}$$

$\boldsymbol{\xi}$ is the material displacement vector, $\boldsymbol{\alpha}$ is the time–space part of the metric perturbation, $\boldsymbol{\beta}$ is the spatial part of the metric perturbation, indices on $h_{\alpha\beta}$ have been raised with the unperturbed metric $\gamma^{\alpha\beta}$ and $\boldsymbol{\xi}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ can all be regarded as solutions of the perturbed Einstein field equations. The 'smoothness' of the Einstein equations at $r = 0$ implies that the Cartesian components of $\boldsymbol{\xi}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ will have power series expansions near $r = 0$ whose leading terms are infinitely differentiable – or, equivalently, whose leading terms are expressible as products of non-negative powers of x^1, x^2, x^3; e.g.

$$\xi^1 = (x^1)^2 (x^2)^{13} (x^3)^4 (1 + \text{terms which vanish as } r \to 0).$$

Using equations (6) and (8) we can write $\boldsymbol{\xi}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as

$$\boldsymbol{\xi} = rY\mathbf{A}, \qquad \boldsymbol{\alpha} = -r\dot{y}\mathbf{A}, \qquad \boldsymbol{\beta} = -e^{-\Phi-\Lambda} Q[\partial_r \otimes \mathbf{A} + \mathbf{A} \otimes \partial_r]; \tag{23a}$$

$$\mathbf{A} \equiv \mathbf{r} \times \boldsymbol{\nabla} P_l(\cos\vartheta). \tag{23b}$$

By writing $P_l(\cos\vartheta)$ in terms of Cartesian coordinates (*cf.* equation 33 below; Section II.C of Thorne 1980) we can bring the Cartesian components of equations (23) near $r = 0$ into the following form:

$$\xi^b = Yr^{-l+1} \mathcal{P}_l^{a_1 \ldots a_l} \epsilon^{bcd} x^c (\partial/\partial x^d)(x^{a_1} \ldots x^{a_l}), \tag{24a}$$

$$\alpha^b = -\dot{y} r^{-l+1} \mathcal{P}_l^{a_1 \ldots a_l} \epsilon^{bcd} x^c (\partial/\partial x^d)(x^{a_1} \ldots x^{a_l}), \tag{24b}$$

$$\beta^{bc} = -Qe^{-\Phi} r^{-l-1} \mathcal{P}_l^{a_1 \ldots a_l} 2x^{(b} \epsilon^{c)df} x^d (\partial/\partial x^f)(x^{a_1} \ldots x^{a_l}). \tag{24c}$$

Here ϵ^{abc} is the Levi–Civita tensor, $\mathcal{P}_l^{a_1 \ldots a_l}$ is a constant, symmetric, trace-free tensor (*cf.* equation 33 below), and the parentheses in the superscript indicate symmetrization.

These Cartesian components will be non-negative products of x^1, x^2 and x^3 near $r = 0$ if and only if

$$Y(t, r) = r^{l-1}\,[\textit{constant} + \text{terms which vanish as } r \to 0], \tag{25a}$$

$$y(t, r) = r^{l-1}\,[\textit{constant} + \text{terms which vanish as } r \to 0], \tag{25b}$$

$$Q(t, r) = r^{l+1}\,[\textit{constant} + \text{terms which vanish as } r \to 0]. \tag{25c}$$

Thus, our boundary conditions near $r = 0$ are

$$Y \sim r^{l-1}; \quad y \sim r^{l-1}; \quad Q \sim r^{l+1} \quad \text{as} \quad r \to 0. \tag{26a}$$

It is straightforward to show that, so long as the unperturbed star is smooth at $r = 0$ (ρ, p, μ and μ' finite there), these asymptotic forms satisfy the equations of motion (19). However, there also exist solutions to (19) which violate these boundary conditions ($Y \sim r^{-l-2}$ and/or $y \sim r^{-l-2}$ and/or $Q \sim r^{-l}$) and which thus are physically unacceptable.

At the star's surface, $r = R$, the normal (radial) components of the stress tensor must vanish (there is no matter outside $r = R$ to support a stress): $T^r_\alpha \to 0$ as $r \to R_- \equiv$ inner edge of stellar surface. Inspection of the stress-energy tensor (equation 17) shows that this condition is satisfied if and only if (i) the unperturbed pressure P approaches 0 as $r \to R_-$, and (ii) the material motions and shear modulus μ satisfy the 'zero-torque-at-surface' condition

$$T^r_\phi = -r^2 e^{-2\Lambda} \mu (Y' - e^{\Lambda - \Phi} Q/r) b_\phi \to 0 \quad \text{as} \quad r \to R_-. \tag{26b}$$

For a star with a solid surface (e.g. iron), μ is finite at $r = R$, so Y' must equal $e^{\Lambda - \Phi} Q/R$ there.

At the star's surface the gravitational potentials y and Q must be sufficiently continuous that (i) the intrinsic geometry of the star's surface

$$ds^2 \equiv -e^{2\Phi} dt^2 + r^2 (d\vartheta^2 + \sin^2\vartheta\, d\phi^2) - 2r^2 \dot{y} b_\phi\, dt\, d\phi$$

is continuous, and (ii) the extrinsic curvature,

$$K_{AB} \equiv e^\Lambda \Gamma^r_{AB} \quad (A, B \text{ ranging over } t, \vartheta, \phi),$$

is continuous (see, e.g. section 21.13 of MTW). Straightforward calculation shows that

$$\mathbf{K} = (e^{2\Phi - \Lambda} \Phi')\, dt^2 + e^{-\Lambda} [(r^2 \dot{y})' - r e^{\Lambda - \Phi} \dot{Q}]\, b_\phi\, dt\, d\phi$$
$$- r e^{-\Phi} Q \sin^2\vartheta\, b^\phi{}_{,\vartheta}\, d\vartheta\, d\phi - r e^{-\Lambda} (d\vartheta^2 + \sin^2\vartheta\, d\phi^2). \tag{27}$$

Therefore, continuity of the intrinsic and extrinsic geometries is satisfied if and only if – in addition to the familiar equilibrium conditions of continuous Φ, Φ' and Λ –

$$y, y' \text{ and } Q \text{ are continuous across } r = R. \tag{26c}$$

At the interfaces of solid regions (crust and/or core) with fluid regions (mantle) the shear modulus μ may go to zero discontinuously. There one must be sure that the zero-torque condition and the continuous intrinsic and extrinsic curvature conditions are satisfied:

$$T^r_\phi = -r^2 e^{-2\Lambda} \mu (Y' - e^{\Lambda - \Phi} Q/r) b_\phi \to 0 \quad \text{at solid–fluid interfaces,} \tag{26d}$$

$$y, y' \text{ and } Q \quad \text{are continuous across solid–fluid interfaces.} \tag{26e}$$

Far from the star, $h_{t\phi}$ and $h_{r\phi}$ must describe outgoing gravitational waves. In this region our equations of motion [(19c) for Q; (A3), which follows from (19a, b, c), for y] become

$$\ddot{Q} = \partial_{r*}\partial_{r*}Q - (1 - 2M/r)\,[l(l+1)/r^2 - 6M/r^3]\,Q, \tag{28a}$$

$$\ddot{y} = r^{-2}\partial_{r*}(rQ), \tag{28b}$$

where

$$r_* \equiv r + 2M\ln(r/2M - 1) \tag{28c}$$

is Wheeler's 'tortoise coordinate', and where no approximations have yet been made. Note that equation (28a) is the Regge–Wheeler (1957) equation for odd-parity gravitational waves. The general outgoing-wave solution to these equations has the asymptotic form at large radii

$$Q = F^{(l+1)}(u) + \frac{l(l+1)}{2r}F^{(l)}(u) + O(r^{-2}) \quad \text{as} \quad r \to \infty, \tag{26f}$$

$$y = -\frac{F^{(l)}(u)}{r} - \frac{(l+2)(l-1)}{2r^2}F^{(l-1)}(u) + O(r^{-3}) \quad \text{as} \quad r \to \infty, \tag{26g}$$

where $u \equiv t - r_*$. Here, $F(u)$ is an arbitrary function of u to be determined by integrating the equations of motion, and $F^{(l)}(u) \equiv d^lF/du^l$ denotes the lth derivative of $F(u)$. We shall see later that $F(u)$, aside from a multiplicative constant, is the star's current l-pole moment. One can show that, in addition to the physically acceptable outgoing-wave solutions (26f, g), the equations of motion (19) possess unacceptable incoming-wave solutions of the form (26f, g) with u replaced by $v \equiv t + r_*$ and with the signs of the second term of Q and first term of y reversed, and also unacceptable solutions with mixtures of outgoing and incoming waves.

3.3 RADIATION FIELD AND ENERGY LOSS RATE

The radiation field far from the star is described by the metric perturbations (obtained by combining equations 6b and 26f, g)

$$h_{t\phi} = h_{\phi t} = [rF^{(l+1)}(u) + 1/2\,(l+2)(l-1)\,F^{(l)}(u)]\,b_\phi + O(r^{-1}); \tag{29a}$$

$$h_{r\phi} = h_{\phi r} = [-(r+2M)F^{(l+1)}(u) - 1/2\,l(l+1)F^{(l)}(u)]\,b_\phi + O(r^{-1}); \tag{29b}$$

all other components vanish. (29c)

The physical components of these perturbations,

$$h_{\hat{t}\hat{\phi}} = e^{-\Phi}(r\sin\vartheta)^{-1}h_{t\phi}, \qquad h_{\hat{r}\hat{\phi}} = e^{-\Lambda}(r\sin\vartheta)^{-1}h_{r\phi},$$

have amplitudes which are independent of r, rather than amplitudes which die out like $1/r$. This is because the Regge–Wheeler gauge is badly behaved in the radiation zone (*cf.* Price & Thorne 1969). A more reasonable behaviour is obtained by making a gauge change (infinitesimal coordinate transformation; Box 18.2 of MTW) with the generating vector

$$\eta_\phi = rF^{(l)}(u)\,b_\phi; \qquad \text{all other } \eta_\mu \text{ vanish.} \tag{30}$$

The metric perturbation in the new gauge,

$$h_{\alpha\beta}^{\text{new}} = h_{\alpha\beta}^{\text{old}} - \eta_{\alpha|\beta} - \eta_{\beta|\alpha},$$

where the bar | denotes covariant derivative with respect to the flat metric, has components

$$h_{t\phi}^{\text{new}} = 1/2\,(l+2)(l-1)F^{(l)}(u)b_\phi + O(r^{-1}); \tag{31a}$$

$$h_{r\phi}^{\text{new}} = -\,1/2\,(l+2)(l-1)F^{(l)}(u)b_\phi + O(r^{-1}); \tag{31b}$$

$$h_{\vartheta\phi}^{\text{new}} = -\,rF^{(l)}(u)\sin^2\vartheta\, b^\phi{}_{,\vartheta}; \tag{31c}$$

all other components vanish. (31d)

To leading order the new metric perturbation is in Lorentz gauge $h^{\text{new}}{}_{\alpha}{}^{\nu}{}_{|\nu} = 0$), and its physical components die off like $1/r$ in the radiation zone.

Any gravitational wave can be characterized in a gauge-invariant way by the transverse-traceless (TT) part of its metric perturbation (see chapter 35 of MTW). Only $h_{\vartheta\phi}^{\text{new}}$ contributes to the TT part of our wave (31). By combining equation (6c) for b_ϕ with (31c) for $h_{\vartheta\phi}^{\text{new}}$, and by converting to covariant notation in the three-dimensional Euclidean space far from the star, we obtain

$$(h_{jk}^{\text{new}})^{\text{TT}} = [-\,2rF^{(l)}(t-r_*)n^p\,\epsilon_{pqj}P_l{}^{|q}{}_{|k}]^{\text{S}}. \tag{32}$$

Here $\mathbf{n} \equiv \mathbf{r}/r$ is the unit radial vector, ϵ_{pqj} is the Levi–Civita tensor, $P_l \equiv P_l(\cos\vartheta)$ is regarded as a scalar field in flat space, | denotes covariant derivative, S means symmetrize on indices j and k, and TT means take the transverse-traceless part using the techniques of Box 35.1 of MTW.

One of the authors has attempted to introduce a standardized formalism for multipole expansions of gravitational radiation fields (Thorne 1980). In that formalism the mass and current multipoles are represented by completely symmetric, trace-free tensors. To make the connection between equation (32) and that formalism we introduce into equation (32) the symmetric, trace-free representation of the Legendre polynomial

$$P_l(\cos\vartheta) = P_l{}^{a_1\cdots a_l}n_{a_1}\cdots n_{a_l} \tag{33}$$

(*cf.* Section II.C of Thorne 1980, where $P_l{}^{a_1\cdots a_l}$ is denoted by $\mathscr{Y}^{l0}_{a_1\ldots a_l}/C^{l0}$) and we then perform the differentiations denoted by $P_l{}^{|q}{}_{|k}$. The result is

$$(h_{jk}^{\text{new}})^{\text{TT}} = r^{-1}\,[2l(l-1)F^{(l)}(t-r_*)\epsilon_{pqj}P_{lk}{}^{pa_3\cdots a_l}n^q\,n_{a_3}\ldots n_{a_l}]^{\text{S}}. \tag{34}$$

Direct comparison with equation (4.8) of Thorne (1980) shows that the radiation field is that of a current l-pole with l-pole moment

$$\mathscr{S}^{a_1\ldots a_l}(t-r_*) = \frac{(l-1)(l+1)!}{4}\,F(t-r_*)P_l^{a_1\cdots a_l}. \tag{35}$$

This radiation field carries off energy at a rate given by (*cf.* Thorne 1980, equation 4.16)

$$\begin{aligned}\frac{dE_{\text{star}}}{dt} &= -\frac{4l(l+2)}{(l-1)(l+1)!\,(2l+1)!!}\,\langle \mathscr{S}^{a_1\ldots a_l(l+1)}\,\mathscr{S}^{a_1\ldots a_l(l+1)}\rangle \\ &= -\frac{(l-1)l(l+1)(l+2)}{4(2l+1)}\,\langle[F^{(l+1)}(t-r_*)]^2\rangle.\end{aligned} \tag{36}$$

where ⟨ ⟩ means averaged over several characteristic periods of the radiation and where equation (2.26a) or (2.5) of Thorne (1980) has been used to obtain the second line.

3.4 ACTION PRINCIPLE AND ENERGY CONSERVATION LAWS

Our star's torsional oscillations are governed by an action principle. The action's Lagrangian density can be derived either by second variation of the Einstein Lagrangian density $(-g)^{1/2}R+\mathscr{L}_{\text{matter}}$ (method of Taub 1969) or by multiplying the star's equations of motion by carefully selected functions and removing a divergence (method of Chandrasekhar 1964a, b; see also Detweiler & Ipser 1973 and Appendix B of this paper). The Lagrangian density is

$$\mathscr{L}=\frac{2\pi l(l+1)}{(2l+1)}\,[(\rho+P)r^4e^{\Lambda-\Phi}(\dot{Y}-\dot{y})^2+(1/16\pi)r^4e^{-\Phi-\Lambda}(\dot{y}'-e^{\Lambda-\Phi}\dot{Q}/r)^2$$
$$+(1/16\pi)(l+2)(l-1)r^2e^{\Lambda-\Phi}\dot{y}^2-\mu r^4e^{\Phi-\Lambda}(Y'-e^{\Lambda-\Phi}Q/r)^2$$
$$-\mu(l+2)(l-1)r^2e^{\Phi+\Lambda}Y^2-(1/16\pi)(l+2)(l-1)e^{\Lambda-\Phi}Q^2], \tag{37}$$

and the action principle is

$$\delta\int_\Omega \mathscr{L}\,dt\,dr=0, \tag{38}$$

where Ω is any compact region of spacetime, and where the functions to be varied (Y, y and Q) must be held fixed on the boundary $\partial\Omega$ (i.e. $\delta Y=\delta y=\delta Q=0$ there). If Ω includes the star's centre or surface or a solid–fluid interface, then Y, y and Q must satisfy the smoothness and continuity equations (26a, b, c, d, e) there. By varying Y, y and Q in this action we obtain, respectively, the perturbed Einstein field equations $\epsilon_T=0$ (equation A.5), $\dot{\epsilon}_{t\phi}=0$ (equation A.2) and $\epsilon_{r\phi}=0$ (equation A.4). Our equations of motion (19) are linear combinations of these equations and their derivatives and time integrals; *cf.* equations (A.7)–(A.9).

Because our Lagrangian density (37) is time-independent, $(\partial\mathscr{L}/\partial t)_{Y,\,y,\,Q\text{ fixed}}=0$, there is a conserved quantity associated with it:

$$S^\alpha{}_{,\alpha}=0, \tag{39}$$

where

$$S^t\equiv\frac{2\pi l(l+1)}{(2l+1)}\left\{(\rho+P)r^4e^{\Lambda-\Phi}(\dot{Y}-\dot{y})^2+\frac{r^4e^{-\Phi-\Lambda}}{16\pi}(\dot{y}'-e^{\Lambda-\Phi}\dot{Q}/r)^2\right.$$
$$+\frac{(l+2)(l-1)}{16\pi}r^2e^{\Lambda-\Phi}\dot{y}^2+\mu r^4e^{\Phi-\Lambda}(Y'-e^{\Lambda-\Phi}Q/r)^2$$
$$\left.+\mu(l+2)(l-1)r^2e^{\Phi+\Lambda}Y^2+\frac{(l+2)(l-1)}{16\pi}e^{\Lambda-\Phi}Q^2\right\}, \tag{40a}$$

and

$$S^r\equiv-\frac{4\pi l(l+1)}{(2l+1)}\left\{\frac{r^4e^{-\Phi-\Lambda}}{16\pi}\dot{y}(\dot{y}'-e^{\Lambda-\Phi}\ddot{Q}/r)+\mu r^4e^{\Phi-\Lambda}\dot{Y}(Y'-e^{\Lambda-\Phi}Q/r)\right\}. \tag{40b}$$

(For a derivation, and for a discussion of how we have selected this specific S^α from among an infinity of such divergence-free quantities, see Appendix B.) Note that the energy density, S^t, is just the Lagrangian density $\mathscr{L}$ with the signs of the potential energy terms converted from minus to plus.

If we regard Q and y as gravitational fields which reside in the unperturbed spacetime and which couple to the matter displacement Y, then we can associate with the perturbations a stress-energy tensor $\tilde{T}^{\mu\nu}$ which resides in the unperturbed spacetime. The law of energy-momentum conservation $\tilde{T}^{\mu\nu}{}_{;\nu} = 0$ (where the semicolon denotes covariant derivative with respect to the unperturbed metric $\gamma_{\mu\nu}$), together with Killing's equation for the generator $\partial/\partial t$ of time translations, guarantees that $\tilde{T} \cdot \partial/\partial t$ has vanishing covariant derivative – i.e. in component notation and in the (t, r, ϑ, ϕ) coordinate system of equation (5a)

$$[(-\gamma)^{1/2}\, \tilde{T}^{\alpha}_{t}\,]_{,\alpha} = 0. \tag{41}$$

Here γ is the determinant of the unperturbed metric components $\gamma_{\mu\nu}$, and $(-\gamma)^{1/2}$ is equal to $r^2 e^{\Phi + \Lambda} \sin\vartheta$. After integrating this equation over angles ϑ and ϕ we obtain the conservation law (39), with

$$S^{\alpha} = - \int \tilde{T}^{\alpha}_{t}\, (-\gamma)^{1/2}\, d\vartheta\, d\phi. \tag{42}$$

The perturbation stress-energy tensor $\tilde{T}^{\mu\nu}$ can be computed in the canonical manner from the Lagrangian for the perturbations (albeit a Lagrangian in which, unlike (37), the angular dependences have not yet been integrated out). There is an infinity of resulting $\tilde{T}^{\mu\nu}$'s depending on the gauge in which the Lagrangian is written (i.e. depending on one's choice of infinitesimal ripples in the perturbed star's coordinate system). If one only wants to know the components $\tilde{T}^{\alpha}_{t}$ one can evaluate them by undoing the angular integrations in equation (42), a process which contains some arbitrariness corresponding to part of the gauge-dependent arbitrariness in $\tilde{T}^{\mu\nu}$. With a choice for this arbitrariness which we regard as optimal, the equations (40a), (42) and

$$\int_0^{\pi} \sin\vartheta\, [\partial_{\vartheta} P_l(\cos\vartheta)]^2 d\vartheta = \frac{2l(l+1)}{(2l+1)},$$

$$\int_0^{\pi} \sin^3\vartheta \left[\partial_{\vartheta}\left(\frac{\partial_{\vartheta} P_l(\cos\vartheta)}{\sin\vartheta}\right)\right]^2 d\vartheta = (l+2)(l-1)\,\frac{2l(l+1)}{2l+1}, \tag{43}$$

give the result

$$\tilde{T}^{\hat{t}\hat{t}} = -\tilde{T}_t{}^{t} = 1/2\,(\rho + P)\,\mathrm{v}^2 + \mu(S_{\hat{j}\hat{k}})^2 + \frac{1}{16\pi}\,(\sigma^{Z}{}_{\hat{j}\hat{k}})^2 + \frac{1}{16\pi}\,(B_{\hat{j}\hat{k}})^2, \tag{44a}$$

$$\tilde{T}^{\hat{t}\hat{r}} = -e^{\Lambda - \Phi}\,\tilde{T}_t{}^{r} = -2\mu S_{\hat{r}\hat{j}} u^{\hat{j}} + \frac{1}{16\pi} A_{\hat{r}\hat{j}\hat{k}}\,\sigma^{Z}{}_{\hat{j}\hat{k}}, \tag{44b}$$

where there is an implied summation over $\hat{j}$ and $\hat{k}$. These equations make use of the orthonormal basis of an observer at rest in the unperturbed star:

$$\mathbf{e}_{\hat{t}} = e^{-\Phi}\,\partial_t, \quad \mathbf{e}_{\hat{r}} = e^{-\Lambda}\,\partial_r, \quad \mathbf{e}_{\hat{\vartheta}} = r^{-1}\,\partial_{\vartheta}, \quad \mathbf{e}_{\hat{\phi}} = (r\sin\vartheta)^{-1}\,\partial_{\vartheta}. \tag{45a}$$

The quantity $\tilde{T}^{\hat{t}\hat{t}}$ (equation 44a) is the energy density measured by this static observer. The term $1/2\,(\rho + P)\,\mathrm{v}^2$ in $\tilde{T}^{\hat{t}\hat{t}}$ is the kinetic energy density of the matter; **v** is the velocity of the matter relative to the ZAMO's (see discussion following equations 6c and 8b),

$$\mathbf{v} = re^{-\Phi}\,(\dot{Y} - \dot{y})\, b_{\hat{\phi}}\, \mathbf{e}_{\hat{\phi}}; \qquad b_{\hat{\phi}} \equiv (\sin\vartheta)^{-1} b_{\phi} = \partial_{\vartheta} P_l(\cos\vartheta). \tag{45b}$$

The term $\mu(S_{\hat{j}\hat{k}})^2$ is the standard expression for the potential energy density of a deformed, elastic solid; $S_{\hat{j}\hat{k}}$ is the shear

$$S_{\hat{r}\hat{\phi}} = S_{\hat{\phi}\hat{r}} = 1/2\, re^{-\Lambda}(Y' - e^{\Lambda-\Phi}Q/r)\, b_{\hat{\phi}},$$
$$S_{\hat{\vartheta}\hat{\phi}} = S_{\hat{\phi}\hat{\vartheta}} = 1/2\, Y \sin\vartheta\, b^{\phi}{}_{,\vartheta} \tag{45c}$$

(equations 15) and by virtue of the stress–strain relation (10)

$$\mu(S_{\hat{j}\hat{k}})^2 = -1/2\; T^{\text{shear}}_{\hat{j}\hat{k}}\, S^{\hat{j}\hat{k}}. \tag{45d}$$

The term $(16\pi)^{-1}(\sigma^Z{}_{\hat{j}\hat{k}})^2$ is the kinetic energy density of the gravitational field; $\sigma^Z{}_{\hat{j}\hat{k}}$ is the rate of shear of the congruence of ZAMO observers (equations 12 with the matter displacement Y replaced by the ZAMO displacement y)

$$\sigma^Z{}_{\hat{r}\hat{\phi}} = \sigma^Z{}_{\hat{\phi}\hat{r}} = 1/2\, re^{-\Phi-\Lambda}(\dot{y}' - e^{\Lambda-\Phi}\dot{Q}/r)\, b_{\hat{\phi}},$$
$$\sigma^Z{}_{\hat{\vartheta}\hat{\phi}} = \sigma^Z{}_{\hat{\phi}\hat{\vartheta}} = 1/2\, e^{-\Phi}\dot{y}\sin\vartheta\, b^{\phi}{}_{,\vartheta}. \tag{45e}$$

The term $(16\pi)^{-1}(B_{\hat{j}\hat{k}})^2$ is the potential energy density of the gravitational field; $B_{\hat{j}\hat{k}}$ is defined to have as its only non-zero components

$$B_{\hat{\vartheta}\hat{\phi}} = B_{\hat{\phi}\hat{\vartheta}} \equiv -1/2\, r^{-1} e^{-\Phi} Q \sin\vartheta\, b^{\phi}{}_{,\vartheta}. \tag{45f}$$

We have not found a simple, physical description of the quantity $B_{\hat{j}\hat{k}}$ whose square is the gravitational potential energy, analogous to the description $\sigma^Z{}_{\hat{j}\hat{k}}$ ($\equiv$ ZAMO rate of shear) of the quantity whose square is the gravitational kinetic energy.

The quantity $\tilde{T}^{\hat{t}\hat{r}}$ (equation 44b) is the energy flux measured by a static observer. The term $-2\mu S_{\hat{r}\hat{j}}u^{\hat{j}} = T^{\text{shear}}_{\hat{r}\hat{j}}u^{\hat{j}}$ is the standard expression for the radial energy flux carried by the matter's shear stress; $u^{\hat{j}}$ is the matter velocity relative to the static observer

$$\mathbf{u} \equiv u^{\hat{\phi}}\mathbf{e}_{\hat{\phi}} = re^{-\Phi}\dot{Y}b^{\hat{\phi}}\,\mathbf{e}_{\hat{\phi}}. \tag{45g}$$

The term $A_{\hat{r}\hat{j}\hat{k}}\sigma^Z{}_{\hat{j}\hat{k}}$ is the radial energy flux carried by the gravitational waves; $A_{\hat{i}\hat{j}\hat{k}}$ is defined to have as its only non-zero components

$$A_{\hat{r}\hat{\vartheta}\hat{\phi}} = A_{\hat{r}\hat{\phi}\hat{\vartheta}} = -\frac{r^2 e^{-2\Phi-\Lambda}}{(l+2)(l-1)}(\ddot{y}' - e^{\Lambda-\Phi}\ddot{Q}/r)\sin\vartheta\, b^{\phi}{}_{,\vartheta}; \tag{45h}$$

and as with $B_{\hat{j}\hat{k}}$ we have not found a simple physical description of $A_{\hat{i}\hat{j}\hat{k}}$.

In the radiation zone the energy density $\tilde{T}^{\hat{t}\hat{t}}$ and energy flux $\tilde{T}^{\hat{t}\hat{r}}$ are carried entirely by the gravitational waves (which we assume to be outgoing):

$$\tilde{T}^{\hat{t}\hat{t}} = \tilde{T}^{\hat{t}\hat{r}} = (32\pi)^{-1}(\dot{h}^{\text{new}}_{\hat{j}\hat{k}})^{\text{TT}}(\dot{h}^{\text{new}}_{\hat{j}\hat{k}})^{\text{TT}}$$
$$= (16\pi r^2)^{-1}[F^{(l+1)}(u)]^2(\sin\vartheta\, b^{\phi}{}_{,\vartheta})^2. \tag{46}$$

Here $(h^{\text{new}}_{\hat{j}\hat{k}})^{\text{TT}}$ is the transverse traceless gravitational-wave field of equations (31) and (32); equation (46) can be derived by combining equations (44), (45), (A.4) with $\epsilon_{r\phi} = 0$, (26) and (31c). When averaged over several wavelengths, expression (46) reduces to the standard Isaacson stress-energy tensor for the waves (see, e.g. Sections 35.7 and 35.15 of MTW).

The differential law of energy conservation $S^{\alpha}{}_{,\alpha} = 0$ or $[(-\gamma)^{1/2}\,\tilde{T}_t{}^{\alpha}]_{,\alpha} = 0$, when spatially integrated over the star's interior and on out to some radius R_∞ in the wave zone, becomes a law of global energy conservation:

$$dE_{\rm star}/dt = -S^r(r=R_\infty) = -\left[\int \tilde{T}^{\hat{t}\hat{r}} r^2\, d\vartheta\, d\phi\right]_{r=R_\infty}$$

$$= -\frac{(l-1)l(l+1)(l+2)}{4(2l+1)}[F^{(l+1)}(t-r^*)]^2. \tag{47a}$$

$$E_{\rm star} = \int_0^{R_\infty} S^t dr = \iint\int_0^{R_\infty} \tilde{T}^{\hat{t}\hat{t}} e^{\Phi}\, d\mathrm{vol}, \tag{47b}$$

where $d\mathrm{vol} \equiv e^{\Lambda} r^2 \sin\vartheta\, d\vartheta\, d\phi\, dr$ is the spatial volume element and e^{Φ} is the gravitational redshift factor.

3.5 STABILITY OF THE OSCILLATING STAR

The law of global energy conservation (47) is a foundation for proving that our oscillating star is stable: So long as the shear modulus μ is non-negative, the energy density $\tilde{T}^{\hat{t}\hat{t}}$ is everywhere positive (equation 44a), and therefore $E_{\rm star}$ (equation 47b) is a positive definite functional of Y, y and Q. Since $dE_{\rm star}/dt < 0$ (equation 47a), no choice of initial conditions $Y(t=0,r)$, $y(t=0,r)$, $Q(t=0,r)$ can produce Y, y, Q which grow arbitrarily large at later times. Therefore our star with outgoing-wave boundary conditions is stable against arbitrary initial perturbations (Liapunov stability; *cf.* LaSalle & Lefschetz 1961).

4 The outgoing-wave normal modes: $l \geqslant 2$

4.1 THE EIGENVALUE PROBLEM

For most applications of the theory developed in this paper one will want to resolve the torsional oscillations into normal modes with complex vibrational frequencies

$$\omega = \sigma + i/2\tau. \tag{48}$$

In a normal mode the perturbation functions have the forms

$$Y(t,r) \equiv Y_\omega(r) e^{i\omega t}; \quad y(t,r) \equiv y_\omega(r) e^{i\omega t}; \quad Q(t,r) \equiv Q_\omega(r) e^{i\omega t}. \tag{49}$$

The real part of the frequency, σ, describes sinusoidal oscillations; the imaginary part, $1/2\tau$, describes damping due to radiation reaction. (The factor 2 appears in $\omega \equiv \sigma + i/2\tau$ so that τ will be the e-folding time of the star's oscillation energy, not of its amplitude.)

For a normal mode the two dynamical equations (19b, c) form a fourth-order system of linear ordinary differential equations for the eigenfunctions $Y_\omega(r)$ and $Q_\omega(r)$ (hereafter we omit the subscript ω):

$$(\mu r^4 e^{\Phi-\Lambda} Y')' - r^4 e^{\Phi+\Lambda}[16\pi(\rho+P) + (l+2)(l-1)r^{-2}]\mu Y$$

$$- (\mu r^3 Q)' + (\rho+P) r^2 (rQ)' = -\omega^2(\rho+P) r^4 e^{\Lambda-\Phi} Y; \tag{50a}$$

$$(e^{\Phi-\Lambda} Q')' - [16\pi e^{\Phi+\Lambda}\mu + (l+2)(l-1) r^{-2} e^{\Phi+\Lambda} - r(r^{-2} e^{\Phi-\Lambda})']Q$$

$$- 16\pi r(\mu e^{2\Phi})' Y = -\omega^2 e^{\Lambda-\Phi} Q. \tag{50b}$$

These equations must be solved subject to the boundary conditions (26a, b, f):

$$Y \sim r^{l-1}, \quad Q \sim r^{l+1} \quad \text{as} \quad r \to 0, \tag{51a}$$

$$\mu(Y' - e^{\Lambda-\Phi} Q/r) \to 0 \quad \text{as} \quad r \to R_-, \tag{51b}$$

and for the physically realistic case of outgoing waves at infinity (outgoing-wave normal mode)

$$Q = (i\omega)^{l+1} F_\omega e^{-i\omega r_*} \quad \text{as} \quad r \to \infty, \tag{51c}$$

where F_ω is the amplitude of the oscillatory l-pole moment at $t = 0$, $F(t) \equiv F_\omega e^{i\omega t}$. Equations (50) and (51) together form an eigenvalue problem for the oscillation frequency ω and eigenfunctions Y, Q. Once the eigenvalue problem has been solved, the remaining metric perturbation function y can be computed most easily from the initial-value equation $\epsilon_{\vartheta\phi} = 0$ (equation A.3), which gives

$$y = \frac{1}{\omega^2}\left[\frac{-e^{\Phi-\Lambda}}{r^2}(rQ)' + 16\pi\mu e^{2\Phi} Y\right]; \tag{52}$$

alternatively (and equivalently) y can be computed from the initial value equation (19a).

In posing the eigenvalue problem (50)–(52) we have omitted some of the boundary conditions (26). It is straightforward to show (*cf.* discussion of equations 26) that, so long as the unperturbed star is well behaved at its centre and surface (ρ, P, μ and μ' finite at $r = 0$; ρ, μ finite but perhaps non-zero and $P \to 0$ as $r \to R_-$), the omitted boundary conditions

$$y \sim r^{l-1} \quad \text{as} \quad r \to 0, \tag{53a}$$

$$\mu(Y' - e^{\Lambda-\Phi} Q/r) \to 0 \text{ at solid–fluid interfaces,} \tag{53b}$$

$$y, y', Q \text{ continuous across } r = R \text{ and across interfaces,} \tag{53c}$$

$$y \sim -(i\omega)^l r^{-1} F_\omega e^{-i\omega r_*} \quad \text{as} \quad r \to \infty \tag{53d}$$

are automatically satisfied by any solution of equations (50)–(52).

In order to understand the spectrum of eigenfrequencies of our torsionally oscillating star, we must first understand the asymptotic behaviours of the solutions of the eigenequations (50) just below the star's surface. Those behaviours depend on the asymptotic forms of the star's density ρ and shear modulus μ. If the star's surface is solid, ρ will be finite; otherwise it may go to zero as a power law. In general μ will go to zero at least as fast as ρ. Hence, it is reasonable to suppose that

$$\rho \sim (R-r)^N, \quad P \sim (R-r)^{N+1}, \quad \mu \sim (R-r)^{N+S}; \quad N \geqslant 0, \quad S \geqslant 0, \tag{54}$$

where the form of P follows from the equation of hydrostatic equilibrium. One can show that, so long as $S < 2$ [i.e. so long as the speed of shear waves $(\mu/\rho)^{1/2}$ goes to zero no faster than $(R - r)$], one solution of the eigenequations (50) will have $\mu(Y' - e^{\Lambda-\Phi} Q/r)$ finite and non-zero at R_- and will thus be physically unacceptable. All other solutions will be acceptable. For $S > 2$ all solutions have $\mu(Y' - e^{\Lambda-\Phi} Q/r)$ zero at R_-, but they also all have Y divergent, which would lead to a breaking of the crystal – a complication we are not prepared to face in this paper. Thus, we shall restrict ourselves henceforth to the case $S < 2$; and we shall impose a similar restriction at interfaces of solid regions with the fluid mantle. In this case the spectrum of eigenfrequencies will be discrete, as the following argument shows.

Imagine a trial integration of the eigenequations (50). One selects a complex trial frequency ω and complex starting values A and B for Y/r^{l-1} and Q/r^{l+1} near $r = 0$. (The eigenequations (50) have the general solution $Y = Ar^{l-1} + Dr^{-l-2}$, $Q = Br^{l+1} + Er^{-l}$ near $r = 0$; one makes sure that D and E vanish.) One then integrates the eigenequations (50) outward from $r = 0$ to the star's surface $r = R$ and examines the value of the complex number $\mu(Y' - e^{\Lambda-\Phi}Q/r)$ there; it will turn out to be non-zero, unless the starting ratio A/B has been chosen to have some special value (or one of a discrete set of special values). That choice must be made. One then continues the integration on outward into the radiation zone, where one finds for Q (general solution of 50b)

$$Q = C^{(O)}e^{-i\omega r_*} + C^{(I)}e^{+i\omega r_*}. \tag{55}$$

To get an outgoing-wave normal mode one must ensure that the complex ingoing-wave amplitude $C^{(I)}$ vanishes. One cannot do so by adjusting the starting product AB; that product merely fixes the overall amplitude and phase of the oscillations. Instead, to make $C^{(I)}$ vanish one must carefully adjust the complex eigenfrequency ω to one of a discrete set of values. Thus, the spectrum is discrete.

The Liapunov proof of stability in Section 3.5 guarantees that the outgoing-wave normal modes are all damped, i.e. all have positive values of $\text{Im}(\omega) = 1/2\tau$.

We now describe five methods for solving the eigenvalue problem (50) and (51): the method of resonances (Section 4.2), the variational method (Section 4.3), the energy method (Section 4.4), the method of the slow-motion approximation (Section 4.5) and the method of the weak-field approximation (Section 4.6).

4.2 METHOD OF RESONANCES

In the method of resonances (Thorne 1969a) one studies the unrealistic problem of an oscillating star inside a large spherical cavity whose walls reflect gravitational waves perfectly. This requires replacing the outgoing-wave boundary condition (51c) by a standing-wave boundary condition. The star and standing wave can oscillate with any desired *real* frequency $\omega \equiv \sigma$. For each value of the frequency ω one can calculate (on a computer) the ratio

$$\mathscr{R} \equiv \frac{\text{(amplitude of star's oscillating motions)}}{\text{(amplitude of waves far from the star)}}. \tag{56}$$

As ω varies, $\mathscr{R}$ will go through a sequence of sharp resonances. These resonances, on the real frequency axis, are induced by nearby complex eigenfrequencies of the discrete, outgoing-wave normal modes; i.e. when ω nears the oscillation frequency ω_n of an outgoing-wave normal mode, the standing gravitational waves will excite the star's fluid into large-amplitude motions. From the locations, half-widths and phase-shifts of the resonances one can compute the complex frequencies $\omega_n = \sigma_n + i/2\tau_n$ of the outgoing-wave normal modes. Thorne (1969a) has discussed these calculations in detail for compressional oscillations; calculations for our case of torsional oscillations would be the same in concept and method.

4.3 VARIATIONAL METHOD

The normal-mode eigenfunctions and eigenfrequencies can be evaluated using a Detweiler–Ipser (1973) type action principle, which is closely related to the Lagrangian density $\mathscr{L}$ of equation (37). The relationship to $\mathscr{L}$ and a derivation of the action principle are sketched in Appendix B. The action principle utilizes integrals from the centre of the star $r = 0$ to a sphere $r = R_\infty$ far out in the radiation zone, and it utilizes complex trial functions Y, Q which are constrained to satisfy the smoothness and continuity conditions (51a,b) and (53b,c) at $r = 0$, on the star's surface $r = R$, and across solid–fluid interfaces. For any choice of such trial functions Y, Q a corresponding complex function y is to be computed by solving the initial-value equation (19a) subject to the smoothness boundary condition (53a) at $r = 0$, and subject to the demand that

$$\chi/y = \text{some fixed value, } (\chi/y)_\infty, \quad \text{at} \quad r = R_\infty; \tag{57a}$$

here

$$\chi \equiv -r^2 e^{-\Phi-\Lambda}(y' - e^{\Lambda-\Phi} Q/r). \tag{57b}$$

(Recall that outside the star $\Lambda = -\Phi$.) The quantity

$$\Omega^2 \equiv B/A \tag{58a}$$

must then be computed, where

$$A \equiv \int_0^{R_\infty} \left[(\rho + P) r^4 e^{\Lambda-\Phi} (Y - y)^2 + \frac{r^4 e^{-\Phi-\Lambda}}{16\pi} (y' - e^{\Lambda-\Phi} Q/r)^2 + \frac{(l+2)(l-1)}{16\pi} r^2 e^{\Lambda-\Phi} y^2 \right] dr + \frac{1}{16\pi} [r^2 y \chi]_{r=R_\infty}, \tag{58b}$$

$$B \equiv \int_0^{R_\infty} \left[\mu r^4 e^{\Phi-\Lambda} (Y' - e^{\Lambda-\Phi} Q/r)^2 + \mu (l+2)(l-1) r^2 e^{\Phi+\Lambda} Y^2 + \frac{(l+2)(l-1)}{16\pi} e^{\Lambda-\Phi} Q^2 \right] dr. \tag{58c}$$

The quantity $\Omega^2 \equiv B/A$ is an action for the normal modes. Those trial functions Q and Y, which make Ω^2 stationary ($\delta\Omega^2 = 0$) with respect to all variations δQ and δY that satisfy our smoothness and continuity conditions, are normal-mode eigenfunctions; and the stationary value of Ω is their complex eigenfrequency ω. (The Euler–Lagrange equations associated with this action principle are our eigenequations 50 with $\omega^2 = \Omega^2$.)

The specific normal modes obtained from this action principle depend on the chosen boundary value $(\chi/y)_\infty$. To obtain standing-wave normal modes, one chooses $(\chi/y)_\infty$ real and all trial functions real. For a given real $(\chi/y)_\infty$ there will be a discrete set of standing-wave modes (analogue of discrete normal modes of a violin string with ends clamped). To obtain the full continuous set of standing-wave modes (one mode for each real ω), the action principle must be used time and again, with various values of $(\chi/y)_\infty$ and fixed R_∞; or with fixed $(\chi/y)_\infty$ and various R_∞ (analogue of changing the clamping location of the violin string).

If one chooses $(\chi/y)_\infty$ complex rather than real and uses complex trial functions, then the action principle (57) and (58) will produce a discrete set of normal modes, each with a different mixture of ingoing and outgoing waves – a mixture that cannot be predicted in advance. Only by an iterative application of the action principle (procedure devised by Detweiler (1975) for compressional oscillations of stars) can one be sure of obtaining a pure outgoing-wave mode. For an outgoing-wave mode, if one knew the complex frequency ω in advance, one could solve the eigenequations (50) and initial-value equation (19a) far from the star to find the asymptotic forms of Q, y and χ:

$$Q = (i\omega)^{l+1} F_\omega \left[1 + \frac{l(l+1)}{2i\omega r} + O\left(\frac{1}{r^2}\right) \right] e^{-i\omega r_*}, \tag{59a}$$

$$y = -\frac{(i\omega)^l}{r} F_\omega \left[1 + \frac{(l+2)(l-1)}{2i\omega r} + O\left(\frac{1}{r^2}\right) \right] e^{-i\omega r_*}, \tag{59b}$$

$$\chi = -(l+2)(l-1)\frac{(i\omega)^{l-1}}{r}F_\omega\left[1+\frac{l(l+1)}{2i\omega r}+O\left(\frac{1}{r^2}\right)\right]e^{-i\omega r_*}, \tag{59c}$$

where the complex number F_ω is the (arbitrary) Fourier amplitude of the l-pole moment; *cf.* equations (26f, g). The corresponding boundary value of χ/y is

$$(\chi/y)_\infty = \frac{(l+2)(l-1)}{i\omega}\left[1+\frac{1}{i\omega R_\infty}+O\left(\frac{1}{R_\infty^2}\right)\right]. \tag{59d}$$

Detweiler's procedure is to guess a value of ω; choose the boundary value $(\chi/y)_\infty$ equal to (59d); apply the action principle using trial functions with the asymptotic forms (59a, b, c), thereby obtaining a stationary Ω; if Ω is equal to ω, stop with joy; if not, reiterate using a new trial value of ω. (One can show that if Ω and ω differ by a small amount, the normal modes of frequency Ω with boundary condition 59d contain a mixture of ingoing and outgoing waves of relative amplitude

$$C^{(I)}/C^{(O)} = (\Omega-\omega)/(\Omega+\omega). \tag{60}$$

This is a measure of the error in an unconverged iteration by Detweiler's procedure.)

As cumbersome as this procedure may seem, it is the best method now known for computing outgoing-wave normal modes from an action principle; and it actually has been made to give reasonably accurate results for compressional oscillations of neutron stars (Detweiler 1975).

4.4 ENERGY METHOD

If one has obtained reasonable approximations to the eigenfunctions Q, Y, y and to the real part σ of the eigenfrequency of a complex normal mode, one can then compute the imaginary part of the eigenfrequency, $i/2\tau$, using the law of energy conservation (39), (40). In integral form, and averaged over time, that law says (*cf.* equations 47 and B.19–B.23):

$$\tau = \frac{\bar{E}_{\rm star}}{\bar{S}^r(r=R_\infty)}, \tag{61}$$

where

$$\begin{aligned}\bar{E}_{\rm star} = \frac{\pi l(l+1)}{(2l+1)}\int_0^{R_\infty}\Big\{&(\sigma^2+1/4\tau^2)\Big[(\rho+p)r^4e^{\Lambda-\Phi}|Y-y|^2+\frac{r^4e^{-\Phi-\Lambda}}{16\pi}|y'-e^{\Lambda-\Phi}Q/r|^2\\&+\frac{(l+2)(l-1)}{16\pi}r^2e^{\Lambda-\Phi}|y|^2\Big]+\mu r^4e^{\Phi-\Lambda}|Y'-e^{\Lambda-\Phi}Q/r|^2\\&+\mu(l+2)(l-1)r^2e^{\Phi+\Lambda}|Y|^2+\frac{(l+2)(l-1)}{16\pi}e^{\Lambda-\Phi}|Q|^2\Big\}\,dr,\end{aligned} \tag{62a}$$

$$\bar{S}^r(r=R_\infty) = -\frac{l(l+1)}{8(2l+1)}\,\mathrm{Im}\left[\left(\sigma+\frac{i}{2\tau}\right)\left(\sigma^2+\frac{1}{4\tau^2}\right)r^2y^*\chi\right]_{R_\infty} \tag{62b}$$

for R_∞ anywhere outside star,

$$= \frac{(l-1)l(l+1)(l+2)}{8(2l+1)}|Q|^2 \text{ for } R_\infty \text{ far out in wave zone.} \tag{62b}$$

Here y^* is the complex conjugate of y.

In applying this energy method one can place R_∞ anywhere one wishes outside the star, even in the near zone, if one uses the first line of equation (62b) for the energy flux.

4.5 SLOW-MOTION METHOD

A generator of gravitational waves is said to be a slow-motion source if and only if the characteristic reduced wavelength of the waves, $\lambdabar \equiv \lambda/2\pi = 1/\sigma$, is much larger than both the source itself and the source's strong-field region:

$$\lambdabar \gg R, \quad \lambdabar \gg 2M \equiv \text{(gravitational radius)}. \tag{63}$$

Thorne (1980) has given a detailed formalism for calculating the gravitational waves from slow-motion sources. Here we specialize that formalism to the case of torsional oscillations of a neutron star. (A forthcoming paper by Thorne will specialize it to g-mode compressional oscillations of a neutron star.)

The discussion in the Introduction of this paper gave reduced wavelengths of $\lambdabar \approx 10^3$ km for crustal oscillations of neutron stars and $\lambdabar \approx 10$ km for core oscillations. Thus the slow-motion approximation is accurate for crustal oscillations but probably not very accurate for core oscillations.

If the slow-motion condition (63) is satisfied, we can neglect retardation of the gravitational fields across the source, i.e. we can neglect $\ddot{x} = -\omega^2 x$ compared to x'', x'/r or x/r^2 ($x \equiv Q$ or y) throughout the interior of the near-zone region

$$r \ll \lambdabar \equiv \sigma^{-1}. \tag{64}$$

(We cannot, of course, neglect retardation of the shear waves; i.e. we cannot neglect $\ddot{x} = -\omega^2 x$ compared to $[\mu/(\rho+P)]\, x''$.) By neglecting gravitational retardation we convert our gravitational variables Q and y into action-at-a-distance potentials analogous to that of Newton; their wave equations become Poisson-like equations.

From $(\mu/\rho) \sim$ (speed of shear waves)$^2 \lesssim (\sigma R)^2$ we learn that

$$\mu \lesssim (R/\lambdabar)^2 \rho \lesssim (R/\lambdabar)^2 r^{-2}; \tag{65a}$$

and from equation (19a) for y and (50b) for Q we learn the relative magnitudes of y, Q and Y in the slow-motion approximation:

$$y \sim (M/R)\, Y, \qquad Q \sim (R/\lambdabar)^2 y \ll y. \tag{65b}$$

Taking account of the extreme smallness of μ compared to ρ and r^{-2} and of the extreme smallness of Q compared to y and Y and neglecting gravitational retardation, we can bring the equations governing normal-mode oscillations into the form

$$(\mu r^4 e^{\Phi-\Lambda} Y')' - (l+2)(l-1) r^2 e^{\Phi+\Lambda} \mu Y = -\omega^2 (\rho+P) r^4 e^{\Lambda-\Phi} (Y-y), \tag{66a}$$

$$(r^4 e^{-\Phi-\Lambda} y')' - (l+2)(l-1) r^2 e^{\Lambda-\Phi} y = -16\pi (\rho+P) r^4 e^{\Lambda-\Phi} (Y-y). \tag{66b}$$

$$(e^{\Phi-\Lambda} Q')' - [(l+2)(l-1) r^{-2} e^{\Phi+\Lambda} - r(r^{-2} e^{\Phi-\Lambda})']\, Q = 16\pi r (\mu e^{2\Phi})' Y. \tag{66c}$$

Equation (66a) is (50a) with (52) used to replace a term involving Q by one involving y; equation (66b) is (19a); and equation (66c) is (50b).

Outside the star, and at radii $M \ll r \ll \lambdabar$ where $\Phi = -\Lambda \simeq 0$ and where the slow-motion approximation is valid, Q and y have power-law fall-offs:

$$Q = \frac{(2l-1)!!}{r^l} i\omega F_\omega, \quad y = -\frac{(l-1)(2l-1)!!}{i\omega r^{l+2}} F_\omega \quad \text{for} \quad M \ll r \ll \lambdabar \quad \text{and} \quad r > R. \tag{67a}$$

Here F_ω is the same l-pole moment used elsewhere in this paper, and $(2l-1)!! \equiv (2l-1)(2l-3)\cdots 1$. These power-law fall-offs are the asymptotic solutions of equations (66b, c). They can also be derived, including the precise coefficients involving l, ω and F_ω, by solving the non-slow-motion, Fourier-decomposed Regge–Wheeler equation (equation 50b with $\rho = \mu = 0$ and $r \gg 2M$), matching to (51c) and (52) to obtain

$$Q = F_\omega \omega^{l+2} r h_l^{(2)}(\omega r), \qquad y = -F_\omega \omega^l r^{-2} \partial_r [r^2 h_l^{(2)}(\omega r)] \tag{68}$$

for all $r \gg M$ and $r > R$, where $h_l^{(2)}$ is the spherical Hankel function, and by then expanding these solutions in powers of ωr in the near zone $\omega r \ll 1$.

By virtue of the smallness of Q in the slow-motion approximation, the no-torque-at-surface boundary condition (51b) reduces to

$$\mu Y' \to 0 \quad \text{as} \quad r \to R_-; \tag{67b}$$

but the smoothness boundary conditions (51a) and (53a) at the star's centre remain unchanged.

$$Y \sim r^{l-1}, \quad y \sim r^{l-1}, \quad Q \sim r^{l+1} \quad \text{as} \quad r \to 0. \tag{67c}$$

The eigenvalue problem in the slow-motion approximation consists of the coupled equations (66a, b) for Y and y (not Y and Q as previously!), which must be solved subject to the boundary conditions (67a, b, c). The resulting eigenfunctions and eigenfrequencies will be real (no damping in slow-motion approximation!) and discrete. They can be derived from (66a, b), (67) by standard techniques, including the following action principle:

Define $\Omega^2 \equiv B/A$ where B and A are the integrals (58b, c) with $R_\infty = \infty$ and with the surface term removed and with Q set to zero. Choose a trial function Y which satisfies the boundary conditions (67), and from it compute y by integrating (66b) subject to the boundary conditions (67). Then insert Y and y into $\Omega^2 \equiv B/A$ and ask whether $\delta\Omega^2 = 0$ for arbitrary variations δY. If $\delta\Omega^2 = 0$, then the trial function Y and the computed function y are eigenfunctions, and their value of Ω is the corresponding eigenfrequency ω.

After the slow-motion eigenvalue problem has been solved, one can use the energy method to compute the tiny imaginary part $i/2\tau$ of ω, which the slow-motion approximation ignores. Specifically, τ will be given by equation (61), where the star's pulsation energy $\bar{E}_{\text{star}}$ is (62a) with $R_\infty = \infty$ and $Q = 0$; and where the energy flux $\bar{S}^r$ is given by the second line of (62b), with $|Q|^2$ replaced by its wave-zone value $|\omega^{l+1} F_\omega|^2$ (equation 51c) and F_ω evaluated from the near-zone expression (67a) for the eigenfunction y.

4.6 WEAK-FIELD METHOD

For a torsionally oscillating star with weak internal gravity,

$$\mu/\rho \lesssim P/\rho \sim \Lambda \sim \Phi \sim M/R \ll 1 \tag{69}$$

(e.g. a white dwarf), the slow-motion approximation is automatically valid, and the slow-motion equations simplify. Most importantly, the fact that $y \sim (M/R)Y \ll Y$ (equation 65b) enables the equation of motion of the matter, (66a), to decouple from all gravitational fields

$$(\mu r^4 Y')' - (l+2)(l-1) r^2 \mu Y = -\omega^2 \rho r^4 Y. \tag{70}$$

This equation, together with the boundary conditions (67b, c)

$$\mu Y' \to 0 \quad \text{as} \quad r \to R_-, \qquad Y \sim r^{l-1} \quad \text{as} \quad r \to 0, \tag{71}$$

forms a Sturm–Liouville eigenvalue problem, which is well known and widely studied in the geophysics literature (e.g. Alterman, Jarosch & Pekeris 1959), and which can be solved by standard techniques. Once it has been solved, the Fourier amplitude of the l-pole moment can be computed from

$$F_\omega = \frac{-16\pi i\omega}{(l-1)(2l+1)!!}\int_0^R r^{l+3}\rho Y dr. \tag{72}$$

(This equation can be derived by setting $\Phi = \Lambda = 0$ in (66c), multiplying by r^{l+1}, integrating from $r = 0$ to $r = \infty$, using the asymptotic form (67a) of Q to evaluate the surface terms, and using the equation of motion (70) to rewrite the integral.) The imaginary part $i/2\tau$ of the eigenfrequency can then be evaluated using the energy method (equations 61, 62, 51c)

$$\tau = \bar{E}_{\text{star}}/\bar{S}^r, \tag{73a}$$

$$\bar{E}_{\text{star}} = \frac{\pi l(l+1)}{(2l+1)}\int_0^R [\omega^2\rho r^4 Y^2 + \mu r^4 Y'^2 + (l+2)(l-1)\mu r^2 Y^2]\, dr, \tag{73b}$$

$$\bar{S}^r = \frac{(l-1)l(l+1)(l+2)}{8(2l+1)}\omega^{2l+2}|F_\omega|^2. \tag{73c}$$

Notice that, aside from an angular factor, $\rho i\omega Yr$ is the density of momentum, i.e. of mass current; consequently F_ω (equation 72) is proportional to $\int r^l \times$ (mass current density) $\times$ (angular factor) $d$vol; i.e. in the language of Thorne (1980, especially equation 5.27b) F_ω is the Fourier amplitude of the star's current l-pole moment.

For the special case of a star with uniform density ρ and radially constant shear modulus μ the eigenequation (70) reduces to the spherical Bessel equation for rY; and consequently

$$Y \propto r^{-1} j_l(kr), \qquad k \equiv (\rho/\mu)^{1/2}\omega, \tag{74}$$

where j_l is the spherical Bessel function. The eigenfrequencies are fixed by the no-torque-at-surface boundary condition $Y'(R) = 0$ (equation 71). Straightforward calculations using standard Bessel-function identities then yield the following formulas for the star's oscillations and gravitational waves, in terms of the star's radius R, mass $M = 4\pi\rho R^3/3$, shear-wave velocity $v_s = (\mu/\rho)^{1/2}$, and amplitude of oscillations

$$\beta \equiv \text{(maximum value of angular displacement function } Y \text{ inside star).} \tag{75}$$

The nth normal mode (of given angular quantum number l) has eigenfrequency and wave-number

$$\omega_n = (v_s/R)x_n, \qquad k_n = x_n/R. \tag{76a}$$

The angular displacement of the star's crystal is

$$\begin{aligned}\delta\phi \equiv \xi^\phi &\equiv Yb^\phi \cos\omega_n t \\ &= \beta\frac{j_l(k_n r)}{\alpha k_n r}\frac{\partial_\vartheta P_l(\cos\vartheta)}{\sin\vartheta}\cos\omega_n t \\ &= -3\beta\frac{j_2(k_n r)}{\alpha k_n r}\cos\vartheta\cos\omega_n t \qquad \text{if} \qquad l = 2.\end{aligned} \tag{76b}$$

The star's energy of oscillation is

$$\bar{E}_{\text{star}} = E_n M v_s^2 \beta^2. \tag{76c}$$

Table 1. Constants governing quadrupole ($l = 2$) torsional oscillations of a star with uniform ρ and μ and with weak gravity.

n	x_n	E_n	G_n	L_n	D_n
1	2.5011	2.030	2.001	15.03	0.3379
2	7.1360	1.461	−1.062	34.45	0.3026
3	10.515	0.7171	0.7274	35.09	0.2148
4	13.772	0.4267	−0.5568	35.28	0.1666
5	16.983	0.2833	0.4519	35.35	0.1361

$\alpha = 0.10403$

The gravitational wave field has as its only non-zero components in an orthonormal, spherical basis

$$h^{\mathrm{TT}}_{\hat{\vartheta}\hat{\phi}} = G_n(M/r)v_s^{l+1}\beta \sin\vartheta\, b^{\phi}{}_{,\vartheta} \cos[\omega_n(t - r_*) + (l+1)\pi/2]$$
$$= 3G_n(M/r)v_s^3\beta \sin^2\vartheta \sin[\omega_n(t - r_*)] \quad \text{if} \quad l = 2. \tag{76d}$$

The power carried off by the waves is

$$\bar{S}^r = -\frac{d\bar{E}_{\text{star}}}{dt} = L_n(M/R)^2 v_s^{2l+4}\beta^2$$
$$= L_n(M/R)^2 v_s^8\beta^2 \quad \text{if} \quad l = 2. \tag{76e}$$

This power loss causes the energy to decay by $1/e$ in a number of oscillations given by

$$\omega_n\tau_n = D_n(M/R)^{-1}v_s^{-(2l+1)}$$
$$= D_n(M/R)^{-1}v_s^{-5} \quad \text{if} \quad l = 2. \tag{76f}$$

Here the constants $\alpha, x_n, E_n, G_n, L_n, D_n$ are given by

$$\alpha \equiv j_l(x_1)/x_1,$$

$$x_n \equiv n\text{th root of } \partial_x[j_l(x)/x] = 0,$$

$$E_n \equiv \frac{3l(l+1)}{4(2l+1)}\left[\frac{j_l(x_n)}{\alpha}\right]^2\left[1 - \frac{(l+2)(l-1)}{x_n^2}\right],$$

$$G_n \equiv \frac{12}{\alpha(l-1)(2l+1)!!}\, x_n^{l-1}\, j_{l+1}(x_n),$$

$$L_n \equiv \frac{18l(l+1)(l+2)}{\alpha^2(l-1)(2l+1)[(2l+1)!!]^2}\,[x_n^l\, j_{l+1}(x_n)]^2,$$

$$D_n \equiv E_n x_n/L_n. \tag{76g}$$

and are tabulated in Table 1 for $l = 2$.

5 Dipole torsional oscillations

We now turn attention to dipole torsional oscillations, i.e. oscillations with $l = 1$ (and, with only trivial loss of generality, $m = 0$). For $l \geq 2$ we used our gauge freedom to annul $h_{\vartheta\phi}$. For $l = 1$ $h_{\vartheta\phi}$ vanishes identically in all gauges because its angular dependence is $\sin^2\vartheta\, b^{\phi}{}_{,\vartheta} \equiv 0$. Thus, we can use our gauge freedom instead to annul $h_{r\phi}$ (i.e. to set $Q = 0$), thereby leaving us with only one non-zero metric perturbation

$$h_{t\phi} = h_{\phi t} = -r^2 \dot{y} b_\phi = r^2 \dot{y} \sin^2 \vartheta \tag{77}$$

(*cf.* equation 6b). The displacement function is defined as for $l \geqslant 2$

$$\xi^r = \xi^\vartheta = 0, \qquad \xi^\phi = Y b^\phi = -Y \tag{78}$$

(*cf.* equation 8b); and the Ricci tensor and stress-energy tensor then also have the same forms as for $l \geqslant 2$ (equations 7 and 17 with specialization to $l = 1$ and $Q = 0$).

Our equations of motion (19) for $l \geqslant 2$ were derived using the Einstein field equation $[\delta R_{\vartheta\phi} - 8\pi\delta(T_{\vartheta\phi} - 1/2\, T g_{\vartheta\phi})]/[\sin^2\vartheta\, b^\phi{}_{,\vartheta}]$ (i.e. $\epsilon_{\vartheta\phi} = 0$; equation A3). Because this equation is invalid for $l = 1$ (it involves dividing by $\sin^2\vartheta\, b^\phi{}_{,\vartheta} \equiv 0$), we cannot obtain the correct $l = 1$ equations by simply setting $Q = 0$ and $l = 1$ in (19). Rather, we must derive our equations of motion directly from the Einstein equations (A.1)–(A.5), with the omission of the $\epsilon_{\vartheta\phi}$ equation. The result is

$$(\rho + P)\, e^{-2\Phi}(\ddot{Y} - \ddot{y}) = r^{-4} e^{-\Phi - \Lambda}(\mu r^4 e^{\Phi - \Lambda} Y')', \tag{79a}$$

$$(\rho + P)\, e^{-2\Phi}(Y - y) = -(16\pi)^{-1} r^{-4} e^{-\Phi - \Lambda}(r^4 e^{-\Phi - \Lambda} y')'. \tag{79b}$$

A third Einstein equation, which is related to these two by the Bianchi identities, is

$$\dot{y}' = -16\pi\mu e^{2\Phi} Y'. \tag{80}$$

The equations of motion (79) are derivable from the action principle (37), (38) in which the Lagrangian density $\mathscr{L}$ is specialized to $l = 1$ and $Q \equiv 0$. The corresponding $l \geqslant 2$ conservation law $S^\alpha{}_{,\alpha} = 0$ is also valid for $l = 1$, with the S^α of equations (40) specialized to $l = 1$ and $Q \equiv 0$; and the proof in Section 3.5 that if $\mu \geqslant 0$ then the star is stable, which is based on the conservation law $S^\alpha{}_{,\alpha} = 0$, remains valid for $l = 1$.

Equation (80) implies that $\dot{y}$ is time-independent outside the star; and equation (79b) says that its radial dependence there is $\dot{y} = A + B/r^3$ (recall that $\Phi + \Lambda = 0$ in vacuum). The constant A is physically unacceptable, while the term B/r^3 describes the dragging of inertial frames by the star's constant angular momentum (see, e.g. Hartle 1967). With only trivial loss of generality we shall set the star's angular momentum to zero (i.e. we shall refuse to consider purely stationary, rotational perturbations), thereby enforcing $y \equiv 0$ everywhere outside the star. As a result, our oscillating star not only will produce no gravitational waves (a consequence of the dipole angular dependence of our perturbations); it will not have any gravitational perturbations whatsoever outside itself.

The eigenvalue problem for normal-mode oscillations with $l = 1$ consists of the coupled differential equations

$$(\mu r^4 e^{\Phi - \Lambda} Y')' = -\omega^2 r^4 e^{\Lambda - \Phi}(\rho + P)(Y - y), \tag{81a}$$

$$(r^4 e^{-\Phi - \Lambda} y')' = -16\pi r^4 e^{\Lambda - \Phi}(\rho + P)(Y - y) \tag{81b}$$

(equations 79 with $Y \propto e^{i\omega t}$ and $y \propto e^{i\omega t}$), together with the boundary conditions of smoothness and zero torque at the origin, the surface, and solid–fluid interfaces

$$Y \sim \text{constant} + O(r^2), \quad y \sim \text{constant} + O(r^2) \text{ near } r = 0, \tag{82a}$$

$$y, y' \quad \text{and} \quad \mu Y' \to 0 \quad \text{as} \quad r \to R_-, \tag{82b}$$

$$\mu Y' \to 0 \quad \text{at solid–fluid interfaces.} \tag{82c}$$

[Equation (82a) rules out the divergent solutions $Y \sim y^{-3}$ and $y \sim r^{-3}$; equation (82b) follows from $y \equiv 0$ outside the star and integrations of (79b) through the star's surface, and from (80) or (51b); equation (82c) follows from (53b) or from integrations of (80) through the interfaces.] The oscillation frequencies ω and eigenfunctions Y, y will be real since there is no gravitational radiation and no energy loss.

Note that the eigenequations (79) for $l = 1$ are identical to those of the $l \geqslant 2$ slow-motion approximation (equations 66a, b). Here the absence of retardation of the gravitational field y is due to its $l = 1$ angular dependence, which forbids gravitational radiation. There the absence of retardation and of waves was due to the slow-motion assumption. Here, as there, an action principle for the eigenvalue problem is given by $\delta\Omega^2 = 0$, where $\Omega^2 \equiv A/B$ with A and B given by expressions (58b, c) with $R_\infty = R_-$, the surface term removed, Q set to zero, and l set to one. For $l = 1$ this action principle does not require slow motion, and a slow-motion assumption produces no simplifications.

For a star with weak internal gravity the dipole eigenvalue problem (81), (82) simplifies to (70), (71) specialized to $l = 1$. When the star is homogeneous with ρ and μ constant, that eigenvalue problem has the analytic solution (74), (75), (76a, b, c, g) specialized to $l = 1$. [For $l = 1$ the gravitational-wave related equations (72), (73a, c), (76d, e, f) are irrelevant and incorrect.]

6 Concluding remarks

It should be straightforward to use the formalisms of this paper to evaluate numerically the characteristics of normal-mode torsional oscillations of neutron star models. Such calculations should be performed both to improve the approximate formulas given in the introduction of this paper (equations 3 and 4) and to discover quantitatively how the physical properties of neutron star matter influence a star's normal-mode frequencies, damping times and gravity-wave strengths.

References

Alterman, Z., Jarosch, H. & Pekeris, C. L., 1959. *Proc. R. Soc.,* **A252**, 80.

Bardeen, J. M., Press, W. H. & Teukolsky, S. A., 1972. *Astrophys. J.,* **178**, 347.

Baym, G. & Pethick, C., 1975. *A. Rev. Nucl. Sci.,* Vol. 25, 27.

Baym, G. & Pethick, C., 1979. *A. Rev. Astr. Astrophys.,* **17**, 415.

Boughn, S. P., Fairbank, W. M., Giffard, R. P., Hollenhorst, J. N., Mapoles, E. R., McAshan, M. S., Michelson, P. F., Paik, H. J. & Taber, R. C., 1982. *Astrophys. J.,* **261**, L19.

Campolattaro, A. & Thorne, K. S., 1970. *Astrophys. J.,* **159**, 847.

Carter, B., 1973a. *Phys. Rev.,* **7**, 1590.

Carter, B., 1973b. *Commun. Math. Phys.,* **30**, 261.

Carter, B. & Quintana, H., 1972. *Proc. R. Soc. Lond.,* **A331**, 57.

Chandrasekhar, S., 1964a. *Phys. Rev. Lett.,* **12**, 114 & 437.

Chandrasekhar, S., 1964b. *Astrophys. J.,* **140**, 417.

Chandrasekhar, S. & Friedman, J., 1973. *Astrophys. J.,* **175**, 379.

Detweiler, S. L., 1975. *Astrophys. J.,* **197**, 203.

Detweiler, S. L. & Ipser, J. R., 1973. *Astrophys. J.,* **185**, 685.

Drever, R. W. P., Hough, J., Munley, A. J., Lee, S.-A., Spero, R., Whitcombe, S. E., Ward, H., Ford, G. M., Hereld, M., Robertson, N. A., Kerr, I., Pugh, J. R., Newton, G. P., Meers, B., Brooks, E. D. & Gursel, Y., 1982. In *Quantum Optics, Experimental Gravitation, and Measurement Theory,* eds Meystre, P. & Scully, M. O. Proceedings of NATO Advanced Study Institute, Bad Windsheim, West Germany, August 17–30, 1981, Plenum Press, London.

Dyson, F., 1972. Unpublished remarks at the Sixth Texas Symposium on Relativistic Astrophysics, New York City, December 1972.

Friedman, J. L. & Schutz, B. F., 1975. *Astrophys. J.*, **200**, 204.
Hansen, C. J., ed., 1974. *Physics of Dense Matter, IAU Symp. No. 53*, Reidel, Dordrecht, Holland.
Hansen, C. J. & Cioffe, D. F., 1980. *Astrophys. J.*, **238**, 740.
Harrison, B. K., Thorne, K. S., Wakano, M. & Wheeler, J. A., 1965. *Gravitation Theory and Gravitational Collapse*, University of Chicago Press, Chicago.
Hartle, J. B., 1967. *Astrophys. J.*, **150**, 1005.
Hartle, J. B. & Thorne, K. S., 1968. *Astrophys. J.*, **153**, 807.
Ipser, J. R., 1971. *Astrophys. J.*, **166**, 175.
Ipser, J. R. & Thorne, K. S., 1973. *Astrophys. J.*, **181**, 181.
Landau, L. D. & Lifshitz, E. M., 1970. *Theory of Elasticity*, Pergamon Press, London.
LaSalle, J. & Lefschetz, S., 1961. *Stability by Liapunov's Direct Method*, Academic Press, New York.
Misner, C. W., Thorne, K. S. & Wheeler, J. A., 1973. *Gravitation*, W. H. Freeman and Co., San Francisco, (MTW).
Moncrief, V., 1974. *Annals of Physics*, **88**, 343.
Pandharipande, V. R., Pines, D. & Smith, R. A., 1976. *Astrophys. J.*, **208**, 550.
Pines, D., Shaham, J. & Ruderman, M. A., 1974. In *Physics of Dense Matter, IAU Symp. No. 53*, p. 189, ed. Hansen, C. J., Reidel, Dordrecht, Holland.
Pirani, F. A. E., 1965. *Lectures on General Relativity*, p. 249, eds Trautman, A., Pirani, F. A. E. & Bondi, H., Brandeis 1964 Summer Institute of Theoretical Physics, Prentice-Hall, New Jersey.
Price, R. H. & Thorne, K. S., 1969. *Astrophys. J.*, **155**, 163.
Regge, T. & Wheeler, J. A., 1957. *Phys. Rev.*, **108**, 1063.
Ruderman, M. A., 1968. *Nature*, **218**, 1128.
Taub, A. H., 1969. *Commun. Math. Phys.*, **15**, 235.
Thorne, K. S., 1969a. *Astrophys. J.*, **158**, 1.
Thorne, K. S., 1969b. *Astrophys. J.*, **158**, 997.
Thorne, K. S., 1980. *Rev. Mod. Phys.*, **52**, 299.
Thorne, K. S. & Campolattaro, A., 1967. *Astrophys. J.*, **149**, 591 and **152**, 673.
Van Horn, H. M., 1980. *Astrophys. J.*, **236**, 899.

Appendix A: Equations of motion

In deriving the equations of motion (19a, b, c) we shall denote by

$$\epsilon_{\alpha\beta} \equiv [\delta R_{\alpha\beta} - 8\pi\delta(T_{\alpha\beta} - 1/2\, Tg_{\alpha\beta})]/f_{\alpha\beta}(r, \vartheta), \qquad \epsilon_T \equiv T^{\alpha}_{\phi;\alpha}/f_T(r, \vartheta) \tag{A.1}$$

the expressions obtained by combining equations (5), (7), (18), (6) and (17) and dividing by the functions

$$f_{t\phi} = -1/2\, r^2 e^{-2\Lambda} b_\phi, \qquad f_{\vartheta\phi} = 1/2\, r^2 e^{-2\Phi} \sin^2\vartheta\, b^{\phi}{}_{,\vartheta},$$

$$f_{r\phi} = -1/2\, r e^{\Lambda - 3\Phi} b_\phi, \quad f_T = r^2 e^{-2\Phi} b_\phi.$$

These expressions are:

$$\epsilon_{t\phi} = \partial_t \left\{ \frac{-e^{\Phi+\Lambda}}{r^4} (r^4 e^{-\Phi-\Lambda} y')' + e^{2\Lambda} \left[16\pi(\rho + P) + \frac{(l+2)(l-1)}{r^2} \right] y \right.$$
$$\left. - 16\pi(\rho + P) e^{2\Lambda} Y + \frac{e^{\Phi+\Lambda}}{r^4} (r^3 e^{-2\Phi} Q)' \right\}; \tag{A.2}$$

$$\epsilon_{\vartheta\phi} = \ddot{y} - \frac{e^{\Phi-\Lambda}}{r^2} (rQ)' + 16\pi\mu e^{2\Phi} Y; \tag{A.3}$$

$$\epsilon_{r\phi} = \ddot{Q} + e^{2\Phi} \left[16\pi\mu + \frac{(l+2)(l-1)}{r^2} \right] Q - r e^{\Phi-\Lambda} \ddot{y}' - 16\pi\mu e^{3\Phi-\Lambda} rY'; \tag{A.4}$$

$$\epsilon_T = (\rho + P)\ddot{Y} - r^{-4} e^{\Phi-\Lambda} (\mu r^4 e^{\Phi-\Lambda} Y')' + (l+2)(l-1) r^{-2} e^{2\Phi} \mu Y$$
$$- (\rho + P)\ddot{y} + r^{-4} e^{\Phi-\Lambda} (\mu r^3 Q)'. \tag{A.5}$$

Note that ϵ_T can be expressed as the following combination of the $\epsilon_{\alpha\beta}$ (Bianchi identity):

$$\epsilon_T = (16\pi)^{-1} [(l+2)(l-1)r^{-2}\epsilon_{\vartheta\phi} - e^{-2\Lambda}\dot{\epsilon}_{t\phi} + r^{-4}e^{\Phi-\Lambda}(r^3 e^{-2\Phi}\epsilon_{r\phi})']. \tag{A.6}$$

The perturbed Einstein field equations are $\epsilon_{\alpha\beta} = 0$; the law of conservation of energy–momentum for the perturbed system, $\delta T^{\alpha\beta}{}_{;\beta} = 0$, reduces to the single equation $\epsilon_T = 0$.

The equations of motion (19a, b, c) used in the text are the following combinations of field equations:

equation (19a), initial-value equation for y:

$$\int \epsilon_{t\phi}\, dt = 0; \tag{A.7}$$

equation (19b), wave-equation for Y:

$$e^{-2\Phi}[\epsilon_T + (\rho + P)\epsilon_{\vartheta\phi}] = 0; \tag{A.8}$$

equation (19c), wave-equation for Q:

$$e^{-2\Phi}[\epsilon_{r\phi} + re^{\Phi-\Lambda}\epsilon_{\vartheta\phi}{}'] = 0. \tag{A.9}$$

We must show that our equations of motion (19) are complete, i.e. that all physically acceptable solutions of (19) also satisfy the full set of perturbed Einstein equations $\epsilon_{\alpha\beta} = 0$ and the equation of energy–momentum conservation $\epsilon_T = 0$. To prove this, we combine the equations of motion (19) with the Bianchi identity (A.6) to obtain the Sturm–Liouville equation

$$r^{-4}e^{\Phi+\Lambda}(r^4 e^{-\Phi-\Lambda}\epsilon_{\vartheta\phi}{}')' - e^{2\Lambda}[16\pi(\rho+P) + (l+2)(l-1)r^{-2}]\epsilon_{\vartheta\phi} = 0 \tag{A.10}$$

for $\epsilon_{\vartheta\phi}$. This equation, together with (19), leads to perturbation functions Y, Q, y which satisfy the physical boundary conditions (26a, d, e) only if $\epsilon_{\vartheta\phi} \sim r^{l-1}$ near $r = 0$ and $\epsilon_{\vartheta\phi} \sim r^{-l-2}$ near $r = \infty$. However, the signs of the terms in (A.10) make it impossible for these two asymptotic formulae to join on to each other except in the case $\epsilon_{\vartheta\phi} = 0$. From this we conclude that our equations of motion and boundary conditions imply $\epsilon_{\vartheta\phi} = 0$; this, together with the equations of motion themselves, implies trivially that all the $\epsilon_{\alpha\beta}$ and ϵ_T vanish (*cf.* equations A.6–A.9). QED.

Appendix B: Foundations for action principles

Friedman & Schutz (1975; their section II) have given an elegant formulation of the general theory of action principles for systems which can radiate waves to infinity. Unfortunately, their analysis was not carried far enough to embrace the Detweiler–Ipser (1973) type of action principle for normal-mode pulsations, which we use in Sections 3.4 and 4.3. In this appendix we extend the Friedman–Schutz analysis to encompass such action principles, and we use it to derive various results presented in the text of the paper. The general theory is presented with full-left margins; the application to torsionally oscillating stars is presented indented.

Consider a system described by functions Z_A (A = 1, 2, ..., n) in a spacetime with coordinates x^α ($\alpha = 0, 1, 2, \ldots, m$). Assume that the equations of motion for Z_A are derivable from an action principle

$$\delta \int_\Omega \mathscr{L} dx^0 \ldots dx^m = 0, \tag{B.1}$$

where $\delta Z_A = 0$ on $\partial\Omega$. Assume that the Lagrangian density $\mathscr{L}$ is quadratic and symmetric, i.e. $\mathscr{L} = L(Z, Z)$ with

$$L(Z^\dagger, Z) \equiv \sum_{k=1}^{p} \sum_{l=1}^{p} \Lambda^{AB\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_l} Z^\dagger_{A,\alpha_1 \ldots \alpha_k} Z_{B,\beta_1 \ldots \beta_l}, \tag{B.2}$$

where $\Lambda^{AB\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_l}$ is a function of the coordinates x^μ which is completely symmetric in the indices $\alpha_1 \ldots \alpha_k$, completely symmetric in the $\beta_1 \ldots \beta_l$ and also symmetric under interchange of $A\alpha_1 \ldots \alpha_k$ with $B\beta_1 \ldots \beta_l$ (so $L(Z^\dagger, Z) = L(Z, Z^\dagger)$). The $Z^\dagger_A$ are a set of functions which have no special relationship to the Z_A, and $Z^\dagger_{A,\alpha_1 \ldots \alpha_k} \equiv \partial^k Z^\dagger_A / \partial x^{\alpha_1} \cdots \partial x^{\alpha_k}$. The quantity p is the maximum number of derivatives that appear in the Lagrangian, and there is an implied summation over repeated function indices A, B as well as coordinate indices $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l$. Define

$$L^A(Z) \equiv \sum_l (-1)^l \partial_{\alpha_1} \ldots \partial_{\alpha_l} \frac{\partial L(Z^\dagger, Z)}{\partial Z^\dagger_{A,\alpha_1 \ldots \alpha_l}}. \tag{B.3}$$

Then the Euler–Lagrange equations are $L^A(Z) = 0$, and the integration-by-parts identity used in deriving these Euler–Lagrange equations from the action principle (B.1) is

$$Z^\dagger_A L^A(Z) = L(Z^\dagger, Z) - \partial_\mu Q^\mu(Z^\dagger, Z). \tag{B.4}$$

The Q^μ are determined only up to a divergence-free vector. Two versions of Q^μ, which differ from each other by a divergence-free vector, are

$$Q^\mu(Z^\dagger, Z) = \sum_{k,l} (-1)^l Z^\dagger_{A,\beta_1 \ldots \beta_k} \partial_{\alpha_1} \ldots \partial_{\alpha_l} \frac{\partial L(Z^\dagger, Z)}{\partial Z^\dagger_{A,\alpha_1 \ldots \alpha_l \beta_1 \ldots \beta_k \mu}}; \tag{B.5a}$$

and

$$Q^0(Z^\dagger, Z) = \sum_{j=1}^{p} \sum_{k=0}^{p} \sum_{l=1}^{j} (-1)^{l-1} \binom{j+k}{j} Z^{\dagger\,(j-l)}_{A,a_1 \ldots a_k} \left(\frac{\partial L}{\partial Z^{\dagger\,(j)}_{A,a_1 \ldots a_k}} \right)^{(l-1)}, \tag{B.5b}$$

$$Q^b(Z^\dagger, Z) = \sum_{j=0}^{p} \sum_{k=1}^{p} \sum_{l=1}^{k} (-1)^{j+k-l} \binom{j+k}{j} Z^\dagger_{A,a_1 \ldots a_{l-1}} \left(\frac{\partial L}{\partial Z^{\dagger\,(j)}_{A,a_1 \ldots a_{k-1} b}} \right)^{(j)}_{,a_l \ldots a_{k-1}}. \tag{B.5c}$$

The second version (equations B5b, c) has the virtue that the time component $Q^0(Z^\dagger, Z)$ contains the lowest possible number of spatial derivatives of the Z_A; it is the version which we use in our analysis of torsional oscillations of stars. In the second version the Latin letters b and $a_1, \ldots, a_p$ denote spatial tensorial indices and run from 1 to m;

$$\binom{j+k}{j}$$

is the binomial coefficient; and superscripts in parentheses denote time derivatives as in the text: $Z^{\dagger(j-l)}_{A,a_1 \ldots a_k} = Z^\dagger_{A,a_1 \ldots a_k 0 \ldots 0}$ with $j-l$ zeros. It is imperative when using equations (B.3), (B.5a–c) and others below that $L(Z^\dagger, Z)$ be properly symmetrized (including making a careful distinction, e.g., between $Z_{A,10}$ and $Z_{A,01}$ and symmetrizing L in them); *cf.* discussion following equation (B.2). Failure to symmetrize will produce in (B.3), (B.5 a–c) multiple counting of second and higher-order derivative terms.

For our torsionally oscillating star the coordinates are $x^0 = t$, $x^1 = r$; the functions Z_A are Y, y, Q; the Lagrangian density $\mathscr{L}$ is equation (37). From the Lagrangian density we can read off

$$L(Z^\dagger, Z) = \frac{2\pi l(l+1)}{(2l+1)}\Big[(\rho+p)r^4 e^{\Lambda-\Phi}(\dot{Y}^\dagger - \dot{y}^\dagger)(\dot{Y}-\dot{y})$$

$$+\frac{r^4 e^{-\Phi-\Lambda}}{16\pi}(\dot{y}^{\dagger\prime} - e^{\Lambda-\Phi}\dot{Q}^\dagger/r)(\dot{y}' - e^{\Lambda-\Phi}\dot{Q}/r) + \frac{(l+2)(l-1)}{16\pi}r^2 e^{\Lambda-\Phi}\dot{y}^\dagger\dot{y}$$

$$-\mu r^4 e^{\Phi-\Lambda}(Y^{\dagger\prime} - e^{\Lambda-\Phi}Q^\dagger/r)(Y' - e^{\Lambda-\Phi}Q/r) - \mu(l+2)(l-1)r^2 e^{\Phi+\Lambda}Y^\dagger Y$$

$$-(1/16\pi)(l+2)(l-1)e^{\Lambda-\Phi}Q^\dagger Q\Big]. \tag{B.6}$$

The Euler–Lagrange expressions $L^A(Z)$, obtained from expression (B.3) (in which one must take careful account of the symmetry properties of L) or by varying the action, are the following:

$$L^Y(Z) = \frac{-2\pi l(l+1)}{(2l+1)} r^4 e^{\Lambda-\Phi}\epsilon_T,$$

$$L^y(Z) = \frac{-l(l+1)}{8(2l+1)} r^4 e^{-\Phi-\Lambda}\dot{\epsilon}_{t\phi}.$$

$$L^Q(Z) = \frac{-l(l+1)}{8(2l+1)} r^2 e^{\Lambda-3\Phi}\epsilon_{r\phi}, \tag{B.7}$$

where ϵ_T, $\epsilon_{t\phi}$ and $\epsilon_{r\phi}$ are the Einstein field-equation expressions given in Appendix A. One of us (BLS) originally derived the Lagrangian density $\mathscr{L}$ by constructing the expression $Z_A^\dagger L^A(Z)$, by adding a perfect divergence (equation B.4) and by then setting $Z_A^\dagger = Z_A$ (method of Chandrasekhar 1964a, b; Detweiler & Ipser 1973). For the quantities Q^μ which appear in the divergence, we shall use expressions (B.5b, c) because they lead to a Q^0 (and subsequently S^0) which contain only first derivatives of Y, y and Q:

$$Q^0(Z^\dagger, Z) = \frac{2\pi l(l+1)}{(2l+1)}\Big[(\rho+p)r^4 e^{\Lambda-\Phi}(Y^\dagger - y^\dagger)(\dot{Y}-\dot{y})$$

$$+\frac{r^4 e^{-\Phi-\Lambda}}{16\pi}(y^{\dagger\prime} - e^{\Lambda-\Phi}Q^\dagger/r)(\dot{y}' - e^{\Lambda-\Phi}\dot{Q}/r) \tag{B.8a}$$

$$+\frac{(l+2)(l-1)}{16\pi}r^2 e^{\Lambda-\Phi}y^\dagger\dot{y}\Big],$$

$$Q^r(Z^\dagger, Z) = \frac{-2\pi l(l+1)}{(2l+1)}\Big[\frac{r^4 e^{-\Phi-\Lambda}}{16\pi}y^\dagger(\ddot{y}' - e^{\Lambda-\Phi}\ddot{Q}/r)$$

$$+\mu r^4 e^{\Phi-\Lambda}Y^\dagger(Y' - e^{\Lambda-\Phi}Q/r\Big]. \tag{B.8b}$$

Whenever the Lagrangian is stationary in the sense that

$$[\partial L(Z^\dagger, Z)/\partial x^0]_{Z^\dagger,\, Z\text{ held fixed}} = 0, \tag{B.9a}$$

the Euler–Langrange equations enforce a law of energy conservation:

$$L^A(Z) = 0 \text{ and (B.9a) imply that } \partial_\mu S^\mu = 0, \tag{B.10a}$$

where

$$S^\mu \equiv 2Q^\mu(\dot{Z}, Z) - \delta_0^\mu L(Z, Z). \tag{B.10b}$$

The arbitrariness in Q^μ (freedom to add any divergence-free vector) produces a corresponding arbitrariness in S^μ.

Our Lagrangian (B.6) is stationary. From our chosen form (B.8) for Q^μ and expression (B.6) for L we derive expressions (40a, b) for our energy density, S^0 and energy flux, S^r. Had we chosen any other Q^μ, the resulting energy density, S^0 would not have been equal to the Lagrangian with sign reversal of the potential energy terms.

We now turn attention to functions Z_A with exponential and sinusoidal time dependence $Z_A(x^\alpha) \equiv z_A(x^j)e^{i\omega t}$ ($j = 1, 2, \ldots, m$; ω a complex frequency) and we decompose L, L^A, and Q^μ into powers of ω:

$$L(z^\dagger e^{-i\omega t}, ze^{i\omega t}) = \omega^n L_n(z^\dagger, z), \qquad L^A(ze^{i\omega t}) = \omega^n L_n^A(z)\, e^{i\omega t},$$
$$Q^\mu(z^\dagger e^{-i\omega t}, ze^{i\omega t}) = \omega^n Q_n^\mu(z^\dagger, z), \tag{B.11}$$

where there is an implied summation over the integer n. In our discussion we shall require that L be stationary (equation B.9a); this guarantees the existence of solutions with $e^{i\omega t}$ time dependence. We shall also require that L_n contain only even powers of ω

$$L_n(z^\dagger, z) = 0 \qquad \text{for} \qquad n \text{ odd}; \tag{B.9b}$$

this, together with symmetry of L $[L(Z^\dagger, Z) = L(Z, Z^\dagger)]$ and definition (B.11) of L_n, implies that L_n is symmetric

$$L_n(z^\dagger, z) = L_n(z, z^\dagger). \tag{B.12}$$

Note that the fundamental identity (B.4) implies that

$$z_A^\dagger L_n^A(z) = L_n(z^\dagger, z) - \partial_j Q_n^j(z^\dagger, z). \tag{B.13}$$

For $Z = ze^{i\omega t}$ our equations of motion $L^A(Z) = 0$ reduce to the eigenequation

$$\omega^n L_n^A(z) = 0 \text{ if and only if } z \text{ is an eigenfunction and } \omega \text{ is its eigenvalue; i.e. if and only if } ze^{i\omega t} \text{ is a normal mode.} \tag{B.14}$$

We shall be interested in normal modes which are defined on a compact region $\mathscr{V}$ of space (not spacetime). Then the identity (B.13) together with the symmetry condition (B.12) implies the following action principle: *Define* $\omega(\mathrm{z})$ *by* $\mathrm{I}(\omega, \mathrm{z}) = 0$ *where*

$$\begin{aligned} I(\omega, z) &= \omega^n \int_{\mathscr{V}} L_n(z, z)\, d^m x - \omega^n \int_{\partial\mathscr{V}} Q_n^j(z, z)\, d^{m-1}\Sigma_j \\ &= \int_{\mathscr{V}} L(ze^{-i\omega t}, ze^{i\omega t})\, d^m x - \int_{\partial\mathscr{V}} Q^j(ze^{-i\omega t} ze^{i\omega t})\, d^{m-1}\Sigma_j \\ &= \omega^n \int_{\mathscr{V}} z_A L_n^A(z)\, d^m x. \end{aligned} \tag{B.15}$$

In general there will be several roots $\omega(\mathrm{z})$. *Consider each root in turn. The eigenfunctions* $\mathrm{z_A}$ *are those for which* $\omega(\mathrm{z})$ *is stationary under small perturbations* $\delta \mathrm{z_A}$, *with*

$$\omega^n \int_{\partial\mathscr{V}} [Q_n^j(z, \delta z) - Q_n^j(\delta z, z)]\, d^{m-1}\Sigma_j = 0. \tag{B.16}$$

Note that constraint (B.16) corresponds to certain combinations of the z_A and their derivatives being held fixed on $\partial \mathscr{V}$. This constraint ensures that the Euler–Lagrange equations associated with the action principle are $\omega^n L_n^A(z) = 0$ (equation B.14).

For our torsionally oscillating star we choose $\mathscr{V}$ to be the interior of a sphere, $r \leqslant R_\infty$ with boundary R_∞ far out in the radiation zone. Then the function $I(\omega, z)$ is easily evaluated from equations (B.6) and (B.8b) for L and Q^r

$$I(\omega, z) = \frac{2\pi l(l+1)}{(2l+1)} (\omega^2 A - B), \tag{B.17}$$

where A and B are expressions (58b, c); and the constraint (B.16) on Q, y, Y is easily evaluated from (B.8b)

$$\omega^2 \frac{2\pi l(l+1)}{(2l+1)} \frac{r^2}{16\pi} y^2 \delta(\chi/y) = 0 \quad \text{at} \quad r = R_\infty. \tag{B.18}$$

Here $\chi = -r^2 e^{-\Phi-\Lambda}(y' - e^{\Lambda-\Phi} Q/r)$ (equation 57b). The action principle thus consists of extremizing $\omega^2 = B/A$ with respect to variations of Q, y, Y, with χ/y held fixed at R_∞.

The initial-value equation (19a) for y is one of the Euler–Lagrange equations of this action principle. Because it is independent of ω, (19a) can be imposed as a constraint on all trial functions before the action is varied. The normal modes obviously will still give stationary ω, and one can verify that this procedure does not introduce any spurious solutions – only the normal modes give stationary ω. This is the version of the action principle presented in the text (Section 4.3).

Assume that $L(Z^\dagger, Z)$ is 'real' in the sense that $L(Z^{\dagger *}, Z^*) = [L(Z^\dagger, Z)]^*$, where * denotes complex conjugation. Then if $Z = z e^{i\omega t}$ is a solution of the Euler–Lagrange equations, $Z^* = z^* e^{-i\omega^* t}$ will also be a solution; and from the complex solution $z e^{i\omega t}$ we can build a real solution

$$Z = 1/2\,(z e^{i\omega t} + z^* e^{-i\omega^* t}), \qquad \omega \equiv \sigma + i/2\tau. \tag{B.19}$$

If we insert this real solution into expression (B.10b) for S^μ we obtain

$$S^\mu = \bar{S}^\mu e^{-t/\tau} + \tilde{S}^\mu \cos(2\sigma t + \vartheta^\mu) e^{-t/\tau}, \tag{B.20}$$

where

$$\bar{S}^0 = \{\mathrm{Im}\,[\omega^* Q^0(z^* e^{-i\omega^* t}, z e^{i\omega t})] - 1/2\,\mathrm{Re}\,[L(z^* e^{-i\omega^* t}, z e^{i\omega t})]\}\, e^{t/\tau}, \tag{B.21a}$$

$$\bar{S}^j = \mathrm{Im}\,[\omega^* Q^j(z^* e^{-i\omega^* t}, z e^{i\omega t})]\, e^{t/\tau}, \tag{B.21b}$$

and where $\tilde{S}^\mu$ is not of interest to us. In the law of energy conservation $\partial_\mu S^\mu = 0$, the pure exponential terms and the sinusoidal terms must be conserved separately. It is the pure exponential terms that interest us; for them, energy conservation says

$$(1/\tau)\bar{S}^0 = \bar{S}^j{}_{,j}; \tag{B.22}$$

integration over the spatial region $\mathscr{V}$ implies

$$(1/\tau) \int_{\mathscr{V}} \bar{S}^0 d^m x = \int_{\partial \mathscr{V}} \bar{S}^j d^{m-1} \Sigma_j. \tag{B23}$$

For our torsional oscillations $\bar{S}^0$ and $\bar{S}^r$, as computed from equations (B.21), (B.6) and (B.8), are the expressions given in equations (62), where $\bar{E}_{\text{star}} = \int \bar{S}^0 dr$.