Torus actions and combinatorics of polytopes

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Introduction

In this work we develop the study of relationship between the algebraic topology of manifolds and the combinatorics of polytopes. Originally, this research was inspired by the results of the toric variety theory. The main object of our study is the smooth manifold defined by the combinatorial structure of a simple polytope. This manifold is equipped with natural action of the compact torus T^m.

We define an n-dimensional *convex polytope* as a bounded set in $\mathbb{R}^n$ that is obtained as the intersection of a finite number of half-spaces. So, any convex polytope is bounded by a finite number of hyperplanes. A convex n-dimensional polytope is called *simple* if there exactly n codimension-one faces (or *facets*) meet at each vertex. The bounding hyperplanes of a simple polytope are in general position at each vertex. A convex polytope could be also defined as the convex hull of a set of points in $\mathbb{R}^n$. In this situation, if points are in general position, the resulting polytope is called *simplicial*, since all its faces are simplices. For each simple polytope there defined its *dual* (or *polar*) simplicial polytope and vise versa (see definition 1.3). Sometimes it is more convenient to study the properties of a simple polytope in terms of its dual simplicial one or even in terms of the corresponding boundary simplicial subdivision of a sphere.

We associate to each simple polytope P^n with m facets a smooth $(m+n)$-dimensional manifold $\mathcal{Z}_P$ with the canonical action of the compact torus T^m. A number of manifolds playing an important role in the different aspects of topology, algebraic and symplectic geometry appear as the special cases of the above manifolds $\mathcal{Z}_P$, or as the quotients $\mathcal{Z}_P/T^k$ for toric subgroups $T^k \subset T^m$ acting on $\mathcal{Z}_P$ freely. It turns out that the maximal rank of a torus subgroup that can act on $\mathcal{Z}_P$ freely equals $m-n$. We call the quotients of $\mathcal{Z}_P$ by tori of maximal possible rank $m-n$ *quasitoric manifolds*. The name refers to the fact that the important class of algebraic varieties known to algebraic geometers as *toric manifolds* fits the above picture. More precisely, one can use the above construction (i.e., the quotient of $\mathcal{Z}_P$ by a torus subgroup) to produce all smooth projective *toric varieties* (cf. [Da]), which we refer to as *toric manifolds*. On a (quasi)toric manifold there is defined the induced action of the torus T^n, whose orbit space is the original simple polytope P^n. However, one can find a simple polytope P that can not be realized as the orbit space for a quasitoric (and also toric) manifold. This means exactly that for this P it is impossible to find a torus subgroup $T^{m-n} \subset T^m$ of rank $m-n$ that acts on the corresponding manifold $\mathcal{Z}_P$ freely. If the manifold $\mathcal{Z}_P$ defined by a polytope P allows the free action of a torus subgroup of rank $m-n$, then the different subgroups of this type can produce different quasitoric manifolds over P^n, and some of them may turn out to be toric manifolds. Originally, the quasitoric manifolds (under the name "toric manifolds") appeared in [DJ], where the different topological properties of them were described.

Our approach to constructing manifolds defined by simple polytopes is based on a construction from algebraic geometry used in [Ba] for studying toric varieties. Namely, the lattice of faces of a simple polytope

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P^n defines a certain affine algebraic set $U(P^n) \subset \mathbb{C}^m$ with the action of algebraic torus $(\mathbb{C}^*)^m$. This set $U(P^n)$ is the complement to a certain collection of affine planes in $\mathbb{C}^m$ defined by the combinatorics of P^n. Toric manifolds appear when one can find a subgroup $D \subset (\mathbb{C}^*)^m$ isomorphic to $(\mathbb{C}^*)^{m-n}$ that acts on $U(P^n)$ freely. The crucial fact in our approach is that it is *always* possible to find a subgroup $R \subset (\mathbb{C}^*)^m$ isomorphic to $(\mathbb{R}_+^*)^{m-n}$ and acting freely on $U(P^n)$. In this case one can define the quotient manifold, which we refer to as the manifold defined by simple polytope P^n. There is the canonical action of the torus T^m on this manifold, namely the one induced from the standard action of T^m on $\mathbb{C}^m$ by diagonal matrices. The another approach to constructing manifolds defined by simple polytopes was proposed in [DJ], where these manifolds where defined as the quotient spaces $\mathcal{Z}_P = T^m \times P^n / \sim$ for an equivalence relation $\sim$. We construct the equivariant embedding i_e of this manifold into $U(P^n) \subset \mathbb{C}^m$ and show that for any subgroup $R \simeq (\mathbb{R}_+^*)^{m-n}$ of the above described type the composition $\mathcal{Z}_P \to U(P^n) \to U(P^n)/R$ of embedding and orbit map is a homeomorphism. Hence, from the topological viewpoint, both approaches produse the same manifold. This is what we will refer to as the manifold defined by simple polytope P^n and denote $\mathcal{Z}_P$.

The analysis of the above constructions shows that one can replace the m-dimensional complex space $\mathbb{C}^m \simeq (\mathbb{R}^2)^m$ with the space $(\mathbb{R}^k)^m$ for arbitrary k. Indeed, we may consider the open subset $U(P^n) \subset (\mathbb{R}^k)^m$ determined by the lattice of faces of P^n as in the case of $\mathbb{C}^m$ (i.e. $U(P^n)$ is obtained by taking off a certain set of affine planes as in definition 2.7). The multiplicative group $(\mathbb{R}_+^*)^m$ acts on $(\mathbb{R}^k)^m$ diagonally (i.e. as the product of m standard diagonal actions of $\mathbb{R}_+^*$ on $\mathbb{R}^k$). As before, it is possible for this action to find a subgroup $R \subset (\mathbb{R}_+^*)^m$ isomorphic to $(\mathbb{R}_+^*)^{m-n}$ that acts on $U(P^n)$ freely. The corresponding quotient $U(P^n)/R$ is now of dimension $(k-1)m+n$ and is invested with the action of the group $O(k)^m$ (the product of m copies of the orthogonal group), which is induced by the diagonal action of this group on $(\mathbb{R}^k)^m$. In the case $k=2$ the above considered action of the torus T^m is exactly the action of the subgroup $SO(2)^m \subset O(2)^m$. In the case $k=1$ we obtain for any simple polytope P^n a smooth n-dimensional manifold $\mathcal{Z}^n$ with an action of the group $(\mathbb{Z}/2)^m$, whose orbit space is P^n. This manifold is known as the universal Abelian cover of P^n regarded as a right-angled Coxeter orbifold (or manifold with corners). The analogue of quasitoric manifolds in the case $k=1$ are the so-called *small covers*. These are manifolds M^n with action of $(\mathbb{Z}/2)^n$ whose orbit space is P^n. The name refers to the fact that any cover of P^n by a manifold must have at least 2^n sheets. All these questions related to the case $k=1$ were detailedly treated in [DJ] along with the quasitoric manifolds. The another case of particular interest is $k=4$, since the space $\mathbb{R}^4$ can be regarded as the one-dimensional quaternionic space. In this work we give the detailed treatment of the case $k=2$ and all constructions below relate to this case.

One of our main goals here is to study the relationship between the combinatorial structure of simple polytopes and topology of the above described manifolds defined by these polytopes. There is a well-known important algebraic invariant of a simple polytope: a graded ring $k(P)$ (here k is any field) called *the face ring* (cf. [St]). This is the quotient ring of the polynomial ring $k[v_1,\ldots,v_m]$ by a certain homogeneous ideal determined by the lattice of faces of a polytope (see definition 1.1). Then one can introduce the corresponding cohomology modules $\operatorname{Tor}^{-i}_{k[v_1,\ldots,v_m]}(k(P),k)$, where $i>0$. These modules are of great interest to algebraic combinatorists; some results about the corresponding Betti numbers $\beta^i(k(P)) = \dim_k \operatorname{Tor}^{-i}_{k[v_1,\ldots,v_m]}(k(P),k)$ can be found in [St]. We show that the bigraded k-module $\operatorname{Tor}_{k[v_1,\ldots,v_m]}(k(P),k)$ can be endowed with a bigraded k-algebra structure and its totalized graded algebra is isomorphic to the cohomology algebra of $\mathcal{Z}_P$. Therefore, the cohomology of $\mathcal{Z}_P$ possesses the canonical *bigraded* algebra structure. To prove all these facts we use the Eilenberg–Moore spectral sequence. This spectral sequence usually appeared in the algebraic topology as a powerfull tool for calculating the cohomology of homogeneous spaces for Lie group actions (see e.g., [Sm]). So, it was interesting for us to discover the quite different application of this spectral sequence. In our situation the E_2 term of the spectral sequence is exactly $\mathrm{Tor}_{k[v_1,\ldots,v_m]}(k(P),k)$ and the spectral sequence collapses in the E_2 term. Using the Koszul complex as a resolution while calculating the E_2 term, we show that the above bigraded algebra is the cohomology algebra of a certain bigraded complex defined in purely combinatorial terms of the polytope P^n. Therefore, our bigraded cohomology algebra of $\mathcal{Z}_P$ carries the whole information about the combinatorics of polytope P^n. In particular, it turns out that the well-known Dehn–Sommerville equations for a simple polytope P^n follow directly from the bigraded Poincaré duality for $\mathcal{Z}_P$. Given the corresponding bigraded Betti numbers one can compute the numbers of faces of P of fixed dimension (the so-called *f-vector* of the polytope). The Upper Bound for the number of faces of simple polytope can be interpreted in terms of the cohomology of the manifold $\mathcal{Z}_P$. There also a lot of other relations between the topology of manifolds and the combinatorics of polytopes.

Moreover, since the homotopy equivalence $\mathcal{Z}_P \simeq U(P^n)$ holds, our calculation of the cohomology is also applicable to the set $U(P^n)$. As it was mentioned above, the set $U(P^n)$ is the complement to a certain collection of affine planes in $\mathbb{C}^m$ defined by the combinatorics of P^n. Hence, here we have a special case of the well-known general problem of calculation the cohomology of the complement to a collection of affine planes. In [GM, part III] there was proved the theorem that reduces this calculation to the calculation of the cohomology of a certain simplicial complex. In fact, our considerations show how the special properties of the collection of affine planes allow to obtain much more explicit description of the corresponding cohomology together with the multiplicative strucrure on it.

The questions considered here were discussed on the first author's talk on the conference "Solitons, Geometry and Topology" devoted to the jubilee of our Teacher Sergey Novikov. The part of these results were announced in [BP].

1 The main constructions and definitions

1.1 Simple polytopes and their face rings.

Let P^n be a simple polytope and denote f_i its number of codimension $(i+1)$ faces, $0 \le i \le n-1$. We refer to the integer vector $(f_0,\ldots,f_{n-1})$ as the *f-vector* of P^n. It is convenient to put also $f_{-1}=1$. Along with f-vector we also consider the *h-vector* $(h_0,\ldots,h_n)$ whose components h_i are defined from the equation

$$h_0t^n+\ldots+h_{n-1}t+h_n=(t-1)^n+f_0(t-1)^{n-1}+\ldots+f_{n-1}. \tag{1}$$

Therefore, we have

$$h_k=\sum_{i=0}^{k}(-1)^{k-i}\binom{n-i}{k-i}f_{i-1}. \tag{2}$$

Now, we fix a commutative ring k, which we refer to as the ground ring. A certain graded ring called *face ring* is associated to the combinatorial type of P^n. More precisely, let P^n be a simple polytope and $\mathcal{F}=(F_1,\ldots,F_m)$ be its set of codimension-one faces, $m=f_0$. Form the polynomial ring $k[v_1,\ldots,v_m]$ where the v_i are regarded as indeterminates corresponding to the facets F_i.

Definition 1.1 *The face ring $k(P)$ of a simple polytope P is defined to be the ring $k[v_1,\ldots,v_m]/I$, where*

$$I=(v_{i_1}\ldots v_{i_s}\mid i_1<i_2<\ldots<i_s,\ F_{i_1}\cap F_{i_2}\cap\cdots\cap F_{i_s}=\varnothing).$$

Originally (see [St]) the face ring was defined for any simplicial complex as follows. Let K be a finite simplicial complex with vertex set $\{v_1, \ldots, v_m\}$. Form a polynomial ring $k[v_1, \ldots, v_m]$ where the v_i are regarded as indeterminates.

Definition 1.2 *The face ring of a simplicial complex K (denoted $k(K)$) is defined to be the quotient ring $k[v_1, \ldots, v_m]/I$, where*

$$I = (v_{i_1} \ldots v_{i_s} \mid i_1 < i_2 < \ldots < i_s,\ \{v_{i_1}, \ldots, v_{i_s}\} \text{ does not span a simplex in } K).$$

We shall regard the indeterminates v_i in $k[v_1, \ldots, v_m]$ as being of degree two; in this way $k(K)$ as well as $k(P)$ becomes a graded ring.

Definition 1.3 *Given a convex polytope $P^n \subset \mathbb{R}^n$, the dual (or polar) polytope $(P^n)^* \subset (\mathbb{R}^n)^*$ is defined as follows:*

$$(P^n)^* = \{x' \in (\mathbb{R}^n)^* \ :\ \langle x', x\rangle \leq 1 \text{ for all } x \in \mathbb{R}^n\}.$$

It can be shown (cf. [Br]), that the above set is indeed a convex polytope. In the case of simple P^n the dual polytope $(P^n)^*$ would be simplicial and the i-dimensional (simplex) faces of $(P^n)^*$ are in one-to-one correspondence with the faces of P^n of codimension $i+1$. The boundary complex of $(P^n)^*$ defines a simplicial subdivision (triangulation) of $(n-1)$-dimensional sphere S^{n-1}, which we will denote K_P. In this situation both definitions 1.1 and 1.2 of the face ring give the same: $k(P) = k(K_P)$. The face rings of simple polytopes have very special algebraic properties. In order to describe them we need to review some commutative algebra.

Now suppose that k is a field and let R be a graded algebra over k. Let n be the maximal number of algebraically independent elements of R (this number is known as the *Krull dimension* of R, denoted $\operatorname{Krull} R$). A sequence $(\lambda_1, \ldots, \lambda_k)$ of homogeneous elements of R is called a *regular sequence*, if λ_{i+1} is not a zero divisor in $R/(\lambda_1, \ldots, \lambda_i)$ for each i (in the other words, the multiplication by λ_{i+1} defines a monomorphism of $R/(\lambda_1, \ldots, \lambda_i)$ into itself). It can be proved that $(\lambda_1, \ldots, \lambda_k)$ is a regular sequence if and only if $\lambda_1, \ldots, \lambda_k$ are algebraically independent and R is a free $k[\lambda_1, \ldots, \lambda_n]$-module. The notion of regular sequence is of great importance for the algebraic topologists (see, for instance, [La] [Sm]). A sequence $(\lambda_1, \ldots, \lambda_n)$ of homogeneous elements of R is called a *homogeneous system of parameters* (hsop), if the Krull dimension of $R/(\lambda_1, \ldots, \lambda_n)$ is zero. The k-algebra R is *Cohen–Macaulay* if it admits a regular sequence $(\lambda_1, \ldots, \lambda_n)$ of $n = \operatorname{Krull} R$ elements (which is then automatically a hsop). It follows from the above that R is Cohen–Macaulay if and only if there exists a sequence $(\lambda_1, \ldots, \lambda_n)$ of algebraically independent homogeneous elements of R such that R is a finite-dimensional free $k[\lambda_1, \ldots, \lambda_n]$-module.

In our case the following statement holds (cf. [St]).

Proposition 1.4 *The face ring $k(P^n)$ of a simple polytope P^n is Cohen–Macaulay.* $\square$

In the sequel we will need two successive generalizations of the notion of a simple polytope. As it was mentioned in the introduction, the bounding hyperplanes of a simple polytope are in general position. First, we define a *simple polyhedron* as any convex set in $\mathbb{R}^n$ (not necessarily bounded) that is obtained as the intersection of a finite number of generally positioned half-spaces. The faces of a simple polyhedron are defined obviously; all of them are simple polyhedra as well. It is also possible to define the $(n-1)$-dimensional simplicial complex K_P dual to the boundary of the simple polyhedron P^n (and again the i-dimensional simplices of K_P are in one-to-one correspondence with the faces of P^n of codimension $i+1$). However, the simplicial complex K_P obtained in such way not necessarily defines a triangulation of $(n-1)$-dimensional sphere S^{n-1}.

Example 1.5 The simple polyhedron

$$\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0\},$$

bounded by n coordinate hyperplanes will appear many times throughout our work. Its dual simplicial complex is $(n-1)$-dimensional simplex Δ^{n-1}.

We note, however, that not any $(n-1)$-dimensional simplicial complex can be obtained as dual to some n-dimensional simple polyhedron. Because of this, we still need to generalize the notion of simple polytope (and simple polyhedron). In this way we come to the notion of a *simple polyhedral complex*. Informally, a simple polyhedral complex of dimension n is "the dual to a general $(n-1)$-dimensional simplicial complex". We take its construction from [DJ]. Let K be a simplicial complex of dimension $n-1$ and let K' be its barycentric subdivision. Hence, the vertices of K' are simplices Δ of the complex K, and the simplices of K' are sets $(\Delta_1, \Delta_2, \ldots, \Delta_k)$, $\Delta_i \in K$, such that $\Delta_1 \subset \Delta_2 \subset \ldots \subset \Delta_k$. For each simplex $\Delta \in K$ denote by F_Δ the subcomplex of K' consisting of all simplices of K' of the form $\Delta = \Delta_0 \subset \Delta_1 \subset \ldots \subset \Delta_k$. If Δ is a $(k-1)$-dimensional simplex, then we refer to F_Δ as a face of codimension k. Let P_K be the cone over K. Then this P_K together with its decomposition into "faces" $\{F_\Delta\}_{\Delta\in K}$ is said to be a *simple polyhedral complex*. Any simple polytope P^n (as well as a simple polyhedron) can be obtained by this construction applied to the simplicial complex K^{n-1} dual to its boundary.

1.2 The topological spaces defined by simple polytopes.

Following [DJ], in this subsection we associate to any simple polyhedral complex P (and, hence, to any simple polytope) two topological spaces $\mathcal{Z}_P$ and B_TP.

Let $T^m = S^1 \times \ldots \times S^1$ be the m-dimensional compact torus. Let $\mathcal{F} = (F_1, \ldots, F_m)$ denote, as before, the set of codimension-one faces of P^n (which coincides with the vertex set of the dual simplicial complex K^{n-1}). We consider a free $\mathbb{Z}$-module $\mathbb{Z}^m$ and fix a one-to-one correspondence between the facets of P^n and the elements of the standard basis $\{e_1, \ldots, e_m\}$ in $\mathbb{Z}^m$. Now we can define the canonical coordinate subgroups $T^k_{i_1,\ldots i_k} \subset T^m$ as the tori corresponding to the coordinate sublattices in $\mathbb{Z}^m$ (i.e., the sublattices spanned by $e_{i_1}, \ldots, e_{i_k}$). Here we identify the torus T^m with the quotient space $\mathbb{R}^m/\mathbb{Z}^m$.

Definition 1.6 *The space Z_P associated to a simple polytope P^n is defined as follows*

$$\mathcal{Z}_P = (T^m \times P^n)/\sim \ \mid (g_1, p) \sim (g_2, q) \Leftrightarrow p = q,\ g_1g_2^{-1} \in T^k_{i_1,\ldots,i_k},$$
$$\text{where } F_{i_1}, \ldots, F_{i_k} \text{ are all facets containing the point } p \in P^n.$$

As it follows from the definition, $\dim \mathcal{Z}_P = m+n$ and the action of the torus T^m on $T^m \times P^n$ descends to the action of T^m on $\mathcal{Z}_P$. In the case of simple polytopes, the orbit space for this action is n-dimensional ball invested with the combinatorial structure of polytope P^n as described by the following proposition.

Proposition 1.7 *Suppose that P^n is a simple polytope. Then the action of T^m on $\mathcal{Z}_P$ has the following properties:*

1. *The isotropy subgroup of any point of $\mathcal{Z}_P$ is a coordinate subgroup in T^m of dimension $\leq n$.*

2. *The isotropy subgroups define the combinatorial structure of the polytope P^n on the orbit space. More precisely, the orbits with same isotropy subgroups define the interiors of faces of the polytope. If this isotropy subgroup is k-dimensional, then the corresponding face has codimension k. In particular, the action is free over the interior of the polytope.*

Proof. This follows easily from the definition of $\mathcal{Z}_P$. $\square$

Now we return to the case of a general simple polyhedral complex P^n. Let ET^m be the contractible space of the universal principal T^m-bundle over $BT^m = (\mathbb{C}P^\infty)^m$. Applying the Borel construction to the T^m-space $\mathcal{Z}_P$, we come to the following definition.

Definition 1.8 *The space B_TP is defined as*

$$B_TP = ET^m \times_{T^m} \mathcal{Z}_P. \tag{3}$$

Hence, the B_TP is the space of bundle with fibre $\mathcal{Z}_P$ associated to the universal bundle via the action of T^m on $\mathcal{Z}_P$. As it follows from the definition, the homotopy type of B_TP is determined by a simple polyhedral complex P^n.

1.3 Toric and quasitoric manifolds.

In the previous subsection we defined for any simple polytope P^n a space $\mathcal{Z}_P$ with action of T^m and the combinatorial structure of P^n in the orbit space (proposition 1.7). As we will see below, this $\mathcal{Z}_P$ turns out to be a smooth manifold. The another class of manifolds possessing the above properties is well known in the algebraic geometry as toric manifolds (or nonsingular projective toric varieties). Below we give a brief review of them. The detailed background material on this subject could be found in [Da, Fu].

Definition 1.9 *A toric variety is a normal algebraic variety M containing the n-dimensional algebraic torus $(\mathbb{C}^*)^n$ as a Zariski open subvariety, with the additional condition that the diagonal action of $(\mathbb{C}^*)^n$ on itself extends to action on the whole M (so, the torus $(\mathbb{C}^*)^n$ is contained in M as a dense orbit).*

On a nonsingular projective toric variety there exists a very ample line bundle whose zero cohomology (i.e., the space of global sections) is generated by the sections corresponding to the points with integer coordinates inside a certain simple polytope with vertices in the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. Conversely, there is an algebraic construction which allows one to produce a projective toric variety M^{2n} of real dimension $2n$ starting from a simple polytope P^n with vertices in $\mathbb{Z}^n$ (see, e.g., [Fu]). However, the resulting variety M^{2n} is not necessarily nonsingular. Namely, one obtains a nonsingular variety via the above construction if and only if for each vertex of P^n the normal covectors of n facets meeting at this vertex form a basis of the dual lattice $(\mathbb{Z}^n)^*$. The toric variety is not uniquely determined by the combinatorial type of a polytope: it depends also on the integral coordinates of vertices. Hence, we see that any number of *nonsingular* projective toric varieties (toric manifolds) can be obtained from the given *combinatorial* simple polytope (and sometimes this number can be zero). The corresponding examples will be discussed below.

The algebraic torus contains the compact torus $T^n \subset (\mathbb{C}^*)^n$, which acts on a toric manifold as well. It can be proved that all isotropy subgroups for this action are tori $T^k \subset T^m$ and the orbit space has the combinatorial structure of the simple polytope P^n as described in the second part of proposition 1.7 (here P^n is the polytope defined by the toric manifold as described above). The action of T^n on M^{2n} is *locally equivalent to the standard action of T^n on $\mathbb{C}^n$* (by the diagonal matrices) in the following sense: every point $x \in M^{2n}$ lies in some T^n-invariant neighbourhood $U \subset M^{2n}$ which is T^n-equivariantly homeomorphic to a certain (T^n-invariant) open subset $V \subset \mathbb{C}^n$. Furthermore, there exists the explicit map $M^{2n} \to \mathbb{R}^n$ (the *moment map*), with image P^n and corresponding orbits as fibres (see [Fu]). The underlying smooth manifolds for a toric manifold M^{2n} can be obtained as the quotient space $T^n \times P^n/\sim$ for some equivalence relation $\sim$ (cf. [DJ]; compare this with the definition 1.6 of the space $\mathcal{Z}_P$). Now, if we are interested only in topological and combinatorial properties, then we should not restrict ourselves to algebraic varieties; in this way, forgetting all algebraic geometry of M^{2n} and the action of the algebraic torus $(\mathbb{C}^*)^n$, we come to the following definition.

Definition 1.10 *A topologically toric (or quasitoric) manifold over a simple polytope P^n is a real orientable $2n$-dimensional manifold M^{2n} with action of the compact torus T^n that is locally isomorphic to the standard action of T^n on $\mathbb{C}^n$ and whose orbit space has the combinatorial structure of P^n (in the sense of the second part of proposition 1.7)*

The quasitoric manifolds were firstly introduced in [DJ] (where they were called simply "toric manifolds"). As it follows from the above discussion, all algebraic toric manifolds are quasitoric as well. The converse is not true: the corresponding examples can be found in [DJ]. One of the most important result on the quasitoric manifolds obtained there is the description of their cohomology rings. As it was shown in [DJ], these cohomologies have the same structure as the cohomologies of toric manifolds (see [Da]). In the rest of this subsection we describe briefly the main constructions with the quasitoric manifolds in order to use them later. The proofs could be found in [DJ].

Suppose M^{2n} is a quasitoric manifold over a simple polytope P^n and $\pi : M^{2n} \to P^n$ is the orbit map. Let F^{n-1} be a codimension-one face of P^n; then for any $x \in \pi^{-1}(\mathrm{int}F^{n-1})$ the isotropy group at x is independent of the choice of x rank-one subgroup $G_F \in T^n$. This subgroup is determined by a primitive vector $v \in \mathbb{Z}^n$. In this way we construct a function λ from the set $\mathcal{F}$ of codimension-one faces of P^n to primitive vectors in $\mathbb{Z}^n$.

Definition 1.11 *The defined above function $\lambda : \mathcal{F} \to \mathbb{Z}^n$ is called the characteristic function of M^{2n}.*

The characteristic function could be also considered as a homomorphism $\lambda : \mathbb{Z}^m \to \mathbb{Z}^n$, where $m = \#\mathcal{F} = f_0$ and $\mathbb{Z}^m$ is the free $\mathbb{Z}$-module spanned by the elements of $\mathcal{F}$.

It follows from the local equivalence of the torus action to the standard one that the characteristic function has the following property: if $F_{i_1}, \ldots, F_{i_n}$ are the codimension-one faces meeting at some vertex, then $\lambda(F_{i_1}), \ldots, \lambda(F_{i_n})$ form a basis in $\mathbb{Z}^n$. For any function $\lambda : \mathcal{F} \to \mathbb{Z}^n$ satisfying this condition there exists a quasitoric manifold $M^{2n}(\lambda)$ over P^n with characteristic function λ, and M^{2n} is determined by its characteristic function up to an equivariant homeomorphism. Nevertheless, there exist simple polytopes that do not admit any characteristic function. One of such examples is the duals to the so-called cyclic polytopes C^n_k for $k \geq 2^n$ (cf. [DJ]). These polytopes can not be realized as the orbit space for any quasitoric manifold (and, hence, this is also true for toric manifolds).

The quasitoric manifolds over a simple polytope P^n are closely related to the spaces $\mathcal{Z}_P$ and B_TP introduced in the previous subsection. Viewing any quasitoric manifold M^{2n} as a T^n-space, we can take the Borel construction $ET^n \times_{T^n} M^{2n}$. It turns out that all these spaces for given P^n are independent of M^{2n} and have the homotopy type B_TP:

$$B_TP \cong ET^n \times_{T^n} M^{2n}. \tag{4}$$

The relationship between the T^m-space $\mathcal{Z}_P$ and the quasitoric manifolds over P^n is described by the following property: for each toric manifold M^{2n} over P^n the orbit map $\mathcal{Z}_P \to P^n$ is decomposed as $\mathcal{Z}_P \to M^{2n} \xrightarrow{\pi} P^n$, where $\mathcal{Z}_P \to M^{2n}$ is a principal T^{m-n}-bundle and $M^{2n} \xrightarrow{\pi} P^n$ is the orbit map for M^{2n}. Therefore, quasitoric manifolds over the polytope P^n correspond to subgroups in T^m isomorphic to T^{m-n} and acting freely on $\mathcal{Z}_P$; and each subgroup of this type produces a quasitoric manifold. As it follows from proposition 1.7, subgroups of rank $\geq m-n$ can not act on $\mathcal{Z}_P$ freely, so the maximally free action of T^m on $\mathcal{Z}_P$ is exactly the case of existence of a quasitoric manifold over P^n. We will discuss this question in more details later.

From (3) we obtain the bundle $p : B_TP \to BT^m$ with fibre $\mathcal{Z}_P$. All the cohomologies below are considered with the coefficients in the ground ring k.

Theorem 1.12 *Let P be a simple polyhedral complex with m codimension-one faces. The map p^* : $H^*(BT^m) \to H^*(B_TP)$ is surjective and after the identification $H^*(BT^m) \cong k[v_1, \dots, v_m]$ it becomes the quotient epimorphism $k[v_1, \dots, v_m] \to k(P)$, where $k(P)$ is the face ring. In particular, $H^*(B_TP) \cong k(P)$.* $\square$

Now let M^{2n} be a quasitoric manifold over a simple polytope P^n with characteristic function λ. The characteristic function is obviously extended to a linear map $k^m \to k^n$. Consider the bundle $p_0 : B_TP \to BT^n$ with fibre M^{2n}.

Theorem 1.13 *The map $p_0^* : H^*(BT^n) \to H^*(B_TP)$ is monomorphic and p_0^* :$H^2(BT^n) \to H^2(B_TP)$ coincides with $\lambda^* : k^n \to k^m$. Furthermore, after the identification $H^*(BT^n) \cong k[t_1, \dots, t_n]$ the elements $\lambda_i = p^*(t_i) \in H^*(B_TP) \cong k(P)$ form a regular sequence of degree-two elements of $k(P)$.* $\square$

Clearly, the quasitoric manifolds in all previous constructions can be replaces by the (algebraic) toric manifolds. The toric manifolds correspond to simple polytopes $P^n \subset \mathbb{R}^n$ whose vertices have integer coordinates. As it follows from the above arguments, the value of the corresponding characteristic function is on the facet $F^{n-1} \in \mathcal{F}$ is its minimal integral normal (co)vector. All characteristic functions corresponding to (algebraic) toric manifolds could be obtained by this method.

2 The geometrical and homotopical properties of $\mathcal{Z}_P$ and B_TP

2.1 The cubical subdivision of a simple polytope.

In this subsection we suppose that P^n is a simple n-dimensional polytope. An *abstract cube* is given by its lattice of faces. We will need the following combinatorial construction.

Definition 2.1 *A cubical complex is a set of abstract cubes of any dimensions such that*

1. *All faces of any cube from the set belong to the set as well;*
2. *The intersection of any two cubes is a face of each.*

We will use also the standard q-dimensional cube $I^q = \{(x_1, \dots, x_q) \in \mathbb{R}^q \mid 0 \le x_i \le 1\}$.

Theorem 2.2 *Any simple polytope P^n with $m = f_0$ facets can be viewed naturally as a cubical complex $\mathcal{C}$, which has $r = f_{n-1}$ n-dimensional cubes I_v^n indexed by the vertices $v \in P^n$. Furthermore, there is a natural embedding i_P of $\mathcal{C}$ into the boundary of standard m-dimensional cube I^m, which takes the cubes of $\mathcal{C}$ to the faces of I^m.*

Proof. Let us fix a point in the interior of each face P^n (we also take all vertices and a point in the interior of the polytope). The resulting set $\mathcal{S}$ of $1 + f_0 + f_1 + \ldots + f_{n-1}$ points is said to be the vertex set of the cubical complex. Since the polytope P^n is simple, the number of k-faces meeting at each vertex is $\binom{n}{k}$, $0 \le k \le n$. In this way we associate to each vertex v of P^n a 2^n-element subset $\mathcal{S}_v$ of $\mathcal{S}$ — one point in the interior of each face containing v (hence, $\mathcal{S}_v$ contains also the vertex v itself and the point in the interior of P^n). We say then that the points from $\mathcal{S}_v$ are the vertices of the cube I^n_v corresponding to v. The faces of I^n_v are defined as follows. We take any two faces F^k_1 and F^l_2 of P^n such that $v \in F^k_1 \subset F^l_2$, $0 \le k = \dim F^k \le l = \dim F^l \le n$. Then there are $\binom{l-k}{i}$ faces F^{k+i} of dimension $k+i$ such that $v \in F^k_1 \subset F^{k+i} \subset F^l_2$, $0 \le i \le l-k$. In the interiors of these faces there are 2^{l-k} points from the set $\mathcal{S}_v \subset \mathcal{S}$. Then we say that all these points span a $(l-k)$-face $I^{l-k}_{F_1,F_2}$ of the cube I^n_v. Now, to finish the definition of the cubical complex $\mathcal{C}$ we need only to check the second condition from definiton 2.1. It is sufficient to check that the intersection of any two cubes I^n_v $I^n_{v'}$ is a face of each. To do this we find a face F^p of P^n of minimal dimension that contains both vertices v and v' (there is obviuosly only one such face). Then it can be easily seen that $I^n_v \cap I^n_{v'} = I^{n-p}_{F^p,P^n}$ is the face of I^n_v and $I^n_{v'}$.

Now let us construct the embedding $\mathcal{C} \hookrightarrow I^m$. First, we describe the images of the vertices of $\mathcal{C}$, i.e. the images of the points from $\mathcal{S}$. To do this we fix the numeration of facets: $F^{n-1}_1, \dots, F^{n-1}_m$. Now, if a point from $\mathcal{S}$ lies inside the facet F^{n-1}_i, then we take it to the vertex $(1, \dots, 0, \dots, 1)$ of the cube I^m, where 0 stands on the i-th place. If a point from $\mathcal{S}$ lies inside a face F^{n-k} of codimension k, we can write $F^{n-k} = F^{n-1}_{i_1} \cap \ldots \cap F^{n-1}_{i_k}$, and then take this point to the vertex of I^m whose $x_{i_1}, \dots, x_{i_k}$ coordinates are zero and the other coordinates are 1. The point of $\mathcal{S}$ in the interior of P^n maps to the vertex of I^m with coordinates $(1, \dots, 1)$. Hence, we constructed the map from the set $\mathcal{S}$ to the vertex set of I^m. This map obviously extends to a map from the cubical complex $\mathcal{C}$ corresponding to P^n to the cubical complex corresponding to I^m. This map can be realized geometrically as follows. We will need a simplicial subdivision $\mathcal{K}$ of P^n with vertex set $\mathcal{S}$ such that for each vertex $v \in P^n$ there exists a simplicial subcomplex $\mathcal{K}_v \subset \mathcal{K}$ with vertex set $\mathcal{S}_v$ isomorphic to the cube I^n_v. The simpliest way to construct such subdivision is to view P^n as the cone over the barycentric subdivision of the complex K^{n-1}_P dual to the boundary ∂P; then $\mathcal{K}_v$ are just the cones over the barycentric subdivisions of the $(n-1)$-simplices of K^{n-1}_P. Now we can extend the map $\mathcal{S} \hookrightarrow I^m$ linearly on each simplex of the triangulation $\mathcal{K}$ to the embedding $i_P : P^n \hookrightarrow I^m$ (which is therefore a piecewise linear map). The picture below illustrates this embedding for $n = 2$, $m = 3$.

The above constructed embedding $i_P : P^n \hookrightarrow I^m$ has the following property:

If $v = F^{n-1}_{i_1} \cap \cdots \cap F^{n-1}_{i_n}$, then the cube $I^n_v \subset P^n$ is mapped onto the n-face of the cube I^m defined by $(m-n)$ equations $x_j = 1$, $j \notin \{i_1, \dots, i_n\}$. (5)

Thus, all cubes of $\mathcal{C}$ map to the faces of I^m, and the proof is finished. $\square$

Lemma 2.3 *The number c_k of k-cubes in the cubical complex $\mathcal{C}$ constructed in the previous theorem for* *a simple polytope P^n can be computed by the following formula:*

$$c_k = \sum_{i=0}^{n-k} f_{n-i-1} \binom{n-i}{k} = f_{n-1} \binom{n}{k} + f_{n-2} \binom{n-1}{k} + \ldots + f_{k-1},$$

where $(f_0, \ldots, f_{n-1})$ is the f-vector of P^n and $f_{-1} = 1$.

Proof. It follows from the fact that k-cubes of $\mathcal{C}$ are in one-to-one correspondence with the pairs (F_1^i, F_2^{i+k}) of faces of P^n such that $F_1^i \subset F_2^{i+k}$ (see the proof of theorem 2.2). $\square$

2.2 $\mathcal{Z}_P$ as a smooth manifold and the equivariant embedding of $\mathcal{Z}_P$ into $\mathbb{C}^m$.

Let us consider the standard polydisc $(D^2)^m \subset \mathbb{C}^m$:

$$(D^2)^m = \{(z_1, \ldots, z_m) \in \mathbb{C}^m | \ |z_i| \leq 1\}.$$

The standard action of T^m on $\mathbb{C}^m$ by the diagonal matrices defines the action of T^m on $(D^2)^m$ with orbit space I^m. The main result of this subsection is the following theorem.

Theorem 2.4 *For any simple polytope P^n with m facets the space $\mathcal{Z}_P$ has the canonical structure of a smooth $(m+n)$-dimensional manifold for which the T^m-action is smooth. Furthermore, there exists a T^m-equivariant embedding $i_e : \mathcal{Z}_P \hookrightarrow (D^2)^m \subset \mathbb{C}^m$.*

Proof. It follows from theorem 2.2 that P^n can be viewed as a union of n-cubes I_v^n indexed by the vertices of P^n. Let $\rho : \mathcal{Z}_P \to P^n$ be the orbit map. It easily follows from the definition of $\mathcal{Z}_P$ that for each cube $I_v^n \subset \mathcal{Z}_P$ we have $\rho^{-1}(I_v^n) = (D^2)^n \times T^{m-n}$, where $(D^2)^n$ is the polydisc in $\mathbb{C}^n$ with the diagonal action of T^n. We see that $\mathcal{Z}_P$ is the union of "blocks" — subsets of the form $B_v \cong (D^2)^n \times T^{m-n}$. These "blocks" are glued together along their boundaries to get the smooth T^m-manifold $\mathcal{Z}_P$.

Now we are going to prove the statement about the equivariant embedding. We fix again a numeration of codimension-one faces of P^n: $F_1^{n-1}, \ldots, F_m^{n-1}$. Let us consider a certain block $B_v \cong (D^2)^n \times T^{m-n}$ corresponding to the vertex $v \in P^n$ (see above). Each factor D^2 or T^1 in B_v corresponds to some codimension-one face of P^n, so one can assign an index i $(1 \leq i \leq m)$ to this factor. Note that the indexes corresponding to the codimension-one faces containing v are assigned to n factors D^2, while other indexes are assigned to $m-n$ factors T^1. Now we numerate the factors $D^2 \subset (D^2)^m$ of the polydisc in any way. Then we embed each block $B_v \subset \mathcal{Z}_P$ into $(D^2)^m$ according to the indexes of its factors. It could be easily seen the set of embeddings $B_v \hookrightarrow (D^2)^m$ define an equivariant embedding $\mathcal{Z}_P \hookrightarrow (D^2)^m$. $\square$

Lemma 2.5 *The equivariant embedding $i_e : \mathcal{Z}_P \hookrightarrow (D^2)^m \subset \mathbb{C}^m$ constructed in theorem 2.4 is pulled back from the standard action of T^m on $(D^2)^m$ by the embedding $i_P : P^n \hookrightarrow I^m$ constructed in theorem 2.2. This can be described by the commutative diagram*

$$\begin{array}{ccc} \mathcal{Z}_P & \xrightarrow{i_e} & (D^2)^m \\ \downarrow & & \downarrow \\ P^n & \xrightarrow{i_P} & I^m. \end{array}$$

Proof. It could be easily seen that the embedding of the face $I^n \subset I^m$ defined by $m-n$ equations of the type $x_j = 1$ (as in (5)) pulls back the equivariant embedding of $(D^2)^n \times T^{m-n}$ into $(D^2)^m$. Then our assertion follows from the representation of $\mathcal{Z}_P$ as a union of blocks $B_v \cong (D^2)^n \times T^{m-n}$ and from the property (5) of the embedding $i_P : P^n \hookrightarrow I^m$. $\square$

The above constructed embedding $i_e : \mathcal{Z}_P \hookrightarrow (D^2)^m \subset \mathbb{C}^m$ allows as to connect the manifold $\mathcal{Z}_P$ with one construction from the theory of toric varieties. Below we describe this construction, following [Ba].

We consider the complex m-dimensional space $\mathbb{C}^m$ whose coordinates $z_1, \ldots, z_m$ are in one-to-one correspondence with m codimension-one faces of P^n.

Definition 2.6 *A subset of facets $\mathcal{P} = \{F_{i_1}, \ldots, F_{i_p}\} \subset \mathcal{F}$ is called a primitive collection if $F_{i_1} \cap \ldots \cap F_{i_p} = \varnothing$, while for all k, $0 \le k < p$, each k-element subset of $\mathcal{P}$ has nonempty intersection. In terms of the simplicial complex K_P dual to the boundary of P^n, the vertex subset $\mathcal{P} = \{v_{i_1}, \ldots, v_{i_p}\}$ is called a primitive collection if $\{v_{i_1}, \ldots, v_{i_p}\}$ does not span a simplex, while for all k, $0 \le k < p$, each k-element subset of $\mathcal{P}$ span a simplex of K_P.*

Now, let $\mathcal{P} = \{F_{i_1}, \ldots, F_{i_p}\}$ be a primitive collection of P^n. Denote by $\mathbf{A}(\mathcal{P})$ the $(m-p)$-dimensional affine subspace in $\mathbb{C}^m$, defined by the equations

$$z_{i_1} = \ldots = z_{i_p} = 0. \tag{6}$$

Since every primitive collection has at least two elements, the codimension of $\mathbf{A}(\mathcal{P})$ is at least 2.

Definition 2.7 *Define the closed algebraic subset $\mathbf{A}(P^n)$ in $\mathbb{C}^m$ as follows*

$$\mathbf{A}(P^n) = \bigcup_{\mathcal{P}} \mathbf{A}(\mathcal{P}),$$

where the union is taken over all primitive collections of facets of P^n. Put

$$U(P^n) = \mathbb{C}^m \setminus \mathbf{A}(P^n).$$

We note that to define $U(P^n)$ we could take the complement in $\mathbb{C}^m$ to the union of *all* planes (6) corresponding to the collections of facets $\{F_{i_1}, \ldots, F_{i_p}\}$ having empty intersection, not only to the primitive collections as in definition 2.7. The resulting set will be the same in both cases. We note also that the open set $U(P^n) \subset \mathbb{C}^m$ is invariant with respect to the action of $(\mathbb{C}^*)^m$ on $\mathbb{C}^m$.

It follows from the property (5) that the image of $\mathcal{Z}_P$ under the embedding $i_e : \mathcal{Z}_P \to \mathbb{C}^m$ (see theorem 2.4) does not intersect $\mathbf{A}(P^n)$, and therefore, $i_e(\mathcal{Z}_P) \subset U(P^n)$.

Let us consider the multiplicative group

$$\mathbb{R}^m_+ = \{(\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^n \mid \lambda_i > 0\}.$$

This group acts on $\mathbb{R}^m$ by dilations (an element $(\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$ takes $(x_1, \ldots, x_m) \in \mathbb{R}^m$ to $(\lambda_1 x_1, \ldots, \lambda_m x_m)$). There is the isomorphism $\exp : \mathbb{R}^m \to \mathbb{R}^m_+$ between the additive and the multiplicative groups, which takes a vector to the dilation generated by it. We fix a basis $e_1, \ldots, e_m$ in $\mathbb{R}^m$ whose elements correspond to the codimension-one faces of P^n.

We will deal with subgroups $R^{m-n}_+ \subset \mathbb{R}^m_+$ of rank $m-n$. Such a subgroup is defined by $(m-n)$ linearly independent vectors $w_i = w_{1i}e_1 + \ldots + w_{mi}e_m \in \mathbb{R}^m$, $1 \le i \le m-n$, which generate $(m-n)$

independent dilations. In the other words, the subgroup R_+^{m-n} is generated by $(m-n)$ one-parameter subgroups

$$a_{w_i} : \mathbb{R}_+ \to \mathbb{R}_+^m, \quad t \to (t^{w_{1i}}, \dots, t^{w_{mi}}).$$

Let us introduce the $m \times (m-n)$-matrix

$$\begin{pmatrix} w_{11} & \dots & w_{1,m-n} \\ \dots & \dots & \dots \\ w_{m1} & \dots & w_{m,m-n} \end{pmatrix}, \tag{7}$$

which correspond to the above subgroup R_+^{m-n}. In our study we consider only subgroups R_+^{m-n} with following property:

All the maximal minors of the matrix (7) that are obtained by deleting n rows whose numbers correspond to codimension-one faces meeting in same vertex are nonzero. (8)

Such matrices (and corresponding subgroups $R_+^{m-n} \subset \mathbb{R}^m$) form an open subset in the Stiefel variety of all $m \times (m-n)$-matrices of full rank. This is the case of particular interest because of the following theorem.

Theorem 2.8 *Any subgroup $R_+^{m-n} \subset \mathbb{R}_+^m$ possessing the property (8) acts freely on the algebraic set $U(P^n) \subset \mathbb{C}^m$ (see definition 2.7). For any such subgroup the composition $\mathcal{Z}_P \to U(P^n) \to U(P^n)/R_+^{m-n}$ of the embedding i_e and the orbit map is a homeomorphism.*

Proof. A point from $\mathbb{C}^m$ could have the non-trivial isotropy subgroup with respect to the action of a subgroup R_+^{m-n} only if it has at least one zero coordinate. As it follows from definition 2.7, if a point of $U(P^n)$ has some zero coordinates, then all of them correspond to facets of P^n having nonempty intersection (i.e., at least one common vertex). So, let $\{i_1, \dots, i_n\}$ be the index set of facets meeting at some vertex and take any point $p \in U(P^n)$ whose corresponding coordinates may vanish. This point could have a non-trivial isotropy subgroup with respect to the action of a subgroup R_+^{m-n} only if some linear combination of $w_1, \dots, w_{m-n}$ lies in the coordinate subspace spanned by $e_{i_1}, \dots, e_{i_n}$. But this is impossible if R_+^{m-n} has property (8). Thus, a subgroup R_+^{m-n} satisfying (8) acts on $U(P^n)$ freely.

Now, let us prove the second part of the theorem. Here we will use both embeddings $i_e : \mathcal{Z}_P \to (D^2)^m \subset \mathbb{C}^m$ from theorem 2.4 and $i_P : P^n \to I^m \subset \mathbb{R}^m$ from theorem 2.2. It is sufficient to prove that each orbit of the action of R_+^{m-n} on $U(P^n)$ intersect the image $i_e(\mathcal{Z}_P)$ in a single point. Since the embedding i_e is equivariant, this is equivalent to the fact that the $(m-n)$-dimensional subspace spanned by $w_1, \dots, w_{m-n}$ is in general position with each n-face of the cube I^m that lies in the image of the embedding i_P (see (5)). But this is exactly the property (8). □

The above theorem gives us a new proof of the fact that $\mathcal{Z}_P$ is a smooth manifold. Furthermore, the following statement holds.

Corollary 2.9 *There exists a smooth equivariant submanifold $\hat{\mathcal{Z}_P} \subset U(P^n) \subset \mathbb{C}^m$ with trivial normal bundle such that the composition $\hat{\mathcal{Z}_P} \to U(P^n) \to U(P^n)/R_+^{m-n}$ is a diffeomorphism. Any such manifold is canonically homeomorphic to $\mathcal{Z}_P$.*

Proof. Let $R_+^{m-n} \subset \mathbb{R}_+^m$ be any subgroup defined by the set of vectors $w_1, \dots, w_{m-n}$ satisfying (8). Consider $(m-n)$ small shifts of the image $i_e(\mathcal{Z}_P)$ along the directions $w_1, \dots, w_{m-n}$. As it follows from the previous theorem, this shifts define $(m-n)$ independent sections of the normal bundle, which is therefore trivial. □

Example 2.10 Let $P^n = \Delta^n$ (n-simplex). It can be easily seen that in this case $m = n+1$, $U(P^n) = \mathbb{C}^{n+1} \setminus \{0\}$ and R_+^{m-n} can be taken to be $\mathbb{R}_+$ with the diagonal action of $\mathbb{C}^{n+1}$. Thus, we have $\mathcal{Z}_P = S^{2n+1}$ (this could be also deduced from definition 1.6).

The property (8) relies only on the combinatorial type of a polytope P^n. At the same time matrices (7) satisfying (8) can be constructed starting from the polytope P^n itself, viewed as a subset in $\mathbb{R}^n$. More precisely, by definition, a polytope P^n is defined as a set of points $x \in \mathbb{R}^n$ satisfying m linear inequalities $\langle v_i, x \rangle \le 1$, $1 \le i \le m$, where $v_i \subset (\mathbb{R}^n)^*$ are the normal (co)vectors of facets. The set of $(\mu_1, \ldots, \mu_m)$ such that $\mu_1 v_1 + \ldots + \mu_m v_m = 0$ form $(m-n)$-dimensional subspace in $\mathbb{R}^m$. We choose a basis $w_i = (w_{1i}, \ldots, w_{mi})$, $1 \le i \le m-n$, in this subspace and form the matrix of the type (7). Then it can be readily checked that this matrix satisfy (8). Indeed, let us take the minor of this matrix that is obtained by deleting the rows $i_1, \ldots, i_n$, where $v = F_{i_1} \cap \ldots \cap F_{i_n}$ for some vertex $v \in P^n$. If this minor vanishes, then one can find a zero nontrivial linear combination of the vectors $v_{i_1}, \ldots, v_{i_n}$. But this is impossible since the simplicity of P^n shows that the set of normal covectors of facets meeting at the same vertex constitute a basis in $\mathbb{R}^n$.

Now suppose that all vertices of P^n belong to the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. Such integral simple polytope P^n defines a projective toric variety M^{2n} (cf. [Fu]). If we take the minimal integral (co)vectors v_i along the corresponding directions as normal (co)vectors to the facets of P^n, then the following criterion for the non-singularity of M^{2n} holds: M^{2n} is smooth if and only if for each vertex $v = F_{i_1} \cap \ldots \cap F_{i_n}$ the vectors $v_{i_1}, \ldots, v_{i_n}$ constitute a basis in $\mathbb{Z}^n$. In this case not only the subgroup $R_+^{m-n} \subset \mathbb{R}_+^m$ defined via the vectors v_i as above acts on $U(P^n)$ freely, but this is also true for the similarly defined subgroup $D \cong (\mathbb{C}^*)^{m-n} \subset (\mathbb{C}^*)^m$ isomorphic to $(\mathbb{C}^*)^{m-n}$. The toric manifold M^{2n} is then the orbit space $U(P^n)/D$ (cf. [Ba]). Thus, for any toric manifold M^{2n} over P^n we have a commutative diagram

$$\begin{array}{ccc} U(P^n) & \xrightarrow{R_+^{m-n}} & \mathcal{Z}_P \\ {\scriptstyle (\mathbb{C}^*)^{m-n}}\downarrow & & \downarrow{\scriptstyle T^{m-n}} \\ M^{2n} & = & M^{2n}. \end{array}$$

The above arguments show that for any subgroup $R_+^{m-n} \subset \mathbb{R}^m$ satisfying (8) the space $U(P^n)$ is homeomorphic to $\mathcal{Z}_P \times R_+^{m-n}$. Thus, all results about the cohomologies of $\mathcal{Z}_P$ exposed below are equally applicable for $U(P^n)$.

2.3 Homotopical properties of $\mathcal{Z}_P$ and $B_T P$.

We start with two simple assertions.

Lemma 2.11 *Let P^n be the product of two simple polytopes, $P^n = P_1^{n_1} \times P_2^{n_2}$. Then $\mathcal{Z}_P = \mathcal{Z}_{P_1} \times \mathcal{Z}_{P_2}$.*

Proof. This follows directly from the definition of $\mathcal{Z}_P$:

$$\mathcal{Z}_P = (T^m \times P^n)/\sim = ((T^{m_1} \times P^{n_1})/\sim) \times ((T^{m_2} \times P^{n_2})/\sim) = \mathcal{Z}_{P_1} \times \mathcal{Z}_{P_2}. \quad \square$$

The next lemma also follows easily from the construction of $\mathcal{Z}_P$.

Lemma 2.12 *If $P_1^{n_1} \subset P^n$ is a face of a simple polytope P^n, then $\mathcal{Z}_{P_1}$ is a submanifold of $\mathcal{Z}_P$.* $\square$

Below we invest the space B_TP defined by a simple polyhedral complex with a canonical cell structure.

We use the standard cell decomposition of $BT^m = (\mathbb{C}P^\infty)^m$ (each $\mathbb{C}P^\infty$ has one cell in every even dimension). This decomposition defines the cell cochain algebra $C^*(BT^m)$, for which we have $C^*(BT^m) = H^*(BT^m) = k[v_1, \ldots, v_m]$.

Theorem 2.13 *The space $B_TP = ET^m \times_{T^m} \mathcal{Z}_P$ defined by a simple polyhedral complex P can be realized as a cell subcomplex in BT^m. This subcomplex is defined as the union of subcomplexes $BT^k_{i_1,\ldots,i_k}$ over all simplices $\Delta = (i_1, \ldots, i_k)$ of the simplicial complex K_P^{n-1} dual to the boundary ∂P^n. In this realization we have $C^*(B_TP) = H^*(B_TP) = k(P)$, and the inclusion $i : B_TP \hookrightarrow BT^m$ induces the quotient epimorphism $C^*(BT^m) = k[v_1, \ldots, v_m] \to k(P) = C^*(B_TP)$ (here $k(P)$ denotes the face ring of P).*

Proof. It follows from the definition of a simple polyhedral complex that P is a cone over the barycentric subdivision of a certain simplicial complex K with m. We will construct the cell embedding $i : B_TP \hookrightarrow BT^m$ by induction on the dimension of K. If $\dim K = 0$, then K is a disjoint union of vertices $v_1, \ldots, v_m$ and P is the cone on K. In this case B_TP is a bouquet of m copies of $\mathbb{C}P^\infty$ and we have the obvious inclusion $i : B_TP \to BT^m = (\mathbb{C}P^\infty)^m$. In degree zero $C^*(B_TP)$ is just k, while in degrees ≥ 1 it is isomorphic to $k[v_1] \oplus \cdots \oplus k[v_m]$. Therefore, $C^*(B_TP) = k[v_1, \ldots, v_m]/I$, where I is the ideal generated by all square free monomials of degree ≥ 2, and i^* is the projection onto the quotient ring. Thus, the theorem holds if $\dim K = 0$.

Now, let $\dim K = k - 1$. By inductive hypothesis, the theorem is true for the $(k-2)$-skeleton $K' \subset K$ and the corresponding simple polyhedral complex P', i.e., $i^*C^*(BT^m) = C^*(B_TP') = k(K') = k[v_1, \ldots, v_m]/I'$. We add $(k-1)$-simplices one at a time. Each added simplex Δ^{k-1} with vertices $v_{i_1}, \ldots, v_{i_k}$ results the adding to $B_TP' \subset BT^m$ all cells of the cell subcomplex $BT^k_{i_1,\ldots,i_k} = BT^1_{i_1} \times \ldots \times BT^1_{i_k} \subset BT^m$. It is clear then that $C^*(B_TP' \cup BT^k_{i_1,\ldots,i_k}) = k(K' \cup \Delta^{k-1}) = k[v_1, \ldots, v_m]/I$, where $I \subset I'$ and $I'/I = (v_{i_1}v_{i_2}\ldots v_{i_k})$. It is also clear that if $i : B_TP' \cup BT^k_{i_1,\ldots,i_k} \hookrightarrow BT^m$ is the natural inclusion, then i^* is the quotient projection. $\square$

In particular, we see that for $K_P = \Delta^{m-1}$ (then $P = \mathbb{R}^m_+$) we have $B_TP = BT^m$.

Now we want to use the cell decomposition of B_TP for the description of the homotopy groups of B_TP and $\mathcal{Z}_P$.

A simple polytope P^n (or simple polyhedral complex) with m codimension-one faces is called *q-neighbourly* [Br] if the $(q-1)$-skeleton of the dual simplicial complex K_P^{n-1} coincides with the $(q-1)$-skeleton of a $(m-1)$-simplex (this means only that any q codimension-one faces of P^n have non-empty intersection). Note that any simple polytope is 1-neighbourly.

Theorem 2.14 *For any simple polyhedral complex P^n with m codimension-one faces we have:*

1. $\pi_1(\mathcal{Z}_P) = \pi_1(B_TP) = 0$;
2. $\pi_2(\mathcal{Z}_P) = 0,\ \pi_2(B_TP) = \mathbb{Z}^m$;
3. $\pi_q(\mathcal{Z}_P) = \pi_q(B_TP)$ *for* $q \geq 3$;
4. *if P^n is q-neighbourly, then $\pi_i(\mathcal{Z}_P) = 0$ for $i < 2q+1$, and $\pi_{2q+1}(\mathcal{Z}_P)$ is a free abelian group with generators corresponding to monomials $v_{i_1} \cdots v_{i_{q+1}} \in I$ (see the definition of the face ring; this monomials correspond to primitive collections of $q+1$ facets).*

Proof. The identities $\pi_1(B_TP) = 0$ and $\pi_2(B_TP) = \mathbb{Z}^m$ follow from the cell decomposition of B_TP described in the previous theorem. In order to calculate $\pi_1(\mathcal{Z}_P)$ and $\pi_2(\mathcal{Z}_P)$ we consider the following fragment of the exact homotopy sequence of the bundle $p : B_TP \to BT^m$ with fibre $\mathcal{Z}_P$:

$$\begin{array}{ccccccc} 0 = \pi_3(BT^m) \to \pi_2(\mathcal{Z}_P) \to & \pi_2(B_TP) & \xrightarrow{p_*} & \pi_2(BT^m) & \to \pi_1(\mathcal{Z}_P) \to \pi_1(B_TP) = 0 \\ & \| & & \| & \\ & \mathbb{Z}^m & \longrightarrow & \mathbb{Z}^m & \end{array}$$

Then it follows from theorem 2.13 that p_* here is an isomorphism, and, hence, $\pi_1(\mathcal{Z}_P) = \pi_2(\mathbb{Z}_P) = 0$. The third assertion of the theorem follows from the consideration of the fragment

$$\pi_{q+1}(BT^m) \to \pi_q(\mathcal{Z}_P) \to \pi_q(B_TP) \to \pi_q(BT^m),$$

in which $\pi_q(BT^m) = \pi_{q+1}(BT^m) = 0$ for $q \geq 3$. Finally, the cell structure of B_TP shows that if P^n is q-neighbourly, then the $(2q+1)$-skeleton of B_TP coincides with the $(2q+1)$-skeleton of BT^m. Thus, $\pi_k(B_TP) = \pi_k(BT^m)$ if $k < 2q + 1$. Now, the last assertion of the theorem follows from the third one and theorem 2.13. $\square$

The above calculation of the homotopy groups of $\mathcal{Z}_P$ and B_TP, allows us to assume that $\mathcal{Z}_P$ can be a first killing space for B_TP, i.e., $\mathcal{Z}_P = B_TP|_3$. It is really true, and in order to see this, we consider the following commutative diagram of bundles obtained from (3):

$$\begin{array}{ccc} \mathcal{Z}_P \times ET^m & \longrightarrow & ET^m \\ \downarrow & & \downarrow \\ B_TP & \xrightarrow{p} & BT^m. \end{array} \tag{9}$$

Since ET^m is contractible, $\mathcal{Z}_P \times ET^m$ is homotopically equivalent to $\mathcal{Z}_P$. On the other hand, since $BT^m = K(\mathbb{Z}^m, 2)$ and $\pi_2(B_TP) = \mathbb{Z}^m$, we deduce that $\mathcal{Z}_P \times ET^m$ is a first killing space for B_TP by definition. Thus, $\mathcal{Z}_P$ has homotopy type of a first killing space $B_TP|_3$ for B_TP.

3 The Eilenberg-Moore spectral sequence.

In [EM] Eilenberg and Moore have developed a spectral sequence which turns out to be of great use in our considerations. We follow [Sm] in the description of this spectral sequence.

Suppose that $\xi_0 = (E_0, p_0, B_0, F)$ is a Serre fibre bundle, B_0 is simply connected and $f : B \to B_0$ is a continuous map. We then can form the diagram

$$\begin{array}{ccc} F & = & F \\ \downarrow & & \downarrow \\ E & \longrightarrow & E_0 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_0} \\ B & \xrightarrow{f} & B_0, \end{array} \tag{10}$$

where $\xi = (E, p, B, F)$ is the induced fibre bundle. Under these assumptions the following theorem holds

Theorem 3.1 (Eilenberg-Moore) *There exists a spectral sequence of commutative algebras* $\{E_r, d_r\}$ *with*

1. $E_r \Rightarrow H^*(E)$ *(the spectral sequence converges to the cohomologies of* E *in the standard sense),*
2. $E_2 = \mathrm{Tor}_{H^*(B_0)}(H^*(B), H^*(E_0))$. $\square$

The Eilenberg-Moore spectral sequence lives in the second quadrant and the differential d_r has bidegree $(r, 1-r)$. In the special case when $B = *$ is a point (in this case $E = F$ is the fibre of ξ) we have

Corollary 3.2 *Let* $F \hookrightarrow E \to B$ *be a fibration over the simply connected space* B*. There exists a spectral sequence of commutative algebras* $\{E_r, d_r\}$ *with*

1. $E_r \Rightarrow H^*(E)$,
2. $E_2 = \mathrm{Tor}_{H^*(B)}(H^*(E), k)$. $\square$

As the first application of the Eilenberg-Moore spectral sequence we will calculate the cohomology ring of a quasitoric manifold M^{2n} over a simple convex polytope P^n (this was already done in [DJ] by means of other methods). Along with the ideal I such that $k(P) = k[v_1, \ldots, v_m]/I$ we define the ideal $J \subset k(P)$ as $J = (\lambda_1, \ldots, \lambda_n)$, where λ_i are the elements of $k(P)$ defined in theorem 1.13 from the characteristic function λ of the manifold M^{2n}. As it follows from theorem 1.13, $\lambda_i = \lambda_{i1}v_1 + \lambda_{i2}v_2 + \ldots + \lambda_{im}v_m$ are algebraically independent elements of degree 2 in $k(P)$ and $k(P)$ is a finite-dimensional free $k[\lambda_1, \ldots, \lambda_n]$-module. The inverse image of the ideal J under the projection $k[v_1, \ldots, v_m] \to k(P)$ is the ideal generated by $\lambda_i = \lambda_{i1}v_1 + \ldots + \lambda_{im}v_m$ considered as elements of $k[v_1, \ldots, v_m]$. This inverse image will be also denoted by J.

Theorem 3.3 *The following isomorphism of rings holds for any quasitoric manifold* M^{2n}*:*

$$H^*(M^{2n}) \cong k(P)/J = k[v_1, \ldots, v_m] \,/\, I+J.$$

Proof. Consider the Eilenberg-Moore spectral sequence of the fibration

$$\begin{array}{ccc} M^{2n} & \longrightarrow & B_TP \\ \downarrow & & \downarrow p_0 \\ * & \longrightarrow & BT^n \end{array}$$

Then theorem 1.13 gives us the monomorphism

$$\begin{array}{rcl} H^*(BT^n) = k[t_1, \ldots, t_n] & \xrightarrow{p_0^*} & H^*(B_TP) = k(P) \\ t_i & \longrightarrow & \lambda_i, \end{array}$$

so that $\mathrm{Im}\, p_0^* = k[\lambda_1, \ldots, \lambda_n] \subset k(P)$. The E_2 term of the Eilenberg-Moore spectral sequence is

$$E_2^{*,*} = \mathrm{Tor}^{*,*}_{H^*(BT^n)}(H^*(B_TP), k) = \mathrm{Tor}^{*,*}_{k[\lambda_1,\ldots,\lambda_n]}(k(P), k).$$

The right hand side above is a bigraded k-module (cf. [Ma, Sm]). The first ("external") grading arises from a projective resolution of $H^*(B_TP)$ as a $H^*(BT^n)$-module used in the definition of the functor Tor. The second ("internal") one arises from the gradings of $H^*(BT^n)$-modules which enter the resolution;

we assume that nonzero elements appear only in even internal degrees (since $\deg \lambda_i = 2$). Since $k(P)$ is a free $k[\lambda_1, \ldots, \lambda_n]$-module, we have

$$\mathrm{Tor}^{*,*}_{k[\lambda_1,\ldots,\lambda_n]}(k(P), k) = \mathrm{Tor}^{0,*}_{k[\lambda_1,\ldots,\lambda_n]}(k(P), k) = k(P) \otimes_{k[\lambda_1,\ldots,\lambda_n]} k = k(P)/J.$$

Therefore, $E_2^{0,*} = k(P)/J$ and $E_2^{-p,*} = 0$ if $p > 0$. From this we deduce that $E_2 = E_\infty$ and $H^*(M^{2n}) = k(P)/J$. $\square$

Corollary 3.4 $H^*(M^{2n}) = \mathrm{Tor}_{k[\lambda_1,\ldots,\lambda_n]}(k(P), k)$. $\square$

4 The calculation of cohomologies of $\mathcal{Z}_P$

In this section we use the Eilenberg–Moore spectral sequence to describe the cohomology ring of $\mathcal{Z}_P$ in terms of the face ring $k(P)$ and also obtain some additional results about this cohomologies in the case where at least one quasitoric manifold exists over the polytope P. Throughout this section we assume that k is a field.

4.1 The additive structure of cohomologies of $\mathcal{Z}_P$.

In this subsection we consider the Eilenberg-Moore spectral sequence of the bundle $p : B_TP \to BT^m$ with fibre $\mathcal{Z}_P$ (see (3)). This spectral sequence defines a decreasing filtration on $H^*(\mathcal{Z}_P)$, which we denote $\{F^{-p}H^*(\mathcal{Z}_P)\}$, such that

$$E_\infty^{-p,n+p} = F^{-p}H^n(\mathcal{Z}_P)/F^{-p+1}H^n(\mathcal{Z}_P).$$

Proposition 4.1 *$F^0H^*(\mathcal{Z}_P) = H^0(\mathcal{Z}_P) = k$ (here k is the ground field).*

Proof. It follows from [Sm, Proposition 4.2] that for the Eilenberg–Moore spectral sequence of arbitrary commutative square (10) one has $F^0H^*(E) = \mathrm{Im}\{H^*(B) \otimes H^*(E_0) \to H^*(E)\}$. In our case we obtain $F^0H^*(\mathcal{Z}_P) = \mathrm{Im}\{H^*(B_TP) \to H^*(\mathcal{Z}_P)\}$. Now, our proposition follows from the consideration of the Leray–Serre spectral sequence of the bundle $p : B_TP \to BT^m$ with fibre $\mathcal{Z}_P$ and the fact that the map $p^* : H^*(BT^m) \to H^*(B_TP)$ is epimorphic (see theorem (1.12)). $\square$

In the Eilenberg–Moore spectral sequence of the bundle $p : B_TP \to BT^m$ one has $E_2 = \mathrm{Tor}_{k[v_1,\ldots,v_m]}(k(P), k)$. Let us consider a free resolution of $k(P)$ as a $k[v_1, \ldots, v_m]$-module:

$$0 \longrightarrow R^{-h} \xrightarrow{d^{-h}} R^{-h+1} \xrightarrow{d^{-h+1}} \cdots \longrightarrow R^{-1} \xrightarrow{d^{-1}} R^0 \xrightarrow{d^0} k(P) \longrightarrow 0. \tag{11}$$

It is convenient for our purposes to assume that R^i are numbered by non-positive integers, i.e. $h > 0$ above.

The minimal number h for which a resolution of the form (11) exists is called the *homological dimension* of $k(P)$ and is denoted by $\mathrm{hd}_{k[v_1,\ldots,v_m]}(k(P))$. By the Hilbert syzygy theorem, $\mathrm{hd}_{k[v_1,\ldots,v_m]}(k(P)) \le m$. At the same time, since $k(P)$ is a Cohen–Macaulay ring, it is known ([Se, Chapter IV]) that

$$\mathrm{hd}_{k[v_1,\ldots,v_m]}(k(P)) = m - n,$$

where n is the Krull dimension (the maximal number of algebraically independent elements) of $k(P)$. In our case $n = \dim P$.

We will use a special free resolution (11) known as *minimal* resolution (cf. [Ad]), which is defined in the following way. Let A be a graded connected commutative algebra and let N, N' be modules over A. Set $I(A) = \sum_{q>0} A_q = \{a \in A | \deg a \neq 0\}$ and $J(N) = I(A) \cdot N$. The map $f : N \to N'$ is called *minimal*, if $\operatorname{Ker} f \subset J(N)$. The resolution (11) is called *minimal*, if all d^i are minimal. One of the ways to construct a minimal resolution is as follows: we construct the free A-modules $R^0, R^{-1}, \ldots, R^{-h}$ successively and once we constructed R^i and d^i we take a minimal set of homogeneous generators of $\operatorname{Ker} d^i$ as a basis for R^{i+1}. For any graded algebra A there is a natural way to choose a minimal set of generators for a A-module R. This is done as follows. Let k_1 is the lowest degree in which R is nonzero. Choose in $(R)^{k_1}$ a vector space basis, say $x_1, \ldots, x_p$. Now let $R_1 = (x_1, \ldots, x_p) \subset R$ be the submodule generated by $x_1, \ldots, x_p$. If $R = R_1$ then we have constructed a minimal set of generators for R. Otherwise, consider the first degree k_2 in which $R \neq R_1$; then in this degree we can choose a direct sum decomposition $R = R_1 \oplus \widehat{R_1}$. Now choose in $\widehat{R_1}$ a vector space basis $x_{p_1+1}, \ldots, x_{p_2}$ and set $R_2 = (x_1, \ldots, x_{p_2})$. If $R = R_2$ we are done, if not just continue to repeat the above process until we obtain the minimal set of generators for R. A minimal set of generators for A-module R possesses the following property: no element x_k could be decomposed as $x_k = \sum a_i x_i$ with $a_i \in A$, $\deg a_i \neq 0$. A minimal resolution is unique up to an isomorphism.

Now let (11) be a minimal resolution of $k(P)$ as a $k[v_1, \ldots, v_m]$-module. Since the resolution is minimal, in (11) we have $h = m - n$ and R^0 is the free $k[v_1, \ldots, v_m]$-module with one generator 1 of degree 0. The set of generators for R^1 consists of elements $v_{i_1 \ldots i_k}$ of degree $2k$ such that $\{v_{i_1}, \ldots, v_{i_k}\}$ does not span a simplex in K, while any proper subset of $v_{i_1}, \ldots, v_{i_k}$ do span a simplex in K. This means exactly that the set $\{v_{i_1}, \ldots, v_{i_k}\}$ is a primitive collection in the sense of definition 2.6 (here as before v_i are regarded as vertices of the simplicial complex K^{n-1} dual to ∂P).

Note that the $k[v_1, \ldots, v_m]$-module structure in k is defined by the homomorphism $k[v_1, \ldots, v_m] \to k$, $v_i \to 0$. Since the resolution (11) is minimal, all differentials d^i in the complex

$$0 \longrightarrow R^{-(m-n)} \otimes_{k[v_1,\ldots,v_m]} k \overset{d^{-(m-n)}}{\longrightarrow} \cdots \longrightarrow R^{-1} \otimes_{k[v_1,\ldots,v_m]} k \overset{d^{-1}}{\longrightarrow} R^0 \otimes_{k[v_1,\ldots,v_m]} k \longrightarrow 0 \tag{12}$$

are trivial. The module $R^i \otimes_{k[v_1,\ldots,v_m]} k$ is a finite-dimensional vector space over k whose dimension is equal to the dimension of R^i as a free $k[v_1, \ldots, v_m]$-module:

$$\dim_k R^i \otimes_{k[v_1,\ldots,v_m]} k = \dim_{k[v_1,\ldots,v_m]} R^i.$$

Therefore, since all differentials in the complex (12) are trivial, the following equality holds for the minimal resolution (11):

$$\dim_k \operatorname{Tor}_{k[v_1,\ldots,v_m]}(k(P), k) = \sum_{i=0}^{m-n} \dim_{k[v_1,\ldots,v_m]} R^{-i}. \tag{13}$$

Now all is ready to describe the additive structure in the cohomologies of $\mathcal{Z}_P$.

Theorem 4.2 *The following isomorphism of graded k-modules holds:*

$$H^*(\mathcal{Z}_P) \cong \operatorname{Tor}_{k[v_1,\ldots,v_m]}(k(P), k).$$

(the bigraded structure in the right hand side is turned to a single grading by taking the total degree). To be more precise, there is a filtration $\{F^{-p}H^*(\mathcal{Z}_P)\}$ in $H^*(\mathcal{Z}_P)$ such that

$$F^{-p}H^*(\mathcal{Z}_P)/F^{-p+1}H^*(\mathcal{Z}_P) = \operatorname{Tor}^{-p}_{k[v_1,\ldots,v_m]}(k(P), k).$$

Proof. First, we show that

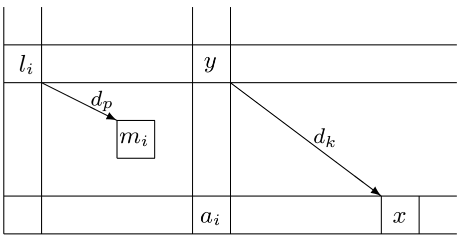
$$\dim_k H^*(\mathcal{Z}_P) \geq \dim_k \operatorname{Tor}_{k[v_1,\ldots,v_m]}(k(P), k). \tag{14}$$

To do this we consider the Leray–Serre spectral sequence of the bundle $p : B_T P \to BT^m$ with fibre $\mathcal{Z}_P$. The first column of the E_2 term of this spectral sequence is the cohomologies of fibre $\mathcal{Z}_P$: $H^*(\mathcal{Z}_P) = E_2^{0,*}$. We will assign to each generator of each $k[v_1, \ldots, v_m]$-free module R^i a certain generator of $H^*(\mathcal{Z}_P)$ in such a way that the different generators of $H^*(\mathcal{Z}_P)$ will correspond to different generators of modules R^i.

By theorem 1.12, nonzero elements can appear in the E_∞ term only in the bottom line, and this bottom line is the ring $k(P) = H^*(B_T P)$:

$$E_\infty^{*,p} = 0,\ p > 0;\ E_\infty^{*,0} = k(P).$$

Therefore, all elements from the kernel of map $d^0 :\ R^0 = k[v_1, \ldots, v_m] = E_2^{*,0} \to E_\infty^{*,0} = k(P)$ from the minimal resolution (11) must be killed by the differentials of the spectral sequence. We denoted this kernel by I (it is an ideal in $k[v_1, \ldots, v_m]$). Let $(x_1, \ldots, x_p)$ be a minimal basis of the ideal I constructed by the described above procedure. Below we prove that the elements x_i can be killed only by the transgression (i.e. by the differentials from the first column). Suppose that the converse is true, so that x is an element of the minimal basis of I that is killed by the non-transgressive differential: $x = d_k y$ for some k, where y lies not in the first column. Then y is a cycle of all differentials up to d_{k-1}. This y arises from some element $\sum_i l_i a_i$ in the E_2 term, $l_i \in E_2^{0,*}$, $a_i \in E_2^{*,0} = k[v_1, \ldots, v_m]$. Assume at the first time that all elements l_i are transgressive (i.e. l_i are cycles of all differentials d_i for $i < k$) and $d_k(l_i) = m_i$, $m_i \in E_k^{*,0}$.



Since all m_i are killed by the differentials, their preimages in E_2 belong to I. Hence, we have $x = d_k y = \sum_i m_i a_i$, $m_i \in I$, which contradicts to the minimality of the basis $(x_1, \ldots, x_p)$. Therefore, our first assumption is false, and there are some non-transgressive elements among l_i, i.e., there exists $p < k$ and i such that $d_p(l_i) = m_i \neq 0$. Then this m_i survives in E_p and, choosing from all such p the minimal one, we obtain $d_p(y) = m_i a_i + \ldots \neq 0$ — contradiction. This means that all elements from the minimal set of generators of I are killed by the transgression, i.e. some (different) elements $l_i^{(1)} \in H^*(\mathcal{Z})$ correspond to them.

Since $E_2 = H^*(\mathcal{Z}_P) \otimes k[v_1, \ldots, v_m]$, the free $k[v_1, \ldots, v_m]$-module generated by the elements $l_i^{(1)}$ is included into the E_2 term as a submodule. Therefore, we have $R^{-1} \subset E_2$ and the map $d^{-1} : R^{-1} \to R^0 = k[v_1, \ldots, v_m]$ is defined by the differentials of the spectral sequence. The kernel of this map $\operatorname{Ker} d^{-1}$ can not be killed by the already constructed differentials. Using the previous argument, we deduce that the generators of the minimal basis of $\operatorname{Ker} d^{-1} \in R^{-1}$ can be killed only by some elements of the first column, say $l_1^{(2)}, \ldots, l_q^{(2)}$. Therefore, the free $k[v_1, \ldots, v_m]$-module generated by the elements $l_i^{(2)}$ is also included into the E_2 term as a submodule, i.e. $R^{-2} \subset E_2$. Proceeding with this procedure, at the end we obtain $\sum_{i=0}^{m-n} \dim_{k[v_1,\ldots,v_m]} R^{-i}$ generators in the first column of the E_2 term. Using (13), we get then required inequality (14).

Now let us consider the Eilenberg–Moore spectral sequence of the bundle $p : B_T P \to BT^m$ with fibre $\mathcal{Z}_P$. For this spectral sequence we have $E_2 = \operatorname{Tor}_{k[v_1,\ldots,v_m]}(k(P), k)$, $E_r \Rightarrow H^*(\mathcal{Z})$. It follows from inequality (14) that $E_2 = E_\infty$, and this concludes the proof of our theorem. $\square$

Let us consider again the Eilenberg-Moore filtration $\{F^{-p} H^*(\mathcal{Z}_P)\}$ in $H^*(\mathcal{Z}_P)$. It turns out that the elements of $F^{-1} H^*(\mathcal{Z})$ have very transparent geometric realization. Namely, the following statement holds:

Theorem 4.3 *The cycles in $H_*(\mathcal{Z}_P)$ Poincaré dual to the elements of $\{F^{-1}H^*(\mathcal{Z}_P)\}$ can be realized as embedded submanifolds of $\mathcal{Z}_P$. Furthermore, this submanifold can be taken a sphere of odd dimension for any generator of $\{F^{-1}H^*(\mathcal{Z}_P,\mathbb{Z})\}$.*

Proof. It follows from theorem 4.2 that

$$F^{-1}H^*(\mathcal{Z}_P)/F^0H^*(\mathcal{Z}_P) = \mathrm{Tor}^{-1}_{k[v_1,\ldots,v_m]}(k(P),k).$$

By proposition 4.1, $F^0H^*(\mathcal{Z}_P) = H^0(\mathcal{Z}_P)$. Take, as in the proof of theorem 4.2, the basis in $\mathrm{Tor}^{-1}_{k[v_1,\ldots,v_m]}(k(P),k)$ consisting of the elements $v_{i_1\ldots i_p}$ of degree $2p$ such that the set of vertices $v_{i_1},\ldots,v_{i_p}$ of the simplicial complex K_P is a primitive collection. Geometrically, this means that the corresponding subcomplex of K (i.e., the subcomplex consisting of all simplices whose vertices are among $v_{i_1},\ldots,v_{i_p}$) is the simplicial complex consisting of all faces of a simplex except one of the highest dimension (i.e., it is the boundary of a simplex). In terms of the simple polytope P the element $v_{i_1\ldots i_p}$ corresponds to the set $\{F_{i_1},\ldots,F_{i_p}\}$ of codimension-one faces such that $F_{i_1}\cap\cdots\cap F_{i_p}=\varnothing$ though any proper subset of $\{F_{i_1},\ldots,F_{i_p}\}$ has non-empty intersection. Note that the cycle corresponding to $v_{i_1\ldots i_p}\in\mathrm{Tor}^{-1}_{k[v_1,\ldots,v_m]}(k(P),k)$ has dimension $2p-1$ in $H_*(\mathcal{Z})$. Now take one point inside each face $F_{i_1}\cap\cdots\cap\widehat{F_{i_r}}\cap\cdots\cap F_{i_p}$, $1\le r\le p$, (F_{i_r} is dropped); then we can embed the simplex Δ^{p-1} on these points into the polytope P so that the boundary $\partial\Delta^{p-1}$ embeds into ∂P (compare this with the construction of the cubical decomposition of P in theorem 2.2). Consider the projection $\rho:\mathcal{Z}_P=(T^m\times P^n)/\sim\,\to P^n$ onto the orbit space; then it is easy to see that $\rho^{-1}(\Delta^{p-1})=(T^p\times\Delta^{p-1})/\sim)\times T^{m-p}=S^{2p-1}\times T^{m-p}$. In this way we obtain an embedding $S^{2p-1}\hookrightarrow\mathcal{Z}_P$ which realize the cycle in $H_*(\mathcal{Z}_P)$ dual to $v_{i_1\ldots i_p}$. $\square$

4.2 The multiplicative structure of cohomologies of $\mathcal{Z}_P$.

Here we describe the ring $H^*(\mathcal{Z}_P)$.

In the previous subsection the bigraded k-module $\mathrm{Tor}_{k[v_1,\ldots,v_m]}(k(P),k)$ was calculated by means of the minimal resolution of the face ring $k(P)$ regarded as a $k[v_1,\ldots,v_m]$-module. Below we use the another approach based on the *Koszul resolution* of the $k[v_1,\ldots,v_m]$-module k. As the result, the bigraded k-module $\mathrm{Tor}_{k[v_1,\ldots,v_m]}(k(P),k)$ becomes the bigraded k-algebra whose total graded k-algebra is isomorphic to the algebra $H^*(\mathcal{Z}_P)$. This approach also gives us the description of $H^*(\mathcal{Z}_P)$ as a cohomology algebra of some differential (bi)graded algebra.

Let $\Gamma=k[y_1,\ldots,y_n]$, $\deg y_i=2$, be a graded polynomial algebra over k, and let $\Lambda[u_1,\ldots,u_n]$ denote an exterior algebra over k on generators $u_1,\ldots,u_n$. Consider the bigraded differential algebra

$$\mathcal{E}=\Gamma\otimes\Lambda[u_1,\ldots,u_n],$$

whose gradings and differential are defined by

$$\begin{array}{rclcrcl} \mathrm{bideg}(y_i\otimes 1) &=& (0,2), & \qquad & d(y_i\otimes 1) &=& 0;\\ \mathrm{bideg}(1\otimes u_i) &=& (-1,2), & & d(1\otimes u_i) &=& y_i\otimes 1. \end{array}$$

and requiring that d be a derivation of algebras. The differential adds $(1,0)$ to bidegree, so the components $\mathcal{E}^{-i,*}$ form a cochain complex that will be also denoted $\mathcal{E}$. It is well known that this complex defines a Γ-free resolution of k (regarded as a Γ-module) called the *Koszul resolution* (cf. [Ma]).

Proposition 4.4 *Let $\Gamma = k[y_1, \dots, y_n]$ and suppose that A is any Γ-module, then*

$$\mathrm{Tor}_\Gamma(A, k) = H[A \otimes \Lambda[u_1, \dots, u_n], d],$$

where d is defined as $d(a \otimes u_i) = (y_i \cdot a) \otimes 1$ for any $a \in A$.

Proof. Let us consider the above Γ-free Koszul resolution $\mathcal{E} = \Gamma \otimes \Lambda[u_1, \dots, u_n]$ of k. Then

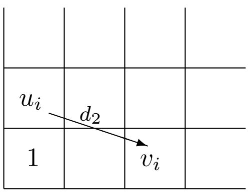
$$\mathrm{Tor}_\Gamma(A, k) = H[A \otimes_\Gamma \Gamma \otimes \Lambda[u_1, \dots, u_n], d] = H[A \otimes \Lambda[u_1, \dots, u_n], d]. \quad \square$$

Now let us consider the principal T^m-bundle $\mathcal{Z}_P \times ET^m \to B_T P$ pulled back from the universal T^m-bundle by the map $p : B_T P \to BT^m$ (see (9)). The following lemma holds.

Lemma 4.5 *The following isomorphism describes the $E_3^{(s)}$ term of the Leray–Serre spectral sequence $\{E_r^{(s)}, d_r\}$ of the bundle $\mathcal{Z}_P \times ET^m \to B_T P$:*

$$E_3^{(s)} \cong \mathrm{Tor}_{k[v_1, \dots, v_m]}(k(P), k).$$

Proof. First, we consider the $E_2^{(s)}$ term of the Leray–Serre spectral sequence of the given bundle. Since $H^*(T^m) = \Lambda[u_1, \dots, u_m]$, $H^*(B_T P) = k(P) = k[v_1, \dots, v_m]/I$, we have



$$E_2^{(s)} = k(P) \otimes \Lambda[u_1, \dots, u_m].$$

It can be easily seen that the differential $d_2^{(s)}$ acts as follows

$$d_2^{(s)}(1 \otimes u_i) = v_i \otimes 1, \quad d_2^{(s)}(v_i \otimes 1) = 0.$$

Now, since $E_3^{(s)} = H[E_2^{(s)}, d_2^{(s)}]$, our assertion follows from proposition 4.4 where we put $\Gamma = k[v_1, \dots, v_m]$, $A = k(P)$. $\square$

Now we are ready to prove our main result on the cohomologies of $\mathcal{Z}_P$.

Theorem 4.6 *The following isomorphism of graded algebras holds:*

$$H^*(\mathcal{Z}_P) = H[k(P) \otimes \Lambda[u_1, \dots, u_m], d],$$
$$\operatorname{bideg} v_i = (0, 2), \quad \operatorname{bideg} u_i = (-1, 2),$$
$$d(1 \otimes u_i) = v_i \otimes 1, \quad d(v_i \otimes 1) = 0.$$

Hence, the Leray–Serre spectral sequence of the T^m-bundle $\mathcal{Z}_P \times ET^m \to B_T P$ collapses in the E_3 term.

Proof. Let us consider the bundle $p : B_T P \to BT^m$ with fibre $\mathcal{Z}_P$. It follows from theorem 2.13 that the correspondent cochain algebras are $C^*(BT^m) = k[v_1, \dots, v_m]$ and $C^*(B_T P) = k(P)$, and the action of $C^*(BT^m)$ on $C^*(B_T P)$ is defined by the quotient projection. It was shown in [Sm, Proposition 3.4] that there is an isomorphism of algebras

$$\theta^* : \mathrm{Tor}_{C^*(BT^m)}(C^*(B_T P), k) \to H^*(\mathcal{Z}_P).$$

But it follows from above arguments and proposition 4.4 that

$$\mathrm{Tor}_{C^*(BT^m)}(C^*(B_T P), k) \cong H[k(P) \otimes \Lambda[u_1, \dots, u_m], d],$$

which concludes the proof. $\square$

4.3 The additional properties of the cohomologies of $\mathcal{Z}_P$ determined by tori actions.

First, we consider the case where the simple polytope P^n can be realized as the orbit space for some quasitoric manifold (see subsection 1.3). The existence of this quasitoric manifold will allow us to reduce the calculation of cohomologies of $\mathcal{Z}_P$ to the calculation of cohomologies of an algebra which is much smaller than that from theorem 4.6.

It was already discussed above that any quasitoric manifold M^{2n} over P^n defines a principal T^{m-n}-bundle $\mathcal{Z}_P \to M^{2n}$. This bundle is induced from the universal T^{m-n}-bundle by a certain map $f : M^{2n} \to BT^{m-n}$.

Theorem 4.7 *Suppose M^{2n} is a quasitoric manifold over a simple polytope P^n; then the Eilenberg-Moore spectral sequences of the following commutative squares*

$$\begin{array}{ccc} \mathcal{Z}_P \times ET^m & \longrightarrow & ET^m \\ \downarrow & & \downarrow \\ B_TP & \xrightarrow{p} & BT^m \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} \mathcal{Z}_P & \longrightarrow & ET^{m-n} \\ \downarrow & & \downarrow \\ M^{2n} & \xrightarrow{f} & BT^{m-n} \end{array}$$

are isomorphic.

Proof. Let $\{E_r, d_r\}$ be the Eilenberg-Moore spectral sequence of the first commutative square and let $\{\bar{E}_r, d_r\}$ be that of the second one. Then, as it follows from the results of [EM, Sm], the natural inclusions $BT^{m-n} \to BT^m$, $ET^{m-n} \to ET^m$, $M^{2n} \to B_TP$ and $\mathcal{Z}_P \to \mathcal{Z}_P \times ET^m$ define a homomorphism of spectral sequences: $g : \{E_r, d_r\} \to \{\bar{E}_r, \bar{d}_r\}$. First, we prove that $g_2 : E_2 \to \bar{E}_2$ is an isomorphism.

The map $f^* : H^*(BT^{m-n}) \to H^*(M^{2n})$ can be viewed in the following way. The ring $H^*(BT^{m-n}) = k[w_1, \ldots, w_{m-n}]$ can be represented as $k[v_1, \ldots, v_m]/J$ with J being the described above ideal, $J = (\lambda_1, \ldots, \lambda_n)$ (this is obtained from the inclusion $BT^{m-n} \hookrightarrow BT^m$). By theorem 3.3 we have $H^*(M^{2n}) = k[v_1, \ldots, v_m] \,/\, I{+}J$. Then $f^* : H^*(BT^{m-n}) = k[v_1, \ldots, v_m]/J \to k[v_1, \ldots, v_m] \,/\, I{+}J = H^*(M^{2n})$ is the quotient epimorphism.

Thus, we have $E_2 = \mathrm{Tor}_{k[v_1,\ldots,v_m]}(k(P), k)$ and $\bar{E}_2 = \mathrm{Tor}_{k[w_1,\ldots,w_{m-n}]}(k(P)/J, k)$.

To proceed further we need the following result.

Proposition 4.8 *Let Λ be an algebra and Γ a subalgebra and set $\Omega = \Lambda//\Gamma$. Suppose that Λ is a free Γ-module and we are given a right Ω-module A and a left Λ-module C. Then there exists a spectral sequence $\{E_r, d_r\}$ with*

$$E_r \Rightarrow \mathrm{Tor}_\Lambda(A, C), \quad E_2^{p,q} = \mathrm{Tor}^p_\Omega(A, \mathrm{Tor}^q_\Gamma(C, k)).$$

Proof. See [CE, p.349]. $\square$

The next proposition is a modification of one assertion from [Sm].

Proposition 4.9 *Suppose $f : k[v_1, \ldots, v_m] \to A$ is an epimorphism of graded algebras, $\deg v_i = 2$, and $J \subset A$ is an ideal generated by a length n regular sequence of degree-two elements of A. Then the following isomorphism holds:*

$$\mathrm{Tor}_{k[v_1,\ldots,v_m]}(A, k) = \mathrm{Tor}_{k[w_1,\ldots,w_{m-n}]}(A/J, k).$$

Proof. Let $J = (\lambda_1, \ldots, \lambda_n)$, $\deg \lambda_i = 2$, and $\{\lambda_1, \ldots, \lambda_n\}$ is a regular sequence. Choose $\hat{\lambda}_i \in k[v_1, \ldots, v_m]$, $1 \le i \le n$, to be preimages of λ_i of degree 2, i.e., $f(\hat{\lambda}_i) = \lambda_i$, $\hat{\lambda}_i = \lambda_{i1}v_1 + \ldots + \lambda_{im}v_m$ and $\mathrm{rk}(\lambda_{ij}) = n$. Let us take elements $w_1, \ldots, w_{m-n}$ of degree two such that

$$k[v_1, \ldots, v_m] = k[\hat{\lambda}_1, \ldots, \hat{\lambda}_n, w_1, \ldots, w_{m-n}],$$

and put $\Gamma = k[\hat{\lambda}_1, \ldots, \hat{\lambda}_n]$. Then $k[v_1, \ldots, v_m]$ is a free Γ-module, and so, by proposition 4.8, we have a spectral sequence

$$E_r \Rightarrow \mathrm{Tor}_{k[v_1,\ldots,v_m]}(A, k), \quad E_2 = \mathrm{Tor}_\Omega(\mathrm{Tor}_\Gamma(A, k), k),$$

where $\Omega = k[v_1, \ldots, v_m]//\Gamma = k[w_1, \ldots, w_{m-n}]$.

Since $\lambda_1, \ldots, \lambda_n$ is a regular sequence, A is a free Γ-module. Therefore,

$$\begin{aligned} \mathrm{Tor}_\Gamma(A, k) &= A \otimes_\Gamma k = A/J \quad \text{and} \quad \mathrm{Tor}^q_\Gamma(A, k) = 0 \quad \text{for} \quad q \ne 0, \\ &\Rightarrow E_2^{p,q} = 0 \quad \text{for} \quad q \ne 0, \\ &\Rightarrow \mathrm{Tor}_{k[v_1,\ldots,v_m]}(A, k) = \mathrm{Tor}_{k[w_1,\ldots,w_{m-n}]}(A/J, k), \end{aligned}$$

which concludes the proof of the proposition. $\square$

Now, we return to the proof of theorem 4.7. Setting $A = k(P)$ in proposition 4.9 we deduce that $g_2 : E_2 \to \bar{E}_2$ is an isomorphism. It follows also from the form of the E_2 terms of both spectral sequences that each of them contains only finite number of non-zero modules in each term. In this situation it is true that if a homomorphism g define an isomorphism in the E_2 terms, then g is the isomorphism of the spectral sequences (cf. [Ma, XI, theorem 1.1]). Thus, theorem 4.7 is proved. $\square$

Corollary 4.10 *$H^*(\mathcal{Z}_P) = \mathrm{Tor}_{k[w_1,\ldots,w_{m-n}]}(H^*(M^{2n}), k)$ for any quasitoric manifold M^{2n} over a simple polytope P^n.*

Proof. By theorem 4.6, $H^*(\mathcal{Z}_P) = \mathrm{Tor}_{k[v_1,\ldots,v_m]}(k(P), k)$. Hence, our assertion follows from the isomorphism between the E_2 terms of the spectral sequences from theorem 4.7. $\square$

Let us turn again to the principal T^{m-n}-bundle $\mathcal{Z}_P \to M^{2n}$ defined for any quasitoric manifold M^{2n}. The following statement similar to lemma 4.5 holds for this bundle (it can be also proved in the similar way).

Lemma 4.11 *For the Leray–Serre spectral sequence of the bundle $\mathcal{Z}_P \to M^{2n}$ the following isomorphism holds:*

$$E_3^{(s)} \cong \mathrm{Tor}_{k[w_1,\ldots,w_{m-n}]}(H^*(M^{2n}), k) = \mathrm{Tor}_{k[w_1,\ldots,w_{m-n}]}(k(P)/J\,, k),$$

where $E_3^{(s)}$ is the E_3 term of the Leray–Serre spectral sequence, and a $k[w_1, \ldots, w_{m-n}]$-module structure in $H^*(M^{2n})$ is defined by the map

$$k[w_1, \ldots, w_{m-n}] = k[v_1, \ldots, v_m]/J \to k[v_1, \ldots, v_m]/I{+}J = H^*(M^{2n}). \quad \square$$

Theorem 4.12 *Suppose M^{2n} is a quasitoric manifold over P^n. Then the Leray–Serre spectral sequence of the principle T^{m-n}-bundle $\mathcal{Z}_P \to M^{2n}$ collapses in the E_3 term, i.e., $E_3 = E_\infty$. Furthermore, the following isomorphism of algebras holds*

$$\begin{gathered} H^*(\mathcal{Z}_P) = H[(k(P)/J) \otimes \Lambda[u_1, \ldots, u_{m-n}], d], \\ \mathrm{bideg}\, a = (0, \deg a), \quad \mathrm{bideg}\, u_i = (-1, 2); \\ d(1 \otimes u_i) = w_i \otimes 1, \quad d(a \otimes 1) = 0, \end{gathered}$$

where $a \in k(P)/J = k[w_1, \ldots, w_{m-n}]/I$ and $\Lambda[u_1, \ldots, u_{m-n}]$ is an exterior algebra.

Proof. The cohomology algebra $H[(k(P)/J)\otimes\Lambda[u_1,\dots,u_{m-n}],d]$ is exactly the E_3 term of the Leray–Serre spectral sequence for the bundle $\mathcal{Z}_P\to M^{2n}$. At the same time, it follows from proposition 4.4 that this cohomology algebra is isomorphic to $\mathrm{Tor}_{k[w_1,\dots,w_{m-n}]}(H^*(M^{2n}),k)$. From corollary 4.10 we deduce that this is exactly $H^*(\mathcal{Z}_P)$. Since the Leray–Serre spectral sequence converges to $H^*(\mathcal{Z}_P)$, it follows that it collapses in the E_3 term. $\square$

The algebra $(k(P)/J)\otimes\Lambda[u_1,\dots,u_{m-n}]$ from theorem 4.12 is significantly smaller than the algebra $k(P)\otimes\Lambda[u_1,\dots,u_m]$ from general theorem 4.6, which allows to calculate the cohomologies of $\mathcal{Z}_P$ more efficiently.

In the rest of this subsection we study free actions on $\mathcal{Z}_P$ of tori of arbitrary rank, not necessarily $m-n$ as in the case of quasitoric manifolds.

The existence of a quasitoric manifold M^{2n} over the simple polytope P^n means that one can find a subgroup $H\subset T^m$ isomorphic to T^{m-n} that acts freely on the corresponding manifold $\mathcal{Z}_P$. (Then $M^{2n}=\mathcal{Z}_P/H$.) In the general case, such a subgroup may fail to exist; however, one still could find a subgroup of dimension less than $m-n$ that acts freely on $\mathcal{Z}_P$. In this case the corresponding quotient $\mathcal{Z}_P/H$ would be a smooth manifold. So, let $H\cong T^k$ acts on $\mathcal{Z}_P$ freely. Then the inclusion $s:H\hookrightarrow T^m$ is defined by an integer $(m\times k)$-matrix $S=(s_{ij})$ such that the $\mathbb{Z}$-module spanned by its columns $s_j=(s_{1j},\dots,s_{mj})$, $j=\overline{1,k}$ is a direct summand in $\mathbb{Z}^m$. Choose any basis $t_i=(t_{i1},\dots,t_{im})$, $i=1,\dots,m-k$ in the kernel of the dual map $s^*:(\mathbb{Z}^m)^*\to(\mathbb{Z}^k)^*$. Then we have the following result describing the cohomology ring of the manifold $\mathcal{Y}_{(k)}=\mathcal{Z}_P/H$ and generalizing simultaneously corollary 3.4 and theorem 4.2.

Theorem 4.13 *The following isomorphism of algebras holds:*

$$H^*(\mathcal{Y}_{(k)})\cong\mathrm{Tor}_{k[t_1,\dots,t_{m-k}]}(k(P),k),$$

where the action $k[t_1,\dots,t_{m-k}]$ on $k(P)=k[v_1,\dots,v_m]/I$ is defined by the map

$$\begin{array}{rcl} k[t_1,\dots,t_{m-k}] & \to & k[v_1,\dots,v_m]\\ t_i & \to & t_{i1}v_1+\ldots+t_{im}v_m. \end{array}$$

Proof. The inclusion of the subgroup $H\simeq T^k\to T^m$ defines a map of classifying spaces $h:BT^k\to BT^m$. Let us consider the bundle pulled back by this map from the bundle $p:B_TP\to BT^m$ with fibre $\mathcal{Z}_P$. It follows directly from the construction of B_TP (see subsection 1.2) that the total space of this bundle has homotopy type $\mathcal{Y}_{(k)}$ (more precisely, it is homeomorphic to $\mathcal{Y}_{(k)}\times ET^k$). Hence, we have a commutative square

$$\begin{array}{ccc} \mathcal{Y}_{(k)} & \longrightarrow & B_TP\\ \downarrow & & \downarrow\\ BT^k & \longrightarrow & BT^m. \end{array}$$

The corresponding Eilenberg–Moore spectral sequence converges to the cohomologies of $\mathcal{Y}_{(k)}$ and has the following E_2 term

$$E_2=\mathrm{Tor}_{k[v_1,\dots,v_m]}(k(P),k[w_1,\dots,w_k]),$$

where the action of $k[v_1,\dots,v_m]$ on $k[w_1,\dots,w_k]$ is defined by the map s^*, i.e., $v_i\to s_{i1}w_1+\ldots+s_{ik}w_k$. Using [Sm, proposition 3.4] in the similar way as in the proof of theorem 4.6, we show that the spectral sequence collapses in the E_2 term and the following isomorphism of algebras holds:

$$H^*(\mathcal{Y}_{(k)})=\mathrm{Tor}_{k[v_1,\dots,v_m]}(k(P),k[w_1,\dots,w_k]). \tag{15}$$

Now set $\Lambda = k[v_1, \dots, v_m]$, $\Gamma = k[t_1, \dots, t_{m-k}]$, $A = k[w_1, \dots, w_k]$ and $C = k(P)$ in the proposition 4.8. Since Λ here is a free Γ-module and $\Omega = \Lambda//\Gamma = k[w_1, \dots, w_k]$, this gives us a spectral sequence $\{E_r, d_r\}$ converging to $\operatorname{Tor}_{k[v_1,\dots,v_m]}(k(P), k[w_1, \dots, w_k])$ whose E_2 term is

$$E_2^{p,q} = \operatorname{Tor}^p_{k[w_1,\dots,w_k]}(A, \operatorname{Tor}^q_{k[t_1,\dots,t_{m-k}]}(k(P), k)).$$

But since A is a free $k[w_1, \dots, w_k]$-module with one generator 1, we have

$$E_2^{p,q} = 0 \text{ for } p \neq 0, \quad E_2^{0,q} = \operatorname{Tor}^q_{k[t_1,\dots,t_{m-k}]}(k(P), k).$$

Thus, the spectral sequence collapses in the E_2 term and we have the isomorphism of algebras:

$$\operatorname{Tor}_{k[v_1,\dots,v_m]}(k(P), k[w_1, \dots, w_k]) \simeq \operatorname{Tor}_{k[t_1,\dots,t_{m-k}]}(k(P), k),$$

which together with the isomorphism (15) proves the theorem. $\square$

Below we characterize subgroups $H \subset T^m$ that act on $\mathcal{Z}_P$ freely.

Let us consider again the integral $(m \times k)$-matrix S defining the subgroup $H \subset T^m$ of rank k. For each vertex $v = F_{i_1} \cap \dots \cap F_{i_n}$ of the polytope P^n we take the $(m-n) \times k$-submatrix $S_{i_1,\dots,i_n}$ of S, which is obtained by deleting the rows $i_1, \dots, i_n$. In this way we construct $r = f_{n-1}$ submatrices of the size $(m-n) \times k$. Then the following criterion for the freeness of the action of H on $\mathcal{Z}_P$ holds.

Lemma 4.14 *The action of the subgroup $H \subset T^m$ defined by an integral $(m\times k)$-matrix S on the manifold $\mathcal{Z}_P$ is free if and only if for any vertex $v = F_{i_1} \cap \ldots \cap F_{i_n}$ of P^n the corresponding $(m-n) \times k$-submatrix $S_{i_1,\dots,i_n}$ defines a direct summand $\mathbb{Z}^k \subset \mathbb{Z}^{m-n}$.*

Proof. It follows from definition 1.6 that the orbits of the action of T^m on $\mathcal{Z}_P$ corresponding to the vertices $v = F_{i_1} \cap \ldots \cap F_{i_n}$ of P^n have maximal (rank n) isotropy subgroups. This isotropy subgroups are the coordinate subgroups $T^n_{i_1,\dots,i_n} \subset T^m$. A subgroup H acts freely on $\mathcal{Z}_P$ if and only if it has only unit in the intersection with each isotropy subgroup. This means that the $m \times (k+n)$-matrix obtained by adding to S of n columns $(0, \dots, 0, 1, 0, \dots, 0)^\top$ with 1 on the place i_j, $j = \overline{1, n}$ defines a direct summand $\mathbb{Z}^{k+n} \subset \mathbb{Z}^m$ (this matrix correspond to the subgroup $H \times T^n_{i_1,\dots,i_n} \subset T^m$). But this is equivalent to the requirements of the lemma. $\square$

In particular, for subgroups of the maximal possible for the free action rank $m-n$ we obtain

Corollary 4.15 *The action of the rank $m-n$ subgroup $H \subset T^m$ defined by an integral $m \times (m-n)$-matrix S on the manifold $\mathcal{Z}_P$ is free if and only if for any vertex $v = F_{i_1} \cap \ldots \cap F_{i_n}$ of P^n the minor of S obtained by deleting the rows $i_1, \dots, i_n$ equals ± 1.* $\square$

This corollary is "an integer analog" of theorem 2.8, which gives the criterion of freeness for the action of a subgroup $R^{m-n}_+ \subset \mathbb{R}^m_+$ on the set $U(P^n) \subset \mathbb{C}^m$. However, unlike the situation of theorem 2.8, the subgroup $H \simeq T^{m-n}$ satisfying the condition of corollary 4.15 may fail to exist.

It can be easily seen that the condition from corollary 4.15 is equivalent to the following: the map $\mathbb{Z}^m \to \mathbb{Z}^n$ defined by factorizing by the image of the inclusion $s : \mathbb{Z}^{m-n} \to \mathbb{Z}^m$ is a characteristic function in the sense of definition 1.11. Thus, we obtained the another interpretation of the fact that quasitoric manifolds exists over P^n if and only if it is possible to find a subgroup $H \simeq T^{m-n}$ that acts on $\mathcal{Z}_P$ freely.

It also follows from lemma 4.14 that the one-dimensional subgroup $H \simeq T^1$ corresponding to the diagonal map $T^1 \subset T^m$ always acts on $\mathcal{Z}_P$ freely. Indeed, in this situation the matrix S is a column of

m units and the condition from lemma 4.14 is obviously satisfied. Theorem 4.13 gives us the following formula for the cohomologies of the corresponding manifold $\mathcal{Y}_{(1)} = \mathcal{Z}_P/H$:

$$H^*(\mathcal{Y}_{(1)}) \simeq \mathrm{Tor}_{k[t_1,\ldots,t_{m-1}]}(k(P), k), \tag{16}$$

where the action of $k[t_1, \ldots, t_{m-1}]$ on $k(P) = k[v_1, \ldots, v_m]/I$ is defined by the homomorphism

$$\begin{array}{rcl} k[t_1, \ldots, t_{m-1}] & \to & k[v_1, \ldots, v_m], \\ t_i & \to & v_i - v_m. \end{array}$$

The principal T^1-bundle $\mathcal{Z}_P \to \mathcal{Y}_{(1)}$ is pulled back from the universal T^1-bundle by a certain map $c : \mathcal{Y}_{(1)} \to BT^1 = \mathbb{C}P^\infty$. Since $H^*(\mathbb{C}P^\infty) = k[v]$, $v \in H^2(\mathbb{C}P^\infty)$, we can consider the element $c^*(v) \in H^2(\mathcal{Y}_{(1)})$. Then, the following statement holds.

Lemma 4.16 *A polytope P^n is q-neighbourly if and only if $(c^*(v))^q \neq 0$.*

Proof. The map c^* takes the ring $k[v]$ of cohomologies of $\mathbb{C}P^\infty$ to the subring $k(P) \otimes_{k[t_1,\ldots,t_{m-1}]} k = \mathrm{Tor}^0_{k[t_1,\ldots,t_{m-1}]}(k(P), k)$ of the cohomology ring of $\mathcal{Y}_{(1)}$ (see (16)). This subring is isomorphic to the quotient ring $k(P)/\{v_1 = \ldots = v_m\}$. Now, the assertion follows from the fact that a polytope P^n is q-neighbourly if and only if the ideal I from the definition of $k(P)$ does not contain monomials of degree less than $q + 1$. $\Box$

Now we return to the general case of a subgroup $H \simeq T^k$ acting on $\mathcal{Z}_P$ freely. For such a subgroup we have

$$B_T P = \mathcal{Z}_P \times_{T^m} ET^m = \big((\mathcal{Z}_P/T^k) \times_{T^{m-k}} ET^{m-k}\big) \times ET^k = (\mathcal{Y}_{(k)} \times_{T^{m-k}} ET^{m-k}) \times ET^k.$$

Hence, there is defined a principal T^{m-k}-bundle $\mathcal{Y}_{(k)} \times ET^m \to B_T P$.

Theorem 4.17 *The Leray–Serre spectral sequence of the T^{m-k}-bundle $\mathcal{Y}_{(k)} \times ET^m \to B_T P$ collapses in the E_3 term, i.e., $E_3 = E_\infty$. Furthermore,*

$$\begin{array}{rcl} H^*(\mathcal{Y}_{(k)}) & = & H[k(P) \otimes \Lambda[u_1, \ldots, u_{m-k}], d], \\ d(1 \otimes u_i) & = & (t_{i1}v_1 + \ldots + t_{im}v_m) \otimes 1,\ d(a \otimes 1) = 0,\ \mathrm{bideg}\, a = (0, \deg a),\ \mathrm{bideg}\, u_i = (-1, 2), \end{array}$$

where $a \in k(P) = k[v_1, \ldots, v_n]/I$ and $\Lambda[u_1, \ldots, u_{m-k}]$ is an exterior algebra.

Proof. We can prove in the similar way as in lemma 4.5 that the E_3 term of the spectral sequence is

$$E_3 = H[k(P) \otimes \Lambda[u_1, \ldots, u_{m-k}], d] = \mathrm{Tor}_{k[t_1,\ldots,t_{m-k}]}(k(P), k).$$

Theorem 4.13 shows that this is exactly $H^*(\mathcal{Y}_{(k)})$. $\Box$

Theorem 4.6 and corollary 3.4 can be obtained from this theorem by setting $k = 0$ and $k = m - n$ respectively.

4.4 The explicit calculation of $H^*(\mathcal{Z}_P)$ for some particular polytopes.

1. Our first example shows how the above methods work in the simple case where P is a product of simplices. So, let us consider $P^n = \Delta^{i_1} \times \Delta^{i_2} \times \ldots \times \Delta^{i_k}$, where Δ^i is a i-simplex and $\sum_k i_k = n$. This P^n has $n+k$ facets, i.e., $m = n+k$. Hence, we deduce from lemma 2.11 that $\mathcal{Z}_P = \mathcal{Z}_{\Delta^{i_1}} \times \ldots \times \mathcal{Z}_{\Delta^{i_k}}$.

The minimal resolution (11) of $k(P_i)$ in the case $P_i = \Delta^i$ is as follows

$$0 \longrightarrow R^{-1} \xrightarrow{d^{-1}} R^0 \xrightarrow{d^0} k(P_i) \longrightarrow 0,$$

where R^0, R^{-1} are free one-dimensional $k[v_1, \ldots, v_{i+1}]$-modules and d^{-1} is the multiplication by $v_1 \cdot \ldots \cdot v_{i+1}$. Hence, we have the isomorphism of algebras

$$\mathrm{Tor}_{k[v_1,\ldots,v_{i+1}]}(k(P_i), k) = \Lambda[a], \quad \operatorname{bideg} a = (-1, 2i+2),$$

where $\Lambda[a]$ is an exterior k-algebra on one generator a. Now, theorem 4.6 shows that

$$H^*(\mathcal{Z}_{\Delta^i}) = \Lambda[a], \quad \deg a = 2i+1,$$

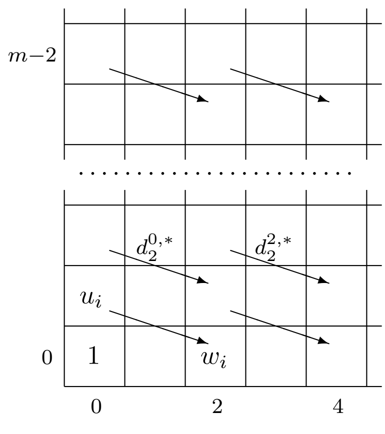
Thus, the cohomologies of $\mathcal{Z}_P = \mathcal{Z}_{\Delta^{i_1}} \times \ldots \times \mathcal{Z}_{\Delta^{i_k}}$ are

$$H^*(\mathcal{Z}_P) = \Lambda[a_1, \ldots, a_k], \quad \deg a_l = 2i_l + 1.$$

Actually, example 2.10 shows that in our situation $\mathcal{Z}_P = S^{2i_1+1} \times \ldots \times S^{2i_k+1}$. However, our calculation of cohomology does not use the geometrical constructions from section 2.

2. In our next example we consider the case of plane polygons, i.e., $n = 2$. Let P^2 be a convex m-gon. Then the corresponding manifold $\mathcal{Z}_P$ has dimension $m+2$. First, we compute the Betti numbers of these manifolds.

Let us consider the E_2 term of the Leray–Serre spectral sequence of the bundle $p : \mathcal{Z}_P \to M^4$ with fibre T^{m-2} for some toric manifold M^4 over P^2 (it could be easily seen that there exists at least one toric manifold over any polygon). From theorem 3.3 we deduce that $H^2(M^4)$ has rank $m-2$, and the ring $H^*(M^4)$ is multiplicatively generated by the basis $w_1, \ldots, w_{m-2}$ of $H^2(M^4)$. At the same time, $H^*(T^{m-2})$ is the exterior algebra on generators $u_1, \ldots, u_{m-2}$ and the second differential of the spectral sequence takes u_i to w_i (more precisely, $d_2(u_i \otimes 1) = 1 \otimes w_i$). Furthermore, the map $p^* : H^*(M^4) \to H^*(\mathcal{Z}_P)$ is zero homomorphism in degrees ≥ 0. This follows from the fact that the map $f^* : H^*(BT^{m-2}) \to H^*(M^4)$ is epimorphic (see the proof of theorem 4.7) and the commutative diagram



$$\begin{array}{ccc} \mathcal{Z}_P & \longrightarrow & ET^{m-2} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_0} \\ M^4 & \xrightarrow{f} & BT^{m-2}. \end{array}$$

Using all these facts and corollary 4.12 (which gives $E_3 = E_\infty$), we deduce that all differentials $d_2^{0,*}$ are monomorphisms, and all differentials $d_2^{2,*}$ are epimorphisms.

Now, using theorem 4.12 we obtain by easy calculations the following formulae for the Betti numbers $b^i(\mathcal{Z}_P)$:

$$\begin{aligned}
&b^0(\mathcal{Z}) = b^{m+2}(\mathcal{Z}) = 1,\\
&b^1(\mathcal{Z}) = b^2(\mathcal{Z}) = b^m(\mathcal{Z}) = b^{m+1}(\mathcal{Z}) = 0,\\
&b^k(\mathcal{Z}) = (m-2)\tbinom{m-2}{k-2} - \tbinom{m-2}{k-1} - \tbinom{m-2}{k-3} = \tbinom{m-2}{k-3}\tfrac{m(m-k)}{k-1}, \quad 3 \le k \le m-1.
\end{aligned} \tag{17}$$

In the small dimensions the above formulae give us the following:

$$\begin{aligned}
m = 3: \quad & b^0(\mathcal{Z}^5) = b^5(\mathcal{Z}^5) = 1 \quad \text{and others are 0},\\
m = 4: \quad & b^0(\mathcal{Z}^6) = b^6(\mathcal{Z}^6) = 1, \quad b^3(\mathcal{Z}^6) = 2 \quad \text{and others are 0}.
\end{aligned}$$

Both this cases are covered by the previous example, since for $m = 3$ we have $P^2 = \Delta^2$, and for $m = 4$ we have $P^2 = \Delta^1 \times \Delta^1$. As it was pointed out above, in this cases $\mathcal{Z}_P^5 = S^5$, $\mathcal{Z}_P^6 = S^3 \times S^3$. Further,

$$\begin{aligned}
m = 5: \quad & b^0(\mathcal{Z}^7) = b^7(\mathcal{Z}^7) = 1,\ b^3(\mathcal{Z}^7) = b^4(\mathcal{Z}^7) = 5, \text{ and others are 0},\\
m = 6: \quad & b^0(\mathcal{Z}^8) = b^8(\mathcal{Z}^8) = 1,\ b^3(\mathcal{Z}^8) = b^5(\mathcal{Z}^8) = 9,\ b^4(\mathcal{Z}^8) = 16, \text{ and others are 0},
\end{aligned}$$

and so on.

Now we want to describe the ring structure in the cohomologies. Theorem 4.6 gives us the isomorphism of algebras

$$H^*(\mathcal{Z}_P^{m+2}) \cong \mathrm{Tor}_{k[v_1,\ldots,v_m]}(k(P^2), k) = H[k(P^2) \otimes \Lambda[u_1, \ldots, u_m], d]. \tag{18}$$

If $m = 3$, then $k(P) = k[v_1, v_2, v_3]/v_1v_2v_3$; if $m > 3$ we have $k(P) = k[v_1, \ldots, v_m]/I$, where I is generated by the monomials v_iv_j such that $i \ne j \pm 1$ (here we use the agreement $v_{m+i} = v_i \;\; v_{i-m} = v_i$). Below we give the complete description of multiplication in the case $m = 5$. The general case is similar but more involved. It is easy to check that five generators of $H^3(\mathcal{Z}_P)$ are represented by the cocycles $v_i \otimes u_{i+2}$, $i = \overline{1,5}$ in the algebra $k(P^2) \otimes \Lambda[u_1, \ldots, u_m]$, and five generators of $H^4(\mathcal{Z}_P)$ are represented by the cocycles $v_j \otimes u_{j+2}u_{j+3}$, $j = \overline{1,5}$. The product of cocycles $v_i \otimes u_{i+2}$ and $v_j \otimes u_{j+2}u_{j+3}$ represents a non-trivial cohomology class in $H^7(\mathcal{Z}_P)$ if and only if the set $\{i, i+2, j, j+2, j+3\}$ is the whole index set $\{1, 2, 3, 4, 5\}$. Hence, for each cohomology class $[v_i \otimes u_{i+2}]$ there is a unique (Poincaré dual) cohomology class (which can be written as $[v_j \otimes u_{j+2}u_{j+3}]$) such that the product with it is non-trivial. This only non-trivial product defines a fundamental cohomology class of $\mathcal{Z}_P$ (for example, this class can be represented by the cocycle $v_1v_2 \otimes u_3u_4u_5$). In the next section we will prove the similar statement in the general case. In our situation all other product in the cohomology algebra $H^*(\mathcal{Z}_P^7)$ are trivial.

5 The cohomologies of $\mathcal{Z}_P$ and the combinatorics of simple polytopes

Theorem 4.6 shows that the cohomologies of $\mathcal{Z}_P$ is naturally a bigraded algebra. The Poincaré duality in $H^*(\mathcal{Z}_P)$ regards this bigraded structure. More precisely, the Poincaré duality has the following combinatorial interpretation.

Lemma 5.1 *In the bigraded differential algebra $[k(P) \otimes \Lambda[u_1, \ldots, u_m], d]$ from theorem 4.6*

1. *the fundamental cocycle of $\mathcal{Z}_P$ is represented by any element of type $v_{i_1} \cdots v_{i_n} \otimes u_{j_1} \cdots u_{j_{m-n}}$, $j_1 < \ldots < j_{m-n}$, where $(i_1, \ldots, i_n)$ is the index set of all codimension-one faces meeting in some vertex $v \in P^n$, and $\{i_1, \ldots, i_n, j_1, \ldots, j_{m-n}\} = \{1, \ldots, m\}$.*

2. *two cocycles $v_{i_1}\cdots v_{i_p}\otimes u_{j_1}\cdots u_{j_r}$ and $v_{k_1}\cdots v_{k_s}\otimes u_{l_1}\cdots u_{l_t}$ represent the Poincaré dual cohomology classes in $H^*(\mathcal{Z}_P)$ if and only if $p+s=n$, $r+t=m-n$, $\{i_1,\ldots,i_p,k_1,\ldots,k_s\}$ is the index set of facets meeting in some vertex $v\in P^n$, and $\{i_1,\ldots,i_p,j_1,\ldots,j_r,k_1,\ldots,k_s,l_1,\ldots,l_t\}=\{1,\ldots,m\}$.*

Proof. The first assertion follows from the fact that the cohomology class described in 1) is a generator of the module $\mathrm{Tor}^{-(m-n),2m}_{k[v_1,\ldots,v_m]}(k(P),k)$, which is isomorphic to $H^{m+n}(\mathcal{Z}_P^{m+n})$ (see theorem 4.6). The second assertion holds since two cohomology classes are Poincar'e dual if and only if their product is the fundamental cohomology class. $\square$

In the sequel we will use the following short notations: $\mathcal{T}^i=\mathrm{Tor}^{-i}_{k[v_1,\ldots,v_m]}(k(P),k)$ and $\mathcal{T}^{i,2j}=\mathrm{Tor}^{-i,2j}_{k[v_1,\ldots,v_m]}(k(P),k)$. We define the *bigraded Betti numbers* of $\mathcal{Z}_P$ as

$$b^{-i,2j}(\mathcal{Z}_P)=\dim_k\mathrm{Tor}^{-i,2j}_{k[v_1,\ldots,v_m]}(k(P),k). \tag{19}$$

Then theorem 4.2 can be reformulated as $b^k(\mathcal{Z}_P)=\sum_{2j-i=k}b^{-i,2j}(\mathcal{Z}_P)$. The second part of lemma 5.1 shows that $b^{-i,2j}(\mathcal{Z}_P)=b^{-(m-n-i),2(m-j)}(\mathcal{Z}_P)$ for all i,j. These equalities can be written as the following identities for the Poincaré series $F(\mathcal{T}^i,t)=\sum_{r=0}^m b^{-i,2r}t^{2r}$ of $\mathcal{T}^i$:

$$F(\mathcal{T}^i,t)=t^{2m}F\left(\mathcal{T}^{m-n-i},\tfrac{1}{t}\right). \tag{20}$$

It is well known in commutative algebra that the above identity holds for the so-called Gorenstein rings (cf. [St]). The face ring of a simplicial subdivision of sphere is a Gorenstein ring. In particular, the ring $k(P^n)$ is Gorenstein for any simple polytope P^n.

A simple combinatorial argument (cf. [St, part II, §1]) shows that for any $(n-1)$-dimensional simplicial complex K the Poincaré series $F(k(K),t)$ can be written as follows

$$F(k(K),t)=1+\sum_{i=0}^{n-1}\frac{f_it^{2(i+1)}}{(1-t^2)^{i+1}},$$

where $(f_0,\ldots,f_{n-1})$ is the f-vector of K. This series can be also expressed in terms of the h-vector $(h_0,\ldots,h_n)$ (see (1)) as:

$$F(k(K),t)=\frac{h_0+h_1t^2+\ldots+h_nt^{2n}}{(1-t^2)^n}. \tag{21}$$

On the other hand, the Poincaré series of the $k[v_1,\ldots,v_m]$-module $k(P)$ (or $k(K)$) can be calculated from any free resolution of $k(P)$. More precisely, the following general theorem holds (see e.g., [St]).

Theorem 5.2 *Let M be a finitely generated graded $k[v_1,\ldots,v_m]$-module, $\deg v_i=2$, and there is given a finite free resolution of M:*

$$0\longrightarrow R^{-h}\xrightarrow{d^{-h}}R^{-h+1}\xrightarrow{d^{-h+1}}\cdots\longrightarrow R^{-1}\xrightarrow{d^{-1}}R^0\xrightarrow{d^0}M\longrightarrow 0.$$

Suppose that the free $k[v_1,\ldots,v_m]$-modules R^{-i} have their generators in dimensions $d_{1i},\ldots,d_{q_ii}$, where $q_i=\dim_{k[v_1,\ldots,v_m]}R^{-i}$. Then the Poincaré series of M can be calculated by the following formula:

$$F(M,t)=\frac{\sum_{i=0}^{-h}(-1)^i(t^{d_{1i}}+\ldots+t^{d_{q_ii}})}{(1-t^2)^m}.\quad\square$$

Now let us apply this theorem to the minimal resolution (11) of $k(P)=k[v_1,\dots,v_m]/I$. Since all differentials of the complex (12) are trivial, we obtain

$$F(k(P),t)=(1-t^2)^{-m}\sum_{i=0}^{m-n}(-1)^iF(\mathcal{T}^i,t). \tag{22}$$

Combining this with (20), we get

$$F(k(P),t)=(1-t^2)^{-m}\sum_{i=0}^{m-n}(-1)^it^{2m}F(\mathcal{T}^{m-n-i},\tfrac{1}{t})=$$
$$=(1-(\tfrac{1}{t})^2)^{-m}\cdot(-1)^m\sum_{j=0}^{m-n}(-1)^{m-n-j}F(\mathcal{T}^j,\tfrac{1}{t})=(-1)^nF(k(P),\tfrac{1}{t}).$$

Substituting here the expressions from the right hand side of (21) for $F(k(P),t)$ and $F(k(P),\frac{1}{t})$, we finally deduce

$$h_i=h_{n-i}. \tag{23}$$

These are the well-known *Dehn–Sommerville equations* [Br] for simple (or simplicial) polytopes.

Thus, we see that the algebraic duality (20) and the combinatorial Dehn–Sommerville equations (23) follows from the Poincaré duality for the manifold $\mathcal{Z}_P$. Furthermore, combining (21) and (22) we obtain

$$\sum_{i=0}^{m-n}(-1)^iF(\mathcal{T}^i,t)=(1-t^2)^{m-n}h(t^2), \tag{24}$$

where $h(t)=\sum_{i=0}^{n}h_it^i$.

Along with the cochain complex $[k(P)\otimes\Lambda[u_1,\dots,u_m],d]$ from theorem 4.6 we will consider its subcomplex $\mathcal{A}$ defined as follows. As a k-module $\mathcal{A}$ is generated by the monomials $v_{i_1}\dots v_{i_p}\otimes u_{j_1}\dots u_{j_q}$ and $1\otimes u_{j_1}\dots u_{j_k}$ such that $\{v_{i_1},\dots,v_{i_p}\}$ *generates* a simplex in K_P and $\{i_1,\dots,i_p\}\cap\{j_1,\dots,j_q\}=\varnothing$. It can be easily checked that $d(\mathcal{A})\subset\mathcal{A}$ and, therefore, $\mathcal{A}$ is a cochain subcomplex. Moreover, $\mathcal{A}$ inherits the bigraded module structure from $k(P)\otimes\Lambda[u_1,\dots,u_m]$ with differential d adding $(1,0)$ to bidegree.

Lemma 5.3 *The cochain complexes $[k(P)\otimes\Lambda[u_1,\dots,u_m],d]$ and $[\mathcal{A},d]$ have same cohomologies. Hence, the following isomorphism of k-modules holds:*

$$H[\mathcal{A},d]\cong\mathrm{Tor}_{k[v_1,\dots,v_m]}(k(P),k).$$

Proof. It is sufficient to prove that any cocycle $\omega=v_{i_1}^{\alpha_1}\dots v_{i_p}^{\alpha_p}\otimes u_{j_1}\dots u_{j_q}$ from $k(P)\otimes\Lambda[u_1,\dots,u_m]$ that does not lie in $\mathcal{A}$ is a coboundary. To do this we note that if there is any $i_k\in\{i_1,\dots,i_p\}\cap\{j_1,\dots,j_q\}$, then $d\omega$ contains the summand $v_{i_1}^{\alpha_1}\dots v_{i_k}^{\alpha_k+1}\dots v_{i_p}^{\alpha_p}\otimes u_{j_1}\dots\widehat{u}_{i_k}\dots u_{j_q}$, hence, $d\omega\neq0$ — a contradiction. Therefore, $\{i_1,\dots,i_p\}\cap\{j_1,\dots,j_q\}=\varnothing$. If ω contains at least one v_k with degree $\alpha_k>1$, then since $d\omega=0$, we have $\omega=\pm d\left(v_{i_1}^{\alpha_1}\dots v_{i_k}^{\alpha_k-1}\dots v_{i_p}^{\alpha_p}\otimes u_{i_k}u_{j_1}\dots u_{j_q}\right)$. Now, our assertion follows from the fact that all nonzero elements of $k(P)$ of the type $v_{i_1}\dots v_{i_p}$ correspond to simplices of K_P. □

Now let us introduce the submodules $\mathcal{A}^{*,2r}\subset\mathcal{A}$, $r=0,\dots,m$ generated by the monomials $v_{i_1}\dots v_{i_p}\otimes u_{j_1}\dots u_{j_q}\in\mathcal{A}$ such that $p+q=r$. Hence, $\mathcal{A}^{*,2r}$ is the submodule in $\mathcal{A}$ consisting of all elements of internal degree $2r$ (i.e., for any $\omega \in \mathcal{A}^{*,2r}$ one has $\operatorname{bideg}\omega = (*, 2r)$; remember that the internal degree corresponds to the second grading). It is clear that $\sum_{r=0}^{2m} \mathcal{A}^{*,2r} = \mathcal{A}$. Since the differential d does not change the internal degree, all $\mathcal{A}^{*,2r}$ are subcomplexes in $\mathcal{A}$. The cohomologies of this complexes are exactly $\mathcal{T}^{i,2r}$ and their dimensions are the bigraded Betti numbers $b^{-i,2r}(\mathcal{Z}_P)$. Let us consider the Euler characteristics of these subcomplexes:

$$\chi_r := \chi(\mathcal{A}^{*,2r}) = \sum_{q=0}^{m}(-1)^q \dim_k \mathcal{A}^{-q,2r} = \sum_{q=0}^{m}(-1)^q b^{-q,2r}(\mathcal{Z}_P),$$

and define

$$\chi(t) = \sum_{r=0}^{m} \chi_r t^{2r}. \tag{25}$$

Then it follows from lemma 5.3 that

$$\begin{aligned}\chi(t) = \textstyle\sum_{r=0}^{m}\sum_{q=0}^{m}(-1)^q \dim_k \mathcal{A}^{-q,2r}t^{2r} = \sum_{q=0}^{m}(-1)^q \sum_{r=0}^{m} \dim_k H^{-q}[\mathcal{A}^{*,2r}]t^{2r} = \\ = \textstyle\sum_{q=0}^{m}(-1)^q \sum_{r=0}^{m} \dim_k \mathcal{T}^{q,2r}t^{2r} = \sum_{q=0}^{m}(-1)^q F(\mathcal{T}^q, t),\end{aligned}$$

where $\mathcal{T}^{q,2r} = H^{-q,2r}[k(P) \otimes \Lambda[u_1, \ldots, u_m], d] = \operatorname{Tor}^{-q,2r}_{k[v_1,\ldots,v_m]}(k(P), k)$. Combining this with formula (24), we get

$$\chi(t) = (1 - t^2)^{m-n} h(t^2). \tag{26}$$

This formula can be also obtained directly from the definition of χ_r. Indeed, it can bee easily seen that

$$\dim_k \mathcal{A}^{-q,2r} = f_{r-q-1}\binom{m-r+q}{q}, \quad \chi_r = \sum_{j=0}^{m}(-1)^{r-j} f_{j-1}\binom{m-j}{r-j}, \tag{27}$$

(here we set $\binom{j}{k} = 0$ if $k < 0$). Then

$$\begin{aligned}\chi(t) = \textstyle\sum_{r=0}^{m} \chi_r t^{2r} = \sum_{r=0}^{m}\sum_{j=0}^{m} t^{2j}t^{2(r-j)}(-1)^{r-j} f_{j-1}\binom{m-j}{r-j} = \sum_{j=0}^{m} f_{j-1}t^{2j}(1 - t^2)^{m-j} = \\ = (1 - t^2)^m \textstyle\sum_{j=0}^{n} f_{j-1}(t^{-2} - 1)^{-j}.\end{aligned} \tag{28}$$

Further, it follows from (1) that

$$t^n h(t^{-1}) = (t - 1)^n \sum_{i=0}^{n} f_{i-1}(t - 1)^{-i}.$$

Substituting here t^{-2} for t and taking into account (28), we finally obtain

$$\frac{\chi(t)}{(1 - t^2)^m} = \frac{t^{-2n}h(t^2)}{(t^{-2} - 1)^n} = \frac{h(t^2)}{(1 - t^2)^n},$$

which is equivalent to (26).

Formula (26) allows to express the h-vector of a simple polytope P^n in terms of the bigraded Betti numbers $b^{-q,2r}(\mathcal{Z}_P)$ of the corresponding manifold $\mathcal{Z}_P$.

Lemma 5.4 *The Poincaré series* $F(\mathcal{A}^{*,*},\tau,t)=\sum_{r,q}\dim_k\mathcal{A}^{-q,2r}\tau^{-q}t^{2r}$ *of the bigraded module* $\mathcal{A}^{*,*}$ *is as follows*

$$F(\mathcal{A}^{*,*},\tau,t)=\sum_j f_{j-1}\left(1+\frac{t^2}{\tau}\right)^{m-j}t^{2j}.$$

Proof. Using formula (27), we calculate

$$\sum_{r,q}\dim_k\mathcal{A}^{-q,2r}\tau^{-q}t^{2r}=\sum_{r,q}f_{r-q-1}\binom{m-r+q}{q}\tau^{-q}t^{2r}=\sum_{r,j}f_{j-1}\binom{m-j}{r-j}\tau^{-(r-j)}t^{2r}=$$
$$=\sum_j f_{j-1}\left(1+\tfrac{t^2}{\tau}\right)^{m-j}t^{2j}.\quad\square$$

The bigraded Betti numbers $b^{-i,2j}(\mathcal{Z}_P)$ can be calculated either by means of theorem 4.6 and the results of subsection 4.3 (as we did before) or by means of the following theorem, which reduces their calculation to the calculation of cohomologies of subcomplexes of the simplicial complex K^{n-1} dual to ∂P^n.

Theorem 5.5 (Hochster, cf. [St, Ho]) *Let* K *be a simplicial complex on the vertex set* $V=\{v_1,\dots,v_m\}$, *let* $k(K)$ *be its face ring, and* $\mathcal{T}^i=\mathrm{Tor}^{-i}_{k[v_1,\dots,v_m]}(k(K),k)$. *Then*

$$F(\mathcal{T}^i,t)=\sum_{W\subseteq V}(\dim_k\tilde{H}_{|W|-i-1}(K_W))t^{2|W|},$$

where K_W *is the subcomplex of* K *consisting of all simplices with vertices in* W. $\square$

However, easy examples show that the calculation based on the above theorem becomes very involved even for small complexes K. It can be shown also that the discussed above result of [GM] applied to $U(P^n)$ (see subsection 2.2) gives the same description of $H^*(U(P^n))$ as that of $H^*(\mathcal{Z}_P)$ given by the Hochster theorem (which of course conforms with our results from subsection 2.2).

Lemma 5.6 *For any simple polytope* P *holds*

$$\mathrm{Tor}^{-q,2r}_{k[v_1,\dots,v_m]}(k(P),k)=0\qquad\text{for }\ 0<r\le q.$$

Proof. This can be seen either directly from the construction of the minimal resolution (11), or from 5.5. $\square$

Theorem 5.7 *1.* $H^1(\mathcal{Z}_P)=H^2(\mathcal{Z}_P)=0$.

2. The rank of the third cohomology group of $\mathcal{Z}_P$ *(i.e., the third Betti number* $b^3(\mathcal{Z}_P)$*) equals to the number of pairs of vertices of the simplicial complex* K^{n-1} *that are not connected by an edge. Hence, if* $f_0=m$ *is the number of vertices of* K *and* f_1 *is the number of edges, then*

$$b^3(\mathcal{Z}_P)=\frac{m(m-1)}{2}-f_1.$$

Proof. It follows from theorem 4.2 and lemma 5.6 that

$$H^3(\mathcal{Z}) = \mathrm{Tor}^{-1,4}_{k[v_1,\dots,v_m]}(k(P),k) = \mathcal{T}^{1,4}.$$

By theorem 5.5,

$$b^{-1,4}(\mathcal{Z}_P) = \dim_k \mathcal{T}^{1,4} = \sum_{W\subseteq V, |W|=2} \dim_k \tilde{H}_0(K_W).$$

Now the theorem follows from the fact that $\dim_k \tilde{H}_0(K_W) = 0$ if K_W is an edge of K, and $\dim_k \tilde{H}_0(K_W) = 1$ if K_W is a pair of disjoint points. $\square$

Remark. Combining theorems 4.2, 5.5 and lemma 5.6 we can also obtain that

$$b^4(\mathcal{Z}) = \dim_k \mathcal{T}^{2,6} = \sum_{W\subseteq V, |W|=3} \dim_k \tilde{H}_0(K_W).$$

Manifolds $\mathcal{Z}_P$ allow to give a nice interpretation not only to the Dehn–Sommerville equations (23) but also to a number of other combinatorial properties of simple polytopes. In particular, using formula (26) one can express the well-known MacMullen inequalities, the upper and lower bound conjectures (see. [Br]) in terms of the cohomologies of $\mathcal{Z}_P$. We review here only two examples.

The first non-trivial MacMullen inequality for a simple polytope P^n can be written as $h_1 \le h_2$, if $n \ge 3$. In terms of the f-vector this means that $f_1 \ge mn - \binom{n+1}{2}$. Theorem 5.7 shows that $b^3(\mathcal{Z}_P) = \binom{m}{2} - f_1$. Hence, we have the following upper bound for $b^3(\mathcal{Z}_P)$:

$$b^3(\mathcal{Z}_P) \le \binom{m-n}{2} \quad \text{if } n \ge 3. \tag{29}$$

The upper bound conjecture for the number of faces of a simple polytope can be formulated in terms of the h-vector as

$$h_i \le \binom{m-n+i-1}{i}. \tag{30}$$

Using the decomposition

$$\left(\frac{1}{1-t^2}\right)^{m-n} = \sum_{i=0}^{\infty} \binom{m-n+i-1}{i} t^{2i},$$

we deduce from (26) and (30) that

$$\chi(t) \le 1. \tag{31}$$

It would be interesting to obtain the purely topological proof of the inequalities (29) and (31).

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