

TORUS-EQUIVARIANT VECTOR BUNDLES ON PROJECTIVE SPACES

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Introduction

For a free Z -module N of rank n , let $T = T_N$ be an n -dimensional algebraic torus over an algebraically closed field k defined by N . Let $X = T_N \text{ emb}(\mathcal{A})$ be a smooth complete toric variety defined by a fan \mathcal{A} (cf. [6]). Then T acts algebraically on X . A vector bundle E on X is said to be an equivariant vector bundle, if there exists an isomorphism $f_t: t^*E \rightarrow E$ for each k -rational point t in T , where $t: X \rightarrow X$ is the action of t . Equivariant vector bundles have T -linearizations (see Definition 1.2 and [2], [4]), hence we consider T -linearized vector bundles.

The n -dimensional projective space P^n has a natural action of T and can be regarded as a toric variety. In [4], we classified indecomposable equivariant vector bundles of rank two on P^2 . When $n > 2$, Hartshorne [3] constructed vector bundles of rank two from codimension two subschemes satisfying certain conditions. Bertin and Elencwajg [2] then used this method to construct equivariant vector bundles of rank two on P^n and showed that there exist no indecomposable equivariant vector bundles of rank two on P^n which are obtained in this way.

In this paper, we generalize our method in [4] to show that there exist no indecomposable equivariant vector bundles of rank r ($1 < r < n$) on P^n (Corollary 3.5) and that indecomposable equivariant vector bundles of rank n on P^n are isomorphic to $E(d)$ or $E^*(d)$ for some integer d , where E is defined by an exact sequence

$$0 \longrightarrow \mathcal{O}_{P^n} \longrightarrow \bigoplus_{i=0}^n \mathcal{O}_{P^n}(a_i) \longrightarrow E \longrightarrow 0$$

for positive integers a_i .

Received August 23, 1986.
 Revised April 28, 1987.

§1. Preliminaries

Let N be a free Z -module of rank n . Let M be the dual Z -module of N . Then there is a natural Z -bilinear map

$$\langle \ , \ \rangle: M \times N \longrightarrow Z.$$

It can naturally be extended to $M_R \times N_R \rightarrow R$, where $M_R = M \otimes_Z R$ and $N_R = N \otimes_Z R$. We denote $\varphi(\xi) = \langle \xi, \varphi \rangle$ for ξ in M_R and φ in N_R . Let $T = T_N$ be an n -dimensional algebraic torus defined by N over an algebraically closed field k . Then we can identify M with the additive group of characters of T . Let $X = T_N \text{ emb}(\Delta)$ be a smooth complete toric variety of dimension n defined by a fan Δ of N for which the reader is referred to [6].

DEFINITION 1.1. An equivariant vector bundle E on X is a vector bundle on X such that there exists an isomorphism $f_t: t^*E \rightarrow E$ for every k -rational point t in T , where $t: X \rightarrow X$ is the action of t on X .

DEFINITION 1.2. An equivariant vector bundle $E = (E, f_t)$ is said to be T -linearized if $f_{t't} = f_{t'} \circ t'^*(f_t)$ holds for every pair of k -rational points t, t' of T , where

$$f_{t't} = f_{t'} \circ t'^*(f_t): (t't)^*E \xrightarrow{t'^*(f_t)} t'^*E \xrightarrow{f_{t'}} E.$$

In [4], we showed that an equivariant vector bundle necessarily has a T -linearization. We also studied how to describe T -linearized vector bundles in terms of fans, as we now recall.

We denote by $\Delta(l)$ the set of l -dimensional cones in Δ . For C in $\Delta(l)$, there exists a finite subset $\{\varphi_1, \dots, \varphi_i\}$ of N such that $C = R_0\varphi_1 + \dots + R_0\varphi_i$, where R_0 is the set of non-negative real numbers. We say that $\{\varphi_1, \dots, \varphi_i\}$ is the fundamental system of generators of C if φ_i are primitive, i.e., φ_i is not a non-trivial integral multiple of any element of N . The fundamental system of generators $\{\varphi_1, \dots, \varphi_i\}$ of C is uniquely determined by C and is denoted by $|C|$. We consider the following:

(I) $m: \{|C'| \mid C' \in \Delta(1)\} \longrightarrow Z^{\oplus r}$
 sending φ to $m(\varphi) = (m(\varphi)_1, \dots, m(\varphi)_r)$, and for every C in $\Delta(n)$

$$m_C: |C| \longrightarrow Z^{\oplus r}$$

so that there exists a permutation $\tau = \tau_C$ such that

$$m_C(\varphi) = (m_C(\varphi)_1, \dots, m_C(\varphi)_r) = (m(\varphi)_{\tau(1)}, \dots, m(\varphi)_{\tau(r)})$$

for every φ in $|C|$.

Let C be an n -dimensional cone in $\Delta(n)$. Then we have a set of characters $\{\xi(C)_1, \dots, \xi(C)_r\}$ in M by solving, for each $1 \leq i \leq r$, the equations $\varphi(\xi(C)_i) = m_C(\varphi)_i$ for every φ in $|C|$. Then it is easy to see that (I) is equivalent to the following:

$$(I') \quad \xi: \Delta(n) \longrightarrow M^{\oplus r}$$

sending C to $\xi(C) = (\xi(C)_1, \dots, \xi(C)_r)$ such that there exists a permutation $\tau = \tau_{C,C'}$ for every pair of cones C and C' in $\Delta(n)$, so that $\varphi(\xi(C)_i) = \varphi(\xi(C')_{\tau(i)})$ for every i and every φ in $|C| \cap |C'|$.

$$(II) \quad P: \Delta(n) \times \Delta(n) \longrightarrow GL_r(k)$$

sending (C, C') to $P(C, C') = (P(C, C')_{ij})$ such that $P(C, C')_{ij} \neq 0$ only if $\varphi(\xi(C)_i) \geq \varphi(\xi(C')_j)$ for every φ in $|C| \cap |C'|$ and that

$$P(C, C')P(C', C'') = P(C, C'')$$

for every C, C', C'' in $\Delta(n)$.

For (m, P) defined by (I) and (II), we denote by $E(m, P)$ the T -linearized vector bundle obtained from (m, P) . We refer the reader to [4] as for the construction of the T -linearized vector bundle $E(m, P)$.

(III) Two pairs (m, P) and (m', P') defined by (I) and (II) are said to be equivalent if there exists a permutation $\tau = \tau_C$ for every C in $\Delta(n)$ such that

$$(m_C(\varphi)_1, \dots, m_C(\varphi)_r) = (m'_C(\varphi)_{\tau(1)}, \dots, m'_C(\varphi)_{\tau(r)})$$

for every φ in $|C|$ and if there exists

$$\sigma: \Delta(n) \longrightarrow GL_r(k)$$

such that $\sigma(C)_{ij} \neq 0$ only if $\varphi(\xi(C)_i) \geq \varphi(\xi(C)_j)$ for every φ in $|C|$ and such that

$$P'(C, C') = \sigma(C)^{-1}P(C, C')\sigma(C')$$

holds for every C and C' in $\Delta(n)$.

THEOREM 1.3 (cf. ([4]). *Let $X = T_N \text{ emb } (\Delta)$ be a smooth complete toric variety defined by a fan Δ . Then the set of T -linearized vector bundles of rank r up to T -isomorphism corresponds bijectively to the set of (I) (or (I')) and (II) up to the equivalence (III).*

Remark 1.4 Let D_φ be the divisor corresponding to the cone R_φ in $\Delta(1)$. Put $m_\varphi = m(\varphi)$ where m is defined by (I) in the case $r = 1$. Let

$P(C, C') = 1$ for every C and C' in $\Delta(n)$. Then the T -linearized vector bundle $E(m, P)$ is the line bundle $\mathcal{O}_X(-\sum m_\varphi D_\varphi)$, where the summation is taken over φ in $\{|C| \mid C \in \Delta(1)\}$.

Remark 1.5. Let $E = E(m, P)$ be the T -linearized vector bundle of rank r defined by (m, P) . Then $E \otimes \mathcal{O}_X(-\sum m_n D_\varphi)$ is T -isomorphic to $E(m', P)$, where

$$m'(\varphi) = (m(\varphi)_1 + m_\varphi, \dots, m(\varphi)_r + m_\varphi)$$

for every φ in $\{|C| \mid C \in \Delta(1)\}$. The dual vector bundle E^* is T -isomorphic to $E(-m, {}^tP^{-1})$, where

$${}^tP^{-1}(C, C') = {}^tP(C, C')^{-1},$$

and

$$-m(\varphi) = (-m(\varphi)_1, \dots, -m(\varphi)_r)$$

for every φ in $\{|C| \mid C \in \Delta(1)\}$.

§2. Some lemmas

LEMMA 2.1. *Let C and C' be two cones in $\Delta(n)$. Suppose $P(C, C')_{ii} \neq 0$ holds for every i . Then $\varphi(\xi(C)_i) = \varphi(\xi(C')_i)$ holds for every i and every φ in $|C| \cap |C'|$.*

Proof. Since $P(C, C')_{ii} \neq 0$ we have $\varphi(\xi(C)_i) \geq \varphi(\xi(C')_i)$ for every i and every φ in $|C| \cap |C'|$. Hence

$$\varphi(\xi(C)_1) + \dots + \varphi(\xi(C)_r) \geq \varphi(\xi(C')_1) + \dots + \varphi(\xi(C')_r).$$

Since the two sets $\{\varphi(\xi(C)_1), \dots, \varphi(\xi(C)_r)\}$ and $\{\varphi(\xi(C')_1), \dots, \varphi(\xi(C')_r)\}$ are the same sets by (I'), we have

$$\varphi(\xi(C)_1) + \dots + \varphi(\xi(C)_r) = \varphi(\xi(C')_1) + \dots + \varphi(\xi(C')_r).$$

Therefore $\varphi(\xi(C)_i) = \varphi(\xi(C')_i), \dots, \varphi(\xi(C)_r) = \varphi(\xi(C')_r)$ for every φ in $|C| \cap |C'|$.

LEMMA 2.2. *Let C and C' be two cones in $\Delta(n)$ such that $C \cap C'$ is in $\Delta(n - 1)$. Then, by rearranging $\{\xi(C)_1, \dots, \xi(C)_r\}$ and $\{\xi(C')_1, \dots, \xi(C')_r\}$ and replacing (m, P) by an equivalent pair, we can reduce the matrix $P(C, C')$ to an upper triangular matrix.*

Proof. Put $P = P(C, C')$. Since $\det(P) \neq 0$, we may assume that $P_{ii} \neq 0$ for every i by rearranging $\{\xi(C)_i\}$ and $\{\xi(C')_i\}$. Suppose that $P_{hk} \neq 0$

and $P_{kh} = 0$ for some h, k ($h > k$). Then, by interchanging $\xi(C)_h$ with $\xi(C)_k$ and $\xi(C')_h$ with $\xi(C')_k$ we have $P_{hk} = 0, P_{kh} \neq 0$. So we may further assume that if $P_{hk} \neq 0$ for $h > k$ then $P_{kh} \neq 0$.

Suppose that P is not an upper triangular matrix. Then we take minimal k such that $P_{hk} \neq 0$ for some $h > k$. Then $P_{ij} = 0$ for $j < k$ and $i > j$. Since $P_{hk} \neq 0$ and $P_{kh} \neq 0$ by assumption, we have

$$\varphi(\xi(C)_h) \geq \varphi(\xi(C')_k) \quad \text{and} \quad \varphi(\xi(C)_k) \geq \varphi(\xi(C')_h)$$

for every φ in $|C| \cap |C'|$. Consequently, for φ in $|C| \cap |C'|$, we have $\varphi(\xi(C)_h) > \varphi(\xi(C')_h)$ if $\varphi(\xi(C)_h) > \varphi(\xi(C)_k)$, while $\varphi(\xi(C)_k) > \varphi(\xi(C')_k)$ if $\varphi(\xi(C)_h) < \varphi(\xi(C)_k)$, a contradiction by Lemma 2.1. Therefore we have

$$\varphi(\xi(C)_h) = \varphi(\xi(C)_k) \quad \text{for every } \varphi \text{ in } |C| \cap |C'|.$$

Since $C \cap C'$ is in $\Delta(n - 1)$, put $|C| - |C| \cap |C'| = \{\psi\}$. Suppose $\psi(\xi(C)_h) < \psi(\xi(C)_k)$. Then we interchange $\xi(C)_h$ and $\xi(C)_k$. This procedure interchanges P_{hi} with P_{ki} for each $1 \leq i \leq r$. Therefore the minimality of k is preserved. Hence we may assume that

$$\varphi(\xi(C)_h) \geq \varphi(\xi(C)_k) \quad \text{for every } \varphi \text{ in } |C|.$$

Now we define $\sigma(C) = (\sigma(C)_{ij})$ by

$$\sigma(C)_{ij} = \begin{cases} 1 & \text{for } i = j, \\ c \neq 0 & \text{for } i = h \text{ and } j = k, \\ 0 & \text{otherwise,} \end{cases}$$

and replace (m, P) by an equivalent pair using this $\sigma(C)$. This is allowed by what we have just seen. In this way, we can reduce ourselves to the case $P_{hk} = 0$. Hence we have $P_{ik} = 0$ for all i ($i > k$). After this replacement, however, P_{ii} may be zero for $i > k$. By rearranging $\{\xi(C)_{k+1}, \dots, \xi(C)_r\}$ and $\{\xi(C')_{k+1}, \dots, \xi(C')_r\}$, we may assume that $P_{ii} \neq 0$ for every i . So we can repeat the same procedure, which will terminate after finitely many steps and leads to an upper triangular matrix P .

LEMMA 2.3. *Let C, C', C'' be three distinct cones in $\Delta(n)$ such that $C' \cap C''$ is in $\Delta(n - 1)$. Then, by rearranging $\{\xi(C)_i\}, \{\xi(C')_i\}$ and $\{\xi(C'')_i\}$ and replacing (m, P) by an equivalent pair, we can reduce ourselves to the situation where $P(C', C'')$ is an upper triangular matrix and $P(C, C')_{ii} \neq 0$ for every i .*

Proof. By Lemma 2.2, $P(C', C'')$ is first reduced to an upper triangular matrix. Since $\det(P(C, C')) \neq 0$, we have $P(C, C')_{ii} \neq 0$ by rearranging $\{\xi(C)_1, \dots, \xi(C)_r\}$.

COROLLARY 2.4. *Let C, C', C'' be three distinct cones in $\Delta(n)$ such that $C' \cap C''$ is in $\Delta(n-1)$. Then by rearranging $\{\xi(C)_i\}$, $\{\xi(C')_i\}$ and $\{\xi(C'')_i\}$ and replacing (m, P) by an equivalent pair, we may assume that*

$$\varphi(\xi(C)_i) = \varphi(\xi(C')_i) \quad \text{for every } \varphi \text{ in } |C| \cap |C'|$$

and

$$\varphi(\xi(C')_i) = \varphi(\xi(C'')_i) \quad \text{for every } \varphi \text{ in } |C'| \cap |C''|$$

hold for every i .

LEMMA 2.5. *Let C, C', C'' be three distinct cones in $\Delta(n)$ such that $C \cap C'$ is in $\Delta(n-1)$. Suppose that $P(C', C'') = I$ is the identity matrix. Then, by rearranging $\{\xi(C)_i\}$, $\{\xi(C')_i\}$ and $\{\xi(C'')_i\}$ and replacing (m, P) by an equivalent pair, we can reduce ourselves to the situation where $P(C, C')$ is an upper triangular matrix and $P(C', C'') = I$.*

Proof. By Lemma 2.2, $P(C, C')$ is reduced to an upper triangular matrix. In this case, $\{\xi(C')_1, \dots, \xi(C')_r\}$ is only rearranged. If we rearrange $\{\xi(C'')_1, \dots, \xi(C'')_r\}$ exactly as $\{\xi(C')_1, \dots, \xi(C')_r\}$ is rearranged, then $P(C', C'')$ remains the identity matrix.

LEMMA 2.6. *Let C and C' be two cones in $\Delta(n)$. Suppose that $P = P(C, C')$ is an upper triangular matrix. Let φ be in $|C| \cap |C'|$. Then, by rearranging $\{\xi(C)_i\}$ and $\{\xi(C')_i\}$, we may assume that $P(C, C')$ is an upper triangular matrix and that*

$$\varphi(\xi(C)_1) \geq \varphi(\xi(C)_2) \geq \dots \geq \varphi(\xi(C)_r)$$

and

$$\varphi(\xi(C')_1) \geq \varphi(\xi(C')_2) \geq \dots \geq \varphi(\xi(C')_r)$$

hold.

Proof. Suppose $\varphi(\xi(C)_n) < \varphi(\xi(C)_{n+1})$. Since $\varphi(\xi(C)_i) = \varphi(\xi(C')_i)$ for every i by Lemma 2.1, we have $P_{h, h+1} = 0$. Hence we have $P_{h, h} \neq 0$, $P_{h+1, h+1} \neq 0$, $P_{h, h+1} = 0$, $P_{h+1, h} = 0$. By interchanging the order of $\xi(C)_h$ and $\xi(C)_{h+1}$ as well as $\xi(C')_h$ and $\xi(C')_{h+1}$, we have $\varphi(\xi(C)_h) > \varphi(\xi(C)_{h+1})$. After a finite repetition of this process we are done.

COROLLARY 2.7. *Let C, C' be two cones in $\Delta(n)$. Suppose that $P(C, C')$ is an upper triangular matrix. Let $\varphi_1, \dots, \varphi_l$ be elements in $|C| \cap |C'|$. Then, by rearranging $\{\xi(C)_i\}$ and $\{\xi(C')_i\}$, we may assume that $P(C, C')$ is an upper triangular matrix and that for every pair of h and k ($h > k$), one of the following conditions holds:*

- (a) $\varphi_i(\xi(C)_k) = \varphi_i(\xi(C)_{k+1}) = \dots = \varphi_i(\xi(C)_h)$ for $1 \leq i \leq l$.
- (b) *There exists v ($\leq l$) such that*

$$\varphi_i(\xi(C)_k) = \varphi_i(\xi(C)_{k+1}) = \dots = \varphi_i(\xi(C)_n) \quad \text{for } 1 \leq i < v$$

and

$$\varphi_v(\xi(C)_k) > \varphi_v(\xi(C)_h).$$

Proof. We first apply Lemma 2.6 to φ_1 . If $\varphi_1(\xi(C)_k) = \dots = \varphi_1(\xi(C)_h)$, then we further apply Lemma 2.6 to φ_2 with respect to $\{\xi(C)_k, \dots, \xi(C)_n\}$ only. Repeating this procedure, we are done.

§3. The case of P^n

From this section on, we restrict ourselves to the case $X = P^n$ and consider a T -linearized vector bundle $E = E(m, P)$ of rank r ($r \geq 2$) on P^n . When $n = 1$, a vector bundle on P^1 is split. Hence we assume $n \geq 2$. Let $\{\varphi_1, \dots, \varphi_n\}$ be a Z -base of N and let $\varphi_0 = -\varphi_1 - \varphi_2 - \dots - \varphi_n$. Let Δ be the fan defined by $\{\varphi_0, \varphi_1, \dots, \varphi_n\}$, i.e., $\Delta(n)$ consists of $C_i = \sum_{j \neq i} R_0 \varphi_j$ ($i = 0, 1, \dots, n$). Then $P^n = T_N \text{emb}(\Delta)$ and $U_{C_i} = \{X_i \neq 0\}$ is the affine open set in P^n corresponding to C_i , where X_0, \dots, X_n are homogeneous coordinates. In this case we note that $C \cap C'$ is an $(n - 1)$ -dimensional cone in Δ for every pair of cones C and C' in $\Delta(n)$.

PROPOSITION 3.1. *Suppose that $P(C', C'') = I$ is the identity matrix for some C' and C'' in $\Delta(n)$ with $C' \neq C''$. Then the T -linearized vector bundle E is a direct sum of T -linearized line bundles, hence, in particular, decomposable.*

Proof. Let C be another cone in $\Delta(n)$. By Lemma 2.5, we may assume that $P = P(C, C')$ is an upper triangular matrix and $P(C', C'') = I$. We show that P can be reduced to the identity matrix. Suppose $P_{hk} \neq 0$ for some h and k ($h < k$). Then $\varphi(\xi(C)_h) \geq \varphi(\xi(C')_k)$ for every φ in $|C| \cap |C'|$, while $\varphi(\xi(C)_k) = \varphi(\xi(C')_k)$ by Lemma 2.1. Hence we have $\varphi(\xi(C)_h) \geq \varphi(\xi(C)_k)$ for every φ in $|C| \cap |C'|$.

Put $\{\psi\} = |C| - |C| \cap |C'|$. Then ψ is in $|C| \cap |C''|$ by the definition of Δ . Assume now that $\psi(\xi(C)_h) < \psi(\xi(C)_k)$. Since $P(C, C'') = P(C, C')$ $P(C', C'') = P$ we have $P(C, C'')_{hk} = P_{hk} \neq 0$, and $\varphi(\xi(C)_h) \geq \varphi(\xi(C'')_k)$ for every φ in $|C| \cap |C''|$. Therefore, since ψ is in $|C| \cap |C''|$, we have $\psi(\xi(C)_k) > \psi(\xi(C)_h) \geq \psi(\xi(C'')_k)$. This is a contradiction to Lemma 2.1 since $P(C, C'') = P$ is an upper triangular matrix and $\det(P) \neq 0$. So we have $\psi(\xi(C)_h) \geq \psi(\xi(C)_k)$. Since $|C| = (|C| \cap |C'|) \cup \{\psi\}$ we have $\varphi(\xi(C)_h) \geq \varphi(\xi(C)_k)$ for every φ in $|C|$. Now we define $\sigma(C) = (\sigma(C)_{ij})$ by

$$\sigma(C)_{ij} = \begin{cases} 1 & \text{for } i = j, \\ c \neq 0 & \text{for } i = h \text{ and } j = k, \\ 0 & \text{otherwise,} \end{cases}$$

and replace (m, P) by an equivalent pair using this $\sigma(C)$. Then we can reduce ourselves to the case $P(C, C')_{hk} = 0$. If this process is repeated for all $P(C, C')_{hk} \neq 0$ ($h \neq k$), then finally $P(C, C')$ will become a diagonal matrix. By taking $\sigma'(C) = (\sigma'(C)_{ij})$ with $\sigma'(C)_{ij} = P(C, C')_{ij}$ and replacing (m, P) by an equivalent pair using this $\sigma'(C)$, we may assume $P(C, C') = I$. Hence $P(C, C'') = I$.

Furthermore, if $n \geq 3$, let C^* be another cone in $\Delta(n)$. We apply the same process to $P(C^*, C')$. Then we may assume $P(C^*, C') = I$. Hence

$$P(C, C^*) = P(C, C')P(C^*, C')^{-1} = I.$$

Therefore all $P(C, C^*)$ can be reduced to the identity matrix. This means that the T -linearized vector bundle E is a direct sum of T -linearized line bundles by the very construction of $E(m, P)$.

PROPOSITION 3.2. *Suppose that $P(C, C')$, $P(C', C'')$ and $P(C'', C)$ are upper triangular matrices for some triple of pairwise distinct cones C, C' and C'' in $\Delta(n)$. Then the T -linearized vector bundle E is a direct sum of T -linearized line bundles.*

Proof. Concerning the first row of P for (m, P) , we suppose that there exists $s > 1$ such that

$$P(C, C')_{1j} = 0, \quad P(C', C'')_{1j} = 0, \quad P(C'', C)_{1j} = 0 \quad \text{for } 1 < j < s$$

and that

$$P(C, C')_{1s} \neq 0, \quad P(C'', C)_{1s} \neq 0.$$

Then, since $P(C, C')_{1s} \neq 0$ and $P(C'', C)_{1s} \neq 0$, we have

$$\varphi(\xi(C)_i) \geq \varphi(\xi(C')_s) \quad \text{for every } \varphi \text{ in } |C| \cap |C'|$$

and

$$\varphi(\xi(C'')_1) \geq \varphi(\xi(C)_s) \quad \text{for every } \varphi \text{ in } |C''| \cap |C|.$$

Since, by Lemma 2.1,

$$\varphi(\xi(C)_s) = \varphi(\xi(C')_s) \quad \text{for every } \varphi \text{ in } |C| \cap |C'|$$

and

$$\varphi(\xi(C)_1) = \varphi(\xi(C'')_1) \quad \text{for every } \varphi \text{ in } |C''| \cap |C|,$$

we have

$$\varphi(\xi(C)_i) \geq \varphi(\xi(C)_s) \quad \text{for every } \varphi \text{ in } |C| \cap |C'|$$

and

$$\varphi(\xi(C)_i) \geq \varphi(\xi(C)_s) \quad \text{for every } \varphi \text{ in } |C''| \cap |C|.$$

Since $|C| = (|C| \cap |C'|) \cup (|C| \cap |C'')$ we have

$$\varphi(\xi(C)_i) \geq \varphi(\xi(C)_s) \quad \text{for every } \varphi \text{ in } |C|.$$

We define $\sigma(C) = (\sigma(C)_{ij})$ by

$$\sigma(C)_{ij} = \begin{cases} 1 & \text{for } i = j, \\ c \neq 0 & \text{for } i = 1 \text{ and } j = s, \\ 0 & \text{otherwise,} \end{cases}$$

and replace (m, P) by an equivalent pair using this $\sigma(C)$. Then we can reduce ourselves to the case $P(C, C')_{1s} = 0$. Furthermore if $P(C'', C)_{1s} \neq 0$ or $P(C', C'')_{1s} \neq 0$ we do the same. Then we are reduced to the case $P(C'', C)_{1s} = 0$, hence $P(C', C'')_{1s} = 0$. Thus we may assume that

$$P(C, C')_{1j} = 0, \quad P(C', C'')_{1j} = 0, \quad P(C'', C)_{1j} = 0 \quad \text{for } j \neq 1.$$

Repeat the same process to the other rows successively. Then $P(C, C')$, $P(C', C'')$ and $P(C'', C)$ become diagonal matrices. We define $\sigma(C) = (\sigma(C)_{ij})$ by $\sigma(C)_{ij} = P(C, C')_{ij}$ and replace (m, P) by an equivalent pair using this $\sigma(C)$. Then we are reduced to the case $P(C, C') = I$. Therefore, by Proposition 3.1, the T -linearized vector bundle E is a direct sum of T -linearized line bundles.

COROLLARY 3.3. *Suppose that $P(C, C')$ and $P(C', C'')$ are upper triangular matrices for some triple of pairwise distinct cones C, C' and C'' in*

$\mathcal{A}(n)$. Then the T -linearized vector bundle E is a direct sum of T -linearized line bundles.

THEOREM 3.4. *Let $r > 1$ and $n \geq 2$. For a pair (m, P) , suppose the corresponding T -linearized vector bundle $E = E(m, P)$ on P^n of rank r is indecomposable. Put $|C| = \{\varphi_1, \dots, \varphi_n\}$ for a cone C in $\mathcal{A}(n)$. Then, for any pair of distinct integers s and t ($1 \leq s, t \leq n$), there exist two integers h and k such that*

$$\varphi_s(\xi(C)_n) < \varphi_s(\xi(C)_k), \quad \varphi_t(\xi(C)_n) > \varphi_t(\xi(C)_k)$$

and

$$\varphi_i(\xi(C)_n) = \varphi_i(\xi(C)_k) \quad \text{for } i \neq s, t.$$

Proof. We prove the assertion only when $s = 1$ and $t = 2$ since the proof in general is the same. Since we are working on P^n , there exist C' and C'' in $\mathcal{A}(n)$ such that

$$|C'| = \{\varphi_0, \varphi_2, \varphi_3, \dots, \varphi_n\}, \quad |C''| = \{\varphi_0, \varphi_1, \varphi_3, \dots, \varphi_n\}$$

for some $\varphi_0 \in N$. By Lemma 2.3, we assume that $P(C, C')_{ii} \neq 0$ for every i and that $P(C', C'')$ is an upper triangular matrix. Hence by Lemma 2.1, we get

$$\varphi(\xi(C)_i) = \varphi(\xi(C')_i) \quad \text{for every } \varphi \text{ in } |C| \cap |C'| = \{\varphi_2, \varphi_3, \dots, \varphi_n\}$$

and

$$\varphi(\xi(C')_i) = \varphi(\xi(C'')_i) \quad \text{for every } \varphi \text{ in } |C'| \cap |C''| = \{\varphi_1, \varphi_3, \dots, \varphi_n\}.$$

Therefore

$$\varphi(\xi(C)_i) = \varphi(\xi(C')_i) = \varphi(\xi(C'')_i)$$

for every φ in $|C| \cap |C'| \cap |C''| = \{\varphi_3, \varphi_4, \dots, \varphi_n\}$. If $n = 2$ then this amounts to nothing since $|C| \cap |C'| \cap |C''| = \emptyset$. When $n \geq 3$, we further apply Corollary 2.7 to $\varphi_3, \dots, \varphi_n$ and $P(C', C'')$ and may assume that $P(C', C'')$ is an upper triangular matrix and that for every pair h and k ($h > k$), one of the following conditions holds:

- (a) $\varphi_i(\xi(C)_k) = \varphi_i(\xi(C)_{k+1}) = \dots = \varphi_i(\xi(C)_n) \quad \text{for } 3 \leq i \leq n.$
- (b) There exists v such that

$$\varphi_i(\xi(C)_k) = \varphi_i(\xi(C)_{k+1}) = \dots = \varphi_i(\xi(C)_n) \quad \text{for } 3 \leq i < v$$

and

$$\varphi_v(\xi(C)_k) > \varphi_v(\xi(C)_h).$$

$P(C, C')$ cannot be reduced to an upper triangular matrix, since, otherwise, E would be decomposable by Corollary 3.3 contradicting our assumption. Thus there exist integers $h > k$ such that $P(C, C')_{hk} \neq 0$. Then we have $\varphi(\xi(C)_h) \geq \varphi(\xi(C)_k)$ for every φ in $|C| \cap |C'|$. Since $\varphi(\xi(C)_k) = \varphi(\xi(C')_k)$ for every φ in $|C| \cap |C'|$, we have $\varphi(\xi(C)_h) \geq \varphi(\xi(C)_k)$ for every φ in $|C| \cap |C'|$, hence for every φ in $\{\varphi_3, \dots, \varphi_n\} = |C| \cap |C'| \cap |C''|$. This means, by (a) and (b) above, that

$$\varphi(\xi(C)_k) = \varphi(\xi(C)_{k+1}) = \dots = \varphi(\xi(C)_n)$$

for every φ in $\{\varphi_3, \dots, \varphi_n\}$. Since φ_2 is in $|C| \cap |C'|$, we have $\varphi_2(\xi(C)_h) \geq \varphi_2(\xi(C)_k)$ as we saw above. Hence we have the following four possibilities:

1. $\varphi_1(\xi(C)_h) \geq \varphi_1(\xi(C)_k)$ and $\varphi_2(\xi(C)_h) = \varphi_2(\xi(C)_k)$,
2. $\varphi_1(\xi(C)_h) < \varphi_1(\xi(C)_k)$ and $\varphi_2(\xi(C)_h) = \varphi_2(\xi(C)_k)$,
3. $\varphi_1(\xi(C)_h) \geq \varphi_1(\xi(C)_k)$ and $\varphi_2(\xi(C)_h) > \varphi_2(\xi(C)_k)$,
4. $\varphi_1(\xi(C)_h) < \varphi_1(\xi(C)_k)$ and $\varphi_2(\xi(C)_h) > \varphi_2(\xi(C)_k)$.

We now show that the case 4 happens for some h and k ($h > k$) such that $P(C, C')_{hk} \neq 0$. Suppose that the case 4 does not happen for any such h, k . Then, by interchanging $\xi(C)_h$ and $\xi(C)_k$ if the case 2 happens, we have

$$\varphi(\xi(C)_h) \geq \varphi(\xi(C)_k) \quad \text{for every } \varphi \text{ in } |C|.$$

Now we take the smallest k such that $P(C, C')_{hk} \neq 0$ for some $h > k$. We define $\sigma(C) = (\sigma(C)_{ij})$ by

$$\sigma(C)_{ij} = \begin{cases} 1 & \text{for } i = j, \\ c \neq 0 & \text{for } i = h \text{ and } j = k, \\ 0 & \text{otherwise,} \end{cases}$$

and replace (m, P) by an equivalent pair using this $\sigma(C)$. Then we can reduce ourselves to the case $P(C, C')_{hk} = 0$. Repeating the same procedure for every h such that $h > k$ and $P(C, C')_{hk} \neq 0$, we have $P(C, C')_{ik} = 0$ for all $i > k$. After this procedure, $P(C, C')_{ii}$ may be zero for some i , but, by rearranging the order of $\{\xi(C)_{k+1}, \dots, \xi(C)_r\}$, we have $P(C, C')_{ii} \neq 0$ for all $i > k$. So we can successively apply the same procedure, and $P(C, C')$ is finally reduced to an upper triangular matrix. By Corollary 3.3, this is a contradiction to the indecomposability of E . Therefore there exist h and k such that

$$\varphi_1(\xi(C)_n) < \varphi_1(\xi(C)_k), \quad \varphi_2(\xi(C)_n) > \varphi_2(\xi(C)_k)$$

and

$$\varphi_i(\xi(C)_n) = \varphi_i(\xi(C)_k) \quad \text{for } i \geq 3.$$

COROLLARY 3.5. *Suppose $1 < r \leq n$ ($n \geq 2$) and let E be an indecomposable T -linearized vector bundle of rank r on P^n . Then we have:*

- (1) $r = n$.
- (2) For every $C \in \mathcal{A}(n)$ and every φ in $|C|$, all except one of $\{\varphi(\xi(C)_1), \dots, \varphi(\xi(C)_n)\}$ are the same integers.
- (3) Let $C = R_0\varphi_1 + \dots + R_0\varphi_n$ be in $\mathcal{A}(n)$. We can tensor a suitable T -linearized line bundle to E and rearrange the order of $\{\xi(C)_1, \dots, \xi(C)_n\}$, so that the following hold for every $i = 1, \dots, n$:

$$\varphi_i(\xi(C)_i) = a_i \quad \text{and} \quad \varphi_i(\xi(C)_j) = 0 \quad \text{for any } j \neq i.$$

In this case, a_1, \dots, a_n are all positive or all negative.

Proof. Let $|C| = \{\varphi_1, \dots, \varphi_n\}$ for a $C \in \mathcal{A}(n)$, and apply Theorem 3.4 to C . For each s , we first see that $\varphi_s(\xi(C)_1), \dots, \varphi_s(\xi(C)_r)$ cannot be all equal, since we can pick $t \neq s$ and apply Theorem 3.4 to the pair (s, t) .

Clearly, Theorem 3.4 gives a one-to-one map from the set $\{(s, t) \mid 1 \leq s < t \leq n\}$ to the set of pairs $\{h, k\}$ of distinct integers between 1 and r . Thus $n(n - 1)/2 \leq r(r - 1)/2$. Since $r \leq n$ by assumption, we have $r = n$, and the above map must be a bijection.

We can so rearrange $\xi(C)_1, \dots, \xi(C)_n$ that for each i , the pair $(1, i)$ is sent to the pair $\{1, i\}$ by the above map. Then for each i , we see that $\varphi_i(\xi(C)_j)$ are equal for all $j \neq i$.

In view of Remark 1.5, we may tensor a T -linearized line bundle to E so that the following holds for each i :

$$\varphi_i(\xi(C)_i) = a_i \quad \text{and} \quad \varphi_i(\xi(C)_j) = 0 \quad \text{for any } j \neq i.$$

By Theorem 3.4, we see that a_s, a_t should have the same sign for all $s \neq t$.

§4. Determination of $P(C, C')$

In this section we consider $P(C, C')$ for an indecomposable T -linearized vector bundle of rank n on P^n ($n \geq 2$). By Corollary 3.5, we may assume that, for every C and every φ in $|C|$, we have $\varphi(\xi(C)_i) \geq 0$ and $\varphi(\xi(C)_i) = 0$ except for one i . For $C, C', C'' \in \mathcal{A}(n)$, let

$$\begin{aligned} |C| &= \{\varphi_1, \varphi_2, \varphi_3, \dots, \varphi_n\}, \\ |C'| &= \{\varphi_0, \varphi_2, \varphi_3, \dots, \varphi_n\}, \\ |C''| &= \{\varphi_0, \varphi_1, \varphi_3, \dots, \varphi_n\}. \end{aligned}$$

By changing the order of $\{\xi(C)_i\}$, $\{\xi(C')_i\}$ and $\{\xi(C'')_i\}$, we assume that

$$\begin{aligned} (\varphi_0(\xi(C')_1), \dots, \varphi_0(\xi(C')_n)) &= (\varphi_0(\xi(C'')_1), \dots, \varphi_0(\xi(C'')_n)) \\ &= (a, 0, 0, \dots, 0) \quad (a > 0), \\ (\varphi_1(\xi(C)_1), \dots, \varphi_1(\xi(C)_n)) &= (b, 0, 0, \dots, 0), \\ (\varphi_1(\xi(C'')_1), \dots, \varphi_1(\xi(C'')_n)) &= (0, b, 0, \dots, 0) \quad (b > 0), \\ (\varphi_2(\xi(C)_1), \dots, \varphi_2(\xi(C)_n)) &= (\varphi_2(\xi(C')_1), \dots, \varphi_2(\xi(C')_n)) \\ &= (0, c, 0, \dots, 0) \quad (c > 0), \end{aligned}$$

and

$$\begin{aligned} (\varphi_i(\xi(C)_1), \dots, \varphi_i(\xi(C)_n)) &= (\varphi_i(\xi(C')_1), \dots, \varphi_i(\xi(C')_n)) \\ &= (\varphi_i(\xi(C'')_1), \dots, \varphi_i(\xi(C'')_n)) \\ &= (0, \dots, 0, d_i, \dots, 0) \end{aligned}$$

for $i \geq 3$, where $d_i > 0$ is the i -th entry.

Then, by (II), we have

$$\begin{aligned} P(C, C') &= \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ p_2 & 1 & 0 & \dots & 0 \\ p_3 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_n & 0 & 0 & \dots & 1 \end{pmatrix}, & P(C', C'') &= \begin{pmatrix} 1 & q_1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & q_3 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & q_n & 0 & \dots & 1 \end{pmatrix}, \\ P(C', C) &= \begin{pmatrix} 0 & -1 & 0 & \dots & 0 \\ 1 & r_2 & 0 & \dots & 0 \\ 0 & r_3 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & r_n & 0 & \dots & 1 \end{pmatrix}. \end{aligned}$$

Since $P(C, C')P(C', C'')P(C'', C) = I$ we have:

LEMMA 4.1. $q_1 = 1, p_2 = -1, r_2 = 1$ and

$$p_i = -q_i = r_i \text{ for } 3 \leq i \leq n.$$

LEMMA 4.2. $p_i \neq 0$ for $2 \leq i \leq n$.

Proof. Fix two cones C and C' and $P(C, C')$. We take another C'' successively and calculate in the above way. Then we have $p_i \neq 0$ for $2 \leq i \leq n$.

LEMMA 4.3. *We may assume that $p_i = 1$ for $1 < i \leq n$.*

Proof. Since $p_i \neq 0$ for $i \geq 2$, we take

$$\sigma(C) = \sigma(C') = \begin{pmatrix} 1 & & & 0 \\ & p_2 & & \\ & & \ddots & \\ 0 & & & p_n \end{pmatrix}$$

and replace (m, P) by an equivalent pair using these $\sigma(C), \sigma(C')$. Then we have

$$P(C, C') = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & & 0 \\ \vdots & & \ddots & \\ \vdots & & & \\ 1 & 0 & & 1 \end{pmatrix}$$

Hence we may assume that $p_i = 1$ for $1 < i \leq n$.

If $P(C, C')$ is in the above form, then $P(C', C'')$ and $P(C'', C)$ are naturally determined if m in (I) is given. Therefore P in (m, P) is determined for every pair of cones in $\mathcal{A}(n)$. Hence, for each indecomposable T -linearized vector bundle of rank n , P is unique up to equivalence (III). Therefore, for any given m in (I) which we know by (3) of Corollary 3.5, an indecomposable T -linearized vector bundle is uniquely determined if it exists.

THEOREM 4.4. *Let E be a T -linearized vector bundle defined by the sequence*

$$(*) \quad 0 \longrightarrow \mathcal{O}_{P^n} \xrightarrow{f} \bigoplus_{i=0}^n \mathcal{O}_{P^n}(a_i) \longrightarrow E \longrightarrow 0$$

such that f sends 1 to $(X_0^{a_0}, X_1^{a_1}, \dots, X_n^{a_n})$, where X_0, \dots, X_n are homogeneous coordinates of P^n and a_0, \dots, a_n are positive integers. Then E is an indecomposable vector bundle.

Proof. Suppose E is decomposable and let $E = E_1 \oplus E_2 \oplus \dots \oplus E_l$ with $l \geq 2$ be a decomposition of E into indecomposable vector bundles. Every indecomposable component E_i is T -equivariant by virtue of the

Krull-Schmidt Theorem (see [1]). Since $\text{rank}(E_i) < n$, E_i is necessarily a line bundle by Corollary 3.5. Hence E is a direct sum of line bundles and we may let

$$E = \mathcal{O}_{P^n}(d_1) \oplus \mathcal{O}_{P^n}(d_2) \oplus \cdots \oplus \mathcal{O}_{P^n}(d_n)$$

for $d_1 \leq d_2 \leq \cdots \leq d_n$. We may assume that $a_0 \leq a_1 \leq \cdots \leq a_n$. By tensoring the sequence (*) with $\mathcal{O}_{P^n}(-k)$ for $k > 0$ we have

$$h^0\left(\bigoplus_{i=0}^n \mathcal{O}_{P^n}(a_i - k)\right) = h^0(E(-k)).$$

We have a contradiction, if we take $k = a_n$ when $a_n > d_n$ while we take $k = d_n$ when $a_n < d_n$. Hence we have $a_n = d_n$. Similarly, we have $a_i = d_i$ for $1 \leq i \leq n$. By (*), we have $\det(E) = \mathcal{O}_{P^n}(\sum_{i=0}^n a_i)$, which is equal to $\mathcal{O}_{P^n}(\sum_{i=1}^n d_i)$. Hence $a_0 = 0$ and (*) is split, a contradiction. Therefore E is indecomposable.

If we take $a_0 = a_1 = \cdots = a_n = 1$ in Theorem 4.4, then the T -linearized vector bundle E is the tangent bundle T_{P^n} for P^n .

COROLLARY 4.5. *T_{P^n} is indecomposable.*

By short calculation, we have

$$m(\varphi_i) = (-a_i, 0, \dots, 0) \quad \text{for } 0 \leq i \leq n$$

for the T -linearized vector bundle E which is defined by (*) in Theorem 4.4. Therefore we have:

THEOREM 4.6. *An indecomposable equivariant vector bundle of rank n on P^n ($n \geq 2$) is isomorphic to $E(d)$ or $E^*(d)$ for some integer d , where E is defined by the sequence (*) in Theorem 4.4 for some positive integers a_i ($0 \leq i \leq n$).*

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