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On Totally Balanced Games and Games of Flow*

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In their 1969 paper, Shapley and Shubik introduced the concept of a <u>totally balanced</u> n-person cooperative game with side payments. These are games in which the core of the game itself as well as those of all its induced subgames are not empty. They showed that the family of totally bananced games can be generated by a certain class of market games with side payments and investigated several properties of the Von Neumann Morgenstern solutions of such games. Scarf [1967] and Billera and Bixby [1974], studied similar issues for the more general class of games without side payments.

In this note we consider totally balanced games with side payments. We show that every game in this class can be expressed as a minimum of a finite collection of trivial games (additive games). Thus, totally balanced games have a very simple structure. Next, we introduced a family of games that are generated by problems of flows in a network and which we call flow games. These games are useful for modeling problems of profit sharing in an integrated production system with alternative production routes. We show that the family of flow games coincides precisely with the family of totally balanced games. The key-elements in our proof are the observation about the structure of totally balanced games and the max flow--min cut theorem of Ford and Fulkerson [1962].

I. Preliminaries and Definitions

For a positive integer n let $N = \{1,...,n\}$ denote a set of players, and $2^N = \{S \quad N : S \neq \emptyset\}$ the set of coalitions of players. An n-person characteristic function game (game for short) is a function $V : 2^N \to R_+$,(the set of nonnegative real numbers). The core of V is defined by

$$\text{CORE}(V) = \{x \in R^n: \sum_{i \in N} x_i = V(N) \text{ and } \sum_{i \in S} x_i \geq V(S) \text{ for every } S \in 2^N\}$$

A game is called balanced if CORE $(V) \neq \emptyset$. For a game V and coalition S consider the obvious $|S|$ - players subgame, V^S, obtained by restricting V to coalitions contained in S. A game V is called totally balanced if CORE $(V^S) \neq \emptyset$ for every $S \in 2^N$.

A game V is called additive if there exists a set of non-negative real numbers $a_1, a_2, ..., a_n$ such that for every $S \in 2^N$ $V(S) = \sum_{i \in S} a_i$. For a collection of n-person games $\{V_t\}_{t \in T}$ we define the minimum game generated by the collection by $(\min V_t)(S) = \min_{t \in T} \{V_t(S)\}$.

It is easy to see that every additive game is totally balanced and that the minimum game of every finite collection of totally balanced games is also totally balanced. It is interesting that the additive games, together with the minimum operation, span the entire class of totally balanced games:

<u>Theorem 1.</u> A game V is totally balanced if and only if it is the minimum game of a finite collection of additive games.

<u>Proof.</u> The "if" is obvious. To show the converse we define a collection of (n-person) games $\{V_S\}_{S\varepsilon 2^N}$ with the properties

(i) V_S is an additive game

(ii) $V_S(T) \geq V(T)$ for every $T \varepsilon 2^N$

(iii) $V_S(S) = V(S)$.

It is obvious that (ii) and (iii) imply that for every coalition T $\min V_S(T) = V_T(T) = V(T)$, i.e. V is the minimum game of the family $\{V_S\}_{S\varepsilon 2^N}$.

To define the games V_S we use the following construction. For every $S \varepsilon 2^N$ let $\{a_i\}_{i\varepsilon S}$ be a point in the core of the ($|S|$ person) game V^S. For $i \varepsilon S$ let a_i be any real number larger than V(N). Define the game V_S to be the additive game generated by $a_1, a_2, \dots. a_n$. It is easy to verify that V_S satisfies properties (i), (ii) and (iii) which completes the proof of Theorem 1.

In Shapley-Shubik (1969) they characterized the totally balanced games to be exactly the games that are generated by a certain type of exchange economies. We shall describe here a family of cooperative games that occur very naturally in problems of revenue sharing in integrated production processes with many alternative production possibilities. We then show that this family also span the family of totally balanced games.

II. Flow Games

Consider a directed Network G with node and arc sets $M = \{1...m\}$, and $L = \{1...\ell\}$ respectively. As before, we let the set of players be $N = \{1,...,n\}$. We associate with each are $i \in L$, a 4-tuple of labels H_i, T_i, U_i and P_i which collectively define the structure of G. The interpretation of the labels is as follows. H_i and T_i describe the "head" and "tail" nodes of the (directed) arc i. Thus, H_i, $T_i \in M$ for every $i \in L$. U_i is a real number which describe the "capacity" of arc i. Thus, U_i may be any non-negative real number. Finally, for each $i \in L$, we let $P_i \in N$ be the identity of the player which "owns" the arc. The reader may note that our formulation allows for multiple arcs between a given pair of nodes.

Two nodes, 1 and m, play a special note in our formulation. We let node 1 be the "initial" or "source" node of the network. Similarly, m is the "terminal" or "sink" node. One may think about the network in terms of a certain "input" which corresponds to (or is available at) node 1 and which is being converted into an output which correspond to node m. The other nodes of G correspond to intermediate "states" in the process of conversion. The transition from state to state is carried out by arcs which are in turn owned by the players. It is customary to model such production systems in terms of a certain "material" or "fluid" which "flows" throughout the network. Naturally, in order to achieve maximum utilization of a given system, one would like to find a flow in the corresponding network which is as large as possible subject to the capacity constraints of the individual arcs. There exist numerous extremely efficient algorithms capable of finding a maximum flow through a given network. For the pioneering work in this context, see Ford and Fulkerson [1962].

To define a game on a network G assume that each unit of flow is worth 1 (say \$). For coalition $S \in 2^N$, let G^S be the Network which is restricted to the arcs owned by the members of S. Let F(S) be the maximal amount of flow throughout that Network. We call a game V a flow game if there exists a network G such that for every coalition $S \in 2^N$ $V(S) = F(S)$. Two lemmas follow immediately.

Lemma 1. Every additive game is a flow game.

Proof. Let V be an additive game generated by the numbers $a_1, a_2, \ldots, a_n$. We construct a network with two nodes 1 and 2 and n arcs connecting 1 to 2 with the i^{th} arc belonging to player i and having capacity a_i. It is obvious that $F(S) = \sum_{i \in S} a_i$ for every coalition S.

Lemma 2. If V and W are two flow games then min (V,W) is a flow game.

Proof. If V is generated by a network G_1 and W by a second network G_2 we combine the two networks in series by identifying the terminal node of G_1 with the initial node of G_2. Otherwise we leave all the arcs, capacities and ownerships of arcs the same. The initial node of the combined network is the initial node of G_1 and the terminal node of the combined network is the terminal node of G_2. Thus, for a coalition S to facilitate flow through the combined network, they must first flow it through G_1 and then through G_2. Thus, the level of flow that they can sustain is the minimum of the levels of flow that they can sustain through the individual networks. It follows that the combined network gives rise to the minimum game.

Theorem 2. A game is totally balanced if and only if it is a flow game.

Proof. Theorem 1 together with lemmas 1 and 2 imply that every totally balanced game is a flow game. To show that every flow game is totally balanced it suffices to show that every flow game has a non-empty core (subgames of a flow game are flow games).

We need to recall some facts from the theory of network flows (see Ford and Fulkerson, [1962]). A cut in a network is a set of nodes C with the property that $1 \in C$ and $m \notin C$. The capacity of a cut C is the sum of the capacities of all the arcs i with $H_i \in C$ and $T_i \in C$. It is known that--

(i) for every cut C the capacity of C is greater than or equal to the maximal flow in the network, and

(ii) there exists at least one cut, called the min cut, with capacity equal to the maximal flow in the network (max flow-min cut theorem).

To show that every flow game V has a non empty core we take a network that gives rise to the game. We fix a min cut, C*, and consider the set of arcs $E^* = \{i \in L : H_i \in C^* \text{ and } T_i \notin C^*\}$. For every player j we let x_j be the sum of the capacities of the arcs in E^* that belong to him. To see that x ε CORE(V) we observe that $\sum_{i \in N} x_i = V(N)$ by the max flow-min cut Theorem. Now let S be an arbitrary coalition and let G^S be its corresponding network. We note that C^* is a cut in G^S with capacity (relative to G^S) given by $U_S = \Sigma\, U_i$ such that $i \in E^*$ and $P_i \in S$. Clearly $U_S \leq \sum_{i \in S} x_i$. By (i) $U_S \geq F(S) = V(S)$. Hence x ε CORE (V), which completes the proof of Theorem 2.

The components of the flow games described above can be generalized in order to facilitate the modeling of more complicated real life situations. For instance, one may wish to include costs per unit of flow on the individual arcs as well as multiplicity of "sources" and "sinks." It can be shown that the games which result from this richer class of network problems are totally balanced. In addition, the connection between core allocations and cuts in the network G is deeper than what is revealed by the proof of Theorem 2. However, these issues require the introduction of different concepts and therefore are left for a later paper.

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