Tôhoku Math. Journ.
38 (1986), 351-356.

TOTALLY GEODESIC FOLIATIONS AND KILLING FIELDS, II

GEN-ICHI OSHIKIRI

(Received May 30, 1985)

1. Introduction. A foliation $\mathscr{F}$ of a Riemannian manifold (M, g) is said to be totally geodesic if every leaf of $\mathscr{F}$ is a totally geodesic submanifold of (M, g). In [6], Johnson and Whitt studied some properties of Killing fields on complete connected Riemannian manifolds admitting codimension-one totally geodesic foliations by compact leaves. In [7], the author studied one of these properties of Killing fields on closed Riemannian manifolds admitting not necessarily compact codimension-one totally geodesic foliations and proved the following: Let (M, g) be a closed connected Riemannian manifold and $\mathscr{F}$ be a codimension-one totally geodesic foliation of (M, g). Then any Killing field Z on (M, g) preserves $\mathscr{F}$, that is, the flow of Z maps each leaf of $\mathscr{F}$ to a leaf of $\mathscr{F}$.

In this paper, we extend this result to higher codimensions by studying Jacobi fields along geodesics on totally geodesic leaves. We prove the following.

THEOREM. *Let (M, g) be a connected complete Riemannian manifold and $\mathscr{F}$ be a totally geodesic foliation of (M, g). Assume that the bundle orthogonally complement to $\mathscr{F}$ is also integrable. Then any Killing field Z on (M, g) with bounded length, i.e., $g(Z, Z) \leqq \text{const.} < \infty$ on M, preserves $\mathscr{F}$.*

The proof will be given in Section 3. In Section 4, we give some examples and study a related topic.

2. Preliminaries. Let (M, g) be a connected complete Riemannian manifold and $\mathscr{F}$ be a codimension-q totally geodesic foliation of (M, g). Denote by D the Riemannian connection of (M, g) and by R the curvature tensor of D. We also denote $g(X, Y)$ by $\langle X, Y\rangle$. Let $c: \boldsymbol{R} \to M$ be a geodesic parametrized by arc length on a totally geodesic leaf L of $\mathscr{F}$ and $Y(t)$ be a Jacobi field along c. Then $Y(t)$ satisfies the Jacobi equation $D_{c'(t)}D_{c'(t)}Y(t) + R_tY(t) = 0$ where $R_tY(t) = R(Y(t), c'(t))c'(t)$. Set $x = c(0)$. We choose an orthonormal basis $\{E_1, \cdots, E_p, X_1, \cdots, X_q\}$ of T_xM with $E_1 = c'(0)$, $E_2, \cdots, E_p \in T_x\mathscr{F}$ and $X_1, \cdots, X_q \in T_x\mathscr{F}^{\perp}$ where $\dim(L) = p$

and $\dim(M) = n = p + q$. By the parallel translation along c, we get a parallel frame field $\{E_i, X_a\} = \{E_1(t), \cdots, E_p(t), X_1(t), \cdots, X_q(t)\}$ along c. As L is a totally geodesic submanifold of (M, g), the frame field $\{E_i, X_a\}$ satisfies the following properties: $E_1(t) = c'(t)$, $E_i(t) \in T_{c(t)}L$ for $i = 1, \cdots, p$ and $X_a(t) \in T_{c(t)}L^\perp$ for $a = 1, \cdots, q$. With respect to this frame field $\{E_i, X_a\}$ we represent $Y(t)$ as $Y(t) = \sum_{i=1}^p u_i(t)E_i(t) + \sum_{a=1}^q v_a(t)X_a(t)$. Note that $\langle R(E_i(t), c'(t))c'(t), X_a(t)\rangle = \langle R(X_a(t), c'(t))c'(t), E_i(t)\rangle = 0$, since L is totally geodesic. Thus $u_i(t)$ and $v_a(t)$ satisfy the following differential equations

$$d^2u_i(t)/dt^2 + \sum_{j=1}^p u_j(t)R_{ij}(t) = 0 \quad \text{for} \quad i = 1, \cdots, p$$

$$d^2v_a(t)/dt^2 + \sum_{b=1}^q v_b(t)R_{ab}(t) = 0 \quad \text{for} \quad a = 1, \cdots, q\,,$$

where $R_{ij}(t) = \langle R(E_i(t), c'(t))c'(t), E_j(t)\rangle$ and $R_{ab}(t) = \langle R(X_a(t), c'(t))c'(t), X_b(t)\rangle$. Hence we have the following.

LEMMA 1. *Let $Y(t)$ be a Jacobi field along c. Then the orthogonal projections $V(t)$ and $H(t)$ of $Y(t)$ to TL and $TL^\perp$ are also Jacobi fields.*

Now assume that the bundle $\mathscr{H} = \{(x, v) \in TM;\ v \perp T_x\mathscr{F},\ x \in M\}$ orthogonally complement to $\mathscr{F}$ is integrable. Then the following is known.

THEOREM (Blumenthal and Hebda [1]). *Let $(M, g, \mathscr{F})$ be as above. Then the universal covering space $\tilde{M}$ of M is topologically a product $L \times H$, where*

(1) L (resp. H) is the universal covering space of the leaves of $\mathscr{F}$ (resp. $\mathscr{H}$),

(2) the canonical lifting $\tilde{\mathscr{F}}$ (resp. $\tilde{\mathscr{H}}$) of $\mathscr{F}$ (resp. $\mathscr{H}$) to $\tilde{M}$ is the foliation by leaves of the form $L \times \{h\}$, $h \in H$ (resp. $\{l\} \times H$, $l \in L$), and

(3) the projection $P: \tilde{M} \to L$ onto the first factor is a Riemannian submersion.

We identify a vector field X on L with the one $\tilde{X}$ on $\tilde{M}$ that is tangent to $\tilde{\mathscr{F}}$ and is P-related to X. We call $\tilde{X}$ the canonical lifting of X. When X is defined only on a subset A of L (e.g., A is a geodesic on L), we also define the canonical lifting $\tilde{X}$ of X to $\tilde{M}$ that is defined only on the subset $P^{-1}(A)$ in $\tilde{M}$ and satisfies the above conditions.

3. Proof of Theorem. Let $\tilde{M}$ be the universal covering space of M and $\tilde{\mathscr{F}}$ (resp. $\tilde{\mathscr{H}}$) be the canonical lifting of $\mathscr{F}$ (resp. $\mathscr{H}$) to $\tilde{M}$. We continue to use the notations in Section 2. Let $L \times \{h\}$, $h \in H$, be a leaf of $\tilde{\mathscr{F}}$ and $c: \boldsymbol{R} \to L \times \{h\}$ be a geodesic parametrized by arc length. By

Lemma 1, any Jacobi field $Y(t)$ along c decomposes into the sum of two Jacobi fields $W(t)+H(t)$, where $W(t)\in T\tilde{\mathscr{F}}$ and $H(t)\in T\tilde{\mathscr{H}}$. Hereafter, we consider only the $T\tilde{\mathscr{H}}$-component $H(t)$ of $Y(t)$ and call it an H-Jacobi field. Note that the dimension of the space of H-Jacobi fields along c is equal to $2q$. Let $\{E_i(t), X_a(t)\}$ be a parallel frame field along c given in Section 2. Denote by $H_{c(t)}$ the leaf of $\tilde{\mathscr{H}}$ passing through $c(t)$, that is, $H_{c(t)}=\{P(c(t))\}\times H$.

LEMMA 2. *There exist q H-Jacobi fields $V_1(t), \cdots, V_q(t)$ along c with the following properties*:

(1) $V_a(0)=X_a(0)$ *for* $a=1, \cdots, q$,

(2) $S_{c'(t)}V_a(t)=V_a'(t)$ *where "$'$" means the covariant differentiation with respect to $c'(t)$ and $S_{c'(t)}$ is the second fundamental form of the leaf $H_{c(t)}$ in the normal direction $c'(t)$ given by* $\langle S_{c'(t)}X, Y\rangle=-\langle c'(t), D_XY\rangle$ *for* $X, Y\in T_{c(t)}H_{c(t)}$, *and*

(3) $V_1(t), \cdots, V_q(t)$ *are linearly independent for all* $t\in\boldsymbol{R}$.

PROOF. For each $a=1, \cdots, q$, take a smooth curve $c_a\colon(-\varepsilon, \varepsilon)\to\tilde{M}$ in $H_{c(0)}$ with $c_a(0)=c(0)$ and $c_a'(0)=X_a(0)$. Identify c with the geodesic $P\circ c$ on L, where $P\colon\tilde{M}\to L$ is the natural projection, and lift $c'(0)$ canonically along curves c_a for $a=1, \cdots, q$. For each $a=1, \cdots, q$ define $F_a\colon(-\varepsilon, \varepsilon)\times\boldsymbol{R}\to\tilde{M}$ by $F_a(s, t)=\exp_{c_a(s)}tc'(0)$, and set $V_a(t)=F_{a*}(\partial/\partial s|_{(0,t)})$. We show that V_a's satisfy the above properties. By the construction, we have $P\circ F_a(s, t)=c(t)$. It follows that $V_a(t)$ is an H-Jacobi field for each a. Clearly V_a satisfies Property (1). For each X_b, we have $\langle S_{c'(t)}V_a, X_b\rangle=-\langle D_{V_a}X_b, c'(t)\rangle=\langle X_b, D_{V_a}c'(t)\rangle=\langle X_b, D_{c'(t)}V_a\rangle$ if we locally extend V_a, X_b and $c'(t)$ to suitable vector fields. On the other hand, for each E_i, $\langle D_{c'(t)}V_a, E_i\rangle=-\langle V_a, D_{c'(t)}E_i\rangle=0$ as $\mathscr{F}$ is totally geodesic. Thus we have $S_{c'(t)}V_a(t)=V_a'(t)$ which is Property (2). Finally we show that $V_a(t)$'s are linearly independent. Suppose not. Then there exist t_0 and $(x_a)\in\boldsymbol{R}^q$ with $(x_a)\neq 0$ and $\sum_{a=1}^q x_aV_a(t_0)=0$. Set $W(t)=\sum_{a=1}^q x_aV_a(t)$, hence $W(t_0)=0$. Further, by Property (2), we have $W'(t_0)=\sum_{a=1}^q x_aV_a'(t_0)=\sum_{a=1}^q x_aS_{c'(t_0)}V_a(t_0)=S_{c'(t_0)}W(t_0)=0$. As $W(t)$ is an H-Jacobi field, we have $W(t)=0$ and $(x_a)=0$, which is a contradiction.

Now represent $V_a(t)$ as $V_a(t)=\sum_{b=1}^q A_{ba}(t)X_b(t)$ and set $S_{ab}=\langle S_{c'(t)}X_a(t), X_b(t)\rangle$. Let $A(t)$ (resp. $S(t)$) be a (q, q)-matrix whose (a, b)-component is $A_{ab}(t)$ (resp. $S_{ab}(t)$). Denote by $A'(t)$ $\left(\text{resp. } \int_a^b A(t)dt\right)$ the componentwise differentiation (resp. integration) with respect to the parameter t. Then, by Lemma 2, (2), we have $A'(t)=S(t)A(t)$. Note that $\det A(t)\neq 0$ by Lemma 2, (3), and $A''(t)+R(t)A(t)=0$, where $R(t)$ is a (q, q)-matix $(R_{ab}(t))$.

The following lemma is proved in Goto [4] and Eschenburg and O'Sullivan [3] ($A(t)$ is a Legendre tensor in the sense of [3]). But we give a proof for convenience. We also refer to these literatures and Eschenburg and O'Sullivan [2] for generalities on Jacobi fields.

LEMMA 3. *Set* $B(t) = A(t)\int_0^t A^{-1}(s)\,{}^*A^{-1}(s)ds$, *where* ${}^*A$ is the transposed matrix *of* A. *Then* $B(t)$ *satisfies the following matrix Jacobi equation*

$$B''(t) + R(t)B(t) = 0 .$$

PROOF. By differentiating $B(t)$ with respect to t, we have $B'(t) = A'(t)\int_0^t A^{-1}(s)\,{}^*A^{-1}(s)ds + {}^*A^{-1}(t)$ and $B''(t) = A''(t)\int_0^t A^{-1}(s)\,{}^*A^{-1}(s)ds + A'(t)A^{-1}(t)\,{}^*A^{-1}(t) + ({}^*A^{-1})'(t)$. As $({}^*A\,{}^*A^{-1})'(t) = {}^*A'(t)\,{}^*A^{-1}(t) + {}^*A(t)({}^*A^{-1})'(t)$, we have $({}^*A^{-1})'(t) = -{}^*A^{-1}(t)\,{}^*A'(t)\,{}^*A^{-1}(t)$. It follows that $B''(t) + R(t)B(t) = {}^*A^{-1}(t)({}^*A(t)A'(t) - {}^*A'(t)A(t))A^{-1}(t)\,{}^*A^{-1}(t) = {}^*A^{-1}(t)({}^*A(t)S(t)A(t) - {}^*A(t)S(t)A(t))A^{-1}(t)\,{}^*A^{-1}(t) = 0$ by the remark preceding Lemma 3.

It follows from Lemma 3 that the space of H-Jacobi fields consists of the elements of the form $A(t)x + B(t)y$ for $x, y \in \boldsymbol{R}^q$.

LEMMA 4. *Let* $Y(t)$ *be an* H-*Jacobi field given by* $A(t)x + B(t)y$ *for* $x, y \in \boldsymbol{R}^q$. *If* $B(t)y \neq 0$ *for some* t, *then the norm* $|Y(t)| = \langle Y(t), Y(t)\rangle^{1/2}$ *of* $Y(t)$ *is unbounded.*

PROOF. Assume that $|Y(t)| \leqq N < \infty$ for $t \in (-\infty, \infty)$. Set

$$h(t) = |(Y(t), {}^*A^{-1}(t)y)| = \left|\left(\int_0^t A^{-1}(s)\,{}^*A^{-1}(s)ds\, y + x, y\right)\right|,$$

where (x, y) denotes the standard inner product of $x, y \in \boldsymbol{R}^q$. By assumption we have $|(Y(t), {}^*A^{-1}(t)y)| \leqq N|{}^*A^{-1}(t)y|$, that is, $h(t) \leqq N|{}^*A^{-1}(t)y|$. Note that ${}^*A^{-1}(t)y \neq 0$ for all $t \in \boldsymbol{R}$ because $y \neq 0$ and $A(t)$ is invertible for all $t \in R$.

Case 1: $(x, y) \geqq 0$. For $t \geqq 0$, we have $h(t) = \int_0^t |{}^*A^{-1}(s)y|^2 ds + (x, y)$. Thus $h(t) > 0$ for $t > 0$. Set $k(t) = 1/h(t)$ for $t > 0$. Then $k'(t) = -|{}^*A^{-1}(t)y|^2/h^2(t)$. Hence we have $k'(t) \leqq -1/N^2 < 0$, which is impossible because $k(t)$ is defined on $(0, \infty)$ and positive everywhere on $(0, \infty)$.

Case 2: $(x, y) < 0$. For $t \in (-\infty, 0)$ we have $h(t) = -\int_0^t |{}^*A^{-1}(s)y|^2 ds - (x, y)$. Then $h(t)$ is positive on $(-\infty, 0)$. Set $k(t) = 1/h(t)$. Then by the same computation as in Case 1, we have $k'(t) = |{}^*A^{-1}(t)y|^2/h^2(t) \geqq 1/N^2 > 0$ which is impossible because $k(t)$ is defined on $(-\infty, 0)$ and positive everywhere on $(-\infty, 0)$.

We now finish the proof of Theorem. Recall that Z preserves $\mathscr{F}$ if and only if $[Z, E] \in \Gamma(T\mathscr{F})$ for all $E \in \Gamma(T\mathscr{F})$. Let Z be a Killing field with bounded length. We denote also by Z the canonical lifting of Z to $\tilde{M}$ and perform the proof on $\tilde{M}$. As Z is a Killing field, the restriction to c is a Jacobi field along c. By Lemma 1, the $\tilde{\mathscr{H}}$-component Z^H of Z is an H-Jacobi field. By the assumption that $\langle Z, Z\rangle$ is bounded on c and by Lemma 4, Z^H is of the form $A(t)u$ for some $u \in \boldsymbol{R}^q$. Thus $Z^H(t) = \sum_{a=1}^q u_a V_a(t)$. Let E be the canonical lifting of a vector field on L. In order to prove that Z preserves $\tilde{\mathscr{F}}$ it suffices to see that $[Z^H, E] = 0$. Now let x be any point of M and c be a geodesic with $c(0) = x$ and $c'(0) = E_x$. We use the same notation as above. Lift $P \circ c'$ canonically on the vertical leaf H_x passing through x and denote it by c', too. Then $E = c'$ along the orbit of the flow generating Z^H and passing through x. It follows that $[Z^H, E] = D_{Z^H}E - D_E Z^H = D_{Z^H}c' - D_{c'}Z^H = [Z^H, c'] = \sum_{a=1}^q u_a[V_a, c'] = 0$ by Lemma 2 and the fact that $[V_a, c'] = F_{a*}([\partial/\partial s, \partial/\partial t]|_{(0,t)}) = 0$.

4. Concluding remarks.

First we give two examples.

EXAMPLE 1. Let $\boldsymbol{E}^2$ be the flat Euclidean plane with coordinates (x, y). Define $\mathscr{F}$ to be the orbits of the flow $\partial/\partial x$. Then $\mathscr{F}$ is a codimension-one totally geodesic foliation of $\boldsymbol{E}^2$. Let Z be a Killing field generated by rotations, e.g., $Z = y\,\partial/\partial x - x\,\partial/\partial y$. Then the function $\langle Z, Z\rangle$ is unbounded and Z does not preserve $\mathscr{F}$. This implies that we cannot drop the assumption on the boundedness of $\langle Z, Z\rangle$.

EXAMPLE 2. Let $\boldsymbol{E}^3$ be the flat Euclidean space with coordinates (x, y, z). Define $\mathscr{F}$ to be the orbits of the flow $\sin(2\pi z)\partial/\partial x + \cos(2\pi z)\partial/\partial y$. Then $\mathscr{F}$ is a one-dimensional totally geodesic foliation of $\boldsymbol{E}^3$. Note that the complementary orthogonal bundle is not integrable. The parallel vector field $Z = \partial/\partial z$ does not preserve $\mathscr{F}$. This implies that we cannot drop the integrability condition of the complementary orthogonal bundle. In this case, we can define V_a as in Lemma 2. But they do not satisfy Property (2) of Lemma 2. Consequently, Lemma 3 no longer holds good.

On the behavior of compact leaves of $\mathscr{F}$ by the flow of a Killing field Z, we have the following under weaker assumptions.

PROPOSITION. *Let (M, g) be a complete connected Riemannian manifold and $\mathscr{F}$ be a minimal foliation with integrable complementary orthogonal bundle. Assume that $\mathscr{F}$ has a compact leaf L_0. Then any flow-generating Killing field maps L_0 to a leaf of $\mathscr{F}$.*

For the proof, we use the notion of calibration introduced by Harvey and Lawson [5]. In this case, the volume form $\chi_{\mathscr{F}}$ of leaves, which is a smooth p-form on M, gives a calibration of $\mathscr{F}$. The existence of a calibration implies the homologically mass-minimizing property of compact leaves. It follows that any flow-generating Killing field maps L_0 to a leaf of $\mathscr{F}$.

Note that the assumption on the integrability of the complementary orthogonal bundle cannot be removed. In fact, we can construct a codimension-2 totally geodesic foliation on the flat torus T^3 from Example 2. This example shows that Proposition does not hold good in this case.

REFERENCES

[1] R. BLUMENTHAL AND J. HEBDA, De Rham decomposition theorems for foliated manifolds, Ann. Inst. Fourier Grenoble 33 (1983), 183–198.

[2] J. ESCHENBURG AND J. O'SULLIVAN, Growth of Jacobi fields and divergence of geodesics, Math. Z. 150 (1976), 221–237.

[3] J. ESCHENBURG AND J. O'SULLIVAN, Jacobi tensors and Ricci curvature, Math. Ann. 252 (1980), 1–26.

[4] M. S. GOTO, Manifolds without focal points, J. Differential Geom. 13 (1978), 341–359.

[5] R. HARVEY AND H. LAWSON, JR., Calibrated foliations, Amer. J. Math. 103 (1981), 411–435.

[6] D. L. JOHNSON AND L. B. WHITT, Totally geodesic foliations, J. Differential Geom. 15 (1980), 225–235.

[7] G. OSHIKIRI, Totally geodesic foliations and Killing fields, Tôhoku Math. J. 35 (1983), 387–392.

DEPARTMENT OF MATHEMATICS
COLLEGE OF GENERAL EDUCATION
TÔHOKU UNIVERSITY
KAWAUCHI, SENDAI 980
JAPAN