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Totally Real Parallel Submanifolds in $P^n(c)$

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Introduction

It is an interesting problem to classify the parallel submanifolds in a specific riemannian symmetric space. Actually, these submanifolds have been classified by D. Ferus [5], [6], [7] when the ambient space is the Euclidean space or the Euclidean sphere, and by M. Takeuchi [17] when the ambient space is the real hyperbolic space. Moreover H. Nakagawa and R. Takagi [10] and M. Takeuchi [16] have classified the parallel Kähler submanifolds in the complex projective space $P^n(c)$ with constant holomorphic sectional curvature c. It is known that parallel non-Kähler submanifolds in $P^n(c)$ are totally real.

In this paper we study n-dimensional complete totally real parallel submanifolds in $P^n(c)$. It is known that a riemannian manifold which admits a parallel isometric immersion into a riemannian symmetric space is a locally symmetric space. Fix an n-dimensional simply connected riemannian symmetric space M^n. Let $\bar{\mathscr{T}}_M$ (resp. $\bar{\mathscr{S}}_M$) be the set of all equivalence classes of totally real parallel isometric immersions of M^n into $P^n(c)$ (resp. of complete totally real parallel submanifolds in $P^n(c)$ with the universal riemannian covering M^n). Moreover, in section 3 we define an equivalence relation among symmetric trilinear forms on a tangent space of M satisfying certain conditions, and denote by $\bar{\mathscr{M}}_M$ the set of all equivalence classes of these trilinear forms. In sections 2, 3, we shall show that there are the natural correspondences among these sets $\bar{\mathscr{T}}_M$, $\bar{\mathscr{S}}_M$, $\bar{\mathscr{M}}_M$. In sections 4, 5, we shall determine the set $\bar{\mathscr{M}}_M$ for a riemannian symmetric space M without Euclidean factor. Moreover, in section 6, we shall study the set $\bar{\mathscr{M}}_M$ for a riemannian symmetric space M with Euclidean factor and an interesting example in the geometry of totally real surfaces in $P^2(c)$.

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§ 1. Preliminaries.

Let $\bar{M}^m$ (resp. M^n) be an m-dimensional (resp. n-dimensional) connected riemannian manifold. Denote by $\bar{\nabla}$ (resp. ∇) the riemannian connection on $\bar{M}^m$ (resp. M^n) and by $\bar{R}$ (resp. R) the riemannian curvature tensor for $\bar{\nabla}$ (resp. ∇). Now let f be an isometric immersion of M^n into $\bar{M}^m$. We denote by the same notation $\langle\ ,\ \rangle$ the riemannian metrics on the both riemannian manifolds. Moreover denote by σ_f the second fundamental form of M^n, by D the normal connection on the normal bundle $N(M)$ of M^n and by $R^\perp$ the curvature tensor for D. For a point p in M and a vector ζ in the normal space $N_p(M)$ at p, the shape operator A_ζ is defined by

$$\langle A_\zeta(X),\ Y\rangle = \langle \sigma_f(X,\ Y),\ \zeta\rangle$$

for all vectors $X, Y \in T_p(M)$. The shape operator A_ζ is a symmetric endomorphism on the tangent space $T_p(M)$ at p. It is also characterized by the equation that

$$\bar{\nabla}_X \zeta = -A_\zeta(X) + D_X \zeta$$

for any tangent vector field X of M and any normal vector field ζ of M.

Now we recall the following fundamental equations, called the equations of Gauss, Codazzi-Mainardi, and Ricci respectively.

$$\begin{aligned} (1.1)\qquad \langle \bar{R}(X,\ Y)Z,\ W\rangle &= \langle R(X,\ Y)Z,\ W\rangle + \langle \sigma_f(X,\ Z),\ \sigma_f(Y,\ W)\rangle \\ &\quad - \langle \sigma_f(X,\ W),\ \sigma_f(Y,\ Z)\rangle \end{aligned}$$

$$(1.2)\qquad \{\bar{R}(X,\ Y)Z\}^\perp = (\nabla^*_X \sigma_f)(Y,\ Z) - (\nabla^*_Y \sigma_f)(X,\ Z)$$

$$(1.3)\qquad \langle \bar{R}(X,\ Y)\zeta,\ \eta\rangle = \langle R^\perp(X,\ Y)\zeta,\ \eta\rangle - \langle [A_\zeta,\ A_\eta](X),\ Y\rangle$$

for all vectors $X, Y, Z, W \in T_p(M)$ and all vectors $\zeta, \eta \in N_p(M)$. Here we denote by $\{*\}^\perp$ the normal component of $*$ and by ∇^* the covariant derivation associated to the isometric immersion $f: M \to \bar{M}$, defined by

$$(\nabla^*_X \sigma_f)(Y,\ Z) = D_X(\sigma_f(Y,\ Z)) - \sigma_f(\nabla_f(\nabla_X Y,\ Z) - \sigma_f(Y,\ \nabla_X Z)$$

for tangent vector fields X, Y, Z of M. The second fundamental form σ_f as well as the isometric immersion f is said to be *parallel* if $\nabla^* \sigma_f =$

0. Moreover when f is an imbedding, the submanifold $f(M)$ is called a *parallel submanifold* in $\bar{M}$. If the second fundamental form σ_f is parallel, we have

$$D_X(\sigma_f(Y, Z)) = \sigma_f(\nabla_X Y, Z) + \sigma_f(Y, \nabla_X Z) \tag{1.4}$$

for all tangent vector fields X, Y, Z of M.

Now let $\bar{M}^{2r} = P^r(c)$ be the r-dimensional complex projective space with constant holomorphic sectional curvature $c(>0)$. The complex structure of $P^r(c)$ will be denoted by J. An isometric immersion $f: M^n \to P^r(c)$ is called *totally real* if $JT_p(M) \subset N_p(M)$ for every point p in M. Moreover when f is an imbedding, the submanifold $f(M)$ is called a *totally real submanifold* in $P^r(c)$. Then we have the following

LEMMA 1.1 (cf. see Lemma 2.4 [11]). *Let f be a totally real isometric immersion of M^n into $P^r(c)$. Then*

$$\langle \sigma_f(X, Y), JZ \rangle = \langle \sigma_f(X, Z), JY \rangle$$

for any point $p \in M$ and all vectors X, Y, $Z \in T_p(M)$.

From now on we assume that the complex dimension r equals n. For a totally real isometric immersion $f: M^n \to P^n(c)$ we define the *associated tensor* $\tilde{\sigma}_f$ of M as follows:

$$\tilde{\sigma}_f(X, Y) = J\sigma_f(X, Y)$$

for vectors X, $Y \in T_p(M)$, $p \in M$. If we identify the tangent space $T_p(M)$ with the cotangent space $T_p^*(M)$ through the riemannian metric on M, the associated tensor $\tilde{\sigma}_f$ is a symmetric covariant tensor of degree 3 on M by Lemma 1.1. For a vector X in $T_p(M)$, we define a symmetric endomorphism $\tilde{\sigma}_f(X)$ of $T_p(M)$ by

$$\tilde{\sigma}_f(X)(Y) = \tilde{\sigma}_f(X, Y)$$

for a vector Y in $T_p(M)$. Since the isometric immersion f is totally real in $P^n(c)$, we have $\bar{R}(X, Y)Z \in T_p(M)$ for all vectors X, Y, $Z \in T_p(M)$ and hence the equation of Gauss reduces to

$$\bar{R}(X, Y)Z = R(X, Y)^Z - [\tilde{\sigma}_f(X), \tilde{\sigma}_f(Y)](Z) \tag{1.5}$$

for all vectors X, Y, $Z \in T_p(M)$. Moreover we have the following

LEMMA 1.2. *Let f be a totally real parallel isometric immersion of M^n into $P^n(c)$. Then $\nabla\tilde{\sigma}_f = 0$, that is,*

$$\nabla_X(\tilde{\sigma}_f(Y, Z))=\tilde{\sigma}_f(\nabla_X Y, Z)+\tilde{\sigma}_f(Y, \nabla_X Z)$$

for all tangent vector fields X, Y, Z of M.

PROOF. Since $J\zeta$ is a tangent vector field of M for any normal vector field ζ along M,

$$J\bar{\nabla}_X J\zeta=J\nabla_X J\zeta+J\sigma_f(X, J\zeta)$$

for every tangent vector field X of M, while

$$J\bar{\nabla}_X J\zeta=-\bar{\nabla}_X\zeta=A_\zeta(X)-D_X\zeta$$

since $J\circ\bar{\nabla}_X=\bar{\nabla}_X\circ J$. Hence, comparing normal components we get

$$JD_X\zeta=\nabla_X J\zeta\ .$$

Thus, substituting $\zeta=\sigma_f(Y, Z)$, together with (1.4) we have

$$\nabla_X(\tilde{\sigma}_f(Y, Z))=\tilde{\sigma}_f(\nabla_X Y, Z)+\tilde{\sigma}_f(Y, \nabla_X Z)$$

for all tangent vector fields X, Y, Z of M. Q.E.D.

Let $\mathfrak{so}(T_p(M))$ be the Lie algebra of all skew symmetric endomorphisms of $T_p(M)$ and $\mathfrak{k}(p)$ the Lie subalgebra in $\mathfrak{so}(T_p(M))$ generated by the set $\{R_p(X, Y);\ X, Y\in T_p(M)\}$. Since the isometric immersion f is parallel, the manifold M is a locally symmetric space* and hence the Lie algebra $\mathfrak{k}(p)$ is spanned by the set $\{R_p(X, Y);\ X, Y\in T_p(M)\}$ and coincides with the holonomy algebra of M at p. Thus, by Lemma 1.2, we have the following

COROLLARY 1.3. *Let f be a totally real parallel isometric immersion of M^n into $P^n(c)$. Then $\mathfrak{k}(p)\cdot\tilde{\sigma}_f=0$, that is,*

$$T(\tilde{\sigma}_f(X, Y))=\tilde{\sigma}_f(T(X), Y)+\tilde{\sigma}_f(X, T(Y))$$

for any endomorphism $T\in\mathfrak{k}(p)$ and all vectors X, $Y\in T_p(M)$.

§ 2. Equivariant immersions associated to trilinear forms.

Assume that the manifold M^n is a simply connected symmetric space and fix a point o in M^n. Put $\mathfrak{p}=T_o(M)$, $\mathfrak{k}=\mathfrak{k}(o)$ and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ and define the bracket product [,] on $\mathfrak{g}$ as follows:

* Symmetric space means riemannian symmetric space in this paper.

$$[T, S]=T\circ S-S\circ T\ , \quad [T, X]=-[X, T]=T(X)\ ,$$
$$[X, Y]=-R_0(X, Y)$$

for endomorphisms T, S in $\mathfrak{k}$ and vectors X, Y in $\mathfrak{p}$. Then $(\mathfrak{g}, [\ ,\ ])$ is a Lie algebra over $\boldsymbol{R}$ and there exists a simply connected Lie group G acting on the symmetric space M isometrically and transitively, such that the Lie algebra of G is isomorphic to $\mathfrak{g}$ and that the Lie subgroup $K=\{g\in G;\ g(o)=o\}$ is connected and has the Lie subalgebra $\mathfrak{k}$ (cf. see [8]). Let $\mathscr{M}_M$ be the set of all $\mathfrak{p}$-valued bilinear forms $\tilde{\sigma}$ on $\mathfrak{p}$ satisfying the following conditions:

(1) $\tilde{\sigma}$ is a symmetric trilinear form on $\mathfrak{p}$ under the canonical identification of $\mathfrak{p}^*\otimes\mathfrak{p}^*\otimes\mathfrak{p}$ with $\mathfrak{p}^*\otimes\mathfrak{p}^*\otimes\mathfrak{p}^*$ through the riemannian metric $\langle\ ,\ \rangle$ on $\mathfrak{p}$,

(2) $\mathfrak{k}\cdot\tilde{\sigma}=0$,

(3) $(c/4)(\langle Y, Z\rangle X-\langle X, Z\rangle Y)=R(X, Y)Z-[\tilde{\sigma}(X), \tilde{\sigma}(Y)](Z)$ for all vectors X, Y, $Z\in\mathfrak{p}$.

Let f be a totally real parallel isometric immersion of M^n into $P^n(c)$. Then

$$\bar{R}(X, Y)Z=(c/4)(\langle Y, Z\rangle X-\langle X, Z\rangle Y)$$

for all vectors X, Y, $Z\in\mathfrak{p}$. Hence we have that $(\tilde{\sigma}_f)_0\in\mathscr{M}_M$ by Lemma 1.1, Corollary 1.3 and (1.5).

Now the riemannian manifold $P^n(c)$ is also a simply connected symmetric space. We denote by $\bar{o}$, $\bar{\mathfrak{p}}$, $\bar{\mathfrak{k}}$, $\bar{\mathfrak{g}}$, $\bar{G}$, $\bar{K}$ the objects for $P^n(c)$ which are generally denoted by o, $\mathfrak{p}$, $\mathfrak{k}$, $\mathfrak{g}$, G, K for M^n. Note that $\bar{G}$ (resp. $\bar{\mathfrak{g}}$) is isomorphic to the compact Lie group $SU(n+1)$ (resp. the compact Lie algebra $\mathfrak{su}(n+1)$) and that $\bar{\mathfrak{k}}$ is given by

$$\bar{\mathfrak{k}}=\mathfrak{u}(\bar{\mathfrak{p}})=\{T\in\mathfrak{so}(\bar{\mathfrak{p}});\ J\circ T=T\circ J\}\ .$$

A linear subspace $\mathfrak{q}$ in $\bar{\mathfrak{p}}$ is called *totally real* if the subspaces $\mathfrak{q}$ and $J\mathfrak{q}$ are orthogonal. Totally real subspaces in $\bar{\mathfrak{p}}$ of the same dimension are conjugate to each other under the natural action of $\bar{K}$ on $\bar{\mathfrak{p}}$. Fix an n-dimensional totally real subspace $\mathfrak{q}$ in $\bar{\mathfrak{p}}$ and set

$$\bar{\mathfrak{k}}_1=\{T\in\bar{\mathfrak{k}};\ T(\mathfrak{q})\subset\mathfrak{q}\}\quad\text{and}\quad\bar{\mathfrak{k}}_2=\{T\in\bar{\mathfrak{k}};\ T(\mathfrak{q})\subset J\mathfrak{q}\}\ .$$

Then $\bar{\mathfrak{k}}_1$ (resp. $\bar{\mathfrak{k}}_2$) is a Lie subalgebra (resp. linear subspace) in $\bar{\mathfrak{k}}$, and $\bar{\mathfrak{k}}$ is the direct sum of $\bar{\mathfrak{k}}_1$ and $\bar{\mathfrak{k}}_2$. In fact, take an orthonormal basis $\{e_1, \cdots, e_n\}$ of $\mathfrak{q}$ and identify $\bar{\mathfrak{p}}$ with $\boldsymbol{C}^n$ by the correspondence:

$$\bar{\mathfrak{p}}\ni(\Sigma x_j e_j)+J(\Sigma y_j e_j)\leftrightarrow(x_j+\sqrt{-1}y_j)\in\boldsymbol{C}^n\ .$$

Then $\bar{\mathfrak{k}}$, $\bar{\mathfrak{k}}_1$ and $\bar{\mathfrak{k}}_2$ are identified with the Lie algebra $\mathfrak{u}(n)$ of all skew hermitian matrices of degree n, the Lie algebra $\mathfrak{so}(n)$ of all real skew symmetric matrices of degree n, and the linear space $\sqrt{-1}S^n(\boldsymbol{R})=\{\sqrt{-1}A;$ A is a real symmetric matrix of degree $n\}$ respectively. This implies the assertion.

Let s be a linear isometry of $\mathfrak{p}$ onto $\mathfrak{q}$. We define an injective Lie homomorphism τ_s of $\mathfrak{so}(\mathfrak{p})$ into $\bar{\mathfrak{k}}_1$ by

$$\tau_s(T)(s(X)+Js(Y))=s(T(X))+Js(T(Y))$$

for $T\in\mathfrak{so}(\mathfrak{p})$ and vectors $X, Y\in\mathfrak{p}$. Next, for an element $\tilde{\sigma}$ in $\mathscr{M}_M$, we define a linear mapping $\mu_{s,\tilde{\sigma}}$ of $\mathfrak{p}$ into $\bar{\mathfrak{k}}_2$ by

$$\mu_{s,\tilde{\sigma}}(X)(s(Y)+Js(Z))=s(\tilde{\sigma}(X, Z))-Js(\tilde{\sigma}(X, Y))$$

for vectors $X, Y, Z\in\mathfrak{p}$. Here note that the condition (1) for $\tilde{\sigma}$ implies that $\mu_{s,\tilde{\sigma}}(X)\in\bar{\mathfrak{k}}$. Now we define a linear mapping $\rho_{s,\tilde{\sigma}}$ of $\mathfrak{g}$ into $\bar{\mathfrak{g}}$ by

$$\rho_{s,\tilde{\sigma}}(T+X)=\tau_s(T)+\mu_{s,\tilde{\sigma}}(X)+s(X)$$

for $T\in\mathfrak{k}$ and $X\in\mathfrak{p}$. Then we have the following

LEMMA 2.1. *The linear mapping $\rho_{s,\tilde{\sigma}}$ of $\mathfrak{g}$ into $\bar{\mathfrak{g}}$ is an injective Lie homomorphism.*

PROOF. At first we shall prove the following three formulas:

$$[\tau_s(T), \mu_{s,\tilde{\sigma}}(X)]=\mu_{s,\tilde{\sigma}}(T(X)) \tag{2.1}$$

$$[\mu_{s,\tilde{\sigma}}(X), \mu_{s,\tilde{\sigma}}(Y)]=\tau_s([\tilde{\sigma}(Y), \tilde{\sigma}(X)]) \tag{2.2}$$

$$\bar{R}(s(X), s(Y))=\tau_s(R(X, Y)-[\tilde{\sigma}(X), \tilde{\sigma}(Y)]) \tag{2.3}$$

for any $T\in\mathfrak{k}$ and all vectors $X, Y\in\mathfrak{p}$. By the condition (2) for $\tilde{\sigma}$ we have

$$\begin{aligned}
&[\tau_s(T), \mu_{s,\tilde{\sigma}}(X)](s(Y)+Js(Z))\\
&\quad=s(T(\tilde{\sigma}(X, Z)))-Js(T(\tilde{\sigma}(X, Y)))+Js(\tilde{\sigma}(X, T(Y)))-s(\tilde{\sigma}(X, T(Z)))\\
&\quad=s(\tilde{\sigma}(T(X), Z))-Js(\tilde{\sigma}(T(X), Y))\\
&\quad=\mu_{s,\tilde{\sigma}}(T(X))(s(Y)+Js(Z))
\end{aligned}$$

for all vectors $Y, Z\in\mathfrak{p}$, and hence (2.1) is proved. Next, by the definitions of τ_s and $\mu_{s,\tilde{\sigma}}$ we have

$$\begin{aligned}
&[\mu_{s,\tilde{\sigma}}(X), \mu_{s,\tilde{\sigma}}(Y)](s(Z)+Js(W)) \\
&\quad = -Js(\tilde{\sigma}(X, \tilde{\sigma}(Y, W)))-s(\tilde{\sigma}(X, \tilde{\sigma}(Y, Z)))+Js(\tilde{\sigma}(Y, \tilde{\sigma}(X, W))) \\
&\quad\quad +s(\tilde{\sigma}(Y, \tilde{\sigma}(X, Z))) \\
&\quad = s([\tilde{\sigma}(Y), \tilde{\sigma}(X)](Z))+Js([\tilde{\sigma}(Y), \tilde{\sigma}(X)](W)) \\
&\quad = \tau_s([\tilde{\sigma}(Y), \tilde{\sigma}(X)])(s(Z)+Js(W))
\end{aligned}$$

for all vectors Z, W in $\mathfrak{p}$, and hence (2.2) is proved. Since the subspace $\mathfrak{q}$ in $\bar{\mathfrak{p}}$ is totally real, we have

$$\bar{R}(s(X), s(Y))s(Z)=(c/4)(\langle Y, Z\rangle s(X)-\langle X, Z\rangle s(Y))$$

for all vectors $X, Y, Z \in \mathfrak{p}$. By the condition (3) for $\tilde{\sigma}$ we have

$$\begin{aligned}
&\bar{R}(s(X), s(Y))(s(Z)+Js(W)) \\
&\quad =\bar{R}(s(X), s(Y))s(Z)+J\bar{R}(s(X), s(Y))s(W) \\
&\quad =s((c/4)(\langle Y, Z\rangle X-\langle X, Z\rangle Y))+Js((c/4)(\langle Y, W\rangle X-\langle X, W\rangle Y)) \\
&\quad =s(R(X, Y)Z-[\tilde{\sigma}(X), \tilde{\sigma}(Y)]Z)+Js(R(X, Y)W-[\tilde{\sigma}(X), \tilde{\sigma}(Y)]W) \\
&\quad =\tau_s(R(X, Y)-[\tilde{\sigma}(X), \tilde{\sigma}(Y)])(s(Z)+Js(W))
\end{aligned}$$

for all vectors $Z, W \in \mathfrak{p}$. Hence (2.3) is proved.

Now by (2.1), (2.2) and (2.3) we have

$$\begin{aligned}
&[\rho_{s,\tilde{\sigma}}(T+X), \rho_{s,\tilde{\sigma}}(S+Y)] \\
&\quad =[\tau_s(T), \tau_s(S)]+[\tau_s(T), \mu_{s,\tilde{\sigma}}(Y)]+[\tau_s(T), s(Y)] \\
&\quad\quad +[\mu_{s,\tilde{\sigma}}(X), \tau_s(S)]+[\mu_{s,\tilde{\sigma}}(X), \mu_{s,\tilde{\sigma}}(Y)]+[\mu_{s,\tilde{\sigma}}(X), s(Y)] \\
&\quad\quad +[s(X), \tau_s(S)]+[s(X), \mu_{s,\tilde{\sigma}}(Y)]+[s(X), s(Y)] \\
&\quad =\tau_s([T, S])+\mu_{s,\tilde{\sigma}}(T(Y))+s(T(Y))-\mu_{s,\tilde{\sigma}}(S(X)) \\
&\quad\quad +\tau_s([\tilde{\sigma}(Y), \tilde{\sigma}(X)])-Js(\tilde{\sigma}(X, Y))-s(S(X))+Js(\tilde{\sigma}(Y, X)) \\
&\quad\quad -\tau_s(R(X, Y)-[\tilde{\sigma}(X), \tilde{\sigma}(Y)]) \\
&\quad =\tau_s([T, S]-R(X, Y))+\mu_{s,\tilde{\sigma}}(T(Y)-S(X))+s(T(Y)-S(X)) \\
&\quad =\rho_{s,\tilde{\sigma}}([T+X, S+Y])
\end{aligned}$$

for all $T, S \in \mathfrak{k}$ and all $X, Y \in \mathfrak{p}$, and hence $\rho_{s,\tilde{\sigma}}$ is a Lie homomorphism of $\mathfrak{g}$ into $\bar{\mathfrak{g}}$. Moreover, since τ_s and s are injective, $\rho_{s,\tilde{\sigma}}$ is injective.

Q.E.D.

Since $\mathfrak{g}=\mathfrak{su}(n+1)$, we have the following

COROLLARY 2.2. *If the set $\mathscr{M}_M$ is not empty, the Lie algebra $\mathfrak{g}$ is the direct sum of an abelian Lie algebra and a Lie algebra of compact type.*

We call $\rho_{s,\tilde{\sigma}}$ the *Lie homomorphism associated to s and $\tilde{\sigma}$.*

Since G is a simply connected Lie group, there exists the unique Lie homomorphism $\hat{\rho}_{s,\tilde{\sigma}}$ of G into $\bar{G}$ such that the differential $d\hat{\rho}_{s,\tilde{\sigma}}$ is $\rho_{s,\tilde{\sigma}}$. The associated homomorphism $\rho_{s,\tilde{\sigma}}$ maps the Lie subalgebra $\mathfrak{k}$ into the Lie subalgebra $\bar{\mathfrak{k}}$ and the isotropy subgroup K is connected. Hence we can define a G-equivariant C^{∞}-mapping $f_{s,\tilde{\sigma}}$ of M^n into $P^n(c)$ by

$$f_{s,\tilde{\sigma}}(g(o))=\hat{\rho}_{s,\tilde{\sigma}}(g)(\bar{o})$$

for $g\in G$. Then we have the following

THEOREM 2.3. *Let M^n be a simply connected symmetric space. Then, for any linear isometry s and any $\tilde{\sigma}\in\mathscr{M}_M$, the associated G-equivariant mapping $f_{s,\tilde{\sigma}}$ of M^n into $P^n(c)$ is a totally real parallel isometric immersion such that*

$$(f_{s,\tilde{\sigma}})_{*o}=s \quad\text{and}\quad (\tilde{\sigma}_{f_{s,\tilde{\sigma}}})_o=\tilde{\sigma}\ .$$

PROOF. Note that $\bar{G}$ divided by the center is the group of all holomorphic isometries of $P^n(c)$. The claim $(f_{s,\tilde{\sigma}})_{*o}=s$ is obvious by the definition of $f_{s,\tilde{\sigma}}$. Now we show that $f_{s,\tilde{\sigma}}$ is a totally real parallel isometric immersion. Since $f_{s,\tilde{\sigma}}$ is G-equivariant, it is sufficient to see our claim at o. The linear mapping s is an isometry and the image $\mathfrak{q}$ of s is a totally real subspace in $\bar{\mathfrak{p}}$. Hence $f_{s,\tilde{\sigma}}$ is a totally real and isometric immersion at o. Moreover, to show that $f_{s,\tilde{\sigma}}$ is parallel, it is sufficient to see that

$$[\rho_{s,\tilde{\sigma}}(X)_{\bar{\mathfrak{k}}},\ [\rho_{s,\tilde{\sigma}}(X)_{\bar{\mathfrak{k}}},\ \rho_{s,\tilde{\sigma}}(X)_{\bar{\mathfrak{p}}}]]\in\mathfrak{q} \tag{2.4}$$

for any vector X in $\mathfrak{p}$ (see Proposition 5.2 in [11]). Here the suffix $\bar{\mathfrak{k}}$ (resp. $\bar{\mathfrak{p}}$) means the $\bar{\mathfrak{k}}$-component (resp. $\bar{\mathfrak{p}}$-component) with respect to the decomposition $\bar{\mathfrak{g}}=\bar{\mathfrak{k}}+\bar{\mathfrak{p}}$. Since

$$\rho_{s,\tilde{\sigma}}(X)_{\bar{\mathfrak{k}}}=\mu_{s,\tilde{\sigma}}(X) \quad\text{and}\quad \rho_{s,\tilde{\sigma}}(X)_{\bar{\mathfrak{p}}}=s(X)\ ,$$

the left hand of (2.4) equals $-s(\tilde{\sigma}(X,\tilde{\sigma}(X,X)))\in\mathfrak{q}$. Now the second fundamental form at o of the G-equivariant immerssion $f_{s,\tilde{\sigma}}$ is given by

$$(\tilde{\sigma}_{f_{s,\tilde{\sigma}}})_o(X,\ Y)=[(\rho_{s,\tilde{\sigma}})(X)_{\bar{\mathfrak{k}}},\ (\rho_{s,\tilde{\sigma}})(Y)_{\bar{\mathfrak{p}}}]_{J\mathfrak{q}} \tag{2.5}$$

for all vectors X, Y in $\mathfrak{p}$ (see Proposition 5.1 in [11]). Here the suffix $J\mathfrak{q}$ means the $J\mathfrak{q}$-component with respect to the decomposition $\bar{\mathfrak{p}}=\mathfrak{q}+J\mathfrak{q}$. Hence we have $(\tilde{\sigma}_{f_{s,\tilde{\sigma}}})_o=-Js(\tilde{\sigma}(X,Y))$. This implies $(\tilde{\sigma}_{f_{s,\tilde{\sigma}}})_o=\tilde{\sigma}$. Q.E.D.

§ 3. Frenet curves and rigidity problems.

Let $\bar{M}$ be a riemannian manifold and $c(t)$ be a C^∞-curve in $\bar{M}$ defined on an open interval I containing 0 which is parametrized by arc-length. The curve $c(t)$ is called a *Frenet curve* in $\bar{M}$ of osculating rank $r(\geqq 1)$ if for all $t \in I$ its higher order derivatives

$$\dot{c}(t)=(\bar{\nabla}^0_{\partial/\partial t}\dot{c})(t),\ (\bar{\nabla}_{\partial/\partial t}\dot{c})(t),\ \cdots,\ (\bar{\nabla}^{r-1}_{\partial/\partial t}\dot{c})(t)$$

are linearly independent but

$$\dot{c}(t)=(\bar{\nabla}^0_{\partial/\partial t}\dot{c})(t),\ (\bar{\nabla}_{\partial/\partial t}\dot{c})(t),\ \cdots,\ (\bar{\nabla}^{r}_{\partial/\partial t}\dot{c})(t)$$

are linearly dependent in $T_{c(t)}(M)$. Then there exist the unique positive C^∞-functions $\kappa_1(t), \cdots, \kappa_{r-1}(t)$ on I and the unique orthonormal C^∞-vector fields $V_1(t), \cdots, V_r(t)$ along the curve $c(t)$ such that

$$\text{(3.1)}\qquad \begin{cases} \dot{c}(t)=V_1(t) \\ (\bar{\nabla}_{\partial/\partial t}V_1)(t)=\kappa_1(t)V_2(t) \\ (\bar{\nabla}_{\partial/\partial t}V_2)(t)=-\kappa_1(t)V_1(t)+\kappa_2(t)V_3(t) \\ \quad\vdots \\ (\bar{\nabla}_{\partial/\partial t}V_j)(t)=-\kappa_{j-1}(t)V_{j-1}(t)+\kappa_j(t)V_{j+1}(t) \\ \quad\vdots \\ (\bar{\nabla}_{\partial/\partial t}V_{r-1})(t)=-\kappa_{r-2}(t)V_{r-2}(t)+\kappa_{r-1}(t)V_r(t) \\ (\bar{\nabla}_{\partial/\partial t}V_r)(t)=-\kappa_{r-1}(t)V_{r-1}(t)\ . \end{cases}$$

Here we call $\kappa_j(t)(1\leqq j\leqq r-1)$ the *Frenet curvature functions* on I, the vector fields $\{V_j(t); 1\leqq j\leqq r\}$ the *Frenet r-frame* along $c(t)$, and the equations (3.1) the *Frenet formulas*. For a given integer $r(\geqq 1)$ and given positive C^∞-functions $\kappa_1(t), \cdots, \kappa_{r-1}(t)$ on I, the Frenet formulas (3.1) may be regarded as a system of differential equations with variables $c, V_1, \cdots, V_r$. It is known that this system of differential equations has the unique local solutions for given initial conditions; a point $c(0)=p\in\bar{M}$ and an orthonormal r-frame $\{V_1(0)=V_1, \cdots, V_r(0)=V_r\}$ of $T_p(\bar{M})$. If the riemannian manifold $\bar{M}$ is complete, the Frenet curve $c(t)$ is defined for $-\infty<t<+\infty$ (cf. see [4] and [15]). Now we have the following

LEMMA 3.1 (W. Strübing [15]). *Let M and $\bar{M}$ be riemannian manifolds and f a parallel isometric immersion of M into $\bar{M}$. Suppose that a curve $c(t)$ defined on I containing 0 is a geodesic in M parametrized by arc-length. Then*

a) *the curve* $(f\circ c)(t)$ *on* I *is a Frenet curve in* $\bar{M}$,

b) *the Frenet curvature functions* $\kappa_1(t), \cdots, \kappa_{r-1}(t)$ *are constant (and positive), where* r *denotes the osculating rank of* $(f\circ c)(t)$,

c) *the integer* $r(\geqq 1)$, *the constant positive numbers* $\kappa_1, \cdots, \kappa_{r-1}$ *and the orthonormal vectors* $V_1 = V_1(0), \cdots, V_r = V_r(0)$ *are determined only by the initial point* $p = c(0)$ *of* $c(t)$, *the initial tangent vector* $X = \dot{c}(0)$ *of* $c(t)$, *the differential* $(f_*)_p$ *at* p, *and the second fundamental form* $(\sigma_f)_p$ *at* p.

Now, by Lemma 3.1, we have the following fundamental lemma.

LEMMA 3.2. *Let* g *and* f *be parallel isometric immersions of a complete riemannian manifold* M *into another riemannian manifold* $\bar{M}$. *If there exists a point* o *in* M *such that*

$$g(o) = f(o) = \bar{o},\ (g_*)_o = (f_*)_o : T_o(M) \to T_{\bar{o}}(\bar{M}),\ (\sigma_g)_o = (\sigma_f)_o\ ,$$

then the mappings g *and* f *coincide on* M.

PROOF. For any point p in M, there exists a geodesic $c(t)$ in M parametrized by arc-length, such that $c(0) = o$ and $c(l) = p$. Then $(g\circ c)(t)$ and $(f\circ c)(t)$ are Frenet curves in $\bar{M}$ by Lemma 3.1, a). By Lemma 3.1, c), the above assumption implies that the Frenet curves $(f\circ c)(t)$ and $(g\circ c)(t)$ are solutions of the same Frenet formulas for the same initial conditions. Hence, by the uniqueness for solutions of the system of differential equations, we have $(f\circ c)(t) = (g\circ c)(t)$ and particularly $f(p) = g(p)$. Q.E.D.

Now let $\mathscr{T}_M$ be the set of all totally real parallel isometric immersions of a simply connected symmetric space M^n into the riemannian manifold $P^n(c)$, $I(M)$ the group of all isometries of M, and $\bar{G}$ the group of all holomorphic isometries of $P^n(c)$. Then we can define an action of $\bar{G}\times I(M)$ on $\mathscr{T}_M$ by

$$(\bar{g}, g)\cdot f = \bar{g}\circ f\circ g^{-1}$$

for $\bar{g}\in\bar{G}$, $g\in I(M)$ and $f\in\mathscr{T}_M$. Let $\bar{\mathscr{T}}_M$ be the set of all orbits of the $\bar{G}\times I(M)$-action on $\mathscr{T}_M$. The orbit $[f]_{\mathscr{T}}$ of f in $\mathscr{T}_M$ is called the *equivalence class* of f.

Secondly, let $\mathscr{S}_M$ be the set of all complete totally real parallel submanifolds whose universal riemannian coverings are M^n. Then we can define an action of $\bar{G}$ on $\mathscr{S}_M$ by

$$\bar{g}\cdot N = \bar{g}(N)$$

for $\bar{g} \in \bar{G}$ and $N \in \mathscr{S}_M$. Let $\bar{\mathscr{S}}_M$ be the set of all orbits of the $\bar{G}$-action on $\mathscr{S}_M$. The orbit $[N]_{\mathscr{S}}$ of N in $\mathscr{S}_M$ is called the *equivalence class* of N.

Finally, set

$$F_o(M)=\{g \in I(M);\ g(o)=o\} .$$

Then we can define an action of $F_o(M)$ on $\mathscr{M}_M$ by

$$(k\cdot\tilde{\sigma})(X,\ Y)=(k_*)_o(\tilde{\sigma}((k_*)_o^{-1}X,\ (k_*)_o^{-1}Y))$$

for $k \in F_o(M)$, $\tilde{\sigma} \in \mathscr{M}_M$ and $X,\ Y \in \mathfrak{p}$. Let $\bar{\mathscr{M}}_M$ be the set of all orbits of the $F_o(M)$-action on $\mathscr{M}_M$. The orbit $[\tilde{\sigma}]_{\mathscr{M}}$ of $\tilde{\sigma}$ in $\mathscr{M}_M$ is called the *equivalence class* of $\tilde{\sigma}$.

Now we study the relations among three kinds of equivalences. At first we have the following

LEMMA 3.3. *For any $\bar{g} \in \bar{G}$, $g \in I(M)$ and $f \in \mathscr{T}_M$, there exists an element $k \in F_o(M)$ such that*

$$(\tilde{\sigma}_{\bar{g}\circ f\circ g^{-1}})_o = k\cdot(\tilde{\sigma}_f)_o .$$

Moreover, if $g \in F_o(M)$, the very same element g can be taken as the above element k.

PROOF. Since $\bar{g}_*$ and J are comutative, we have

$$(3.2)\qquad (\tilde{\sigma}_{\bar{g}\circ f\circ g^{-1}})_o(X,\ Y)=(\tilde{\sigma}_{f\circ g^{-1}})_o(X,\ Y) \\ = g_*((\tilde{\sigma}_f)_{g^{-1}(o)}((g_*)^{-1}X,\ (g_*)^{-1}Y))$$

for all vectors $X,\ Y \in \mathfrak{p}$. Let $\gamma(t)$ be a geodesic joining o to $g^{-1}(o)$. Since M is a symmetric space, there exists some $h \in I(M)$ such that $h(o)=g^{-1}(o)$ and that $h^{-1}\cdot(\tilde{\sigma}_f)_{h(o)}$ is the parallel translation of $(\tilde{\sigma}_f)_{h(o)}$ along the geodesic $\gamma(t)$, where

$$h^{-1}\cdot(\tilde{\sigma}_f)_{h(o)}(X,\ Y)=h_*^{-1}((\tilde{\sigma}_f)_{h(o)}(h_*X,\ h_*Y))$$

for all vectors $X,\ Y \in \mathfrak{p}$ (cf. see [8]). Putting $k=g\circ h$, we have $k \in F_o(M)$. Since $\tilde{\sigma}_f$ is parallel by Lemma 1.2, we have

$$\text{the last term of (3.2)} \\ = k_*(h_*^{-1}((\tilde{\sigma}_f)_{h(o)}(h_*(k_*^{-1}X),\ h_*(k_*^{-1}Y))) \\ = k_*((\tilde{\sigma}_f)_o(k_*^{-1}X,\ k_*^{-1}Y))=(k\cdot(\tilde{\sigma}_f)_o)(X,\ Y) .$$

The second assertion is clear from the above proof. Q.E.D.

Now we define a mapping i_M of $\bar{\mathscr{T}}_M$ into $\bar{\mathscr{M}}_M$ by

$$i_M([f]_{\mathscr{T}})=[(\tilde{\sigma}_f)_o]_{\mathscr{M}}$$

for f in $\mathscr{T}_M$. By Lemma 3.3 the mapping i_M is well-defined. Then we have the following

THEOREM 3.4. *The mapping i_M of $\bar{\mathscr{T}}_M$ into $\bar{\mathscr{M}}_M$ is bijective.*

PROOF. By Theorem 2.3 it is obvious that i_M is onto. We show that the mapping i_M is injective. Take two mappings f_1, f_2 in $\mathscr{T}_M$ and suppose that $(\tilde{\sigma}_{f_1})_o=k\cdot(\tilde{\sigma}_{f_2})_o$ for some $k\in F_o(M)$. Then, putting $f_3=f_2\circ k^{-1}$, we have $(\tilde{\sigma}_{f_1})_o=(\tilde{\sigma}_{f_3})_o$ by Lemma 3.3. Since f_1 and f_3 are totally real, there exists some $\bar{g}\in\bar{G}$ such that

$$(\bar{g}\circ f_3)(o)=f_1(o)=\bar{o} \quad \text{and} \quad (\bar{g}\circ f_3)_*(T_o(M))=(f_1)_*(T_o(M))=\mathfrak{q}\ .$$

Moreover, since any linear isometry of the totally real subspace $\mathfrak{q}$ is the differential at $\bar{o}$ of some holomorphic isometry of $P^n(c)$, we may assume that $(\bar{g}\circ f_3)_{*o}=(f_1)_{*o}$. Here note that $(\tilde{\sigma}_{\bar{g}\circ f_3})_o=(\tilde{\sigma}_{f_3})_o$ by Lemma 3.3. Hence, by Lemma 3.2, we have $\bar{g}\circ f_3=f_1$ on M and thus $[f_1]_{\mathscr{T}}=[f_3]_{\mathscr{T}}=[f_2]_{\mathscr{T}}$.
Q.E.D.

THEOREM 3.5. *Any totally real parallel isometric immersion of M^n into $P^n(c)$ is G-equivariant.*

PROOF. Let f be a totally real parallel isometric immersion and put $f(o)=\bar{o}$. Then we have $f=f_{(f_*)_o,(\tilde{\sigma}_f)_o}$ by Theorem 2.3 and Lemma 3.2. This implies the theorem. Q.E.D.

Now let j_M be a mapping of $\bar{\mathscr{T}}_M$ into $\bar{\mathscr{S}}_M$ defined by

$$j_M([f]_{\mathscr{T}})=[f(M)]_{\mathscr{S}}$$

for $f\in\mathscr{T}_M$. Here note that the image $f(M)$ is a submanifold in $P^n(c)$ by Theorem 3.5. Then we have the following

THEOREM 3.6. *The mapping j_M of $\bar{\mathscr{T}}_M$ into $\bar{\mathscr{S}}_M$ is bijective.*

PROOF. It is obvious that j_M is onto. We show that the mapping j_M is injective. Take two mappings $f_1, f_2\in\mathscr{T}_M$ and suppose that $f_1(M)=\bar{g}(f_2(M))$ for some $\bar{g}\in\bar{G}$. Put $\bar{o}=f_1(o)$ and $N=f_1(M)$. Taking some $g\in I(M)$ and putting $f_3=\bar{g}\circ f_2\circ g$, we have

$$f_1(o)=f_3(o)=\bar{o} \quad \text{and} \quad f_1(M)=f_3(M)=N\ .$$

Let $(\sigma_N)_{\bar{o}}$ be the second fundamental form at $\bar{o}$ of the submanifold N. Then we have

$$(\sigma_N)_{\bar{o}}(\bar{X}, \bar{Y})=(\sigma_{f_1})_o((f_1)_*^{-1}\bar{X}, (f_1)_*^{-1}\bar{Y})$$
$$=(\sigma_{f_3})_o((f_3)_*^{-1}\bar{X}, (f_3)_*^{-1}\bar{Y})$$

for all vectors $\bar{X}, \bar{Y}\in T_{\bar{o}}(N)$. Hence we have

$$(\tilde{\sigma}_{f_3})_o(X, Y)=((f_3)_*^{-1}\circ(f_1)_*)((\tilde{\sigma}_{f_1})_o((f_1)_*^{-1}\circ(f_3)_*X, (f_1)_*^{-1}\circ(f_3)_*Y))$$

for all vectors $X, Y\in T_o(M)$. Note that $f_3^{-1}\circ f_1$ defines a local isometry of M around o. Since M is a simply connected symmetric space, there exists a unique element $k\in F_o(M)$ that coincides with $f_3^{-1}\circ f_1$ around o. Hence we have $(\tilde{\sigma}_{f_3})_o=k\cdot(\tilde{\sigma}_{f_1})_o$. By Theorem 3.4 we have $[f_3]_{\mathscr{T}}=[f_1]_{\mathscr{T}}$ and thus $[f_2]_{\mathscr{T}}=[f_1]_{\mathscr{T}}$. Q.E.D.

§ 4. The set $\mathscr{M}_M$ for a simply connected symmetric space M without Euclidean factor.

In this section we assume that M^n is a simply connected symmetric space without Euclidean factor; thus, M is decomposed as a riemannian manifold as follows:

$$M^n=M_1^{n_1}\times\cdots\times M_r^{n_r}\left(n=\sum_{j=1}^r n_j\right)$$

where $M_j^{n_j}$ is an n_j-dimensional irreducible simply connected symmetric space for each j. Then the tangent space $T_o(M)=\mathfrak{p}$ (resp. the holonomy algebra $\mathfrak{k}$) is decomposed as follows:

$$\mathfrak{p}=\sum_{j=1}^r \mathfrak{p}_j \quad \left(\text{resp. } \mathfrak{k}=\sum_{j=1}^r \mathfrak{k}_j\right)$$

where the subspace $\mathfrak{p}_j\subset\mathfrak{p}$ (resp. the subalgebra $\mathfrak{k}_j\subset\mathfrak{k}$) denotes the tangent space $T_o(M_j)$ (resp. the holonomy algebra of M_j). For a $\mathfrak{p}$-valued symmetric bilinear form $\tilde{\sigma}$ on $\mathfrak{p}$ and any ordered triple $\{i, j, k\}(1\leqq i, j, k\leqq r)$, a mapping $\tilde{\sigma}_{ij}^k\colon \mathfrak{p}_i\times\mathfrak{p}_j\to\mathfrak{p}_k$ is defined by

$$\tilde{\sigma}_{ij}^k(X_i, Y_j)=\text{the } \mathfrak{p}_k\text{-component of } \tilde{\sigma}(X_i, Y_j)$$

for $X_i\in\mathfrak{p}_i$ and $Y_j\in\mathfrak{p}_j$. Then we may write symbolically as

$$\tilde{\sigma}=\sum_{i,j,k=1}^r \tilde{\sigma}_{ij}^k\ .$$

Assume that $\tilde{\sigma}\in\mathscr{M}_M$. Since each holonomy algebra $\mathfrak{k}_j(1\leqq j\leqq r)$ acts on the subspace $\mathfrak{p}_j$ irreducibly and on the other subspaces $\mathfrak{p}_k(j\neq k)$ trivially, the condition (2) for $\tilde{\sigma}$ implies that

$$\tilde{\sigma}=\sum_{j=1}^{r}\tilde{\sigma}_{jj}^{j}\ . \tag{4.1}$$

Now we have the following

LEMMA 4.1. *Assume that the set $\mathscr{M}_M$ is not empty. Then the simply connected symmetric space M without Euclidean factor is irreducible and of compact type.*

PROOF. Suppose that $r\geqq 2$ and $\tilde{\sigma}\in\mathscr{M}_M$. In the condition (3) for $\tilde{\sigma}$, let X be a nonzero vector in $\mathfrak{p}_j$ and $Y=Z$ a nonzero vector in $\mathfrak{p}_k$ with $j\neq k$. Then, by (4.1), we have

$$\begin{aligned}(c/4)\langle Y, Y\rangle X=R(X, Y)Y-[\tilde{\sigma}(X), \tilde{\sigma}(Y)]Y=-[\tilde{\sigma}(X), \tilde{\sigma}(Y)]Y\\ =\tilde{\sigma}(Y, \tilde{\sigma}(X, Y))-\tilde{\sigma}(X, \tilde{\sigma}(Y, Y))=0\ .\end{aligned}$$

This is a contradiction. Hence we have $r=1$.

Sicce M has not an Euclidean factor, the Lie algebra $\mathfrak{g}$ is semi-simple. Hence Corollary 2.2 implies that M is of compact type. Q.E.D.

Hereafter we assume that M is a simply connected compact irreducible symmetric space. Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$ and W the Weyl group of M relative to $\mathfrak{a}$. Denote by $S^3(\mathfrak{p})$ (resp. $S^3(\mathfrak{a})$) the vector space of all symmetric trilinear forms on $\mathfrak{p}$ (resp. on $\mathfrak{a}$). Then it is known that the vector subspace $\{\tilde{\sigma}\in S^3(\mathfrak{p});\ \mathfrak{k}\cdot\tilde{\sigma}=0\}$ is isomorphic to the vector subspace $\{\tilde{\lambda}\in S^3(\mathfrak{a});\ w\cdot\tilde{\lambda}=\tilde{\lambda}$ for all $w\in W\}$ by the restriction to the subspace $\mathfrak{a}$. Since the Weyl group W acts on $\mathfrak{a}$ irreducibly, W-invariant polynomials on $\mathfrak{a}$ of degree 3 are irreducible. Hence a basis of the vector subspace is given by all the fundamental W-invariant polynomials of degree 3. The Weyl group W for M is of types A_l, B_l, C_l, D_l, E_l, F_4, G_2, or type B_lC_l by the Araki's table [1]. Then, by N. Bourbaki [2], only the Weyl groups W of type $A_l(l\geqq 2)$ have one fundamental W-invariant polynomial of degree 3 and the other Weyl groups have nothing. Hence we have the following

LEMMA 4.2. *Let M be a simply connected compact irreducible symmetric space and set $\mathrm{d}_M=\dim\{\tilde{\sigma}\in S^3(\mathfrak{p});\ \mathfrak{k}\cdot\tilde{\sigma}=0\}$. Then $\mathrm{d}_M=1$ if M is one of the following spaces and $\mathrm{d}_M=0$ otherwise:*

$$SU(n)/SO(n)(n\geqq 3),\ SU(2n)/Sp(n)(n\geqq 3),\ SU(n)(n\geqq 3),\ E_6/F_4\ \cdot$$

Now we determine the set $\bar{\mathscr{M}}_M$.

PROPOSIITON 4.3. *Let M^n be a simply connected compact irreducible symmetric space satisfying $\mathrm{d}_M=0$. Assume that the set $\bar{\mathscr{M}}_M$ is not empty.*

Then the riemannian manifold M^n is the sphere $S^n(c/4)$ with constant sectional curvature $c/4$ and the set $\bar{\mathscr{M}}_M$ consists of one point. Moreover the unique element in $\bar{\mathscr{M}}_M$ corresponds to the natural totally geodesic isometric immersion $f: S^n(c/4) \to P^n(c)$.

PROOF. Take $\tilde{\sigma} \in \mathscr{M}_M$. Then the assumption that $d_M = 0$ implies that $\tilde{\sigma} = 0$. Hence, by the condition (3) for $\tilde{\sigma}$, we have

$$R(X, Y)Z = (c/4)(\langle Y, Z\rangle X - \langle X, Z\rangle Y)$$

for all vectors $X, Y, Z \in \mathfrak{p}$. This implies that M^n has constant sectional curvatures $c/4$. The other assertions are obvious. Q.E.D.

Now we consider the case when $\mathrm{d}_M = 1$. Then we have the following

PROPOSITION 4.4. *Let M^n be a simply connected compact irreducible symmetric space satisfying $\mathrm{d}_M = 1$. Assume that the set $\bar{\mathscr{M}}_M$ is not empty. Then the metric of M^n is determined uniquely by the constant c and the set $\bar{\mathscr{M}}_M$ consists of one point.*

PROOF. Let $(M, \langle\, ,\rangle_1)$ and $(M, \langle\, ,\rangle_2)$ be symmetric spaces with the same underlying manifold M. Suppose that $\bar{\mathscr{M}}_{(M,\langle,\rangle_1)}$ and $\bar{\mathscr{M}}_{(M,\langle,\rangle_2)}$ are not empty, and take $\tilde{\sigma}_j \in \mathscr{M}_{(M,\langle,\rangle_j)}$ for $j = 1, 2$. Then, noting that M is not a sphere, we can see by the same way as for Proposition 4.3 that each $\tilde{\sigma}_j$ is nonzero. Since M is irreducible, we have $\langle\, ,\rangle_2 = \alpha\langle\, ,\rangle_1$ for some $\alpha > 0$. Moreover the assumption that $d_M = 1$ implies that $\tilde{\sigma}_2 = \beta\tilde{\sigma}_1$ for some β. By the condition (3) for $\tilde{\sigma}_j (j = 1, 2)$, we have

$$(c/4)(\langle Y, Z\rangle_j X - \langle X, Z\rangle_j Y) = R(X, Y)Z - [\tilde{\sigma}_j(X), \tilde{\sigma}_j(Y)](Z)$$

and thus

$$(c/4)(\beta^2 - \alpha)(\langle Y, Z\rangle_1 X - \langle X, Z\rangle_1 Y) = (\beta^2 - 1)R(X, Y)Z$$

for all vectors $X, Y, Z \in \mathfrak{p}$. Since M is not a sphere, we have $\beta^2 = 1$ and $\alpha = 1$. Hence we have $\langle\, ,\rangle_1 = \langle\, ,\rangle_2$ and $\tilde{\sigma}_2 = \pm\tilde{\sigma}_1$. Note that the symmetry $\phi \in F_o(M)$ at o acts on the set $S^3(\mathfrak{p})$ by $\phi \cdot \tilde{\sigma} = -\tilde{\sigma}$ for any $\tilde{\sigma} \in S^3(\mathfrak{p})$. Then we can see that the set $\bar{\mathscr{M}}_{(M,\langle,\rangle_1)} = \bar{\mathscr{M}}_{(M,\langle,\rangle_2)}$ consists of one point. Q.E.D.

In the next section we shall construct a model of a totally real parallel isometric immersion of M^n into $P^n(c)$ for M^n satisfying $\mathrm{d}_M = 1$. Hence, summing up Lemma 4.1 and Propositions 4.3, 4.4, we have the following

THEOREM 4.5. *Let M^n be a simply connected symmetric space without*

Euclidean factor. Then the set $\bar{\mathscr{M}}_M$ *is not empty if and only if the symmetric space* M^n *is one of the followings*:

$$SU(n)/SO(n)\ (n\geqq 3),\quad SU(2n)/Sp(n)\ (n\geqq 3),\quad SU(n)\ (n\geqq 3)\ ,$$
$$E_6/F_4,\quad SO(n+1)/SO(n)\ (n\geqq 2)\ .$$

In this case, the metric on the manifold M^n *is determined uniquely by the constant* c *and the set* $\bar{\mathscr{M}}_M$ *consists of one point.*

§5. Models of totally real parallel isometric immersions.

Let V be an $(n+1)$-dimensional complex vector space furnished with a positive definite hermitian inner product $(\ ,\ )$. Then we can define the associated inner product $\langle\ ,\ \rangle_V$ on V as follows:

$$\langle X,\ Y\rangle_V=\mathrm{Re}(X,\ Y)$$

for vectors $X,\ Y\in V$. Let $P(V)$ be the complex projective space associated to V, furnished with the Kähler metric $\langle\ ,\ \rangle$ with constant holomorphic sectional curvature c, and S the unit sphere in V furnished with the following riemannian metric $\langle\ ,\ \rangle_S$:

$$\langle X,\ Y\rangle_S=(c/4)\langle X,\ Y\rangle_V$$

for tangent vectors X, Y of S. Then the Hopf fibring $\pi\colon S\to P(V)$ is a riemannian submersion. For a point $p\in S$, the horizontal subspace H_p at p is given by

$$H_p=\{X\in V;\ \langle X,\ p\rangle_V=\langle X,\ \sqrt{-1}\cdot p\rangle_V=0\}\ .$$

Here note that the linear mapping $\pi_*\colon H_p\to T_{\pi(p)}(P(V))$ is a linear isometry satisfying $\pi_*(\sqrt{-1}X)=J(\pi_*X)$ for any $X\in H_p$. Let $\gamma(t)$ be a curve in S. Then a vector field Z_t along $\gamma(t)$ is called *horizontal* if $Z_t\in H_{\gamma(t)}$ for all t. The curve $\gamma(t)$ is called *horizontal* if $\dot\gamma(t)$ is a horizontal vector field along $\gamma(t)$. Moreover an isometric immersion $\hat f$ of a riemannian manifold M into S is called *horizontal* if $\hat f_*(T_p(M))\subset H_{\hat f(p)}$ for any point p in M. And a horizontal isometric immersion $\hat f$ is called *totally real* if the subspaces $\hat f_*(T_p(M))$ and $\sqrt{-1}\hat f_*(T_p(M))$ are orthogonal. Let ∇^S be the riemannian connection on S for the riemannian metric $\langle\ ,\ \rangle_S$. Then we have the following

LEMMA 5.1 (K. Nomizu [12] and B. O'Neill [13]). *Let* $\gamma(t)$ *be a horizontal curve in* S *parametrized by arc-length. Then* $(\nabla^S_t\dot\gamma)(t)$ *is a horizontal vector field along* $\gamma(t)$. *Moreover*

$$\bar{\nabla}_t(\pi_* Z_t) = \pi_*(\nabla_t^S Z_t)$$

for any horizontal vector field Z_t along $\gamma(t)$.

Let $\hat{f}$ be a horizontal (resp. horizontal and totally real) isometric immersion of an n-dimensional riemannian manifold M^n into S. Then the mapping $f = \pi \circ \hat{f}: M^n \to P(V)$ is an isometric immersion (resp. a totally real isometric immersion). Now we have the following

LEMMA 5.2. *Let $\gamma(t)$ be a geodesic in M parametrized by arc-length. If the horizontal part of $(\nabla_t^S)^2 \hat{f}_*(\dot{\gamma}(t))$ is contained in $\hat{f}_*(T_{\gamma(t)}(M))$, the normal vector $(\nabla_t^* \sigma_f)(\dot{\gamma}(t), \dot{\gamma}(t))$ at $f(\gamma(t))$ equals zero.*

PROOF. Since the vector field $\nabla_t^S \hat{f}_*(\dot{\gamma}(t))$ is horizontal and $\pi_*(\nabla_t^S \hat{f}_*(\dot{\gamma}(t))) = \bar{\nabla}_t(f_*(\dot{\gamma}(t))) = \sigma_f(\dot{\gamma}(t), \dot{\gamma}(t))$ by Lemma 5.1, we have by Lemma 5.1 again

$$\pi_*((\nabla_t^S)^2 \hat{f}_*(\dot{\gamma}(t))) = \bar{\nabla}_t(\sigma_f(\dot{\gamma}(t), \dot{\gamma}(t))) . \tag{5.1}$$

Note that

$$(\nabla_t^* \sigma_f)(\dot{\gamma}(t), \dot{\gamma}(t)) = D_t(\sigma_f(\dot{\gamma}(t), \dot{\gamma}(t)))$$
$$= \text{the normal component of } \bar{\nabla}_t(\sigma_f(\dot{\gamma}(t), \dot{\gamma}(t))) .$$

By (5.1) and the assumption, the vector field $\bar{\nabla}_t(\sigma_f(\dot{\gamma}(t), \dot{\gamma}(t)))$ is a tangent vector field of M and thus $(\nabla_t^* \sigma_f)(\dot{\gamma}(t), \dot{\gamma}(t)) = 0$. Q.E.D.

Now we give the models of totally real parallel isometric immersions into $P^n(c)$ of irreducible compact simply connected symmetric spaces M satisfying $d_M = 1$.

MODEL 1. Let M be the manifold $SU(n)/SO(n) (n \geqq 3)$ and V the complex vector space $S^n(\boldsymbol{C})$ of all complex symmetric matrices of degree n, furnished with the hermitian inner product:

$$(X, Y) = \mathrm{Tr}\, XY^*$$

for $X, Y \in V$. An imbedding $\hat{f}: M \to S$ is defined by

$$\hat{f}(g \cdot SO(n)) = (1/\sqrt{n})\, {}^t g \cdot g$$

for $g \in SU(n)$ and thus the manifold M is furnished with the riemannian metric induced from that of S. Let e_n be the identity element of $SU(n)$ and put $o = e_n \cdot SO(n) \in M$. Now we can easily see the following facts:

(1) The tangent space $T_o(M)$ at o is identified with the space $\mathfrak{p} = \{\sqrt{-1}A; A \in S^n(\boldsymbol{R}), \mathrm{Tr}\, A = 0\}$ and the following set $\mathfrak{a}$ is a maximal abelian

subspace in $\mathfrak{p}$:

$$\mathfrak{a}=\left\{\sqrt{-1}\begin{bmatrix} -\Sigma x_j & & & 0 \\ & x_1 & & \\ & & \ddots & \\ 0 & & & x_{n-1} \end{bmatrix}; x_j \in \boldsymbol{R}\right\}.$$

(2) The isometric imbedding $\hat{f}$ is equivariant relative to the representation $\rho: SU(n) \to SU(V)$ defined by

$$\rho(g)(X) = {}^t gXg$$

for $g \in SU(n)$ and $X \in V$.

(3) $\hat{f}(o) = (1/\sqrt{n})e_n$ and $(\hat{f}_*)_o(\mathfrak{p}) = \mathfrak{p}$. Hence $\hat{f}$ is horizontal and totally real at o.

Then the riemannian metric of M is invariant under $SU(n)$ by (2) and hence M is a symmetric space, and the isometric imbedding $\hat{f}$ is horizontal and totally real by (2) and (3). Hence $f = \pi \circ \hat{f}$ is a totally real isometric immersion.

Now we show that the isometric immersion f has the parallel second fundamental form. Since f is totally real in $P(V)$, the equation of Codazzi-Mainardi implies that $\nabla^* \sigma_f$ is a normal bundle valued symmetric tensor of degree 3. Note that f is equivariant by (2), and that maximal abelian subspaces in $\mathfrak{p}$ are conjugate to each other under the natural action of $K = SO(n)$ on $\mathfrak{p}$. Hence it is sufficient for our claim to see that $(\nabla_X^* \sigma_f)(X, X) = 0$ for any unit vector

$$X = \sqrt{-1} \cdot \begin{bmatrix} -\Sigma x_j & & & 0 \\ & x_1 & & \\ & & \ddots & \\ 0 & & & x_{n-1} \end{bmatrix} \in \mathfrak{a}.$$

Let $\gamma(t)$ be the geodesic in M such that $\gamma(0) = o$ and $\dot{\gamma}(0) = X$. Then we have

$$\hat{f}(\gamma(t)) = (1/\sqrt{n}) \cdot \begin{bmatrix} e^{-2t(\Sigma x_j)\sqrt{-1}} & & & 0 \\ & e^{2tx_1\sqrt{-1}} & & \\ & & \ddots & \\ 0 & & & e^{2tx_{n-1}\sqrt{-1}} \end{bmatrix}$$

and

$$\hat{f}_*(\dot{\gamma}(t))=(1/\sqrt{n})\cdot\begin{bmatrix} -2\sqrt{-1}(\Sigma x_j)e^{-2t(\Sigma x_j)\sqrt{-1}} & & & 0 \\ & 2x_1\sqrt{-1}e^{2tx_1\sqrt{-1}} & & \\ & & \ddots & \\ 0 & & & 2x_{n-1}\sqrt{-1}e^{2tx_{n-1}\sqrt{-1}} \end{bmatrix}.$$

Note that $\nabla_t^S Z_t = dZ_t/dt + (c/4)\langle \hat{f}_*(\dot{\gamma}(t)), Z_t\rangle_S \hat{f}(\gamma(t))$ for any vector field Z_t along $f(\gamma(t))$. Thus we have

$$\nabla_t^S \hat{f}_*(\dot{\gamma}(t))=(1/\sqrt{n})\cdot\begin{bmatrix} (c/4-4(\Sigma x_j)^2)e^{-2t(\Sigma x_j)\sqrt{-1}} & & & 0 \\ & (c/4-4x_1^2)e^{2tx_1\sqrt{-1}} & & \\ & & \ddots & \\ 0 & & & (c/4-4x_{n-1}^2)e^{2tx_{n-1}\sqrt{-1}} \end{bmatrix}$$

and

$$(\nabla_t^S)^2 \hat{f}_*(\dot{\gamma}(t))|_{t=0}=(2\sqrt{-1}/\sqrt{n})\cdot\begin{bmatrix} -(c/4-4(\Sigma x_j)^2)(\Sigma x_j) & & & 0 \\ & (c/4-4x_1^2)x_1 & & \\ & & \ddots & \\ 0 & & & (c/4-4x_{n-1}^2)x_{n-1} \end{bmatrix}.$$

Hence the horizontal part of $(\nabla_t^S)^2 \hat{f}_*(\dot{\gamma}(t))|_{t=0}$ is given by

$$(\nabla_t^S)^2 \hat{f}_*(\dot{\gamma}(t))|_{t=0}-\frac{\langle(\nabla_t^S)^2 \hat{f}_*(\dot{\gamma}(t))|_{t=0}, \sqrt{-1}\hat{f}(\gamma(0))\rangle_S}{|\sqrt{-1}\hat{f}(\gamma(0))|_S^2}\cdot\sqrt{-1}\hat{f}(\gamma(0))$$

$$=(\sqrt{-1}/\sqrt{n})\cdot\begin{bmatrix} -2(\Sigma x_j)(c/4-4(\Sigma x_j)^2)-\lambda\sqrt{c}/2 & & & 0 \\ & 2x_1(c/4-4x_1^2)-\lambda\sqrt{c}/2 & & \\ & & \ddots & \\ 0 & & & 2x_{n-1}(c/4-4x_{n-1}^2)-\lambda\sqrt{c}/2 \end{bmatrix}$$

where $\lambda=(16/n\sqrt{c})((\Sigma x_j)^3-(\Sigma x_j^3))$. Here note that the trace of the above matrix equals zero. Hence the horizontal part of $(\nabla_t^S)^2 \hat{f}_*(\dot{\gamma}(t))|_{t=0}$ is contained in $\mathfrak{p}$. This implies that $(\nabla^*\sigma_f)(\dot{\gamma}(0), \dot{\gamma}(0), \dot{\gamma}(0))=0$ by Lemma 5.2. Hence f is a totally real parallel isometric immersion of M into $P(V)$.

MODEL 2. Let M be the manifold $SU(2n)/Sp(n)(n\geqq 3)$ and V the complex vector space $\mathfrak{so}(2n; C)$ of all complex skew symmetric matrices of degree $2n$, furnished with the hermitian inner product:

$$(X, Y)=\mathrm{Tr}\, XY^*$$

for vectors $X, Y\in V$. An imbedding $\hat{f}: M\to S$ is defined by

$$\hat{f}(g\cdot Sp(n))=(1/\sqrt{2n})^t g J_n g$$

for $g \in SU(2n)$, where $J_n=\begin{bmatrix} 0 & -e_n \\ e_n & 0 \end{bmatrix} \in V$, and thus the manifold M is furnished with the riemannian metric induced from that of S. Put $o=e_{2n}\cdot Sp(n) \in M$. Now we can easily see the following facts:

(1) The tangent space $T_o(M)$ at o is identified with the space

$$\mathfrak{p}=\left\{\begin{bmatrix} Z & W \\ \bar{W} & {}^tZ \end{bmatrix};\ Z \in su(n),\ W \in \mathfrak{so}(n; C)\right\}$$

and the following set $\mathfrak{a}$ is a maximal abelian subspace in $\mathfrak{p}$:

$$\mathfrak{a}=\left\{\sqrt{-1}\cdot\begin{bmatrix} -(\Sigma x_j) & & & & & & \\ & x_1 & & & & & \\ & & \ddots & & & & \\ & & & x_{n-1} & & & \\ & & & & -(\Sigma x_j) & & \\ & & & & & x_1 & \\ & & & & & & \ddots \\ & & & & & & & x_{n-1} \end{bmatrix};\ x_j \in \boldsymbol{R}\right\}.$$

(2) The isometric imbedding $\hat{f}$ is equivariant relative to the representation $\rho: SU(2n) \to SU(V)$ defined by

$$\rho(g)(X)={}^tgXg$$

for $g \in SU(2n)$ and $X \in V$.

(3) $\hat{f}(o)=(1/\sqrt{2n})J_n$ and $(\hat{f}_*)_o(\mathfrak{p})=\left\{\begin{bmatrix} -\bar{W} & -{}^tZ \\ Z & W \end{bmatrix};\ Z \in \mathfrak{su}(n),\ W \in \mathfrak{so}(n; C)\right\}$. Hence $\hat{f}$ is horizontal and totally real at o.

Then, by the same way as in Model 1, we can see that $f=\pi\circ\hat{f}$ is a totally real parallel isometric immersion.

MODEL 3. Let M be the manifold $SU(n)\times SU(n)/SU(n)(n\geqq 3)$ and V the complex vector space $M_n(C)$ of all complex matrices of degree n, furnished with the hermitian inner product:

$$(X, Y)=\mathrm{Tr}\, XY^*$$

for vectors $X, Y \in V$. An imbedding $\hat{f}: M \to S$ is defined by

$$\hat{f}((g, h)\cdot SU(n))=(1/\sqrt{n})gh^{-1}$$

for $g, h \in SU(n)$ and thus the manifold M is furnished with the riemannian metric induced from that of S. Put $o=(e_n, e_n)\cdot SU(n) \in M$. Now we can easily see the following facts:

(1) The tangent space $T_o(M)$ at o is identified with the space $\mathfrak{p}=\{(X, -X);\ X\in \mathfrak{su}(n)\}$ and the following set $\mathfrak{a}$ is a maximal abelian subspace in $\mathfrak{p}$:

$$\mathfrak{a}=\{(X, -X)\in \mathfrak{p};\ X \text{ is diagonal}\}\ .$$

(2) The isometric imbedding $\hat{f}$ is equivariant relative to the representation ρ: $SU(n)\times SU(n)\to SU(V)$ defined by

$$\rho((g, h))(X)=gXh^{-1}$$

for $g, h\in SU(n)$ and $X\in V$.

(3) $\hat{f}(o)=(1/\sqrt{n})e_n$ and $(\hat{f}_*)_o(\mathfrak{p})=\mathfrak{su}(n)$. Hence $\hat{f}$ is horizontal and totally real at o.

Then, by the same way as in Model 1, we can see that $f=\pi\circ\hat{f}$ is a totally real parallel isometric immersion.

MODEL 4. Let $\mathscr{C}$ be the Cayley algebra over $\boldsymbol{R}$ furnished with the canonical conjugation $-$, and set $\mathscr{F}=\{X\in M_3(\mathscr{C});\ {}^t\bar{X}=X\}$. On the real vector space $\mathscr{F}$, we define the Jordan product $\circ$, the inner product $((\ ,\ ))$, the cross product $\times$, and the determinant det as follows respectively:

$$X\circ Y=(1/2)(XY+YX),\ ((X, Y))=\mathrm{Tr}(X\circ Y)\ ,$$
$$X\times Y=(1/2)(2X\circ Y-\mathrm{Tr}(X)Y-\mathrm{Tr}(Y)X+(\mathrm{Tr}(X)\mathrm{Tr}(Y)-\mathrm{Tr}(X\circ Y))e_3)\ ,$$
$$\det(X)=(1/3)((X\times X, X))$$

for $X, Y\in\mathscr{F}$. Let V be the complexification of the real vector space $\mathscr{F}$ and extend these $\circ$, $((\ ,\ ))$, $\times$, det $\boldsymbol{C}$-linearly and naturally on V. Denote by τ the complex conjugate on V with respect to $\mathscr{F}$. Then $(X, Y)=((\tau X, Y))$ is a positive definite hermitian inner product on V. We define

$$E_6=\{g\in \mathrm{GL}_C(V);\ \det(g(X))=\det(X),\ (gX, gY)=(X, Y) \text{ for any } X, Y\in V\}$$

and

$$F_4=\{g\in E_6;\ g(e_3)=e_3\}\ .$$

Then E_6 (resp. F_4) is a simply connected compact simple Lie group of type E_6 (resp. of type F_4). (cf. O. Shukugawa-I. Yokota [14])

Let M be the manifold E_6/F_4. An imbedding $\hat{f}$: $M\to S$ is defined by

$$\hat{f}(g\cdot F_4)=(1/\sqrt{3})g(e_3)$$

for $g\in E_6$ and thus the manifold M is furnished with the riemannian metric induced from that of S. Put $o=e_3\cdot F_4\in M$ and set $\mathscr{F}_0=\{X\in\mathscr{F};\ \mathrm{Tr}\,X=0\}$. Now we can easily see the following facts:

(1) Define the right translation R_X on $\mathscr{F}$ for $X\in\mathscr{F}$ by $R_X(Y)=Y\circ X$ for $Y\in\mathscr{F}$. The tangent space $T_o(M)$ at o is identified with the space $\mathfrak{p}=\{\sqrt{-1}R_X\in\mathfrak{gl}(V);\ X\in\mathscr{F}_0\}$ and the following set $\mathfrak{a}$ is a maximal abelian subspace in $\mathfrak{p}$:

$$\mathfrak{a}=\{\sqrt{-1}R_X\in\mathfrak{gl}(V);\ X\in\mathscr{F}_0,\ X \text{ is diagonal}\}\ .$$

(2) The isometric imbedding $\hat{f}$ is equivariant relative to the representation $\rho: E_6\to SU(V)$ defined by $\rho(g)(x)=g(x)$ for $g\in E_6$ and $X\in V$.

(3) $\hat{f}(o)=(1/\sqrt{3})e_3$ and $(\hat{f}_*)_o(\mathfrak{p})=\sqrt{-1}\mathscr{F}_0$. Hence $\hat{f}$ is horizontal and totally real at o.

Then, by the same way as in Model 1, we can see that $f=\pi\circ\hat{f}$ is a totally real parallel isometric immersion.

REMARK 5.3. It is known that the isometric imbeddings $\hat{f}: M\to S$ in the above models are minimal. Since the imbeddings $\hat{f}$ are horizontal, the isometric immersions f are minimal.

REMARK 5.4. We can see easily that the above isometric immersion $f: M\to P(V)$ is $(\sqrt{c}/2\sqrt{2})$-isotropic (that is, $|\sigma_f(X, X)|=\sqrt{c}/2\sqrt{2}$ for any unit tangent vector X of M) if the symmetric space M is of rank two. Hence these isometric immersions f are examples of Theorem 4.13 in [11].

§6. The set $\bar{\mathscr{M}}_M$ for a simply connected symmetric space M with Euclidean factor.

In this section we assume that M^n is a simply connected symmetric space with Euclidean factor; thus, M is decomposed as a riemannian manifold as follows:

$$M^n=\mathbf{R}^{n_0}\times M_1^{n_1}\times\cdots\times M_r^{n_r}\quad\left(n=\sum_{j=0}^{r}n_j,\ n_0>0\right)$$

where $M_j^{n_j}$ is an n_j-dimensional irreducible simply connected symmetric space for each j. Then the tangent space $T_o(M)=\mathfrak{p}$ (resp. the holonomy algebra $\mathfrak{k}$) is decomposed as follows:

$$\mathfrak{p}=\mathfrak{p}_0+\sum_{j=1}^{r}\mathfrak{p}_j\quad\left(\text{resp. } \mathfrak{k}=\sum_{j=1}^{r}\mathfrak{k}_j\right)$$

where the subspaces $\mathfrak{p}_j$ and $\mathfrak{p}_0$ in $\mathfrak{p}$ (resp. the subalgebra $\mathfrak{k}_j$ in $\mathfrak{k}$) denote the tangent spaces $T_o(M_j)$ and $T_o(\mathbf{R}^{n_0})$ (resp. the holonomy algebra of M_j).

For a $\mathfrak{p}$-valued symmetric bilinear form $\tilde{\sigma}$ on $\mathfrak{p}$ and any ordered triple $\{i, j, k\}(0 \leqq i, j, k \leqq r)$, a mapping $\tilde{\sigma}_{ij}^{k}: \mathfrak{p}_i \times \mathfrak{p}_j \to \mathfrak{p}_k$ is defined as in the section 4. Assume that $\tilde{\sigma} \in \mathscr{M}_M$. Since each holonomy algebra $\mathfrak{k}_j (1 \leqq j \leqq r)$ acts on the subspace $\mathfrak{p}_j$ irreducibly and on the other spaces $\mathfrak{p}_k (j \neq k)$ trivially, the condition (2) for $\tilde{\sigma}$ implies that

$$\tilde{\sigma} = \sum_{j=0}^{r} \tilde{\sigma}_{jj}^{j} + \sum_{j=1}^{r} \tilde{\sigma}_{jj}^{0} + \sum_{j=1}^{r} \tilde{\sigma}_{0j}^{j} + \sum_{j=1}^{r} \tilde{\sigma}_{j0}^{j} . \tag{6.1}$$

Now we define the *Euclidean j-th mean curvature vector* $H_j (1 \leqq j \leqq r)$ in $\mathfrak{p}_0$ by

$$H_j = (1/n_j) \operatorname{Tr} \tilde{\sigma}_{jj}^{0} = (1/n_j) \sum_{k=1}^{n_j} \tilde{\sigma}_{jj}^{0}(e_{jk}, e_{jk})$$

where $\{e_{jk}\}_{k=1}^{n_j}$ denotes an orthonormal basis of $\mathfrak{p}_j$, and call the length h_j of the vector H_j the *Euclidean j-th mean curvature*. Then we have the following

LEMMA 6.1. *Let $\tilde{\sigma} \in \mathscr{M}_M$. Then*

$$\tilde{\sigma}_{jj}^{0}(X_j, X_j) = \langle X_j, Y_j \rangle H_j$$
$$\tilde{\sigma}_{j0}^{j}(X_j, Z_0) = \tilde{\sigma}_{0j}^{j}(Z_0, X_j) = \langle Z_0, H_j \rangle X_j$$

for any $j (1 \leqq j \leqq r)$ and $Z_0 \in \mathfrak{p}_0$, $X_j, Y_j \in \mathfrak{p}_j$.

PROOF. Since $\mathfrak{k}_j \cdot \tilde{\sigma} = 0$, we have

$$\tilde{\sigma}_{jj}^{0}(T_j X_j, Y_j) + \tilde{\sigma}_{jj}^{0}(X_j, T_j Y_j) = 0 \tag{6.2}$$

and

$$\tilde{\sigma}_{jj}^{j}(T_j X_j, Y_j) + \tilde{\sigma}_{jj}^{j}(X_j, T_j Y_j) = T_j(\tilde{\sigma}_{jj}^{j}(X_j, Y_j)) \tag{6.3}$$

for any $T_j \in \mathfrak{k}_j$ and all vectors $X_j, Y_j \in \mathfrak{p}_j$. Let $\{e_a\}_{a=1}^{n_0}$ be an orthonormal basis of $\mathfrak{p}_0$. Since M_j is irreducible, the condition (6.2) implies that

$$\langle \tilde{\sigma}_{jj}^{0}(X_j, Y_j), e_a \rangle = c_j^a \langle X_j, Y_j \rangle$$

for some $c_j^a \in \boldsymbol{R}$ and thus

$$\tilde{\sigma}_{jj}^{0}(X_j, Y_j) = \langle X_j, Y_j \rangle \left(\sum_{a=1}^{n_0} c_j^a e_a \right) = \langle X_j, Y_j \rangle H_j$$

for all vectors $X_j, Y_j \in \mathfrak{p}_j$.

The second equality is obtained by the symmetry condition (1) for $\tilde{\sigma}$ and the first equality. Q.E.D.

We denote by $\mathscr{M}_M^d$ the set defined in the same way as $\mathscr{M}_M$ by replacing the number $c/4$ in the condition (3) with the number d. Then we have the following

LEMMA 6.2. *Let* $\tilde{\sigma}\in\mathscr{M}_M$. *Then* $\tilde{\sigma}_{jj}^j\in\mathscr{M}_{M_j}^{c/4+h_j^2}$ *for each* j.

PROOF. The conditions (1) and (2) for $\mathscr{M}_{M_j}^{c/4+h_j^2}$ is obvious by the condition (1) for $\tilde{\sigma}$ and (6.3). We show that $\tilde{\sigma}_{jj}^j$ satisfies the condition (3) for $\mathscr{M}_{M_j}^{c/4+h_j^2}$. Denote by R^{M_j} the curvature tensor of M_j. Then, by the condition (3) for $\tilde{\sigma}$,

$$(c/4)(\langle Y_j, Z_j\rangle X_j-\langle X_j, Z_j\rangle Y_j)=R^{M_j}(X_j, Y_j)Z_j-[\tilde{\sigma}(X_j), \tilde{\sigma}(Y_j)]Z_j$$

for all vectors $X_j, Y_j, Z_j\in\mathfrak{p}_j$. By (6.1) and Lemma 6.1, the second term of the right hand side is calculated as follows:

$$\begin{aligned}[\tilde{\sigma}(X_j), \tilde{\sigma}(Y_j)]Z_j=&[\tilde{\sigma}_{jj}^j(X_j), \tilde{\sigma}_{jj}^j(Y_j)]Z_j\\&+h_j^2(\langle Y_j, Z_j\rangle X_j-\langle X_j, Z_j\rangle Y_j)\,.\end{aligned}$$

Hence $\tilde{\sigma}_{jj}^j$ satisfies the condition (3) for $\mathscr{M}_{M_j}^{c/4+h_j^2}$. Q.E.D.

LEMMA 6.3. *Let* $\tilde{\sigma}\in\mathscr{M}_M$. *Then* $\tilde{\sigma}_{00}^0\in\mathscr{M}_{R^{n_0}}$ *and*

$$\tilde{\sigma}_{00}^0(X_0, H_j)=\langle X_0, H_j\rangle H_j-(c/4)X_0$$

for any $X_0\in\mathfrak{p}_0$. *Moreover* $\langle H_j, H_k\rangle=-c/4$ *for distinct indeces* $j, k(1\leq j, k\leq r)$.

PROOF. Note that the condition (2) for $\mathscr{M}_{R^{n_0}}$ is obvious since R^{n_0} is flat. Moreover by the conditions (1) and (3) for $\tilde{\sigma}$ we can see easily that $\tilde{\sigma}_{00}^0$ satisfies the conditions (1) and (3) for $\mathscr{M}_{R^{n_0}}$. Put $X=X_0\in\mathfrak{p}_0$, $Y=Y_j$, $Z=Z_j\in\mathfrak{p}_j$ in the condition (3) for $\tilde{\sigma}$. Then we have

$$(c/4)\langle Y_j, Z_j\rangle X_0=-[\tilde{\sigma}(X_0), \tilde{\sigma}(Y_j)]Z_j\,.$$

The right hand side is calculated by (6.1) and Lemma 6.2 as follows:

$$-[\tilde{\sigma}(X_0), \tilde{\sigma}(Y_j)]Z_j=\langle X_0, H_j\rangle\langle Y_j, Z_j\rangle H_j-\langle Y_j, Z_j\rangle\tilde{\sigma}_{00}^0(X_0, H_j)\,.$$

Hence we have

$$(c/4)X_0=\langle X_0, H_j\rangle H_j-\tilde{\sigma}_{00}^0(X_0, H_j)\,.$$

Now, putting $X=X_j\in\mathfrak{p}_j$ and $Y=Y_k$, $Z=Z_k\in\mathfrak{p}_k(1\leq j\neq k\leq r)$ in the condition (3) for $\tilde{\sigma}$, we have

$$(c/4)\langle Y_k, Z_k\rangle X_j=-\langle Y_k, Z_k\rangle\langle H_j, H_k\rangle X_j$$

by (6.1) and Lemma 6.2, and thus $\langle H_j, H_k\rangle = -c/4$. Q.E.D.

Summing up Lemmas 6.1, 6.2 and 6.3, we have the claim (A) in the following

THEOREM 6.4. *Let M^n be a simply connected symmetric space with Euclidean factor decomposed as $M^n = R^{n_0}\times\prod_{j=1}^r M_j^{n_j}$ and $n=\sum_{j=0}^r n_j$. Then the following claims are true:*

(A) *Let $\tilde\sigma \in \mathscr{M}_M$. Then*

$$\tilde\sigma = \sum_{j=0}^r \tilde\sigma_{jj}^j + \sum_{j=1}^r \tilde\sigma_{jj}^0 + \sum_{j=1}^r \tilde\sigma_{j0}^j + \sum_{j=1}^r \tilde\sigma_{0j}^j \tag{1}$$

$$\tilde\sigma_{jj}^j \in \mathscr{M}_{M_j}^{c/4+h_j^2} \tag{2}$$

$$\tilde\sigma_{00}^0 \in \mathscr{M}_{R^{n_0}},\ \langle H_j, H_k\rangle = -c/4 \quad (1\leqq j\neq k\leqq r)\,, \tag{3}$$
$$\tilde\sigma_{00}^0(Z_0, H_j) = \langle Z_0, H_j\rangle H_j - (c/4)Z_0$$

$$\tilde\sigma_{j0}^j(X_j, Z_0) = \tilde\sigma_{0j}^j(Z_0, X_j) = \langle Z_0, H_j\rangle X_j\,, \tag{4}$$
$$\tilde\sigma_{jj}^0(X_j, Y_j) = \langle X_j, Y_j\rangle H_j$$

for any $Z_0\in\mathfrak{p}_0$ and all vectors $X_j, Y_j\in\mathfrak{p}_j$.

(B) *Conversely any p-valued bilinear form $\tilde\sigma$ on $\mathfrak{p}$ satisfying the conditions* (1), (2), (3), (4) *of* (A) *is an element in $\mathscr{M}_M$.*

Here the proof of our claim (B) is omitted since it is straightforward.

REMARK 6.5. Let M^n be a simply connected symmetric space with Euclidean factor. Changing the metric on M^n componentwise, we can construct infinitely many elements in $\mathscr{M}_M$. In fact, decompose M as above and suppose that $n_0 = r\geqq 1$. First we shall show that there exist a basis $\{H_i\}_{i=1}^r$ of R^r and an R^r-valued bilinear form $\tilde\sigma_{00}^0$ on R^r satisfying the condition (3) of (A). If there exist such basis and R^r-valued form, by Theorem 6.4, (B) an element in $\mathscr{M}_M$ can be constructed. Let $\{e_j\}_{j=1}^r$ be an orthonormal basis of R^r and set $H_i = \sum_{j=1}^r a_i^j e_j$, $A=(a_i^j)$. Moreover, for positive real numbers $h_1, \cdots, h_r$, we set

$$S(h_1,\cdots,h_r) = \begin{bmatrix} h_1^2 & -c/4 & \cdots & -c/4 \\ -c/4 & h_2^2 & & \vdots \\ \vdots & & \ddots & -c/4 \\ -c/4 & \cdots & -c/4 & h_r^2 \end{bmatrix}.$$

Then the condition for that $\{H_i\}$ is a basis of R^r such that $|H_j| = h_j$ $(1\leqq j\leqq r)$ and $\langle H_j, H_k\rangle = -c/4 (j\neq k)$ is written as follows:

(6.4) $$\det A\neq 0,\ A^tA=S(h_1, \cdots, h_r)\ .$$

Since the matrix $S(h_1, \cdots, h_r)$ is symmetric, for sufficiently large numbers $h_1, \cdots, h_r$, there exists a positive definite symmetric matrix A satisfying the condition (6.4). Then we define an $\boldsymbol{R}^r$-valued bilinear form $\tilde{\sigma}^0_{c0}$ on $\boldsymbol{R}^r$ as follows:

$$\tilde{\sigma}^0_{c0}(H_j, H_k)=\langle H_j, H_k\rangle H_k-(c/4)H_j\ .$$

By easy calculations, we can see that the $\boldsymbol{R}^r$-valued bilinear form $\tilde{\sigma}^0_{c0}$ on $\boldsymbol{R}^r$ satisfies the condition (3) of (A). Thus we get infinitely many elements in $\mathscr{M}_M$ by taking suitable metrics on $M_j (1\leqq j\leqq r)$.

Now, in the case when $M=\boldsymbol{R}^2$, we have the following

THEOREM 6.6. *There exists a unique complete totally real parallel flat minimal surface M^2 in $P^2(c)$ (up to holomorphic isometries of $P^2(c)$). The norm $|\sigma|$ of the second fundamental form σ of M^2 is given by $|\sigma|^2=(1/2)c$.*

PROOF. Let $\{e_1, e_2\}$ be an orthonormal basis of $\boldsymbol{R}^2$. Then the condition $\tilde{\sigma}\in\mathscr{M}_{R^2}$ is equivalent to the condition that

(6.5) $$\begin{cases}\tilde{\sigma}(e_1, e_1)=\alpha e_1+\beta e_2\\ \tilde{\sigma}(e_1, e_2)=\beta e_1+\gamma e_2\\ \tilde{\sigma}(e_2, e_2)=\gamma e_1+\delta e_2\end{cases},\quad \text{and}\quad c/4=\beta^2+\gamma^2-\alpha\gamma-\beta\delta\ .$$

Suppose that the totally real parallel immersion of $\boldsymbol{R}^2$ corresponding to $\tilde{\sigma}$ is minimal. Then $\alpha+\gamma=\beta+\delta=0$ and thus $\beta^2+\gamma^2=c/8$ by the second equality of (6.5). Put $\beta=\sqrt{c/8}\cos\theta$ and $\gamma=\sqrt{c/8}\sin\theta$ for some θ and define a linear isometry g of $\boldsymbol{R}^2$ by

$$(g(e_1), g(e_2))=(e_1, e_2)\begin{bmatrix}\cos(\theta/3) & \sin(\theta/3)\\ -\sin(\theta/3) & \cos(\theta/3)\end{bmatrix}.$$

Then we have

$$(g\cdot\tilde{\sigma})(e_1, e_1)=-(g\cdot\tilde{\sigma})(e_2, e_2)=\sqrt{c/8}e_2,\ (g\cdot\tilde{\sigma})(e_1, e_2)=\sqrt{c/8}e_1\ .$$

Hence all elements in $\mathscr{M}_{R^2}$ corresponding to minimal immersions belong to the same equivalence class. Now by Theorem 3.4 and 3.6 we get our first claim. The second claim follows from $|g\cdot\tilde{\sigma}|^2=(1/2)c$. Q.E.D.

REMARK 6.7. S.T.Yau [18] has shown that if M^2 is a complete non-negative curved totally real minimal surface in $P^2(c)$, M^2 is totally geodesic or flat, and moreover in the second case the second fundamental form is parallel. The minimal surface of Theorem 6.6 gives a unique example of surfaces in the flat case. This has been constructed concretely in the author's paper [11] and it is compact.

REMARK 6.8. B. Y. Chen and K. Ogiue [3] has shown that if M^n is a compact totally real minimal submanifold in $P^n(c)$ such that $|\sigma_p|^2<(n(n+1)/4(2n-1))c$ for any point p in M, then M^n is totally geodesic. Suppose that $|\sigma_p|^2=(n(n+1)/4(2n-1))c$ for any point $p\in M$. Then, along their proof, the second fundamental form is parallel. In the case when $n=2$ (then $(n(n+1)/4(2n-1))c=(1/2)c$), the universal covering of the compact totally real parallel minimal surface M^2 has Euclidean factor and thus is flat. Hence our minimal surface in $P^2(c)$ of Theorem 6.6 is a unique compact totally real minimal surface M^2 in $P^2(c)$ such that $|\sigma_p|^2=(1/2)c$ for any point $p\in M^2$.

REMARK 6.9. In the next paper together with M. Takeuchi the complete classification of n-dimensional complete totally real parallel submanifolds in $P^n(c)$ shall be given by a different way.

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