Totally Real Submanifolds in a 6-Sphere

SHARIEF DESHMUKH

1. Introduction

On a 6-dimensional unit sphere S^6, one can construct an almost complex structure using the properties of the Cayley division algebra; we refer to [3] for this construction. Further, it is known that this almost complex structure on S^6 is not integrable and that it is a nearly Kaehler structure on S^6. Regarding submanifolds of S^6, it is known that S^6 has no 4-dimensional complex submanifolds [4]. However, S^6 has 3-dimensional totally real submanifolds, which are minimal and orientable [3]. For a compact 3-dimensional totally real submanifold M of S^6, in [2] it is shown that if the sectional curvatures k of M satisfy $1/16 < k \le 1$, then $k=1$, that is, M is totally geodesic. However, there are 3-dimensional compact totally real submanifolds of S^6 some of whose sectional curvatures are greater than 1 (cf. [1, p. 436]).

The object of the present paper is to prove the following.

THEOREM. *Let M be a compact 3-dimensional totally real submanifold of S^6. If k_0 is the infimum of the sectional curvatures of M, then either $4k_0 \le 1$ or M is totally geodesic.*

2. Totally Real Submanifolds of S^6

Let J be the almost complex structure defined on S^6 by the properties of the Cayley division algebra, and let g be the standard metric of constant curvature 1 on S^6. Then we have

(2.1) $$g(JX, JY) = g(X, Y), \quad (\bar{\nabla}_X J)(X) = 0, \quad X, Y \in \chi(S^6),$$

where $\bar{\nabla}$ is the Riemannian connection on S^6 with respect to g and $\chi(S^6)$ is the Lie algebra of vector fields on S^6.

Define a tensor field G of the type $(1,2)$ on S^6 by $G(X, Y) = (\bar{\nabla}_X J)(Y)$, $X, Y \in \chi(S^6)$. This tensor field has the following properties:

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$$G(X,Y)+G(Y,X)=0; \tag{2.2}$$

$$G(X,JY)+JG(X,Y)=0; \tag{2.3}$$

$$(\bar{\nabla}_X G)(Y,Z)=g(Y,JZ)X+g(X,Z)JY-g(X,Y)JZ, \quad X,Y,Z\in\chi(S^6). \tag{2.4}$$

A 3-dimensional submanifold M of S^6 is called a *totally real* submanifold of S^6 if $JTM=T^{\perp}M$, where TM is the tangent bundle and $T^{\perp}M$ is the normal bundle of M. In [3] Ejiri proved that a 3-dimensional totally real submanifold M of S^6 is orientable and minimal, and that $G(X,Y)$ is orthogonal to M for $X,Y\in\chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields on M. We denote by ∇ and $\nabla^{\perp}$ the Riemannian connection on M and the connection in the normal bundle $T^{\perp}M$ induced by the connection $\bar{\nabla}$. The formulae of Gauss and Weingarten are given by

$$\bar{\nabla}_X Y=\nabla_X Y+h(X,Y) \tag{2.5}$$

and

$$\bar{\nabla}_X \xi=-A_\xi X+\nabla^{\perp}_X\xi, \tag{2.6}$$

where X and Y are vector fields on M and ξ is a normal vector field on M. The second fundamental form h is related to A_ξ by

$$g(h(X,Y),\xi)=g(A_\xi X,Y), \tag{2.7}$$

$$A_{JX}Y=-Jh(X,Y), \tag{2.8}$$

and

$$g(h(X,Y),JZ)=g(h(Y,Z),JX). \tag{2.9}$$

If we denote the curvature tensors of ∇ and $\nabla^{\perp}$ by R and $R^{\perp}$, respectively, then the equation of Codazzi gives

$$(\bar{\nabla}h)(X,Y,Z)=(\bar{\nabla}h)(Y,Z,X), \tag{2.10}$$

and the equations (2.4), (2.8), and (2.9) give

$$\begin{aligned} g(R^{\perp}(X,Y)JZ,JW)&=g(R(X,Y)Z,W)+g(Z,X)g(Y,W)\\ &\quad -g(Z,Y)g(X,W), \end{aligned} \tag{2.11}$$

where $X,Y,Z,W\in\chi(M)$ and $\bar{\nabla}h$ is defined by

$$(\bar{\nabla}h)(X,Y,Z)=\nabla^{\perp}_X h(Y,Z)-h(\nabla_X Y,Z)-h(Y,\nabla_X Z).$$

Using equations (2.4) and (2.10), by a straightforward computation we obtain

$$\begin{aligned} &g((\bar{\nabla}h)(X,Y,Z),JW)-g((\bar{\nabla}h)(X,Y,W),JZ)\\ &\quad =g(h(Y,W),G(X,Z))-g(h(Y,Z),G(X,W)). \end{aligned} \tag{2.12}$$

We also define $\bar{\nabla}^2 h$ by

$$\begin{aligned} (\bar{\nabla}^2 h)(X,Y,Z,W)&=\nabla^{\perp}_X(\bar{\nabla}h)(Y,Z,W)-(\bar{\nabla}h)(\nabla_X Y,Z,W)\\ &\quad -(\bar{\nabla}h)(Y,\nabla_X Z,W)-(\bar{\nabla}h)(Y,Z,\nabla_X W). \end{aligned}$$

Then $\bar{\nabla}^2 h$ satisfies the following equation:

$$(2.13)\quad \begin{aligned}(\bar{\nabla}^2 h)(X,Y,Z,W) &= (\bar{\nabla}^2 h)(Y,X,Z,W) + R^{\perp}(X,Y)h(Z,W)\\ &\quad - h(R(X,Y)Z,W) - h(Z,R(X,Y)W).\end{aligned}$$

For a unit vector $v \in TM$, we can choose an orthonormal basis $\{e_1, e_2, e_3\}$, with $e_3 = v$, such that $G(e_1,e_2) = Je_3$, $G(e_2,e_3) = Je_1$, and $G(e_3,e_1) = Je_2$ (cf. [3]). Then, using minimality of M and (2.12), we have

$$(2.14)\quad \begin{aligned}&\sum_{i=1}^{3} g(G(e_i,v), (\bar{\nabla}h)(e_i,v,v))\\ &\quad = g(-Je_2, (\bar{\nabla}h)(e_1,v,v)) + g(Je_1, (\bar{\nabla}h)(e_2,v,v))\\ &\quad = g((\bar{\nabla}h)(v,v,e_2), Je_1) - g((\bar{\nabla}h)(v,v,e_1), Je_2)\\ &\quad = -g(h(v,e_1),e_1) - g(h(v,e_2),e_2)\\ &\quad = -g(h(e_1,e_1) + h(e_2,e_2), Jv)\\ &\quad = g(h(v,v), Jv).\end{aligned}$$

3. Proof of the Theorem

Let UM be the unit tangent bundle of M and let UM_p be the fiber over $p \in M$. Define a smooth function $f\colon UM \to R$ by $f(v) = g(h(v,v), Jv)$. Since UM is compact, f attains a maximum at a unit vector v tangent to M at a point p. For any $u \in UM_p$, let $\alpha(t) = (\gamma(t), v(t))$, $t \in (-\delta, \delta)$ be a smooth curve in UM such that $\gamma(t)$ is the unique geodesic in M with $\gamma(0) = p$ and $\gamma'(0) = u$; let $v(t)$ be the parallel vector field along γ with $v(0) = v$. Then we have

$$\begin{aligned}0 = df_v(u) &= \left(\frac{d}{dt}\right)_{t=0} g(h(v(t),v(t)), Jv(t))\\ &= g((\bar{\nabla}h)(u,v,v), Jv) + g(h(v,v), G(u,v))\end{aligned}$$

and

$$(3.1)\quad \begin{aligned}0 \geq d_v^2 f(u,u) &= g((\bar{\nabla}^2 h)(u,u,v,v), Jv)\\ &\quad + 2g((\bar{\nabla}h)(u,v,v), G(u,v)) + g(h(v,v), (\bar{\nabla}_u G)(u,v)),\end{aligned}$$

where we have used the fact that $\nabla_X Y = \bar{\nabla}_X Y - h(X,Y)$ and that $G(h(X,Y),Z)$ is tangent to M for X, Y, Z tangent to M. Now, using (2.4), (2.9), (2.10), (2.11), and (2.13), we have

$$(3.2)\quad \begin{aligned}d^2 f_v(u,u) &= g((\bar{\nabla}^2 h)(v,v,u,u), Jv) + 2R(u,v;v,Jh(u,v))\\ &\quad + R(u,v;u,Jh(v,v)) + g(h(u,u),Jv)\\ &\quad - g(u,v)g(h(u,v),Jv) + 2g((\bar{\nabla}h)(u,v,v), G(u,v))\\ &\quad + g(u,v)g(h(v,v),Ju) - g(h(v,v),Jv).\end{aligned}$$

The function f restricted to the fiber UM_p attains a maximum at v. Thus, if $\beta(t)$, $t \in (-\delta, \delta)$, is a curve in UM_p with $\beta(0) = v$, $\|\beta'(t)\| = 1$, and $\beta'(0) = u$, then realising UM_p as S^2 and using (2.9) we obtain

$$0 = d(f|_{UM_p})_v(u) = \left(\frac{d}{dt}\right)_{t=0} g(h(\beta(t), \beta(t)), J\beta(t)) \tag{3.3}$$
$$= 3g(h(v, v), Ju)$$

and

$$0 \geq d^2(f|_{UM_p})(u, u) = 6g(h(\beta'(0), v), J\beta'(0)) + 3g(h(v, v), J\beta''(0)) \tag{3.4}$$
$$= 6g(h(u, v), Ju) - 3g(h(v, v), Jv)).$$

Since (3.3) is true for any unit vector u orthogonal to v, we have $h(v, v) = f(v)Jv$ and thus, in light of (2.9), we see that v is an eigenvector of A_{Jv} corresponding to the eigenvalue $f(v)$. Now we can choose an orthonormal basis $\{u_1, u_2, u_3\}$ of T_pM (the tangent space of M at p) which diagonalizes A_{Jv} such that $u_3 = v$. If $A_{Jv}u_i = \rho_i u_i$, $i = 1, 2$, then using $h(v, v) = f(v)Jv$ and $h(u_i, v) = \rho_i Ju_i$ in (3.4) yields

$$f(v) - 2\rho_i \geq 0, \quad i = 1, 2. \tag{3.5}$$

Now, adding the equations (3.2) over basis vectors $\{u_1, u_2, v\}$ and using (2.9), (2.14), and (3.1) as well as minimality, we obtain

$$0 \geq \sum_{i=1}^{3} d^2 f_v(u_i, u_i) = \sum_{i=1}^{2} K(v, u_i)(f(v) - 2\rho_i) - f(v), \tag{3.6}$$

where $K(v, u_i)$ is the sectional curvature of the plane section spanned by $\{v, u_i\}$. If k_0 is the infimum of the sectional curvatures of M, then using (3.5) together with $\rho_1 + \rho_2 = -f(v)$ in (3.6) yields $(4k_0 - 1)f(v) \leq 0$. Also, since $f(v)$ is the maximum value of the cubic function f, we have $f(v) \geq 0$ (this also follows from (3.5) and the minimality).

Thus we get that either $4k_0 \leq 1$ or $f(v) = 0$. In case $f(v) = 0$, as $f(v)$ is the maximum value of f, we get $f(u) \leq 0$ for all $u \in UM$, and this together with the formula $f(-u) = -f(u)$ gives $f(u) = 0$, $u \in UM$. Using the standard polarization formula

$$6g(h(u_1, u_2), Ju_3) = f(u_1 + u_2 + u_3) - f(u_2 + u_3) - f(u_1 + u_3)$$
$$- f(u_1 + u_2) + f(u_1) + f(u_2) + f(u_3),$$

which is valid for any symmetric cubic form, we obtain $g(h(u, v), Jw) = 0$, $u, v, w \in UM$, that is, M is totally geodesic.

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Department of Mathematics
College of Science
King Saud University
Riyadh 11451
Saudi Arabia