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KODAI MATH. J.
26 (2003), 49–68

TOTALLY UMBILICAL LIGHTLIKE SUBMANIFOLDS

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Abstract

This paper provides new results on a class of totally umbilical lightlike submanifolds in semi-Riemannian manifolds of constant curvature. We prove that the induced Ricci tensor of any such submanifold is symmetric if and only if its screen distribution is integrable.

1. Introduction

The theory of submanifolds of a Riemannian or semi-Riemannian manifold is well known (see for example, Chen [4] and O'Neill [12]. However, its counter part of lightlike (null) submanifolds (for which the local and global geometry is completely different than the non-degenerate case) is relatively new and in a developing stage ([1, 3, 5–9, 11]). In 1996, the first author and Bejancu published their work (see Chapters 4 and 5 of [8]) on lightlike submanifolds M of semi-Riemannian manifolds. They constructed structure equations for four possible cases of M, proved the fundamental existence theorem for lightlike submanifolds and found some geometric conditions for the induced connection on M to be a metric connection. Much of their study was restricted to totally geodesic lightlike submanifolds of semi-Riemannian manifolds. In this paper we study further the geometry of totally umbilical lightlike submanifolds M.

In Sections 2 and 3, we recall some results for lightlike submanifolds and their structure equations. In Section 4, we prove several new theorems on M in semi-Riemannian manifolds of constant curvature. Finally, in Section 5, we find conditions for the induced Ricci curvature tensor of M to be symmetric. The paper contains several simple examples.

2. Lightlike submanifolds

Let $(\overline{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) an m-dimensional

2000 *Mathematics Subject Classification*: 53B25, 53C40, 53C50.
Keywords: Degenerate metric, totally umbilical submanifolds.
Received April 12, 2002; revised October 17, 2002.

submanifold of $\overline{M}$. In case $\bar{g}$ is degenerate on the tangent bundle TM of M we say that M is a lightlike submanifold of $\overline{M}$ [8]. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E (same notation for any other vector bundle) over M. The following range of indices is used:

$$i, j, k, \ldots \in \{1, \ldots, r\}; \quad a, b, c, \ldots \in \{r+1, \ldots, m\};$$
$$A, B, C, \ldots \in \{1, \ldots, m\}; \quad \alpha, \beta, \gamma, \ldots \in \{r+1, \ldots, n\}.$$

For a degenerate tensor field g on M, there exists locally a vector field $\xi \in \Gamma(TM)$, $\xi \neq 0$, such that $g(\xi, X) = 0$, for any $X \in \Gamma(TM)$. Then, for each tangent space T_xM we have

$$T_xM^{\perp} = \{u \in T_x\overline{M} : \bar{g}(u, v) = 0, \ \forall v \in T_xM\},$$

which is a degenerate n-dimension subspace of $T_x\overline{M}$. The radical (null) subspace of T_xM, denoted by Rad T_xM, is defined by

$$\text{Rad } T_xM = \{\xi_x \in T_xM; g(\xi_x, X) = 0, X \in T_xM\}.$$

The dimension of Rad $T_xM = T_xM \cap T_xM^{\perp}$ depends on $x \in M$. The submanifold M of $\overline{M}$ is said to be r-lightlike submanifold if the mapping

$$\text{Rad } TM : x \in M \to \text{Rad } T_xM$$

defines a smooth distribution on M of rank $r > 0$, where Rad TM is called the radical (null) distribution on M. Following are four possible cases:

Case 1. r-lightlike submanifold. $1 \leq r < \min\{m, n\}$.

Case 2. Co-isotropic submanifold. $1 \leq r = n < m$.

Case 3. Isotropic submanifold. $1 \leq r = m < n$.

Case 4. Totally lightlike submanifold. $1 \leq r = m = n$.

We refer [8] for notations and details not mentioned in this paper. For Case 1, there exists a non-degenerate screen distribution $S(TM)$ which is a complementary vector subbundle to Rad TM in TM. Therefore,

$$TM = \text{Rad } TM \oplus S(TM). \tag{2.1}$$

Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM/\text{Rad } TM$. Denote an r-lightlike submanifold by $(M, g, S(TM), S(TM^{\perp}))$, where $S(TM^{\perp})$ is a complementary vector subbundle to Rad TM in $TM^{\perp}$. For the dependence of all the induced geometric objects, of M, on $\{S(TM), S(TM^{\perp})\}$ we refer [8]. Let $\text{tr}(TM)$ and $\text{ltr}(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\overline{M}\,|\,M$ and to Rad TM in $S(TM^{\perp})$ respectively. Then, we obtain

$$\text{(2.2)} \qquad \operatorname{tr}(TM) = \operatorname{ltr}(TM) \oplus S(TM^{\perp}),$$

$$\text{(2.3)} \qquad \begin{aligned} T\overline{M}|_M &= TM \oplus \operatorname{tr}(TM) \\ &= (\operatorname{Rad} TM \oplus \operatorname{ltr}(TM)) \oplus S(TM) \oplus S(TM^{\perp}). \end{aligned}$$

Consider the following local quasi-orthonormal field of frames of $\overline{M}$ along M:

$$\text{(2.4)} \qquad \{\xi_1, \dots, \xi_r, N_1, \dots, N_r, X_{r+1}, \dots, X_m, W_{r+1}, \dots, W_n\}$$

where $\{\xi_1, \dots, \xi_r\}$ is a lightlike basis of $\Gamma(\operatorname{Rad} TM)$, $\{N_1, \dots, N_r\}$ a lightlike basis of $\Gamma(\operatorname{ltr}(TM))$, $\{X_{r+1}, \dots, X_m\}$ and $\{W_{r+1}, \dots, W_n\}$ orthonormal basis of $\Gamma(S(TM) \mid \mathcal{U})$ and $\Gamma(S(TM^{\perp}) \mid \mathcal{U})$ respectively.

Example 1. Consider a surface (M, g) in R^4_2 given by the equations

$$x^3 = \frac{1}{\sqrt{2}}(x^1 + x^2); \quad x^4 = \frac{1}{2}\log(1 + (x^1 - x^2)^2),$$

where $(x^1, \dots, x^4)$ is a local coordinate system for R^4_2. Using a simple procedure of linear algebra, we choose a set of vectors $\{U, V, \xi, W\}$ given by

$$\begin{aligned} U &= \sqrt{2}(1 + (x^1 - x^2)^2)\partial_1 + (1 + (x^1 - x^2)^2)\partial_3 + \sqrt{2}(x^1 - x^2)\partial_4, \\ V &= \sqrt{2}(1 + (x^1 - x^2)^2)\partial_2 + (1 + (x^1 - x^2)^2)\partial_3 - \sqrt{2}(x^1 - x^2)\partial_4, \\ \xi &= \partial_1 + \partial_2 + \sqrt{2}\partial_3, \\ W &= 2(x^2 - x^1)\partial_2 + \sqrt{2}(x^2 - x^1)\partial_3 + (1 + (x^1 - x^2)^2)\partial_4, \end{aligned}$$

so that TM and $TM^{\perp}$ are spanned by $\{U, V\}$ and $\{\xi, W\}$ respectively. By direct calculations it follows that $\operatorname{Rad} TM$ is a distribution on M of rank 1 and spanned by the lightlike vector ξ. Choose $S(TM)$ and $S(TM^{\perp})$ spanned by the timelike vector V and the spacelike vector W respectively. Then,

$$\begin{aligned} \operatorname{ltr}(TM) &= \operatorname{Span}\left\{N = -\frac{1}{2}\partial_1 + \frac{1}{2}\partial_2 + \frac{1}{\sqrt{2}}\partial_3\right\}, \\ \operatorname{tr}(TM) &= \operatorname{Span}\{N, W\}, \end{aligned}$$

where N is a lightlike vector such that $g(N, \xi) = 1$. Thus, M is a 1-lightlike submanifold of Case 1, with basis $\{\xi, N, V, W\}$ of R^4_2 along M.

For Case 2, we have $\operatorname{Rad} TM = TM^{\perp}$. Therefore, $S(TM^{\perp}) = \{0\}$ and from (2.2) $\operatorname{tr}(TM) = \operatorname{ltr}(TM)$. Thus, (2.3) and (2.4) reduce to

$$\text{(2.5)} \qquad T\overline{M}|_M = TM \oplus \operatorname{tr}(TM) = (TM^{\perp} \oplus \operatorname{ltr}(TM)) \oplus S(TM)$$

$$\text{(2.6)} \qquad \{\xi_1, \dots, \xi_r, N_1, \dots, N_r, X_{r+1}, \dots, X_m\}.$$

Example 2. Consider the unit pseudo sphere S_1^3 of Minkowski space R_1^4 given by the equation $-t^2 + x^2 + y^2 + z^2 = 1$. Cut S_1^3 by the hypersurface $t - x = 0$ and obtain a lightlike surface (M, g) of S_1^3 with Rad TM spanned by a lightlike vector $\xi = \partial_t + \partial_x$. Clearly, Rad $TM = TM^\perp$ and, therefore, this example belongs to Case 2. Consider a screen distribution $S(TM)$ spanned by a spacelike vector $X = z\partial_y - y\partial_z$. Then, we obtain a lightlike transversal vector bundle $\mathrm{tr}(TM) = \mathrm{ltr}(TM)$ spanned by $N = (-1/2)\{(1 + t^2)\partial_t + (t^2 - 1)\partial_x + 2ty\partial_y + 2tz\partial_z\}$ such that $g(N, \xi) = 1$, with a basis $\{\xi, N, X\}$ for S_1^3 along M.

For Case 3, we have Rad $TM = TM$. Therefore, $S(TM) = \{0\}$. Therefore, (2.3) and (2.4) reduce to

$$T\overline{M}|_M = TM \oplus \mathrm{tr}(TM) = (TM \oplus \mathrm{ltr}(TM)) \oplus S(TM^\perp) \tag{2.7}$$

$$\{\xi_1, \ldots, \xi_r, N_1, \ldots, N_r, W_{r+1}, \ldots, W_n\}. \tag{2.8}$$

Example 3. Suppose (M, g) is a surface of R_2^5 given by equations

$$x^3 = \cos x^1, \quad x^4 = \sin x^1, \quad x^5 = x^2.$$

We choose a set of vectors $\{\xi_1, \xi_2, U_1, U_2\}$ given by

$$\xi_1 = \partial_2 + \partial_5, \quad \xi_2 = \partial_1 - \sin x^1 \partial_3 + \cos x^1 \partial_4,$$

$$U_1 = -\sin x^1 \partial_1 + \partial_3, \quad U_2 = \cos x^1 \partial_1 + \partial_4,$$

so that Rad $TM = TM = \mathrm{Span}\{\xi_1, \xi_2\}$, $TM^\perp = \mathrm{Span}\{\xi_1, U_1, U_2\}$. Therefore, M belongs to Case 3. Construct two null vectors

$$N_1 = \frac{1}{2}\{-\partial_2 + \partial_5\},$$

$$N_2 = \frac{1}{2}\{-\partial_1 - \sin x^1 \partial_3 + \cos x^1 \partial_4\},$$

such that $g(N_i, \xi_j) = \delta_{ij}$ for $i, j \in \{1, 2\}$ and $\mathrm{ltr}(TM) = \mathrm{Span}\{N_1, N_2\}$. Let $W = \cos x^1 \partial_3 + \sin x^1 \partial_4$ be a spacelike vector such that $S(TM^\perp) = \mathrm{Span}\{W\}$. Thus, $\{\xi_1, \xi_2, N_1, N_2, W\}$ is a basis of R_2^5 along M.

For Case 4, Rad $TM = TM = TM^\perp$, $S(TM) = S(TM^\perp) = \{0\}$. Therefore, (2.3) and (2.4) reduce to

$$T\overline{M}|_M = TM \oplus \mathrm{ltr}(TM) \tag{2.9}$$

$$\{\xi_1, \ldots, \xi_r, N_1, \ldots, N_r\}. \tag{2.10}$$

Example 4. Suppose (M, g) is a surface of R_2^4 given by the equations

$$x^3 = \frac{1}{\sqrt{2}}(x^1 + x^2), \quad x^4 = \frac{1}{\sqrt{2}}(x^1 - x^2).$$

We choose a set of vectors $\{\xi_1, \xi_2, U, V\}$ given by

$$\xi_1 = \partial_1 + \frac{1}{\sqrt{2}}\partial_3 + \frac{1}{\sqrt{2}}\partial_4, \quad \xi_2 = \partial_2 + \frac{1}{\sqrt{2}}\partial_3 - \frac{1}{\sqrt{2}}\partial_4,$$
$$U = \partial_1 + \partial_2 + \sqrt{2}\partial_3, \quad V = \partial_1 - \partial_2 + \sqrt{2}\partial_4,$$

so that TM and $TM^\perp$ are spanned by $\{\xi_1, \xi_2\}$ and $\{U, V\}$ respectively. By direct calculations we check that $\mathrm{Span}\{\xi_1, \xi_2\} = \mathrm{Span}\{U, V\}$, that is, $TM = TM^\perp$. Finally, the two lightlike transversal vector bundles are:

$$N_1 = \partial_1 + \sqrt{2}\partial_3 + \sqrt{2}\partial_4, \quad N_2 = \partial_2 + \sqrt{2}\partial_3 - \sqrt{2}\partial_4,$$

such that $g(N_i, \xi_j) = \delta_{ij}$, $i, j = 1, 2$. Thus, M is of Case 4, with a basis $\{\xi_1, \xi_2, N_1, N_2\}$ of R_2^4 along M.

On the existence of a local quasi-orthonormal field of frames of $\overline{M}$ along M we state (see Chapter 5 of [8] for its proof) the following main result:

Theorem 2.1 [8]. *Let $(M, g, S(TM), S(TM^\perp))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\overline{M}, \bar{g})$. Then there exists a complementary vector bundle* $\mathrm{ltr}(TM)$ *of* Rad TM *in $S(TM^\perp)^\perp$ and a basis of $\Gamma(\mathrm{ltr}(TM)|_{\mathscr{U}})$ consisting of smooth sections $\{N_i\}$ of $S(TM^\perp)^\perp|_{\mathscr{U}}$, where $\mathscr{U}$ is a coordinate neighborhood of M, such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \tag{2.11}$$

where $\{\xi_1, \dots, \xi_r\}$ is a lightlike basis of $\Gamma(\mathrm{Rad}\, TM)$.

Define locally r differential 1-forms $\{\eta_i\}$ on $\Gamma(TM)$ by

$$\eta_i(X) = \bar{g}(X, N_i), \quad \forall X \in \Gamma(TM). \tag{2.12}$$

Let P the projection of TM on $S(TM)$ with respect to (2.1). Then,

$$X = PX + \sum_{i=1}^{r} \eta_i(X)\xi_i, \tag{2.13}$$

for every $X \in \Gamma(TM)$. According to (2.3) we put

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.14}$$

$$\overline{\nabla}_X V = -A(V, X) + \nabla_X^\perp V, \quad \forall X, Y \in \Gamma(TM), \tag{2.15}$$

$V \in \Gamma(\mathrm{tr}(TM))$, $\{\nabla_X Y, A(V, X)\}$ and $\{h(X, Y), \nabla_X^\perp V\}$ belong to $\Gamma(TM)$ and $\Gamma(\mathrm{tr}(TM))$ respectively. Here $\overline{\nabla}$ is the metric connection on $\overline{M}$ but ∇ (torsion-free) and $\nabla^\perp$ are linear connections on M and $\mathrm{tr}(TM)$ respectively.

Suppose $S(TM^\perp) \neq \{0\}$, that is, M is either an r-lightlike or a isotropic submanifold of $\overline{M}$. According to (2.3) we consider the projection morphisms L and S of $\mathrm{tr}(TM)$ on $\mathrm{ltr}(TM)$ and $S(TM^\perp)$ respectively. Then (2.14) and (2.15) become

$$\overline{\nabla}_X Y = \nabla_X Y + h^{\ell}(X, Y) + h^s(X, Y), \tag{2.16}$$

$$\overline{\nabla}_X V = -A_V X + D_X^{\ell} V + D_X^s V, \tag{2.17}$$

where we put

$$h^{\ell}(X, Y) = L(h(X, Y)); \quad h^s(X, Y) = S(h(X, Y)); \quad A_V X = A(V, X),$$
$$D_X^{\ell} V = L(\nabla_X^{\perp} V) = D^{\ell}(X, V); \quad D_X^s V = S(\nabla_X^{\perp} V) = D^s(X, V).$$

As h^{ℓ} and h^s are $\Gamma(\mathrm{ltr}(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued respectively, we call them the lightlike second fundamental form and the screen second fundamental form of M. In particular, we derive

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\ell} N + D^s(X, N), \tag{2.18}$$

$$\overline{\nabla}_X W = -A_W X + \nabla_X^s W + D^{\ell}(X, W), \tag{2.19}$$

for any $X \in \Gamma(TM)$, $N \in \Gamma(\mathrm{ltr}(TM))$ and $W \in \Gamma(S(TM^{\perp}))$.

Next, suppose $S(TM^{\perp}) = \{0\}$, that is, M is either co-isotropic or totally lightlike. Then, (2.16) and (2.17) become

$$\overline{\nabla}_X Y = \nabla_X Y + h^{\ell}(X, Y), \tag{2.20}$$

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\ell} N, \tag{2.21}$$

for any $X, Y \in \Gamma(TM)$. We call (2.14), (2.16), (2.20) the Gauss formulae and (2.15), (2.17)–(2.21) the Weingarten formulae for all cases of a lightlike submanifold M. Using (2.16)–(2.21), (2.3), (2.5), (2.7) and (2.9), we obtain

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^{\ell}(X, W)) = g(A_W X, Y), \tag{2.22}$$

$$\bar{g}(h^{\ell}(X, Y), \xi) + \bar{g}(Y, h^{\ell}(X, \xi)) + g(Y, \nabla_X \xi) = 0, \tag{2.23}$$

$$\bar{g}(A_N X, N') + \bar{g}(A_{N'} X, N)) = 0, \tag{2.24}$$

for any $\xi \in \Gamma(\mathrm{Rad}\, TM)$, $W \in \Gamma(S(TM^{\perp}))$ and $N, N' \in \Gamma(\mathrm{ltr}(TM))$.

Next, suppose $S(TM) \neq \{0\}$, that is, M is either r-lightlike or co-isotropic. Then according to (2.1) we set

$$\nabla_X PY = \nabla_X^* PY + h^*(X, PY), \tag{2.25}$$

$$\nabla_X \xi = -A^*(\xi, X) + \nabla_X^{*t} \xi, \tag{2.26}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}\, TM)$, where $\{\nabla_X^* PY, A^*(\xi, X)\}$ and $\{h^*(X, PY), \nabla_X^{*t} \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(\mathrm{Rad}\, TM)$ respectively. It follows that ∇^* and ∇^{*t} are linear connections on $S(TM)$ and $\mathrm{Rad}\, TM$ respectively. By using (2.16), (2.21), (2.25) and (2.26) we obtain

$$\bar{g}(h^{\ell}(X, PY), \xi) = \bar{g}(A_{\xi}^* X, PY) \tag{2.27}$$

$$\bar{g}(h^*(X, PY), N) = \bar{g}(A_N X, PY), \quad \forall X, Y \in \Gamma(TM) \tag{2.28}$$

Theorem 2.2 [8]. *Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold* *or a co-isotropic submanifold of a semi-Riemannian manifold* $(\overline{M}, \bar{g})$. *Then the following assertions are equivalent*:

(1) $S(TM)$ *is integrable.*
(2) h^* *is symmetric on* $\Gamma(S(TM))$.
(3) A_N *is self-adjoint on* $\Gamma(S(TM))$ *with respect to* g.
(4) ∇^* *is torsion-free linear connection.*

Example 5. Let $(R_1^{d+1}, \bar{g})$ be a Minkowski spacetime, where

$$\bar{g}(x, y) = -x^0 y^0 + \sum_{i=1}^{d} x^i y^i, \quad \forall x, y \in R^{d+1}.$$

Consider a smooth function $f : D \to R$, where D is an open set of R^d. Then

$$M = \{(x^0, \dots, x^n) \in R_1^{d+1}; x^0 = f(x^1, \dots, x^d)\},$$

is a hypersurface of R_1^{d+1} which is called a *Monge hypersurface*. Let natural parameterization on M be given by

$$x^0 = f(v^0, \dots, v^{d-1}); \quad x^{\alpha+1} = v^\alpha, \quad \alpha \in \{0, \dots, n-1\}.$$

Hence, the natural frames field on M is globally defined by

$$\partial_{v^\alpha} = f'_{x^{\alpha+1}} \partial_{x^0} + \partial_{x^{\alpha+1}}, \quad \alpha \in \{0, \dots, d-1\}.$$

Then, it follows that $TM^\perp$ is spanned by a global vector

$$\xi = \partial_{x^0} + \sum_{i=1}^{d} f'_{x^i} \partial_{x^i}. \tag{2.29}$$

It is known [8] that M is a lightlike hypersurface if $TM^\perp = \operatorname{Rad} TM$. This means that ξ, given by (2.29), must be a null vector field. Hence, there exists a lightlike Monge hypersurface M, if the function f is a solution of the differential equation $\sum_{i=1}^{d} (f'_{x^i})^2 = 1$. The null transversal vector is given by $N = (1/2)\{-\partial_{x^0} + \sum_{i=1}^{d} f'_{x^i} \partial_{x^i}\}$, $\bar{g}(N, \xi) = 1$. Let $\overline{\nabla}$ be the Levi-Civita connection, with respect to the metric $\bar{g}$, on R_1^{d+1}. Then, for any two vectors $X, Y \in \Gamma(S(TM))$, the Lie bracket $[X, Y] \in \Gamma(S(TM))$. Indeed,

$$\begin{aligned}\bar{g}([X, Y], N) &= \bar{g}(\overline{\nabla}_X Y - \overline{\nabla}_Y X, \partial_{x^0}) \\ &= -\{\bar{g}(X, \overline{\nabla}_Y \partial_{x^0}) - \bar{g}(Y, \overline{\nabla}_X \partial_{x^0})\} = 0.\end{aligned}$$

Hence, $S(TM)$ is integrable. Other equivalent assertions follow easily.

3. Structure equations

Let $(M, g, S(TM), S(TM^\perp))$ be an m-dimensional r-lightlike submanifold of $(m+n)$-dimensional semi-Riemannian manifold $(\overline{M}, \bar{g})$. Denote by $\bar{R}, R$ and R^ℓ the curvature tensors of $\overline{\nabla}, \nabla$ and ∇^ℓ respectively. We need following structure equations (see [8] for details on a complete set of equations):

$$(3.1)\quad \bar{R}(X, Y)Z = R(X, Y)Z + A_{h^\ell(X,Z)}Y - A_{h^\ell(Y,Z)}X + A_{h^s(X,Z)}Y - A_{h^s(Y,Z)}X + (\nabla_X h^\ell)(Y,Z) - (\nabla_Y h^\ell)(X,Z) + D^\ell(X, h^s(Y,Z)) - D^\ell(Y, h^s(X,Z)) + (\nabla_X h^s)(Y,Z) - (\nabla_Y h^s)(X,Z) + D^s(X, h^\ell(Y,Z)) - D^s(Y, h^\ell(X,Z)),$$

for any $X, Y, Z \in \Gamma(TM)$. Consider the curvature tensor $\bar{R}$ of type $(0,4)$.

$$(3.2)\quad \bar{R}(X, Y, PZ, PU) = g(R(X, Y)PZ, PU) + \bar{g}(h^*(Y, PU), h^\ell(X, PZ)) - \bar{g}(h^*(X, PU), h^\ell(Y, PZ)) + \bar{g}(h^s(Y, PU), h^s(X, PZ)) - \bar{g}(h^s(X, PU), h^s(Y, PZ)),$$

$$(3.3)\quad \bar{R}(X, Y, \xi, PU) = g(R(X, Y)\xi, PU) + \bar{g}(h^*(Y, PU), h^\ell(X, \xi)) - \bar{g}(h^*(X, PU), h^\ell(Y, \xi)) + \bar{g}(h^s(Y, PU), h^s(X, \xi)) - \bar{g}(h^s(X, PU), h^s(Y, \xi))$$
$$= \bar{g}((\nabla_Y h^\ell)(X, PU) - (\nabla_X h^\ell)(Y, PU), \xi) + \bar{g}(h^s(Y, PU), h^s(X, \xi)) - \bar{g}(h^s(X, PU), h^s(Y, \xi)),$$

$$(3.4)\quad \bar{R}(X, Y, N, PU) = -\bar{g}(R(X, Y)PU, N) + \bar{g}(A_N Y, h^\ell(X, PU)) - \bar{g}(A_N X, h^\ell(Y, PU)) + \bar{g}(h^s(Y, PU), D^s(X, N)) - \bar{g}(h^s(X, PU), D^s(Y, N))$$
$$= \bar{g}((\nabla_Y A)(N, X) - (\nabla_X A)(N, Y), PU) + \bar{g}(h^s(Y, PU), D^s(X, N)) - \bar{g}(h^s(X, PU), D^s(Y, N)),$$

$$(3.5)\quad \bar{R}(X, Y, W, PU) = \bar{g}((\nabla_Y A)(W, X) - (\nabla_X A)(W, Y), PU) + \bar{g}(h^*(Y, PU), D^\ell(X, W)) - \bar{g}(h^*(X, PU), D^\ell(Y, W))$$
$$= \bar{g}((\nabla_Y h^s)(X, PU) - (\nabla_X h^s)(Y, PU), W) + \bar{g}(h^\ell(X, PU), A_W Y) - \bar{g}(h^\ell(X, PU), A_W X),$$

$$(3.6)\quad \bar{R}(X, Y, N, \xi) = \bar{g}(R^\ell(X, Y)N, \xi) + \bar{g}(h^\ell(Y, A_N X), \xi) - \bar{g}(h^\ell(X, A_N Y), \xi) + \bar{g}(D^s(X, N), h^s(Y, \xi)) - \bar{g}(D^s(Y, N), h^s(X, \xi))$$

$$= -\bar{g}(R(X, Y)\xi, N)$$
$$+ \bar{g}(A_N Y, h^\ell(X, \xi)) - (A_N Y, h^\ell(Y, \xi))$$
$$+ \bar{g}(D^s(X, N), h^s(Y, \xi)) - \bar{g}(D^s(Y, N), h^s(X, \xi)),$$

$X, Y, U \in \Gamma(TM)$. Let R^{*t} be the curvature tensor of ∇^{*t}. Then,

$$g(R(X, Y)\xi, PU) = g((\nabla_Y A^\star)(\xi, X) - (\nabla_X A^\star)(\xi, Y), PU), \tag{3.7}$$

$$g(R(X, Y)\xi, N = \bar{g}(R^{*t}(X, Y)\xi, N) + g(A_N Y, A_\xi^* X) - g(A_N X, A_\xi^* Y). \tag{3.8}$$

$$\begin{aligned} g(R(X, Y)PU, N) &= \bar{g}((\nabla_X A)(N, Y) - (\nabla_Y A)(N, X), PU) \\ &\quad + \bar{g}(h^\ell(X, PU), A_N Y) - \bar{g}(h^\ell(Y, PU), A_N X) \\ &= \bar{g}((\nabla_X h^*)(Y, PU) - (\nabla_Y h^*)(X, PU), N). \end{aligned} \tag{3.9}$$

Finally, from (3.6), by using (2.23) and (2.25) we deduce

$$\bar{g}(R(X, Y)\xi, N) + \bar{g}(R^\ell(X, Y)N, \xi) = g(A_\xi^* X, A_N Y) - g(A_\xi^* Y, A_N X). \tag{3.10}$$

Remark 1. For structure equations of Case 2, delete all the components involving $S(TM^\perp)$. Similarly, one can find the structure equations of the other two cases.

Remark 2. In the sequel we denote by (M, g) a lightlike submanifold for which the results hold for all its four cases. Any result which does not hold for all the cases will be so specified.

4. Totally umbilical lightlike submanifold

Let $\{N_i, W_\alpha\}$ be a basis of $\Gamma(\text{tr}(TM)|_{\mathscr{U}})$ on a coordinate neighborhood $\mathscr{U}$ of M, where $N_i \in \Gamma(\text{ltr}(TM)|_{\mathscr{U}})$ and $W_\alpha \in \Gamma(S(TM^\perp)|_{\mathscr{U}})$. Then (2.16) becomes

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^\ell(X, Y)N_i + \sum_{\alpha=r+1}^{n} h_\alpha^s(X, Y)W_\alpha, \tag{4.1}$$

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{m<n} h_i^\ell(X, Y)N_i + \sum_{\alpha=m+1}^{n} h_\alpha^s(X, Y)W_\alpha, \tag{4.2}$$

for an r-lightlike or an isotropic submanifold respectively. (2.20) becomes

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{n<m} h_i^\ell(X, Y)N_i, \tag{4.3}$$

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{n=m} h_i^\ell(X, Y)N_i, \tag{4.4}$$

for a co-isotropic and a totally lightlike submanifold respectively. We call $\{h_i^\ell\}$ and $\{h_\alpha^s\}$ the local lightlike second fundamental forms and the local screen second fundamental forms of M on $\mathscr{U}$. Also (2.18) and (2.19) become

$$(4.5)\quad \begin{aligned} \overline{\nabla}_X N_i &= -A_{N_i}X + \sum_{j=1}^{r} \rho_{ij}(X)N_j + \sum_{\alpha=r+1}^{n} \tau_{i\alpha}(X)W_\alpha, \\ \overline{\nabla}_X W_\alpha &= -A_{W_\alpha}X + \sum_{i=1}^{r} v_{\alpha i}(X)N_i + \sum_{\beta=r+1}^{n} \theta_{\alpha\beta}(X)W_\beta, \\ \overline{\nabla}_X N_i &= -A_{N_i}X + \sum_{j=1}^{m<n} \rho_{ij}(X)N_j + \sum_{\alpha=m+1}^{n} \tau_{i\alpha}(X)W_\alpha, \\ \overline{\nabla}_X W_\alpha &= -A_{W_\alpha}X + \sum_{i=1}^{m<n} v_{\alpha i}(X)N_i + \sum_{\beta=m+1}^{n} \theta_{\alpha\beta}(X)W_\beta, \end{aligned}$$

for an r-lightlike and an isotropic submanifold respectively, where

$$(4.6)\quad \begin{aligned} \rho_{ij}(X) &= \bar{g}(\nabla_X^\ell N_i, \xi_j), \quad \varepsilon_\alpha\tau_{i\alpha}(X) = \bar{g}(D^s(X, N_i), W_\alpha), \\ v_{\alpha i}(X) &= \bar{g}(D^\ell(X, W_\alpha), \xi_i), \quad \varepsilon_\beta\theta_{\alpha\beta}(X) = \bar{g}(\nabla_X^s W_\alpha, W_\beta), \end{aligned}$$

and ε_α is the signature of W_α. Similarly, (2.21) becomes

$$(4.7)\quad \begin{aligned} \overline{\nabla}_X N_i &= -A_{N_i}X + \sum_{j=1}^{r} \rho_{ij}(X)N_j, \\ \overline{\nabla}_X N_i &= -A_{N_i}X + \sum_{j=1}^{m<n} \rho_{ij}(X)N_j, \end{aligned}$$

for a co-isotropic and a totally lightlike submanifold respectively. Then, (2.25) and (2.26) become

$$(4.8)\quad \begin{aligned} \nabla_X PY &= \nabla_X^* PY + \sum_{i=1}^{r} h_i^*(X, PY)\xi_i, \\ \nabla_X \xi_i &= -A_{\xi_i}^* X + \sum_{j=1}^{r} \mu_{ij}(X)\xi_j, \end{aligned}$$

where $h_i^*(X, PY) = \bar{g}(h^*(X, PY), N_i)$ and $\mu_{ij}(X) = \bar{g}(\nabla_X^{*t}\xi_i, N_j)$. Using the equations (2.11) and (4.5)–(4.8) we obtain $\mu_{ij}(X) = -\rho_{ji}(X)$. Thus,

$$(4.9)\quad \nabla_X \xi_i = -A_{\xi_i}^* X - \sum_{j=1}^{r} \rho_{ji}(X)\xi_j.$$

Definition 1. A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\overline{M}, \bar{g})$ is said to be totally umbilical in $\overline{M}$ if there is a smooth transversal vector field $\mathscr{H} \in \Gamma(\mathrm{tr}(TM))$ on M, called the transversal curvature vector field of M, such that, for all $X, Y \in \Gamma(TM)$,

$$h(X, Y) = \mathscr{H}\bar{g}(X, Y) \tag{4.10}$$

Using (2.16) and (4.1) it is easy to see that M is totally umbilical, if and only if on each coordinate neighborhood $\mathscr{U}$ there exist smooth vector fields $H^\ell \in \Gamma(\mathrm{ltr}(TM))$ and $H^s \in \Gamma(S(TM^\perp))$, and smooth functions $H_i^\ell \in F(\mathrm{ltr}(TM))$ and $H_i^s \in F(S(TM^\perp))$ such that

$$\begin{aligned} h^\ell(X, Y) &= H^\ell \bar{g}(X, Y), \quad h^s(X, Y) = H^s \bar{g}(X, Y) \\ h_i^\ell(X, Y) &= H_i^\ell \bar{g}(X, Y), \quad h_\alpha^s(X, Y) = H_\alpha^s \bar{g}(X, Y) \end{aligned} \tag{4.11}$$

for any $X, Y \in \Gamma(TM)$. Above definition does not depend on the screen distribution and the screen transversal vector bundle of M. On the other hand, from the equation (2.22) we obtain the following equation

$$g(A_{W_\alpha}X, Y) = \varepsilon_\alpha h_\alpha^s(X, Y) + \sum_{i=1}^{r} D_i^\ell(X, W_\alpha)\eta_i(Y). \tag{4.12}$$

Now replace Y by ξ_j and obtain

$$D_i^\ell(X, W_\alpha) = -\varepsilon_\alpha h_\alpha^s(\xi_i, X). \tag{4.13}$$

Using (2.22), (2.27), (4.11) and (4.13), we conclude (the relations (4.11) trivially hold in case $S(TM)$ and or $S(TM^\perp)$ vanish)

THEOREM 4.1. *Let (M, g) be a lightlike submanifold of $(\overline{M}, \bar{g})$. Then M is totally umbilical, if and only if, on each coordinate neighborhood $\mathscr{U}$ there exist smooth vector fields H^ℓ and H^s such that*

$$\begin{aligned} D^\ell(X, W) &= 0, \quad A_\xi^* X = H^\ell PX, \quad P(A_W X) = \varepsilon H^s PX, \\ D_i^\ell(X, W_\alpha) &= 0, \quad A_{\xi_i}^* X = H_i^\ell PX, \quad P(A_{W_\alpha}X) = \varepsilon_\alpha H_\alpha^s PX, \end{aligned} \tag{4.14}$$

for any $X \in \Gamma(TM)$, where ε is the signature of $W \in \Gamma(S(TM^\perp))$.

Example 6. Let M be a surface of R_2^4, of Example 1, given by

$$x^3 = \frac{1}{\sqrt{2}}(x^1 + x^2); \quad x^4 = \frac{1}{2}\log(1 + (x^1 - x^2)^2),$$

where $(x^1, \dots, x^4)$ is a local coordinate system for R_2^4. As explained in Example 1, M is a 1-lightlike surface of Case 1 having a local quasi-orthonormal field of frames $\{\xi, N, V, W\}$ along M. Denote by $\overline{\nabla}$ the Levi-Civita connection on R_2^4. Then, by straightforward calculations, we obtain

$$\overline{\nabla}_V V = 2(1 + (x^1 - x^2)^2)\{2(x^2 - x^1)\partial_2 + \sqrt{2}(x^2 - x^1)\partial_3 + \partial_4\},$$
$$\overline{\nabla}_{\xi_1} V = 0, \quad \overline{\nabla}_X \xi_1 = \overline{\nabla}_X N = 0, \quad \forall X \in \Gamma(TM).$$

For this example, the equations (4.11) reduce to

$$h^1(X, Y) = H^1\bar{g}(X, Y); \quad h^2(X, Y) = H^2\bar{g}(X, Y)$$

where h^1 and h^2 are $\Gamma(\mathrm{ltr}(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued bilinear forms (see equation (2.16)). Using the Gauss and Weingarten formulae we infer

$$h^1 = 0; \quad A_{\xi_1} = 0; \quad A_N = 0; \quad \nabla_X \xi_1 = 0; \quad \rho_i(X) = 0;$$

where for the symbol ρ_i see the equation (4.7). $h^2(X, \xi) = 0$;

$$H^2(V, V) = 2; \quad \nabla_X V = \frac{2\sqrt{2}(x^2 - x^1)^3}{1 + (x^1 - x^2)^2} X^2 V,$$

$\forall X = X^1\xi_1 + X^2 V \in \Gamma(TM)$. Since $\bar{g}(V, V) = -(1 + (x^1 - x^2)^4)$ we get

$$h^2(V, V) = H^2\bar{g}(V, V), \quad H^2 = -\frac{2}{(1 + (x^1 - x^2)^4)}.$$

Therefore, M is totally umbilical 1-lightlike submanifold of R_2^4.

Note that in case M is totally umbilical, then due to (2.27)

$$h^{\ell}(X, \xi) = 0, \quad h^s(X, \xi) = 0, \quad A^*_{\xi}\xi' = 0, \quad A_W \xi = 0. \tag{4.15}$$

THEOREM 4.2. *Let (M, g) be an m-dimensional totally umbilical lightlike submanifold of an $(m+n)$-dimensional semi-Riemannian manifold of constant curvature $(\overline{M}(\bar{c}), \bar{g})$. Then, the functions H_i^{ℓ}, H_{α}^s from* (4.11) *satisfy the following partial differential equations*

$$(4.16)\quad \begin{aligned}
&\xi_j(H_i^{\ell}) - H_i^{\ell} H_j^{\ell} + \sum_{k=1}^{r} H_k^{\ell}\rho_{ki}(\xi_j) = 0,\\
&\xi_j(H_{\alpha}^s) - H_{\alpha}^s H_j^{\ell} + \sum_{i=1}^{r} H_i^{\ell}\tau_{i\alpha}(\xi_j) + \sum_{\beta=r+1}^{n} H_{\beta}^s\theta_{\beta\alpha}(\xi_j) = 0,\\
&R(X, Y)Z = \left\{\bar{c}X + \sum_{i=1}^{r} H_i^{\ell} A_{N_i} X + \sum_{\alpha=r+1}^{n} H_{\alpha}^s A_{W_\alpha} X\right\} g(Y, Z)\\
&\qquad - \left\{\bar{c}Y + \sum_{i=1}^{r} H_i^{\ell} A_{N_i} Y + \sum_{\alpha=r+1}^{n} H_{\alpha}^s A_{W_\alpha} Y\right\} g(X, Z),
\end{aligned}$$

for any $X, Y \in \Gamma(TM)$. Moreover,

$$(4.17)\quad \begin{aligned}
&PX(H_k^{\ell}) + \sum_{i=1}^{r} H_i^{\ell}\rho_{ik}(PX) = 0,\\
&PX(H_{\alpha}^s) + \sum_{i=1}^{r} H_i^{\ell}\tau_{i\alpha}(PX) + \sum_{\alpha=r+1}^{n} H_{\beta}^s\theta_{\beta\alpha}(PX) = 0.
\end{aligned}$$

Proof. Taking account of (4.9) in (3.3) and (3.5), and using the fact that $\overline{M}$ is a space of constant curvature we obtain

$$\left\{X(H_k^\ell) - H_k^\ell \sum_{i=1}^r H_i^\ell \eta_i(X) + \sum_{i=1}^r H_i^\ell \rho_{ik}(X)\right\} g(Y, PU) - \left\{Y(H_k^\ell) - H_k^\ell \sum_{i=1}^r H_i^\ell \eta_i(Y) + \sum_{i=1}^r H_i^\ell \rho_{ik}(Y)\right\} g(X, PU) = 0,$$

$$\left\{X(H_\alpha^s) - H_\alpha^s \sum_{i=1}^r H_i^\ell \eta_i(X) + \sum_{i=1}^r H_i^\ell \tau_{i\alpha}(X) + \sum_{\beta=r+1}^n H_\beta^s \theta_{\beta\alpha}(X)\right\} g(Y, PU) - \left\{Y(H_\alpha^s) - H_\alpha^s \sum_{i=1}^r H_i^\ell \eta_i(Y) + \sum_{i=1}^r H_i^\ell \tau_{i\alpha}(Y) + \sum_{\beta=r+1}^n H_\beta^s \theta_{\beta\alpha}(Y)\right\} g(X, PU) = 0, \tag{4.18}$$

for any $X, Y, U \in \Gamma(TM)$. Take $X = \xi_j$ and $U = Y \in \Gamma(S(TM))$ such that $g(Y, Y) \neq 0$ on $\mathscr{U}$ and using (2.12) we obtain (4.16). Then, (4.17) follows from (3.1), (4.18), $\overline{M}$ a space of constant curvature and (4.16). Setting $X = PX$ and $Y = PY$ in (4.18) and using (2.12) we obtain

$$\left\{PX(H_k^\ell) + \sum_{i=1}^r H_i^\ell \rho_{ik}(PX)\right\} PY = \left\{PY(H_k^\ell) + \sum_{i=1}^r H_i^\ell \rho_{ik}(PY)\right\} PX,$$

$$\left\{PX(H_\alpha^s) + \sum_{i=1}^r H_i^\ell \tau_{i\alpha}(PX) + \sum_{\alpha=r+1}^n H_\beta^s \theta_{\beta\alpha}(PX)\right\} PY = \left\{PY(H_\alpha^s) + \sum_{i=1}^r H_i^\ell \tau_{i\alpha}(PY) + \sum_{\alpha=r+1}^n H_\beta^s \theta_{\beta\alpha}(PY)\right\} PX,$$

Now suppose there exists a vector field $X_o \in \Gamma(TM)$ such that

$$PX_o(H_k^\ell) + \sum_{i=1}^r H_i^\ell \rho_{ik}(PX_o) \neq 0,$$

$$PX_o(H_\alpha^s) + \sum_{i=1}^r H_i^\ell \tau_{i\alpha}(PX_o) + \sum_{\alpha=r+1}^n H_\beta^s \theta_{\beta\alpha}(PX_o) \neq 0$$

at each point $u \in M$. Then from the last equations it follows that all vectors from the fiber $(S(TM))_u$ are collinear with $(PX_o)_u$. This is a contradiction as $\dim((S(TM))_u) = n - r$. In particular, if $r = n$, that is, if $S(TM)$ vanishes, then also we have a trivial contradiction. Hence the equations (4.18) in theorem are true at any point of $\mathscr{U}$, which completes the proof.

From (3.6), (3.8), (3.10) and $\overline{M}$ of constant curvature we get

$$2d(\mathrm{Tr}(\rho_{ij}))(X, Y) + \sum_{i=1}^{r} H_i^{\ell}\{g(Y, A_{N_i}X) - g(X, A_{N_i}Y)\} = 0$$

where $\mathrm{Tr}(\rho_{ij})$ is the trace of the matrix (ρ_{ij}). If (M, g) is an isotropic or a totally light submanifold, then, we have $g(Y, A_{N_i}X) = g(X, A_{N_i}Y) = 0$ for every $X, Y \in \Gamma(TM)$. Thus, the following holds:

Lemma 1. *Let (M, g) be an isotropic or a totally lightlike submanifold of a semi-Riemannian manifold $(\overline{M}(\bar{c}), \bar{g})$ of constant curvature. Then, the trace of each ρ_{ij}, defined by* (4.6), *is closed, i.e.*, $d(\mathrm{Tr}(\rho_{ij})) = 0$.

In case $H_i^{\ell} \neq 0$ and $H_\alpha^s \neq 0$ on $\mathscr{U}$ we say that M is proper totally umbilical. From Theorem 2.2 and the last equation we obtain

Theorem 4.3. *Let $(M, g, S(TM))$ be a proper totally umbilical r-lightlike or a co-isotropic submanifold of a semi-Riemannian manifold $(\overline{M}(\bar{c}), \bar{g})$ of constant curvature $\bar{c}$. Then $S(TM)$ is integrable, if and only if, each* 1*-form* $\mathrm{Tr}(\rho_{ij})$ *induced by $S(TM)$ is closed, i.e.*, $d(\mathrm{Tr}(\rho_{ij})) = 0$.

Remark 3. In view of Lemma 1, $d(\mathrm{Tr}(\rho_{ij})) = 0$ trivially holds for a proper totally umbilical isotropic or a totally lightlike submanifold (M, g).

Definition 2. Let $(M, g, S(TM))$ be either an r-lightlike or a co-isotropic submanifold of a semi-Riemannian manifold $(\overline{M}, \bar{g})$. Then, the screen distribution $S(TM)$ is said to be totally umbilical in M if there is a smooth vector field $\mathscr{K} \in \Gamma(\mathrm{Rad}\, TM)$ on M, such that

$$h^*(X, PY) = \mathscr{K} g(X, PY) \quad \forall X, Y \in \Gamma(TM).$$

$S(TM)$ is totally umbilical, if and only if, on any coordinate neighborhood $\mathscr{U} \subset M$, there exist smooth functions K_i such that

$$h_i^*(X, PY) = K_i g(X, PY) \quad \forall X, Y \in \Gamma(TM). \tag{4.19}$$

It follows that h^* is symmetric on $\Gamma(S(TM))$ and hence from Theorem 2.2, $S(TM)$ is integrable. In case $\mathscr{K} = 0$ ($\mathscr{K} \neq 0$) on $\mathscr{U}$ we say that $S(TM)$ is totally geodesic (proper totally umbilical). (2.13) and (4.11) imply

$$P(A_{N_i}X) = K_i PX, \quad h^*(\xi, PX) = 0, \quad \forall X \in \Gamma(TM). \tag{4.20}$$

In case $S(TM)$ is totally umbilical, we have from (2.1), (2.24) and (4.20)

$$A_{N_i}X = K_i PX + \sum_{i\neq j=1}^{r} \eta_j(A_{N_i}X)\xi_j, \quad \eta_i(A_{N_j}X) = -\eta_j(A_{N_i}X). \tag{4.21}$$

Theorem 4.4. *Let $(M, g, S(TM))$ be either an r-lightlike or a co-isotropic submanifold of a semi-Riemannian manifold $(\overline{M}(\bar{c}), \bar{g})$ of constant curvature $\bar{c}$, with a totally umbilical screen distribution $S(TM)$. If M is also totally umbilical, then, the mean curvature vectors K_i of $S(TM)$ are a solution of the following partial differential equations*

$$X(K_i) - \sum_{j=1}^{r} K_j\rho_{ji}(X) - K_i \sum_{j=1}^{r} H_j^{\ell}\eta_j(X) + \sum_{j=1}^{r} H_j^{\ell}\eta_i(A_{N_j}X) + \sum_{\alpha=r+1}^{n} H_\alpha^s \eta_i(A_{W_\alpha X}) - \bar{c}\eta_i(X) = 0.$$

Proof. Taking account of (4.19) and (4.21) into (3.4) and using (2.12), (2.22), (2.24) and $\overline{M}$ a space of constant curvature we obtain

$$\begin{aligned}&\left\{X(K_i) - \sum_{j=1}^{r} K_j\rho_{ji}(X) - K_i \sum_{j=1}^{r} H_j^{\ell}\eta_j(X) + \sum_{j=1}^{r} H_j^{\ell}\eta_i(A_{N_j}X) + \sum_{\alpha=r+1}^{n} H_\alpha^s \eta_i(A_{W_\alpha X}) - \bar{c}\eta_i(X)\right\} g(Y, PU)\\ &= \left\{Y(K_i) - \sum_{j=1}^{r} K_j\rho_{ji}(Y) - K_i \sum_{j=1}^{r} H_j^{\ell}\eta_j(Y) + \sum_{j=1}^{r} H_j^{\ell}\eta_i(A_{N_j}Y) + \sum_{\alpha=r+1}^{n} H_\alpha^s \eta_i(A_{W_\alpha Y}) - \bar{c}\eta_i(Y)\right\} g(X, PU).\end{aligned}$$

Thus by the method of Theorem 4.2 we have the equation in theorem.

Corollary 1. *Let $(M, g, S(TM))$ be a totally umbilical co-isotropic lightlike submanifold of a semi-Riemannian manifold $(\overline{M}(\bar{c}), \bar{g})$. If $S(TM)$ is totally geodesic, then $\bar{c} = 0$, i.e., $\overline{M}$ is semi-Euclidean.*

Corollary 2. *Under the hypothesis of Corollary 1, ∇ is a metric connection on M, if and only if, the mean curvature vectors K_i of $S(TM)$ are a solution of the following partial differential equations*

$$X(K_i) - \sum_{j=1}^{r} K_j\rho_{ji}(X) - \bar{c}\eta_i(X) = 0.$$

By the method of Theorem 4.4, using (4.19) and (4.21) into (3.9), and then (2.12), (2.22), (2.24) and M a space of constant curvature, we obtain

THEOREM 4.5. *Let $(M(c), g, S(TM))$ be a totally umbilical r-lightlike or a co-isotropic submanifold of constant curvature c of a semi-Riemannian manifold $(\overline{M}, \bar{g})$. If $S(TM)$ is totally umbilical, then the mean curvature vectors K_i of $S(TM)$ are a solution of the partial differential equations*

$$X(K_i) - \sum_{j=1}^{r} K_j \rho_{ji}(X) - K_i \sum_{j=1}^{r} H_j^{\ell} \eta_j(X) - c\eta_i(X) = 0.$$

COROLLARY 3. *Let $(M(c), g, S(TM))$ be an r-lightlike or a co-isotropic submanifold of constant curvature c of a semi-Riemannian manifold $(\overline{M}, \bar{g})$. If $S(TM)$ is totally geodesic, then $c = 0$, i.e., M is semi-Euclidean.*

COROLLARY 4. *Under the hypothesis of Theorem 4.5, ∇ on M is metric connection, if and only if, the mean curvature vectors K_i of $S(TM)$ are a solution of the following partial differential equations*

$$X(K_i) - \sum_{j=1}^{r} K_j \rho_{ji}(X) - c\eta_i(X) = 0.$$

Using (3.6), the symmetries of the operators $A^*_{\xi_i}$ and Lemma 1, we have

THEOREM 4.6. *Let $(M(c), g)$ be a lightlike submanifold of constant curvature c of a semi-Riemannian manifold $(\overline{M}, \bar{g})$, such that $S(TM)$ is proper totally umbilical or $S(TM)$ vanishes. Then, $d(\mathrm{Tr}(\rho_{ij})) = 0$.*

Let $x \in M$ and ξ be a null vector of $T_x\overline{M}$. A plane Π of $T_x\overline{M}$ is called a null plane directed by ξ if it contains ξ, $\bar{g}_x(\xi, W) = 0$ for any $W \in \Pi$ and there exists $W_o \in \Pi$ such that $\bar{g}(W_o, W_o) \neq 0$. Following [2], define the null sectional curvature of Π with respect to ξ and $\overline{\nabla}$, as a real number

$$\overline{K}_{\xi}(\Pi) = \frac{\overline{R}(W, \xi, \xi, W)}{g(W, W)} \tag{4.22}$$

where W is an arbitrary non-null vector in Π. Similarly, define the null sectional curvature $K_{\xi}(\Pi)$ of the null plane Π of the tangent space T_xM with respect to ξ and ∇, as a real number

$$K_{\xi}(\Pi) = \frac{R(W, \xi, \xi, W)}{g(W, W)}. \tag{4.23}$$

Taking into account that both null sectional curvatures do not depend on the vector W and by using (3.3) and (3.7) we obtain

$$\overline{K}_{\xi}(\Pi_i) = \xi_i(H_i^{\ell}) - (H_i^{\ell})^2 + \sum_{k=1}^{r} H_k^{\ell}\rho_{ki}(\xi_i) = K_{\xi}(\Pi_i), \tag{4.24}$$

where Π_i is a null plane directed by ξ_i. Thus we have

THEOREM 4.7. *Let (M, g) be a totally umbilical lightlike submanifold of a semi-Riemannian manifold $(\overline{M}, \bar{g})$. Then, both the null sectional curvature functions $\overline{K}_{\xi}(\Pi_i)$ and $K_{\xi}(\Pi_i)$ vanish, if and only if, H_i^{ℓ} is a solution of the partial differential equation*

$$\xi_i(H_i^{\ell}) - (H_i^{\ell})^2 + \sum_{k=1}^{r} H_k^{\ell}\rho_{ki}(\xi_i) = 0.$$

From the equation (4.16) in Theorem 4.2 and Theorem 4.7, we obtain

THEOREM 4.8. *Let (M, g) be a totally umbilical lightlike submanifold of a semi-Riemannian manifold of constant curvature $(\overline{M}, \bar{g})$. Then, both the null sectional curvature functions $\overline{K}_{\xi}(\Pi_i)$ and $K_{\xi}(\Pi_i)$ vanish.*

5. Induced Ricci tensor

Consider an m-dimensional lightlike submanifold (M, g) of an $(m+n)$-dimensional semi-Riemannian manifold $(\overline{M}, \bar{g})$. Note that h_i^{ℓ}, ρ_{ij} and $\tau_{i\alpha}$ depend on the section $\xi \in \Gamma(\mathrm{Rad}\, TM)$. Indeed, take $\xi_i^* = \sum_{j=1}^{r} \alpha_{ij}\xi_j$, where α_{ij} are smooth functions with $\Delta = \det(\alpha_{ij}) \neq 0$ and A_{ij} be the co-factors of α_{ij} in the determinant of Δ. It follows that $N_i^* = (1/\Delta)\sum_{j=1}^{r} A_{ij}N_j$. Hence by straightforward calculation and using (4.1)–(4.4) and (4.6) we obtain $h_i^{\ell *} = \sum_{j=1}^{r} \alpha_{ij}h_j^{\ell}$. Denote ρ_{ij}^* and $\tau_{i\alpha}^*$ by affinely combinations of ρ_{ij} and $\tau_{i\alpha}$ with coefficients α_{ij}, A_{ij} and $X(A_{ij})$. Moreover,

$$\mathrm{Tr}(\rho_{ij})(X) = \mathrm{Tr}(\rho_{ij}^*)(X) + X(\log \Delta), \quad \forall X \in \Gamma(TM).$$

Thus, using the formula $d\rho(X, Y) = (1/2)\{X(\rho(Y)) - Y(\rho(X)) - \rho([X, Y])\}$ of a differential 2-form, we obtain

THEOREM 5.1. *Let (M, g) be a lightlike submanifold of a semi-Riemannian manifold $(\overline{M}, \bar{g})$. Suppose $\mathrm{Tr}(\rho_{ij})$ and $\mathrm{Tr}(\rho_{ij}^*)$ are 1-forms on $\mathscr{U}$ with respect to ξ_i and ξ_i^*. Then $d(\mathrm{Tr}(\rho_{ij}^*)) = d(\mathrm{Tr}(\rho_{ij}))$ on $\mathscr{U}$.*

To find local expression of Ricci tensor of M, consider the frames field

$$\{\xi_1, \ldots, \xi_r; N_1, \ldots, N_r; X_{r+1}, \ldots, X_m; W_{r+1}, \ldots, W_n\}$$

on $\overline{M}$. Denote by $\{F_A\} = \{\xi_1, \ldots, \xi_r, X_{r+1}, \ldots, X_m\}$ the induced frames field on M. Then,

$$\bar{R}_{ABCD} = \bar{g}(\bar{R}(F_D, F_C)F_B, F_A), \quad R_{ABCD} = g(R(F_D, F_C)F_B, F_A),$$
$$\bar{R}_{iBCD} = \bar{g}(\bar{R}(F_D, F_C)F_B, N_i), \quad R_{iBCD} = \bar{g}(R(F_D, F_C)F_B, N_i),$$
$$\bar{R}_{\alpha BCD} = \bar{g}(\bar{R}(F_D, F_C)F_B, W_\alpha), \quad R_{\alpha BCD} = \bar{g}(R(F_D, F_C)F_B, W_\alpha),$$
$$\bar{R}_{i\alpha CD} = \bar{g}(\bar{R}(F_D, F_C)W_\alpha, N_i), \quad R_{i\alpha CD} = \bar{g}(R(F_D, F_C)W_\alpha, N_i).$$

Using above we obtain the following local expression for the Ricci tensor:

$$\mathrm{Ric}(X, Y) = \sum_{a,b=r+1}^{m} g^{ab} g(R(X, X_a)Y, X_b) + \sum_{i=1}^{r} \bar{g}(R(X, \xi_i)Y, N_i).$$

By using the symmetries of curvature tensor and the first Bianchi identity and taking into account (3.2) and (3.9) we obtain

$$\begin{aligned}
&\mathrm{Ric}(X, Y) - \mathrm{Ric}(Y, X) \\
&\quad = \sum_{a,b=r+1}^{m} g^{ab}\{\bar{g}(h^*(X, X_b), h^\ell(Y, X_a)) - \bar{g}(h^*(Y, X_b), h^\ell(X, X_a))\} \\
&\qquad + \sum_{i=1}^{r}\{g(A^*_{\xi_i}X, A_{N_i}Y) - g(A^*_{\xi_i}Y, A_{N_i}X) + \bar{g}(R^{*t}(X, Y)\xi_i, N_i)\}.
\end{aligned}$$

Replacing X, Y by X_A, X_B respectively, using (2.27), (2.28), (4.9) and

$$\begin{aligned}
\sum_{i=1}^{r} \bar{g}(R^{*t}(X, Y)\xi_i, N_i) &= -2\sum_{i,j=1}^{r} \bar{g}(d(\rho_{ji})(X, Y)\xi_j, N_i) \\
&= -2d(\mathrm{Tr}(\rho_{ij})(X, Y)),
\end{aligned}$$

we have

$$R_{AB} - R_{BA} = 2d(\mathrm{Tr}(\rho_{ij}))(X_A, X_B)$$

where $R_{AB} = \mathrm{Ric}(X_B, X_A)$. Thus, using Theorem 5.1, we conclude

THEOREM 5.2. *Let $(M, g, S(TM))$ be an r-lightlike or a co-isotropic submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the Ricci tensor of the induced connection ∇ on M is symmetric, if and only if, each 1-forms $\mathrm{Tr}(\rho_{ij})$ induced by $S(TM)$ is closed, i.e., on any $\mathcal{U} \subset M$,*

$$d(\mathrm{Tr}(\rho_{ij})) = 0.$$

Using Theorems 4.3, 4.6 and 5.2, we obtain the following theorem:

THEOREM 5.3. *Let $(M, g, S(TM))$ be a proper totally umbilical r-lightlike or a co-isotropic submanifold of a semi-Riemannian manifold $(\bar{M}(\bar{c}), \bar{g})$ of a constant curvature $\bar{c}$. Then, the induced Ricci tensor on M is symmetric, if and only if its screen distribution $S(TM)$ is integrable.*

COROLLARY 5. *Let $(M(c), g, S(TM))$ be an r-lightlike or a co-isotropic submanifold of constant curvature c of a semi-Riemannian manifold $(\overline{M}, \bar{g})$, such that $S(TM)$ is proper totally umbilical. Then the Ricci tensor of the induced connection ∇ on M is symmetric.*

Suppose the Ricci tensor of ∇ is symmetric. Theorem 5.3 and Poincare lemma implies $\mathrm{Tr}(\rho_{ij}(X)) = X(f)$, where f is a smooth function. Let $\triangle = \exp f$ and obtain $\mathrm{Tr}(\rho_{ij}^*(X)) = 0 \ \forall X \in \Gamma(TM|_{\mathscr{U}})$. Thus we have

THEOREM 5.4. *Let $(M, g, S(TM))$ be an r-lightlike or a co-isotropic submanifold of a semi-Riemannian manifold $(\overline{M}, \bar{g})$. Then Ricci tensor of M is symmetric and there exists a pair of frames field $\{\xi_i^*, N_i^*\}$ on $\mathscr{U}$ such that the corresponding 1-forms $\mathrm{Tr}(\rho_{ij}^*)$ induced by $S(TM)$ vanishes.*

Remark 4. Lemma 1 and Theorem 5.2 imply that the induced Ricci tensor of either an isotropic or a totally lightlike M of $\overline{M}(\bar{c})$ is always symmetric. This clarifies the fact that Theorems 5.1–5.4 will trivially hold for an isotropic or a totally lightlike M, since for these two cases $d(\mathrm{Tr}(\rho_{ij})) = 0$.

Examples. Minkowski [3], de Sitter [1], Schwarzchild and Robertson-Walker spacetimes (see [10] and pages 225–230 of [8]) all have lightlike hypersurfaces with an integrable 2-dimensional screen distribution.

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