Czechoslovak Mathematical Journal

P. M. Gadea; A. Montesinos Amilibia

Totally umbilical pseudo-Riemannian submanifolds of the paracomplex projective space

Czechoslovak Mathematical Journal, Vol. 44 (1994), No. 4, 741–756

Persistent URL: http://dml.cz/dmlcz/128493

Czechoslovak Mathematical Journal, 44 (119) 1994, Praha

TOTALLY UMBILICAL PSEUDO-RIEMANNIAN SUBMANIFOLDS OF THE PARACOMPLEX PROJECTIVE SPACE*

P. M. Gadea, Madrid and A. Montesinos Amilibia, Burjasot

(Received December 31, 1992)

1. Introduction

Para-Kaehlerian manifolds were introduced by Rasevskii [14] and Libermann [12], and studied by several authors (see Bejan [2] and the long list of references therein). An interesting class of para-Kaehlerian manifolds is the class of para-Hermitian symmetric spaces. Kaneyuki and Kozai [10] gave the infinitesimal classification in the case of semisimple group. A particular type is given by the paracomplex projective spaces, introduced by the authors in [4]. These spaces are harmonic symmetric spaces ([1], [5], [6]), and models of spaces of constant non vanishing paraholomorphic sectional curvature, which have a rich family of para-Kaehlerian space forms ([4], [8], [9]). These spaces have also been studied in [2] and [7].

Totally umbilical submanifolds of a given manifold, provided they exist, constitute one of the most natural and useful families of submanifolds. They are known for several classes of important manifolds (see Chen [3]). In the present paper we determine all of the totally umbilical pseudo-Riemannian submanifolds of the paracomplex projective spaces. Let $P(E\oplus E^*)$ be the paracomplex projective space naturally associated to the finite dimensional real vector space E. We prove that its non totally geodesic, totally umbilical pseudo-Riemannian submanifolds are of constant (ordinary) sectional curvature. In fact, if h is any non-degenerate symmetric bilinear form in E and $S_h = \{x \in E\colon h(x,x) = 1\}$ is the corresponding sphere, then S_h can be isometrically immersed as a totally geodesic submanifold of $P(E\oplus E^*)$ (cf. [7]). We prove that the *parallels* of S_h, that is its intersections with affine subspaces of E, are then isometrically immersed as totally umbilical submanifolds of $P(E\oplus E^*)$, and

* Work partially supported by the DGICYT (Spain) grants n. PB 89-0004 and PB 90-0014-C03-01.

that every non totally geodesic, totally umbilical pseudo-Riemannian submanifold of $P(E \oplus E^*)$ of dimension greater that 1 is part of such an immersed parallel.

2. Preliminaries

Let E be an $(r+1)$-dimensional real vector space, and E^* its dual. Typically, we shall write $x + \alpha$ to denote an element of $E \oplus E^*$. On the space $E \oplus E^*$ there exist a natural non-degenerate bilinear form $\langle\ ,\ \rangle$ given by

$$\langle x + \alpha, y + \beta \rangle = \tfrac{1}{2}\big(\alpha(y) + \beta(x)\big),$$

and a linear automorphism J such that

$$J\big|_E = \mathrm{id}_E, \quad J\big|_{E^*} = -\,\mathrm{id}_{E^*}.$$

We introduce in

$$(E \oplus E^*)_+ = \big\{x + \alpha \in E \oplus E^* : \langle x + \alpha, x + \alpha \rangle = \alpha(x) > 0\big\}$$

the equivalence relation $\sim$ such that $x + \alpha \sim ax + b\alpha$ whenever $0 < a, b \in \mathbb{R}$, and define the paracomplex projective space $P(E \oplus E^*)$ by

$$P(E \oplus E^*) = (E \oplus E^*)_+ / \sim .$$

Let p denote the natural projection $p\colon (E \oplus E^*)_+ \to P(E \oplus E^*)$. We define the vector fields $\mathbf{n}$, $\mathbf{v}$ in $E \oplus E^*$ by $\mathbf{n}_{x+\alpha} = x + \alpha$, $\mathbf{v}_{x+\alpha} = x - \alpha$, so that $J\mathbf{n} = \mathbf{v}$. The pseudosphere in $E \oplus E^*$ is defined as

$$S = \big\{x + \alpha \in (E \oplus E^*)_+ : \langle x + \alpha, x + \alpha \rangle = \alpha(x) = 1\big\}.$$

Then $\mathbf{n}$ is the unit normal to S. We have a principal bundle $p\colon S \to P(E \oplus E^*)$ with group $\mathbb{R}^+$. This group acts on the right upon S by $(x + \alpha)a = ax + a^{-1}\alpha$, for $a \in \mathbb{R}^+$. If S is given the pseudo-Riemannian metric induced by that of $E \oplus E^*$, then $\mathbb{R}^+$ acts on S by isometries. Thus, it induces a pseudo-Riemannian metric g on $P(E \oplus E^*)$ so that p is a pseudo-Riemannian submersion. The vector field $\mathbf{v}$, when restricted to S is parallel to the fibres of p. Therefore, a vector tangent to S is *p-horizontal* iff it is orthogonal to $\mathbf{v}$. Also, J passes to the quotient and gives an almost product structure J on $P(E \oplus E^*)$ such that $J^2 = 1$ and $g(JX, Y) = -g(X, JY)$. If $\widetilde{\nabla}$ is the Levi-Civita connection on $P(E \oplus E^*)$, then $\widetilde{\nabla} J = 0$. Thus $P(E \oplus E^*)$ is a para-Kaehlerian manifold, and if $r > 1$ it is simply connected. Also, it has constant para-holomorphic sectional curvature (equal to 4) [4], that is the Riemann-Christoffel tensor field is given by

$$(1) \quad \widetilde{R}(X,Y,Z,W) = g(X,Z)g(Y,W) - g(X,W)g(Y,Z) - g(X,JZ)g(Y,JW) + g(X,JW)g(Y,JZ) - 2g(X,JY)g(Z,JW).$$

where we define the Riemann-Christoffel tensor field by

$$R(X,Y,Z,W) = g\big(R(X,Y)Z,W\big)$$

and the curvature operator by $R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$.

We shall study (regular) pseudo-Riemannian submanifolds of $P(E \oplus E^*)$, that is imbedded submanifolds $i\colon V \to P(E \oplus E^*)$ such that i^*g is non-degenerate. Let $1 < s = \dim V$. If $m \in V$ then we shall put

$$\mathscr{N}_m = (T_m V)^{\perp}, \quad \mathscr{N} = \bigcup_{m \in V} \mathscr{N}_m.$$

Thus $T_m P(E \oplus E^*) = T_m V \perp \mathscr{N}_m$, and we shall denote by τ and ν the corresponding projectors to $T_m V$ and $\mathscr{N}_m$. Let $P = \tau \circ J$, $Q = \nu \circ J$. Then if $X, Y \in \mathscr{X}(V)$ and $\eta, \mu \in \Gamma(\mathscr{N})$ we have $g(X,PY) = -g(PX,Y)$, $g(Q\eta,\mu) = -g(\eta,Q\mu)$, and if ∇ denotes the Levi-Civita connection on V we put

$$\nabla_X Y = \tau\widetilde{\nabla}_X Y, \quad \alpha(X,Y) = \nu\widetilde{\nabla}_X Y,$$
$$A_\eta X = -\tau\widetilde{\nabla}_X \eta, \quad D_X \eta = \nu\widetilde{\nabla}_X \eta.$$

We have

$$g(A_\eta X, Y) = g\big(\alpha(X,Y),\eta\big).$$

We say that V is *totally umbilical* iff there exists $\xi \in \Gamma(\mathscr{N})$ such that

$$(2) \qquad \alpha(X,Y) = g(X,Y)\xi$$

for every $X, Y \in \mathscr{X}(V)$. Then, ξ is called the *normal curvature vector field.*

3. Totally umbilical submanifolds of $P(E \oplus E^*)$ either are totally geodesic or have constant curvature

In the following, V will be a totally umbilical pseudo-Riemannian submanifold of $P(E \oplus E^*)$ with normal curvature vector field ξ. Let $X, Y, Z \in \mathscr{X}(V)$. Codazzi's equation [11, Vol. II, p. 25] reads

$$-\nu \widetilde{R}(X,Y)Z = (\hat{\nabla}_X \alpha)(Y,Z) - (\hat{\nabla}_Y \alpha)(X,Z),$$

where $\hat{\nabla}\alpha$ is defined by

$$(\hat{\nabla}_X \alpha)(Y,Z) = D_X\big(\alpha(Y,Z)\big) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z).$$

Having in mind (2), that is

$$(\hat{\nabla}_X \alpha)(Y,Z) = D_X\big(g(Y,Z)\xi\big) - g(\nabla_X Y, Z)\xi - g(Y, \nabla_X Z)\xi = g(Y,Z) D_X \xi.$$

Then, Codazzi's equation is

$$\begin{aligned} g(X,PZ)g(Y,P\eta) - g(Y,PZ)g(X,P\eta) + 2g(X,PY)g(Z,P\eta) \\ = g(Y,Z)g(D_X\xi,\eta) - g(X,Z)g(D_Y\xi,\eta), \end{aligned} \tag{3}$$

where $\eta \in \Gamma(\mathscr{N})$.

Let R_D be the curvature of the connection D in $\mathscr{N}$. Then Ricci's equation [15, Vol. 4, p. 60] is

$$\nu \widetilde{R}(X,Y)\eta = R_D(X,Y)\eta - \alpha(A_\eta X, Y) + \alpha(A_\eta Y, X).$$

Since $g(A_\eta X, Y) = g\big(\alpha(X,Y),\eta\big) = g(X,Y)g(\xi,\eta)$, we have $A_\eta X = g(\xi,\eta)X$ and $\alpha(A_\eta X, Y) = g(\xi,\eta)g(X,Y)\xi$. Ricci's equation reduces thus to

$$\nu \widetilde{R}(X,Y)\eta = R_D(X,Y)\eta. \tag{4}$$

We take the trace of (3) in the arguments X, Z. Let $\{e_i\}$ be a g-orthonormal local reference for V, in the sense that $e_i \in \mathscr{X}(U)$, $U \subset V$, $g(e_i,e_j) = \varepsilon_i \delta_{ij}$, $\varepsilon_i = \pm 1$. Then

$$\begin{aligned} 0 &= \sum_{i=1}^{s} \varepsilon_i \Big(g(e_i,Pe_i)g(Y,P\eta) - g(Y,Pe_i)g(e_i,P\eta) + 2g(e_i,PY)g(e_i,P\eta) \\ &\qquad - g(Y,e_i)g(D_{e_i}\xi,\eta) + g(e_i,e_i)g(D_Y\xi,\eta)\Big) \\ &= (s-1)g(D_Y\xi,\eta) - 3g(QPY,\eta). \end{aligned}$$

Since $g\big|_{\mathcal{N}}$ is non-degenerate and $\eta \in \Gamma(\mathcal{N})$ is arbitrary, we conclude that

$$D_Y \xi = \frac{3}{s-1} QPY. \tag{5}$$

If we bring (5) to (3), we get

$$\begin{aligned} &g(X, PZ)g(Y, P\eta) - g(Y, PZ)g(X, P\eta) + 2g(X, PY)g(Z, P\eta) \\ &\qquad + \frac{3}{s-1}\big(g(Y, Z)g(PX, P\eta) - g(X, Z)g(PY, P\eta)\big) = 0. \end{aligned} \tag{6}$$

If we put $Y = Z$, then

$$g(X, PZ)g(Z, P\eta) + \frac{1}{s-1}\big(g(Z, Z)g(PX, P\eta) - g(X, Z)g(PZ, P\eta)\big) = 0. \tag{7}$$

Since X is arbitrary and i^*g is non-degenerate, we have

$$g(Z, P_\eta)PZ - \frac{1}{s-1} g(Z, Z)P^2\eta - \frac{1}{s-1} g(PZ, P\eta)Z = 0.$$

Finally, we put $Z = P\eta$, and have

$$(s-2)g(P\eta, P\eta)P^2\eta = 0 \tag{8}$$

for any $\eta \in \Gamma(\mathcal{N})$. Thus, it is clear that we must separate the case $s = 2$ from the others. Assume first that $s > 2$. Then, (8) reads $g(P\eta, P\eta)P^2\eta = 0$ for any $\eta \in \Gamma(\mathcal{N})$. Assume that we have chosen such a field η and that in some open subset U of the submanifold V we have $P^2\eta \neq 0$. Then $g(P\eta, P\eta) = 0$ in U. Putting $Y = P\eta$ in (6) we obtain

$$g(P^2\eta, Z)g(P\eta, X) + \frac{2s-5}{s-1} g(P\eta, Z)g(P^2\eta, X) = 0.$$

Since X, Z are arbitrary, we conclude that

$$P^2\eta \otimes P\eta + \frac{2s-5}{s-1} P_\eta \otimes P^2\eta = 0.$$

This implies that $P\eta$ and $P^2\eta$ are linearly dependent, but this is absurd because $1 + (2s-5)/(s-1) = 3(s-2)/(s-1) \neq 0$ and $P^2\eta \neq 0$. Therefore we have proved that $P^2\eta = 0$ for every $\eta \in \Gamma(\mathcal{N})$. Then, by (7) we have $g(P\eta, Z)g(PX, Z) = 0$, and by polarization $g(P\eta, Y)g(PX, Z) + g(P\eta, Z)g(PX, Y) = 0$, from which

$$P\eta \otimes PX + PX \otimes P\eta = 0. \tag{9}$$

Lemma 1. *Let V be a totally umbilical pseudo-Riemannian submanifold of $P(E \oplus E^*)$ with $s = \dim V > 2$ and let ξ be its normal curvature vector field. Let $X, Y, Z \in \mathscr{X}(V)$ and $\eta \in \Gamma(\mathscr{N})$. Then:*

(i) $\nu \widetilde{R}(X,Y)Z = 0$;

(ii) $D_X \xi = 0$;

(iii) $\widetilde{R}(X, Y, \eta, \xi) = 0$.

Proof. From (9) we see that at each point $m \in V$ we have that either $P(T_m V) = 0$ or $P(\mathscr{N}_m) = 0$. Then if we multiply (5) by η we have

$$g(D_Y \xi, \eta) = -\frac{3}{s-1} g(PY, P\eta) = 0,$$

and (ii) follows. Then the right hand side of Codazzi's equation vanishes identically and this is (i). From (ii) we have $R_D(X,Y)\xi = 0$. Hence, by (4) we have (iii). □

Assume now that $s = \dim V = 2$. Let $m \in V$ and let v_m, w_m be an orthonormal base of $T_m V$, that is $g(v_m, v_m) = a$, $g(w_m, w_m) = b$, $g(v_m, w_m) = 0$, $a^2 = b^2 = 1$. For u in a neighborhood of 0, let $\gamma(u)$ be the geodesic in V with initial condition (m, w_m). Let $v(u)$ be the V-parallel displacement of v_m along γ. Let $t \mapsto \varphi(t,u)$ be the geodesic in V with initial condition $(\gamma(u), v(u))$. We thus have a local chart $(t,u) \mapsto \varphi(t,u)$ of V defined in a neighborhood of $0 \in \mathbb{R}^2$. We define two local vector fields v, w as follows: if $m_1 = \varphi(t_1, u_1)$, then we put

$$v_{m_1} = \left.\frac{\partial \varphi}{\partial t}\right|_{(t_1, u_1)}$$

and w_{m_1} is defined as the V-parallel displacement of $\dot{\gamma}(u_1)$ along the curve $t \mapsto \varphi(t, u_1)$ up to the point m_1. By this construction, it is clear that $g(v,v) = a$, $g(w,w) = b$, $g(v,w) = 0$, and that

$$\nabla_v v = 0, \quad \nabla_v w = 0, \quad (\nabla_w v) \circ \gamma = 0, \quad (\nabla_w w) \circ \gamma = 0.$$

Let us call $f = g(v, Jw)$. Then

$$\begin{aligned}
QPv &= Q(\tau J v) = Q\big(ag(v, Jv)v + bg(w, Jv)w\big) \\
&= -bfQw = -bf\big(Jw - ag(v, Jw)v\big) = bf(afv - Jw), \\
QPw &= af(Jv + bfw), \\
\widetilde{\nabla}_v v &= \nabla_v v + \alpha(v,v) = a\xi, \quad \widetilde{\nabla}_v w = g(v,w)\xi = 0, \\
\widetilde{\nabla}_v \xi &= -A_\xi v + D_v \xi = -g(\xi,\xi)v + 3QPv = -g(\xi,\xi)v + 3bf(afv - Jw), \\
\widetilde{\nabla}_w \xi &= -g(\xi,\xi)w + 3af(Jv + bfw), \\
(\widetilde{\nabla}_w w) \circ \gamma &= b\xi \circ \gamma, \quad (\widetilde{\nabla}_w v) \circ \gamma = 0, \\
v(f) &= \widetilde{\nabla}_v g(v, Jw) = ag(\xi, Jw), \quad w(f) \circ \gamma = bg(v, J\xi) \circ \gamma.
\end{aligned}$$

Thus, as computation shows,

$$\big(\widetilde{R}(v,w)\xi\big)\circ\gamma = \Big(-3g(v,J\xi)Jw + 3g(w,J\xi)Jv - 6g(v,Jw)J\xi + 12f\big(ag(J\xi,v)v + bg(J\xi,w)w\big)\Big)\circ\gamma,$$

whereas by (1) we have

$$\widetilde{R}(v,w)\xi = g(v,J\xi)Jw - g(w,J\xi)Jv + 2g(v,Jw)J\xi.$$

Therefore

$$\Big(g(J\xi,w)Jv - g(J\xi,v)Jw - 2g(v,Jw)J\xi + 3f\big(ag(J\xi,v)v + bg(J\xi,w)w\big)\Big)\circ\gamma = 0.$$

If we apply J and then make the inner product by v we have along γ:

$$ag(J\xi,w) + 3bfg(J\xi,w)g(v,Jw) = g(J\xi,w)(a+3bf^2) = 0.$$

Assume that $g(J\xi,w)_m \neq 0$. Then, $f\circ\gamma$ is constant in a neighborhood of 0 and equal to $\sqrt{-\frac{1}{3}ab}$. But then, by the preceding formulae, we would have $d(f\circ\gamma)/du = w(f)\circ\gamma = bg(v,J\xi)\circ\gamma = 0$ in that neighborhood. In particular, $g(J\xi,v)_m = 0$. Then $P\xi_m = bg(J\xi,w)_m w_m$. Since f is real we have that $-ab$ is positive, so that $a=-b$. Let c be an arbitrary real number and put $v'_m = v_m\cosh c + w_m\sinh c$, $w'_m = v_m\sinh c + w_m\cosh c$. Then $g(v'_m,v'_m) = a$, $g(w'_m,w'_m) = b$, $g(v'_m,w'_m) = 0$, so that we have another orthonormal base of T_mV. Then $P\xi_m = ag(J\xi_m,v'_m)v'_m + bg(J\xi_m,w'_m)w'_m = g(J\xi,w)_m(v'_m a\sinh c + w'_m b\cosh c)$. If $c\neq 0$ we have an orthonormal base of T_mV on which both components of $P\xi_m$ are non-zero. Since the whole construction could have been done starting from the new base, we have reached a contradiction. We conclude that $g(J\xi,w)_m = g(J\xi,v)_m = 0$ and as a consequence, if $\xi_m\neq 0$ one has moreover $g(v,Jw)_m = 0$. Since m is arbitrary, the same holds in the whole V. Then, if $\xi\neq 0$, we have $f=0$, $D\xi=0$, $J(TV)\subset\mathscr{N}$, $J\xi\in\Gamma(\mathscr{N})$, $\nu\widetilde{R}(X,Y)Z = 0$, $\widetilde{R}(X,Y,\eta,\xi) = 0$ and $g(\xi,\xi)$ is constant.

Theorem 2. *Let V be a connected totally umbilical pseudo-Riemannian submanifold of $P(E\oplus E^*)$ with $\dim V > 1$ and let $\mathscr{N}$ be the bundle orthogonal to TV. Then, either V is totally geodesic or $J(TV)\subset\mathscr{N}$ and in this case V is a pseudo-Riemannian manifold with constant sectional curvature.*

Proof. Let $s>2$. Then, we put

$$\mathscr{A} = \big\{m\in V\colon (P\circ\nu)\big|_{T_mP(E\oplus E^*)} = 0\big\},\quad \mathscr{B} = \big\{m\in V\colon (P\circ\tau)\big|_{T_mP(E\oplus E^*)} = 0\big\}.$$

Clearly, these subsets are closed in V. By (9), $\mathscr{A} \cup \mathscr{B} = V$. If $m \in \mathscr{A} \cap \mathscr{B}$, then $P = \tau \circ J = 0$ on $T_m P(E \oplus E^*)$, and this is absurd because J is an isomorphism. Then $\mathscr{A} \cap \mathscr{B} = \emptyset$, and therefore either $\mathscr{A} = V$ or $\mathscr{B} = V$. Assume that $\mathscr{A} = V$. Then, by (1) and Lemma 1, (iii) we have

$$\widetilde{R}(X, Y, \eta, \xi) = -2g(X, JY)g(\eta, J\xi) = 2g(X, JY)g(J\eta, \xi) = 0. \tag{10}$$

Now $g(J\eta, X) = g(P\eta, X) = 0$, whence $J(\mathscr{N}) \subset \mathscr{N}$. Then, applying (10) to $J\eta$ instead of η, and having in mind that X, Y are arbitrary, we conclude that $g(\eta, \xi) = 0$, that is $\xi = 0$, and so V is totally geodesic.

Thus, assume that $\mathscr{B} = V$. Then $J(TV) \subset \mathscr{N}$. By Gauss' equation we have directly

$$\begin{aligned} R(X, Y, Z, W) &= \widetilde{R}(X, Y, Z, W) + g\big(\alpha(X, Z), \alpha(Y, W)\big) - g\big(\alpha(Y, Z), \alpha(X, W)\big) \\ &= (1 + l)\big(g(X, Z)g(Y, W) - g(Y, Z)g(X, W)\big), \end{aligned}$$

where $l = g(\xi, \xi)$, which by Lemma 1, (ii), is a constant. The same results hold obviously when $s = 2$. □

4. Parallels as totally umbilical submanifolds of $P(E \oplus E^*)$

Let F, Λ be subspaces of E and E^*, respectively, such that the pairing $F \times \Lambda \to \mathbb{R}$ given by $(x, \alpha) \mapsto \alpha(x)$ is non-degenerate. Let $f \colon F \to \Lambda$ be an isomorphism such that $f(x, y) \equiv f(x)(y) = f(y, x)$ for any $x, y \in F$. We shall use the following notation

$$F^\perp = \{\alpha \in E^* \colon \alpha(x) = 0, \text{ if } x \in F\}, \quad \Lambda^\perp = \{x \in E \colon \alpha(x) = 0, \text{ if } \alpha \in \Lambda\}.$$

We put

$$\Sigma = \{x \in F \colon f(x, x) = a\}, \quad 0 \neq a \in \mathbb{R},$$

and consider it as a pseudo-Riemannian sphere defined by the pseudo-Riemannian metric f on F. Let $x_0 + \alpha_0$ be some fixed element of $E \oplus E^*$ such that

$$\alpha_0 \in F^\perp, \quad x_0 \in \Lambda^\perp, \quad \alpha_0(x_0) + a = 1. \tag{11}$$

We map F into $E \oplus E^*$ by means of $j \colon F \to E \oplus E^*$ defined by

$$j(x) = x + x_0 + f(x) + \alpha_0.$$

It is clear that since $j_*(X) = X + f(X)$, j is an isometry. Let $x \in \Sigma$; then $\langle j(x), j(x)\rangle = f(x, x + x_0) + \alpha_0(x + x_0) = a + \alpha_0(x_0) = 1$. Thus, $j(\Sigma) \subset S$. Also, if $X \in T_x\Sigma$ we have

$$\begin{aligned}\langle j_*(X), \mathbf{v}_{j(x)}\rangle &= \langle X + f(X), x + x_0 - f(x) - \alpha_0\rangle \\ &= \tfrac{1}{2}\big(f(X, x + x_0) - f(x, X) - \alpha_0(X)\big) = 0\end{aligned}$$

because $X \in F$. Therefore, $j_*(X)$ is p-horizontal and, as a consequence, $p \circ j\colon \Sigma \to P(E \oplus E^*)$ is an isometry. Let us prove that $V = p\big(j(\Sigma)\big)$ is a totally umbilical submanifold of $P(E \oplus E^*)$.

Let $\widetilde{X} \in \mathscr{X}\big(j(\Sigma)\big)$. Then $\widetilde{X}$ is p-horizontal and there are fields $X \in \mathscr{X}(V)$, $\hat{X} \in \mathscr{X}(\Sigma)$ such that

$$j_* \circ \hat{X} = \widetilde{X} \circ j, \quad p_* \circ \widetilde{X} = X \circ p, \quad j(\hat{X}, \hat{X}) = \langle \widetilde{X}, \widetilde{X}\rangle \circ j = g(X, X) \circ p \circ j.$$

We shall also consider fields $\hat{Y}$, $\widetilde{Y}$, Y with the analogous properties. We denote by $\hat{X}(\hat{Y})$ and $\widetilde{X}(\widetilde{Y})$ the canonical covariant derivative in E and in $E \oplus E^*$. Let ∇^Σ, ∇^S, $\widetilde{\nabla}$, ∇ be the Levi-Civita connections in Σ, S, $P(E \oplus E^*)$ and V, respectively. We have

$$\nabla^S_{\widetilde{X}}\widetilde{Y} = \widetilde{X}(\widetilde{Y}) + \langle \widetilde{X}, \widetilde{Y}\rangle \mathbf{n}.$$

Also, $\langle \widetilde{X}(\widetilde{Y}), \mathbf{v}\rangle \circ j = -\langle \widetilde{Y}, \widetilde{X}(v)\rangle \circ j = -\langle \widetilde{Y}, J\widetilde{X}\rangle \circ j = -\langle \hat{Y} + f(\hat{Y}), \hat{X} - f(\hat{X})\rangle = -\frac{1}{2}\big(f(\hat{Y}, \hat{X}) - f(\hat{X}, \hat{Y})\big) = 0$. Since $\mathbf{n}$ is also orthogonal to $\mathbf{v}$, we have that $\nabla^S_{\widetilde{X}}\widetilde{Y}$ is p-horizontal. Let $x(t) \in \Sigma$ be an integral curve of $\hat{X}$; then, $j\big(x(t)\big)$ is an integral curve of $\widetilde{X}$. If $x = x(0)$, then

$$\begin{aligned}\big(\widetilde{X}(\widetilde{Y})\big)_{j(x)} &= \frac{d}{dt}\Big|_{t=0} \widetilde{Y}_{j(x(t))} = \frac{d}{dt}\Big|_{t=0} j_*\hat{Y}_{x(t)} = \frac{d}{dt}\Big|_{t=0} \big(\hat{Y}_{x(t)} + f(\hat{Y}_{x(t)})\big) \\ &= \Big(\hat{X}(\hat{Y}) + f\big(\hat{X}(\hat{Y})\big)\Big)_x.\end{aligned}$$

Therefore, if $H\widetilde{U}$ denotes the p-horizontal part of $\widetilde{U} \in \mathscr{X}(S)$, we have

$$(H\nabla^S_{\widetilde{X}}\widetilde{Y}) \circ j = \hat{X}(\hat{Y}) + f\big(\hat{X}(\hat{Y})\big) + f(\hat{X}, \hat{Y})(\mathbf{n} \circ j).$$

Since $p\colon S \to P(E \oplus E^*)$ is a pseudo-Riemannian submersion, we know [13, p. 212] that

$$p_* \circ (H\nabla^S_{\widetilde{X}}\widetilde{Y}) = (\widetilde{\nabla}_X Y) \circ p.$$

Therefore

$$(\widetilde{\nabla}_X Y) \circ p \circ j = p_* \circ \Big(\hat{X}(\hat{Y}) + f\big(\hat{X}(\hat{Y})\big) + f(\hat{X}, \hat{Y})(\mathbf{n} \circ j)\Big).$$

On the other hand, since $p \circ j\colon \Sigma \to V$ is an isometry, we have

$$(p \circ j)_* \nabla^{\Sigma}_{\hat{X}} \hat{Y} = (\nabla_X Y) \circ p \circ j.$$

Since $(\widetilde{\nabla}_X Y - \nabla_X Y) \circ p \circ j = (\nu \widetilde{\nabla}_X Y) \circ p \circ j = \alpha(X, Y) \circ p \circ j$ defines the second fundamental form of V, we need only to calculate $\nabla^{\Sigma}_{\hat{X}} \hat{Y}$. But as it is well known about pseudo-spheres, we have

$$\nabla^{\Sigma}_{\hat{X}} \hat{Y} = \hat{X}(\hat{Y}) - \frac{1}{a} f\big(\hat{X}(\hat{Y}), \mathbf{x}\big)\mathbf{x} = \hat{X}(\hat{Y}) + \frac{1}{a} f(\hat{X}, \hat{Y})\mathbf{x},$$

where $\mathbf{x}$ denotes the vector field whose value at x is x. Thus

$$\begin{aligned} \alpha(X, Y) \circ p \circ j &= p_* \circ \Big(\hat{X}(\hat{Y}) + f\big(\hat{X}(\hat{Y})\big) + f(\hat{X}, \hat{Y})(\mathbf{n} \circ j) - \hat{X}(\hat{Y}) \\ &\qquad - f\big(\hat{X}(\hat{Y})\big) - f(\hat{X}, \hat{Y}) \frac{\mathbf{x} + f(\mathbf{x})}{a} \Big) \\ &= \big(g(X, Y) \circ p \circ j\big) p_* \circ \Big(\frac{a-1}{a} \big(\mathbf{x} + f(\mathbf{x})\big) + x_0 + \alpha_0 \Big), \end{aligned}$$

and this proves that $V = p\big(j(\Sigma)\big)$ is a totally umbilical submanifold of $P(E \oplus E^*)$ with normal curvature vector field ξ given by $\xi \circ p \circ j = p_*\big(\frac{a-1}{a}\big(\mathbf{x} + f(\mathbf{x})\big) + x_0 + \alpha_0\big)$. We have

$$\Big\langle \frac{a-1}{a}\big(\mathbf{x} + f(\mathbf{x})\big) + x_0 + \alpha_0, \mathbf{v} \Big\rangle \circ j = \Big\langle \frac{a-1}{a}\big(\mathbf{x} + f(\mathbf{x})\big) + x_0 + \alpha_0, \mathbf{x} - f(\mathbf{x}) + x_0 - \alpha_0 \Big\rangle = 0$$

because of (11). Thus, $\frac{a-1}{a}\big(\mathbf{x} + f(\mathbf{x})\big) + x_0 + \alpha_0$ is p-horizontal. Hence

$$l = g(\xi, \xi) = \Big\langle \frac{a-1}{a}\big(\mathbf{x} + f(\mathbf{x})\big) + x_0 + \alpha_0, \frac{a-1}{a}\big(\mathbf{x} + f(\mathbf{x})\big) + x_0 + \alpha_0 \Big\rangle = \frac{1-a}{a}.$$

Let us suppose that $l = 0$. Then $a = 1$ and by (11) we have $\alpha_0(x_0) = 0$. Hence, if $\dim F^{\perp} = \operatorname{codim} F = 1$, the vector field ξ, which in this case would be given by $\xi \circ p = p_*(x_0 + \alpha_0)$, must be an eigen-vector field of J. In fact, the assumption $x_0 \neq 0$ would imply then that $F \oplus \mathbb{R}x_0 = E$. Since $\alpha_0 \in F^{\perp}$ and $\alpha_0(x_0) = 0$, we conclude $\alpha_0 = 0$ and $J\xi = \xi$. If $x_0 = 0$, then $J\xi = -\xi$.

Note that $a = 1/(1 + l)$. Therefore, this construction cannot yield the case $l = -1$. To deal with it, let $0 \neq z \in F$ be such that $f(z, z) = 0$ and put $\mu = f(z)$. We put

$$\Sigma = \{x \in F\colon f(x, x) = 1,\ \mu(x) = 1\}.$$

If $x \in \Sigma$ and $v \in T_x F$, then $v \in T_x \Sigma$ iff $f(x, v) = \mu(v) = 0$, so that x, z span the orthogonal space to $T_x \Sigma$ in F. The orthogonal projection of a vector $v \in T_x F$ upon $T_x \Sigma$ is given by $v \mapsto v + \big(\mu(v) - f(x, v)\big)z - \mu(v)x$. Then, if $\hat{X}, \hat{Y} \in \mathscr{X}(\Sigma)$, we have

$$\nabla^{\Sigma}_{\hat{X}} \hat{Y} = \hat{X}(\hat{Y}) + \Big(\mu\big(\hat{X}(\hat{Y})\big) - f\big(\mathbf{x}, \hat{X}(\hat{Y})\big)\Big)z - \mu\big(\hat{X}(\hat{Y})\big)\mathbf{x} = \hat{X}(\hat{Y}) + f(\hat{X}, \hat{Y})z,$$

because

$$f(\mathbf{x}, \hat{X}(\hat{Y})) = \hat{X}(f(\mathbf{x}, \hat{Y})) - f(\hat{X}(\mathbf{x}), \hat{Y}) = -f(\hat{X}, \hat{Y}),$$
$$\mu(\hat{X}(\hat{Y})) = \hat{X}(\mu(\hat{Y})) = 0.$$

We map Σ into S by

$$j(x) = x + f(x).$$

As in the other case, this is an isometry and $j(\Sigma)$ is p-horizontal, so that $p \circ j$ is an isometry. The only change in the computations lies in the connection ∇^{Σ}. By using its new formula, we have immediately with the same notations:

$$\begin{aligned}\alpha(X,Y) \circ p \circ j &= p_* \circ \Big(\hat{X}(\hat{Y}) + f(\hat{X}(\hat{Y})) + f(\hat{X}, \hat{Y})(\mathbf{n} \circ j) \\ &\qquad - \hat{X}(\hat{Y}) - f(\hat{X}(\hat{Y})) - f(\hat{X}, \hat{Y})(z + f(z))\Big) \\ &= (g(X,Y) \circ p \circ j) p_* \circ (x - z + f(x) - \mu).\end{aligned}$$

Thus, $p(j(\Sigma))$ is a totally umbilical submanifold of $P(E \oplus E^*)$ with normal curvature vector field given by $\xi \circ p \circ j = p_* \circ (\mathbf{x} - z + f(\mathbf{x}) - \mu)$. We have $\langle x - z + f(x) - \mu, \mathbf{v}_{j(x)} \rangle = -\langle z + \mu, x - f(x) \rangle = -\frac{1}{2}(-\mu(x) + \mu(x)) = 0$. Therefore $l = g(\xi, \xi) = (f(\mathbf{x}) - \mu)(\mathbf{x} - z) = 1 - \mu(\mathbf{x}) - \mu(\mathbf{x}) = -1$, as desired. We shall call *parallels of* $P(E \oplus E^*)$ the totally umbilical submanifolds defined in this section.

5. Construction of all the totally umbilical submanifolds of $P(E \oplus E^*)$

Until near the end, we shall assume in this section that V is a non totally geodesic, totally umbilical submanifold of $P(E \oplus E^*)$ so that $\xi \neq 0$. First of all we shall prove that the inclusion $J(TV) \subset \mathcal{N}$ is strict. From (1), we have now, for $X, Y \in \mathscr{X}(V)$ and $\eta, \mu \in \Gamma(\mathcal{N})$, that

$$\widetilde{R}(X, Y, \eta, \mu) = g(JX, \mu) g(JY, \eta) - g(JX, \eta) g(JY, \mu),$$

and this is zero if $\mu = \xi$, that is

$$g(JX, \xi) g(JY, \eta) - g(JX, \eta) g(JY, \xi) = 0. \tag{12}$$

Assume that $J(TV) = \mathcal{N}$. Then we can put $\beta = JX$, $\mu = JY$ and consider that they are arbitrary sections of $\mathcal{N}$. Thus $g(\beta, \xi) g(\mu, \eta) - g(\beta, \eta) g(\mu, \xi) = 0$, that is

$\eta \otimes \xi = \xi \otimes \eta$ for every $\eta \in \Gamma(\mathscr{N})$, and this would imply $s = \dim V = \operatorname{rank} \mathscr{N} = 1$, which is contrary to the assumption $s \geqslant 2$.

Now, we prove that $J\xi \in \Gamma(\mathscr{N})$. In fact, since in (12) η is arbitrary we have $g(JX,\xi)JY - g(JY,\xi)JX = 0$. By multiplication by JX we get $g(JX,\xi)JX \wedge JY = 0$. Since J is an isomorphism and $s \geqslant 2$, this implies $g(JX,\xi) = -g(X,J\xi) = 0$, that is $J\xi \in \Gamma(\mathscr{N})$. From this, we can prove that if ξ is not an eigen-vector field of J and $l = 0$ then $s \leqslant r - 1$. On these assumptions, let us consider the subbundle of $\mathscr{N}$ generated by $J(TV)$, ξ and $J\xi$, and suppose that there is some vector in the intersection of $J(TV)$ with the subbundle generated by ξ and $J\xi$, namely $JX = a\xi + bJ\xi$, with $X \in TV$. Then $X = aJ\xi + b\xi$, whence $X = 0$. Therefore $\operatorname{rank} \mathscr{N} = 2r - s \geqslant s + 2$. Now, $g(\xi, J\xi) = g(\xi,\xi) = g(\xi, JX) = -g(J\xi, X) = 0$ for every $X \in TV$. The equal sign in $2r - s \geqslant s + 2$ would then imply that $g|_{\mathscr{N}}$ be degenerate, for ξ would be orthogonal to the whole $\mathscr{N}$; so, $2r - s > s + 2$, that is $s < r - 1$.

The identity tensor field I can be decomposed into two projectors on the eigenspaces of J as $I = \frac{1}{2}(I+J) + \frac{1}{2}(I-J)$. Let $\pi_1 = \frac{1}{2}(I+J)\big|_{TV}$, $\pi_2 = \frac{1}{2}(I-J)\big|_{TV}$, $v \in T_mV$ and suppose that $\pi_1(v) = 0$. Then $Jv = -v \in \mathscr{N}_m \cap T_mV$, whence $v = 0$. Thus, if $M_1 = \pi_1(TV)$ and $M_2 = \pi_2(TV)$, we have that $\pi_1 : TV \to M_1$ and $\pi_2 : TV \to M_2$ are isomorphisms. Let h be the isomorphism $h\colon M_1 \to M_2$ given by $h = \pi_2 \circ \pi_1^{-1}$. We claim that

$$T_mV = \{v + hv\colon v \in (M_1)_m\}.$$

In fact, if $v \in (M_1)_m$, then $v = \pi_1 w$, for some $w \in T_mV$. Thus $w = \pi_1 w + \pi_2 w = \pi_1 w + \pi_2 \circ \pi_1^{-1} \circ \pi_1 w = \pi_1 w + h(\pi_1 w) = v + hv$. Moreover, h is self-adjoint, that is $g(hX,Y) = g(X,hY)$ for every $X, Y \in \Gamma(M_1)$, and the bilinear symmetric tensor field given by $X, Y \in \Gamma(M_1) \mapsto g(hX,Y)$ is non-degenerate at each point. To show this, we note that since $Y + hY \in \mathscr{X}(V)$ we have $J(Y + hY) \in \Gamma(\mathscr{N})$, that is $g\big(X + hX, J(Y + hY)\big) = g(X + hX, Y - hY) = g(hX,Y) - g(X,hY) = 0$. Also, $g(X + hX, Y + hY) = 2g(hX,Y)$ by the above result, and the non-degeneracy of this bilinear symmetric tensor field follows from that of i^*g.

Let $l + 1 \neq 0$, so that V is not flat. Then, given a point $m = p(y + \beta) \in V$, with $y + \beta \in S$, we want to show that there is some parallel of $P(E \oplus E^*)$, $j(\Sigma)$, defined as in Section 4, that passes by m having T_mV as tangent space at m and ξ_m as normal curvature vector at m. With the notations of Section 4, we want to determine x, x_0,

$f(x)$, α_0, a such that if $u+\gamma$ is the p-horizontal lift of ξ_m to $y+\beta$, then

$$x + x_0 + f(x) + \alpha_0 = y + \beta,$$
$$\frac{a-1}{a}x + x_0 + \frac{a-1}{a}f(x) + \alpha_0 = u + \gamma,$$
$$\frac{1-a}{a} = l.$$

The solution of this system is the following

$$a = \frac{1}{1+l}, \quad x = \frac{y-u}{1+l}, \quad x_0 = \frac{ly+u}{1+l}, \quad f(x) = \frac{\beta-\gamma}{1+l}, \quad \alpha_0 = \frac{l\beta+\gamma}{1+l}.$$

We have
$$\beta(y) = 1, \quad \gamma(u) = l, \quad \gamma(y) = 0, \quad \beta(u) = 0,$$
formulae that express that $y+\beta \in S$, $g(\xi,\xi) = l$, $u+\gamma \in T_{y+\beta}S$ and $u+\gamma$ is p-horizontal. Let the superscript H denote the p-horizontal lift of T_mV to $T_{y+\beta}S$. This lift preserves J and the inner product. Thus, as before we can see that $(T_mV)^H = \{v + f(v) \colon v \in M_1^H \equiv \pi_1((T_mV)^H)\}$, where $f = \pi_2 \circ \pi_1^{-1}$ with the obvious meaning. We put $F = M_1^H + \mathbb{R}x$.

The formula for $f(x)$, that until now was just a form, gives $f(x)(x) = (\beta-\gamma)(y-u)/(1+l)^2 = 1/(1+l) = a$. Also, $f(x)(v) = 0$ if $v \in M_1^H$. In fact, we have then $v+f(v) \in (T_mV)^H$. But $\xi, J\xi \in \Gamma(\mathscr{N})$, whence $\langle v+f(v), u+\gamma\rangle = f(v)(u)+\gamma(v) = 0$ and $\langle v+f(v), u-\gamma\rangle = f(v)(u) - \gamma(v) = 0$. Therefore $f(v)(u) = \gamma(v) = 0$, and $f(x)(v) = (\beta(v) - \gamma(v))/(1+l) = \beta(v)/(1+l)$ and this is zero. In fact, $v+f(v)$ is p-horizontal and tangent to S, so that $\langle v+f(v), y+\beta\rangle = \langle v+f(v), y-\beta\rangle = 0$, whence $\beta(v) = f(v)(y) = 0$. As a consequence, $f(x)$ allows us to extend f to F by putting $f(x,x) = f(x)(x) = a$, $f(x,v) = f(x)(v) = 0$, and we have $x \in \Sigma$, with Σ defined as in the preceding section. To complete our construction, we need to show that $\alpha_0 \in F^\perp$, $x_0 \in f(F)^\perp$, and this is easily done using the same techniques used for proving that $f(x)(v) = 0$.

Let $l+1 = 0$, so that V is flat. Now, we want to determine x, z, $f(x)$, μ such that if $u+\gamma$ is the p-horizontal lift of ξ_m to $y+\beta$, then

$$x + f(x) = y + \beta,$$
$$x - z + f(x) - \mu = u + \gamma.$$

Clearly, this implies

$$z = y - u, \quad x = y, \quad \mu = \beta - \gamma, \quad f(x) = \beta.$$

F and f are defined as before. Then, $f(x)(x) = \beta(y) = 1$, $\mu(z) = (\beta - \gamma)(y - u) = 1 - 1 = 0$. As in the other case, one can easily verify that this construction gives the desired parallel of $P(E \oplus E^*)$.

Theorem 3. *Let V be a connected totally umbilical pseudo-Riemannian submanifold of $P(E \oplus E^*)$ with $s = \dim V > 1$ and assume that it is not totally geodesic. Then, V is contained in a parallel of $P(E \oplus E^*)$ of the same dimension.*

Proof. As proved above, if $m \in V$, there is a parallel of $P(E \oplus E^*)$, $p(j(\Sigma))$, that passes by m having $T_m V$ as tangent space at m and ξ_m as normal curvature vector at m. Let γ be a geodesic of V. Then, we have

$$\begin{aligned}\widetilde{\nabla}_{\dot\gamma}\dot\gamma &= \nabla_{\dot\gamma}\dot\gamma + g(\dot\gamma,\dot\gamma)(\xi\circ\gamma) = g(\dot\gamma,\dot\gamma)(\xi\circ\gamma),\\ \widetilde{\nabla}_{\dot\gamma}\xi &= D_{\dot\gamma}\xi - A_\xi\dot\gamma = -l\dot\gamma.\end{aligned}$$

Thus, if we put $\chi = \xi \circ \gamma$, we have a curve χ in $TP(E \oplus E^*)$, with projection γ on $P(E \oplus E^*)$, that satisfies the following differential equations

$$\begin{aligned}\widetilde{\nabla}_{\dot\gamma}\dot\gamma &= g(\dot\gamma,\dot\gamma)\chi,\\ \widetilde{\nabla}_{\dot\gamma}\chi &= -l\dot\gamma,\end{aligned}$$

where l is a constant. To convince oneself that this is a well posed system of ordinary differential equations, we can write it locally as

$$\begin{aligned}\ddot x^i + (\Gamma^i_{jk}\circ\gamma)\dot x^j\dot x^k - (g_{jk}\circ\gamma)\dot x^j\dot x^k\chi^i &= 0,\\ \dot\chi^i + (\Gamma^i_{jk}\circ\gamma)\chi^j\dot x^k + l\dot x^i &= 0.\end{aligned}$$

Since the geodesics of both V and $p(j(\Sigma))$ starting from m satisfy the same system with the same initial conditions, we conclude that there is some open neighborhood of m where both submanifolds coincide. By a standard argument, we have our claim. □

As for totally geodesic submanifolds of $P(E \oplus E^*)$, we can separate them in three classes [6]. First, totally geodesic submanifolds with a degenerate metric i^*g, which are of no interest here in the context of umbilical pseudo-Riemannian submanifolds.

The second consists of the paracomplex projective subspaces. Let $E = F \oplus G$ be a splitting of E in two subspaces, and let $E^* = F^* \oplus G^*$ be the corresponding splitting for E^*. Then, the inclusion $i\colon F \oplus F^* \to E \oplus E^*$ passes to the quotient and gives the paracomplex projective subspace $P(F \oplus F^*) \hookrightarrow P(E \oplus E^*)$, which is a totally geodesic pseudo-Riemannian submanifold.

Submanifolds V of the third class are such that for each point $m \in V$, $T_mV = \{v + h_mv: v \in (T_mV)_1\}$, where $(T_mV)_1 = (I+J)(T_mV)$ and h_m is a symmetric isomorphism from $(T_mV)_1$ to $(T_mV)_2 = (I-J)(T_mV)$. These are parallels of $P(E \oplus E^*)$ given by the preceding formulae for the non-flat case when $\xi = 0$. Then $l = 0$, $a = 1$, $x_0 + \alpha_0 = 0$, and $p \circ j: \Sigma \to P(E \oplus E^*)$ is a totally geodesic isometric immersion of the pseudo-Riemannian sphere $\Sigma = \{x \in F: f(x,x) = 1\}$. Let us call *meridians* these submanifolds $p(j(\Sigma))$.

Theorem 4. *Let V be a connected totally umbilical pseudo-Riemannian submanifold of the paracomplex projective space $P(E \oplus E^*)$ with $s = \dim V > 1$. Then:*

(1) *If V is not totally geodesic, it is contained in a parallel of $P(E \oplus E^*)$ of the same dimension s, and then V has constant sectional curvature.*

(2) *If V is totally geodesic, then either it is contained in a paracomplex projective subspace $P(F \oplus F^*)$ of $P(E \oplus E^*)$ with $\dim F = \frac{1}{2}s + 1$ and then V has constant para-holomorphic sectional curvature, or it is contained in a meridian of $P(E \oplus E^*)$ of the same dimension s and then V has constant sectional curvature.*

References

[1] *C. Allamigeon*: Espaces homogènes symétriques à groupe semi-simple. C.R. Acad. Sci. Paris *243* (1956), 121–123.

[2] *C. Bejan*: Structuri hiperbolici pe diverse spatii fibrate. Ph. D. thesis Math. Iasi, 1990.

[3] *B. Y. Chen*: Totally umbilical submanifolds. Soochow J. Math. *5* (1979), 9–37.

[4] *P. M. Gadea and A. Montesinos Amilibia*: Spaces of constant para-holomorphic sectional curvature. Pacific Journal of Mathematics *136(1)* (1989), 85–101.

[5] *P. M. Gadea and A. Montesinos Amilibia*: Some geometric properties of para-Kaehlerian space forms. Rend. Sem. Mat. Univ. Cagliari *59* (1989), 131–145.

[6] *P. M. Gadea and A. Montesinos Amilibia*: The paracomplex projective spaces as symmetric and natural spaces. Indian J. Pure Appl. Math. To appear.

[7] *P. M. Gadea and A. Montesinos Amilibia*: The paracomplex projective model. The para-Grassmannian manifolds. "Jubilee volume Prof. J. J. Etayo Miqueo". Madrid, 1992, to appear.

[8] *P. M. Gadea and J. Muñoz Masqué*: Classification of homogeneous para-Kaehlerian space forms. Nova J. Alg. Geom. To appear.

[9] *P. M. Gadea and J. Muñoz Masqué*: Classification of nonflat para-Kaehlerian space forms. Preprint.

[10] *S. Kaneyuki and M. Kozai*: Paracomplex structures and affine symmetric spaces. Tokyo J. Math. *8(1)* (1985), 81–98.

[11] *S. Kobayashi and K. Nomizu*: Foundations of differential geometry. Interscience Publishers, New York, 1969.

[12] *P. Libermann*: Sur le problème d'équivalence de certaines structures infinitésimales. Ann. Mat. Pura Appl. *36(1)* (1954), 27–120.

[13] *B. O'Neill*: Semi-Riemannian geometry. Academic Press, New York, 1983.

[14] *P. K. Rasevskii*: The scalar field in a stratified space. Trudy Sem. Vekt. Tenz. Anal. *6* (1948), 225–248.

[15] *M. Spivak*: Differential geometry. Publish or Perish, Inc., Boston, 1975.

Authors' addresses: *P. M. Gadea*, Instituto de Matemáticas y Física Fundamental, C.S.I.C., Serrano 123, 28006 Madrid, Spain; *A. Montesinos Amilibia*, Departamento de Geometría y Topología, Universidad de Valencia, Dr. Moliner 50, 46100 Burjasot (Valencia), Spain.