Toward the interpretation of non constructive reasoning as non-monotonic learning

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Abstract

We study an abstract representation of the learning process, which we call *learning sequence*, aiming at a constructive interpretation of classical logical proofs, that we see as learning strategies, coming from Coquand's game theoretic interpretation of classical logic. Inspired by Gold's notion of limiting recursion and by the Limit-Computable Mathematics by Hayashi, we investigate the idea of learning in the limit in the general case, where both guess retraction and resumption are allowed. The main contribution is the characterization of the limits of non-monotonic learning sequences in terms of the extension relation between guesses.

Key words: logic, proof theory, game theory, learning theory.

1 Introduction

A logical proof is constructive if it embodies a description of some witness of its thesis, which e.g. in case of Π^0_2 statements is a computable function. This fails when non constructive proof methods are allowed. However, if we relax the effectiveness requirement about the proof content asking for an effective search method of learning the witness of a proved statement, we conceive a more powerful notion of constructive content. Indeed the shift from algorithms to effective learning processes is a longstanding and fascinating proposal underlying research work to extract constructive content from non constructive proofs.

In his seminal papers [17, 18], Gold introduced the idea of learning in the limit: a function or a language is learnable in the limit if we can identify a correct grammar or a correct program by means of an infinite sequence of "guesses" at the grammar or at the program, that are correct from some step on (called Ex-learning in the literature). The mathematical notion used here is that of the limit of a sequence in a discrete topology, which exists if and only if the sequence stabilizes. Gold justified his concept of learning by the fact that a learner does not need to know when her guess is correct; rather it suffices to know that she will be acting correctly after some finite time.

Since then, learning in the limit has been largely exploited in the studies of machine learning theory, e.g. [11, 3] where it is called inductive inference, and it is still considered as a cornerstone in the field. More recently Hayashi [19] shows that Gold's view of learning is suitable to interpret certain non constructive arguments in mathematics. He observes that Hilbert's celebrated basis theorem as well as many other relevant mathematical results have been proved by means of non constructive arguments, which although are good examples of learning in the limit. Therefore it makes sense to speak of Limit-Computable Mathematics as a reasonable widening of constructive mathematics, especially in view of applications to interactive and semi-automatic tools for theorem proving.

A limitation of the approach, however, lays in the use of negative information. When the "learner" thinks that her latest guesses have been wrong, her only possibility is to backtrack to some previous stage and to forget definitely about the retracted guesses (see [17], page 33, proof of theorem 1). This introduces a dramatic simplification in the structure of the memory of the learner (just a stack), which is responsible for the limitation to 1-recursive problems, or equivalently to Δ^0_2 sets, as it was clearly recognized in Gold's work.

A natural extension of learning in the limit can be achieved by means of iterated limits, as introduced in [26]. In [14] it is proved that for all k, Δ^0_{k+1} sets coincide with k-limiting recursive sets, namely sets whose characteristic function is definable by k-iterated limits of some total recursive function. The problem with such an extension is that iterated limits completely blur the structure of computation, which is in the end a brute force searching algorithm, and so not a candidate for the interpretation of proofs, which include instead ingenuous search methods.

The concept of a strategy in the learning activity has its natural setting in game theory. It was Lorenzen who first proposed in [22] to see logical formulas as games, and proofs as winning strategies. The idea, grown through the works of Lorenz, Hintikka and many others (see e.g. [16]), has been used in [13] by Coquand to provide a new interpretation of the finitist standpoint in arithmetic. Now, if we look at the notion of debate and at the example 2.4 on page 330 of [13], we feel fully entitled to say that the winning strategy for the formula $\exists y \geq x \forall z \geq x. f(y) \leq f(z)$ (where f is some function on integers given as a parameter) is actually a strategy learning a function which yields a y depending on x.

A breakthrough in the study of games and strategies in logic and computer science has been the solution of the full abstraction problem for PCF [2, 21, 23], derived from the game theoretic analysis of Linear Logic [10, 1], and from earlier Kleene's and Gandy's work on computable functionals (see [21] for an historical reconstruction and references). In this context, as well as in that of Coquand's work [12, 13], plays have been modeled by *pointing sequences*: each new move in a play has in fact a pointer to some previous move, in such a way that the new move "is justified" by the previous move. It is this pointer structure that defines the view of each player in a play at a given stage, which in turn is all the information at disposal of the players to decide their next move (a property of strategies called "innocency" in [21]). By looking at the move sequence as the successive guesses of a learner, and by retaining the pointer structure only, we get a picture of the memory of the learner in the learning process, arriving at the concept of *learning sequence* we investigate in the present paper.

According to this model, a learning process is some countable sequence of guesses indexed over natural numbers (the time). It is equipped with a regressive function f, telling for each time x which is the last guess of a sequence extended by the guess at x. For historical reasons we sometime use the game theoretic terminology by speaking of the "justification" relation in a sequence, though its meaning here is different, since a guess at time x actually "extends" the entire line of thought ending with the guess at $f(x)$, and it is not in general a reply to it, nor it is authorized by that position in a game. Now if any guess in a sequence just extends its immediate predecessor, then we say that the process is retraction-free. Otherwise the learner is allowed to backtrack to some guess earlier than the immediate predecessor, implicitly retracting all intermediate guesses. If we impose that the guess to which the learner goes back has not been itself retracted, then we have the same situation as Gold's learning in the limit, which is called *monotonic learning* in the literature. If instead retraction is not definite and guess resumption is allowed so that we can suppress some previous retractions, then learning is *non-monotonic*, and the model is much more powerful and complex. Following [8] we say that a learning process is 0-backtracking if it is retraction-free; 1-backtracking (speaking also of "simple backtracking") if no retracted guess is ever resumed; α-backtracking for $\alpha > 1$ in the general case.

The problem we face is then: which is the proper notion of limit in the case of an infinite learning sequence in which the same guess can be retracted and resumed infinitely many times? This question is clearly preliminary to any further investigation, and it is quite challenging. Our understanding of the limit of a learning sequence is the sequence which is obtained by repeatedly removing all guesses which are definitely retracted. This removal can be analyzed into several stages, which we formalize via a reduction relation. This is not an elementary concept; indeed (identifying guesses with time) the removal of a retracted guess x can make another guess y removable, e.g. in the case y is retracted and x is the only guess resuming y: after removing x, the guess y is retracted and never resumed, so that y becomes removable. Indeed the removal of a guess might require the removal of infinitely many other guesses before. We prove however that the limit of a learning sequence always exists and that it is unique; moreover the reduction of a sequence attains its limit, providing a motivation for the latter definition.

We further investigate the structure of limits, by discovering that the notions of active and inactive edges, corresponding to established and retracted guesses respectively, make sense even in the infinite case. In this latter case a third possibility exists, namely of those guesses which are neither retracted nor resumed definitely no matter how many other guesses are removed; we call them unstable. The good limits are then such that the set of unstable guesses is empty, so that only definitely approved guesses are left in the sequence, which represents the learned object. This leads to a relation, we call the "undo" relation, which we prove to be well founded if and only if there are no unstable guesses. In such a case the minimum ordinal α needed in the definition of active and inactive edges expresses a measure of the logical complexity of the learning processes. Indeed there should be a relation between the ordinal α and the logical complexity of the set which is described by the limit. When $\alpha = 1$, our notion of limit clearly coincides with Gold's notion of limit, which is 1-backtracking, which in turn can be seen as a description of Σ^0_1 sets. Also there should be a correspondence between the ordinal α (when it is finite) and iterated limits in the sense of Schulte, and hence the arithmetical hierarchy by the result in [14]. In general we conjecture that the limits without unstable guesses are exactly the Δ^1_1-sets: if this is the case, then the ordinal α is a possible answer to the quest for a complexity measure of infinite learning processes found in [14], going even behind the arithmetical sets.

The paper is organized as follows. In Section 2 we introduce learning sequences in which we abstract from the nature of the elements we are learning. Pointing sequences are actually interpretable as possibly infinite plays of any of the formalisms proposed in the game theoretic investigations of logic mentioned above. Because of this we treat here and in the subsequent Sections 3 and 5 just of pointing sequences, while the interpretation in terms of learning processes is proposed in Section 4 to make explicit the motivation of our work, and to illustrate applications of the concept of limit through examples. We then establish our main results in Section 6, where the structure of limits is characterized in terms of the undo relation. We report on related works in Section 7 and eventually conclude in Section 8.

$$\begin{tikzcd}
0 & 1 \arrow[l] & 2 \arrow[l] & 3 \arrow[lll, bend right] & 4 \arrow[l] & 5 \arrow[l] & \cdots
\end{tikzcd}$$

Figure 1: Initial segment of a pointing sequence.

2 Pointing Sequences

We think of the elements of a sequence $s_0 s_1 s_2 \ldots$, possibly infinite, as "guesses" of a learning process. We restrict our attention to countable sequences. A pointing structure is a backward mapping f over the indexes, relating each element s_i but the first one to some previous element $s_{f(i)}$. We think of the guess s_i as some "extension" of the list of guesses $s_{f(i)}, s_{f^2(i)}, \ldots$ (here listed in the opposite order w.r.t. their indexes) . We also imagine that all guesses in between $s_{f(i)}$ and s_i are "retracted" at the step i. In this section the central notion is *pointing sequence*, in which we focus on the pointer structure of the sequence, so that the actual nature of the elements s_i is immaterial.

Definition 2.1 (Countable pointing sequences) *A* countable pointing sequence *is a pair* (X, f) *such that:*

1. $X \subseteq \mathbb{N}$ *is a non empty set ordered by the restriction of the ordering over* $\mathbb{N}$*;*
2. $f : \mathbb{N} \setminus \{0\} \to \mathbb{N}$*, is a mapping such that* $f(x) < x$ *for all* $x \neq 0$*, and the image of* $X \setminus \{0\}$ *under* f *is included in* X *(in which case we also say that* X *is* f-closed*).*

When $f(x) = y$ *we say that* x extends y*. We call* the tree of (X, f) *the tree of domain* X*, having root* 0 *and as father/child relation the extension relation.*

Countable pointing sequences are just "pointing sequences" in the rest of the paper. X is the set of indexes. Since it helps to think of X as a straight line, we sometime call its elements *points*. We take as set of indexes subsets and not initial segments of $\mathbb{N}$, in order to fix a linear ordering of indexes and to avoid a tedious re-indexing of a sequence when subtracting some of its "points".

If we interpret $f(x)$ as the parent of x, then (X, f) is just a tree (whence the definition of the tree of a sequence); however the relevant facts about a pointing sequence concern the interplay between the linear (or "chronological") ordering of the indexes and the tree ordering induced by the "extension relation" $f(x) = y$. To fix terminology and notation: given $x \in X \setminus \{0\}$, we call the pair $(f(x), x)$ the *edge* from x, or just "the edge x" for short (we can think of any $x > 0$ in X both as an edge and as a point). We write:

$$]f(x), x[_X = \{y \in X \mid f(x) < y < x\}$$

for the *interior* of the edge from x. We say that an edge x is *empty* if its interior is, i.e., if $f(x) = x - 1$. We represent edges by curved lines, or by straight lines if their interior is empty, as shown in Figure 1. We shall also use the notations $[f(x), x]_X = \{y \in X \mid f(x) \leq y \leq x\}$ and $[f(x), x[_X = \{y \in X \mid f(x) \leq y < x\}$. The subscript X will be omitted when it is clear from the context.

Definition 2.2 (Crossing Edges) *In a pointing sequence* (X, f) *we say that an edge from* $y \in X$ crosses *the edge from* $x \in X$*, and we write* $y \curvearrowright x$ *or* $x \curvearrowleft y$*, if*

$$f(x) < f(y) < x < y,$$

which is in a picture:

$$\begin{tikzcd} f(x) & f(y) & x \arrow[ll, bend right] & y. \arrow[ll, bend right] \end{tikzcd}$$

By definition $y \curvearrowright x$ if and only if $y > x$ and $f(y) \in \,]f(x), x[$. If either $x = f(y)$ or $y = f(x)$, instead, we say that the edges x, y are *adjacent*. If $[f(x), x] \cap [f(y), y] = \emptyset$ we say they are *disjoint*. For any two edges $x \neq y$, we distinguish the mutual positions of x, y into three disjoint situations.

Lemma 2.3 *Given two edges x, y of (X, f), with $x < y$, all possible mutual positions of x, y are:*

1. $[f(x), x] \subset [f(y), y[$ *(x is strictly included in y):*

$$\begin{tikzcd} f(y) & \leq & f(x) & \cdots & x \arrow[ll] & y \arrow[lllll, bend right] \end{tikzcd}$$

2. $[f(x), x] \not\subseteq [f(y), y[$, *but* $x \in \,]f(y), y[$ *(hence $y \curvearrowright x$):*

$$\begin{tikzcd} f(x) & f(y) & x \arrow[ll, bend right] & y \arrow[ll, bend right] \end{tikzcd}$$

3. $[f(x), x] \cap [f(y), y] \subseteq \{x\}$ *(hence x, y are adjacent or disjoint):*

$$\begin{tikzcd} f(x) & \cdots & x \leq f(y) \arrow[ll] & \cdots & y \arrow[ll] \end{tikzcd}$$

Proof. Immediate by definition. We just remark w.r.t. usual interval inclusion over the real line, that the situation $[x', x] \subset [y', y]$ with $x = y$ is impossible, since in our setting we would have $x' = f(x) = f(y) = y'$.

■

Assume that (X, f) is a pointing sequence, and denote by f^i the i-time self composition of f; f^0 is just identity. Then the sequence of powers $f^i(x)$ is strictly decreasing and it ends in 0:

$$0 = f^i(x) < \ldots < f^2(x) < f(x) < x$$

We call this sub-sequence "the thread of x". Intuitively, each thread is a different line of thought within a learning process.

Definition 2.4 (Thread) *Let (X, f) be any pointing sequence.*

1. *If $x \in X$, then the* thread of x *in (X, f) is* $0 = f^i(x) < \ldots < f^2(x) < f(x) < x$.
2. *Any $Y \subseteq X$ is a* thread of X *if Y is finite and equal to the thread of its last element, or Y is infinite and all its finite initial segments are threads.*
3. *When $Y = X$ we say that X itself is a thread.*
4. *If y is in the thread of $x - 1$ but not in the thread of x, then we say that y is* retracted *by x.*
5. *If y is in the thread of x but not in the thread of $x - 1$, then we say that y is* resumed by *x.*

As an immediate consequence of the definition, a thread is f-closed, therefore it is a sub-sequence of (X, f). A thread is a pointing sequence in which each point extends the point immediately before, therefore threads included in (X, f) are exactly the branches of

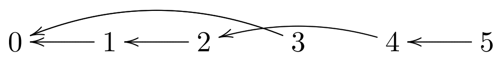

Figure 2: A sequence with two threads of maximal length.

the tree of (X, f). Pointing sequences are not single threaded in general, exactly as trees are not single branched in general.

Figure 2 illustrates a case of a sequence with two maximal threads, namely $\{0, 3\}$ and $\{0, 1, 2, 4, 5\}$. The latter might be considered as the main thread of the sequence since it originates with the last element: intuitively, this thread is a line of thought prevailing on all the other ones within the learning process. We face the problem of extending such a notion of the "main" thread to the infinite case, leading to the notion of limit (see Section 3).

By definition, if y retracts x then x does not belong to the same thread of y, so that if y belongs to the main thread, x does not. On the other hand if y retracts x but y does not belong to the main thread, it is still possible for x to be in the main thread, provided that y is itself retracted by some z in the main thread. This is, indeed, the case of $x = 2, y = 3, z = 4$ in Figure 2. In this case, according to our definition, we say that x is resumed by z. An edge y retracting some x can be "inactivated" by some other edge z crossing it; but y can be "activated" if some other edge t crosses z, and consequently y retracts x at time t once again. If X is infinite, this might happen infinitely many times. This is why it is so difficult to define the limit as the "main thread" of an infinite X. In order to solve the problem, we introduce two sets of edges, namely *active* and *inactive*, inspired by the situation in Figure 2. Active and inactive edges are defined by mutual recursion: an edge is active if all edges crossing it are inactive, and inactive if some edge crossing it is active. Formally:

Definition 2.5 (Active and Inactive Edges) *Given a pointing sequence (X, f) the sets of* active *and* inactive *edges of X, are respectively:*

$$\begin{aligned} A^X &= \{x \in X \mid \forall y \in X\ (y \curvearrowright x \Rightarrow y \in I^X)\}, \\ I^X &= \{x \in X \mid \exists y \in X\ (y \in A^X \ \&\ y \curvearrowright x)\}. \end{aligned}$$

When X is finite, a backward induction shows that A^X and I^X are well defined. In the example in Figure 3 the active edges are 4 and 2. The edge 4 is active since it is the last one, hence no edge can cross it. The edge 2 is active since, even if $3 \curvearrowright 2$, we also have $4 \curvearrowright 3$, therefore the edge 4 "inactivates" 3, thus 3 does not "inactivate" 2 any more. According to the proposed definition, the last edge (if any) is always active; by backward induction we can also prove that we have defined a partition of all edges of a finite (X, f) into a set A of active edges and a set I of inactive ones.

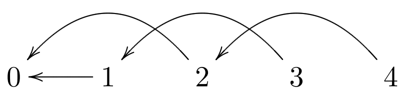

Figure 3: Active and inactive edges in the finite case.

In the next section we will define the limit of an infinite pointing sequence (X, f) as the set of points which are not in the interior of any active edge. But before that, we have to define the sets A^X, I^X of active and inactive edges when X is infinite. We use Knaster-Tarski fixed point theorem. First of all, let us consider the structure $(\mathcal{P}(X) \times \mathcal{P}(X), \sqsubseteq)$, where $\sqsubseteq$ is componentwise inclusion. It is a complete lattice, where sups are (arbitrary) componentwise unions, and infs (arbitrary) componentwise intersections. Let us define the operator $\Phi : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X) \times \mathcal{P}(X)$ by $\Phi(A, I) = (A', I')$ where:

$$\begin{array}{rcl} A' & = & \{x \in X \setminus \{0\} \mid \forall y \in X(y \curvearrowright x \Rightarrow y \in I)\}, \\ I' & = & \{x \in X \setminus \{0\} \mid \exists y \in X(y \curvearrowright x \ \ \& \ \ y \in A)\}. \end{array}$$

Φ is easily seen to be monotonic w.r.t. $\sqsubseteq$, hence by Knaster-Tarski theorem Φ has a least fixed point that defines (A^X, I^X), even when X is infinite.

Unravelling this construction, we write the following more explicit definition, which will be of use in the technical development of Section 3:

Definition 2.6 *Let (X, f) be a pointing sequence; we define by mutual induction, the families $\{A^X_\alpha\}_{\alpha \in Ord}$ and $\{I^X_\alpha\}_{\alpha \in Ord}$ of subsets of X as follows:*

$$\begin{array}{rcl} A^X_\alpha & = & \{x \in X \setminus \{0\} \mid \forall y \in X(y \curvearrowright x \Rightarrow y \in I^X_{<\alpha})\}, \\ I^X_\alpha & = & \{x \in X \setminus \{0\} \mid \exists y \in X(y \curvearrowright x \ \ \& \ \ y \in A^X_{<\alpha})\}, \end{array}$$

where $I^X_{<\alpha} = \bigcup_{\beta<\alpha} I^X_\beta$ and similarly for $A^X_{<\alpha}$. Then we set

$$(A^X, I^X) = (A^X_\alpha, I^X_\alpha),$$

where $\alpha \in Ord$ is the minimal one such that $(A^X_\alpha, I^X_\alpha) = (A^X_{<\alpha}, I^X_{<\alpha})$, if any.

We say that A^X and I^X are the set of active *and of* inactive *edges in X respectively.*

Let us see the first terms of the series $\{A^X_\alpha\}_{\alpha \in Ord}$ and $\{I^X_\alpha\}_{\alpha \in Ord}$:

$$\begin{array}{rcl} A^X_0 & = & \{x \in X \setminus \{0\} \mid \forall y \in X(y \curvearrowright x \Rightarrow y \in I^X_{<0})\} \\ & = & \{x \in X \setminus \{0\} \mid \forall y \in X(y \curvearrowright x \Rightarrow y \in \emptyset)\} \\ & = & \{x \in X \setminus \{0\} \mid \forall y \in X(y \not\curvearrowright x)\} \end{array}$$

since $I^X_{<0} = \bigcup_{\beta<0} I^X_\beta = \emptyset$. In words, A^X_0 is the set of edges such that no other edge (no matter whether active or inactive) crosses them.

Since $A^X_{<0} = \emptyset$ and the definition of I^X_0 asks for the existence of some edge $y \in A^X_{<0}$, we immediately have $I^X_0 = \emptyset$. This implies $A^X_1 = A^X_0$, but

$$\begin{array}{rcl} I^X_1 & = & \{x \in X \mid \exists y \in X(y \curvearrowright x \ \ \& \ \ y \in A^X_{<1})\} \\ & = & \{x \in X \mid \exists y \in X(y \curvearrowright x \ \ \& \ \ y \in A^X_0)\} \end{array}$$

needs not to be empty, and indeed it is the set of those edges which are inactivated by edges, which are not crossed by any other edge. Since I^X_1 can be nonempty, any edge which is inactivated by some edge in I^X_1, will be in $A^X_2 \setminus A^X_1$. We now check that the two definitions of active and inactive edges coincide.

Proposition 2.7 *Given any pointing sequence (X, f), the pair (A^X, I^X) exists, is $(A^X_{<\alpha}, I^X_{<\alpha})$ for some countable ordinal α, and it coincides with the least fixed point of Φ.*

Proof. First observe that if $\alpha < \beta$ then both $A_\alpha \subseteq A_\beta$ and $I_\alpha \subseteq I_\beta$: hence $A_{<\alpha} = \bigcup_{\beta<\alpha} A_\beta \subseteq A_\alpha$, and similarly $I_{<\alpha} \subseteq I_\alpha$.

As a consequence, the subsets $A_\alpha \setminus A_{<\alpha}$ of X are pairwise disjoint. Indeed, if $\alpha < \beta$ then $A_\alpha \subseteq A_{<\beta}$ is disjoint with $A_\beta \setminus A_{<\beta}$. Since $X \subseteq \mathbb{N}$ is countable, only countably many sets $A_\alpha \setminus A_{<\alpha}$ are not empty. Thus, for some $\alpha_0 < \omega_1$ and all $\alpha \geq \alpha_0$ we have $A_\alpha \setminus A_{<\alpha} = \emptyset$, hence $A_\alpha = A_{<\alpha}$. By a similar reasoning, for some $\beta_0 < \omega_1$ and all $\beta \geq \beta_0$ we have $I_\beta = I_{<\beta}$. Therefore if we take $\gamma_0 = \max(\alpha_0, \beta_0)$, we obtain $(A_{\gamma_0}, I_{\gamma_0}) = (A_{<\gamma_0}, I_{<\gamma_0})$.

To see that $(A_{\gamma_0}, I_{\gamma_0})$ is the least fixed point of Φ, define $\Phi^{<\alpha}$ by:

$$
\begin{array}{lcll}
\Phi^{<0} & = & (\emptyset, \emptyset) & \\
\Phi^{<\alpha+1} & = & \Phi(\Phi^{<\alpha}) & \\
\Phi^{<\lambda} & = & \bigsqcup_{\beta<\lambda} \Phi^{<\beta} & \text{for limit } \lambda.
\end{array}
$$

and Φ^α as $\Phi^{<\alpha+1}$. Now the least fixed point of Φ is Φ^β where β is the least ordinal such that $\Phi^\beta = \Phi^{<\beta}$. It is immediately seen, by induction on α, that $\Phi^{<\alpha} = (A_{<\alpha}, I_{<\alpha})$ and $\Phi^\alpha = (A_\alpha, I_\alpha)$ for any α. We conclude that $(A_{\gamma_0}, I_{\gamma_0})$ is the least fixed point of Φ.

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As a matter of fact, since $X \subseteq \mathbb{N}$, we know that the minimal α such that $A^X_\alpha = A^X_{<\alpha}$ has to be countable, hence $\alpha < \omega_1$.

3 Limit of a pointing sequence

In the light of Definition 2.6 and Proposition 2.7 we know that a pointing sequence (X, f) might contain redundant information, namely the interior of active edges. We call the interior of active edges the retracted part of the sequence, and we extract the "main thread" of the pointing sequence by eliminating this retracted part: we call the result the *limit* of the sequence. We choose the limit terminology since our notion of limit turns out to be a generalization of the notion of limit by Gold [17].

Definition 3.1 (Limit) *Given a pointing sequence (X, f) with active-inactive sets of edges (A^X, I^X), define:*

1. $RF^X = \{x \in X \setminus \{0\} \mid]f(x), x[= \emptyset\}$*, the set of* retraction-free *edges in X (i.e., the set of edges with empty interior),*
2. $U^X = X \setminus (A^X \cup I^X \cup \{0\})$*, the* unstable *edges in X,*
3. $R^X = \bigcup_{x \in A^X}]f(x), x[_X$ *the* retracted *part of X,*
4. $L^X = X \setminus R^X$ *the* limit *of X.*

Unstable edges are neither active nor inactive. The limit L^X of a sequence (X, f) determines a new sequence: $(L^X, f \upharpoonright L^X)$. Indeed it is a well defined pointing sequence since, as we shall see in Theorem 3.6, this set is f-closed. We write $(L^X, f \upharpoonright L^X)$ simply as (L^X, f), or just L^X when f is understood. We shall also drop the superscript X when it is clear from the context.

In the example in Figure 4 we have $RF = \{1\} \subset \{1, 3\} = A$, while $I = \{2\}$. Active and inactive edges do form a partition of $X \setminus \{0\}$ in the finite case, so that $U^X = \emptyset$. However this is not true in general.

Lemma 3.2

1. *No edge is both active and inactive: $A^X \cap I^X = \emptyset$ for any (X, f),*

$$\begin{tikzcd} 0 & 1 \arrow[l] & 2 \arrow[ll, bend right] & 3 \arrow[ll, bend right] \end{tikzcd}$$

Figure 4: In this sequence the set of retraction free points is just $RF = \{1\}$.

2. *Assume in the step 0 of definition of A^X we only get empty edges. Then all active edges are empty, and there is no inactive edge: if $A_0^X = RF^X$, then $A^X = RF^X$ and $I^X = \emptyset$.*
3. *There exists some infinite (X, f) such that $U^X \neq \emptyset$.*

Proof.

1. We show, by induction over α, that $A_\alpha \cap I_\alpha$ is empty for all α. Suppose that $x \in A_\alpha \cap I_\alpha$: then $x \in I_\alpha$ implies that there exists y such that $y \curvearrowright x$ and $y \in A_{<\alpha}$; since $y \curvearrowright x$ and $x \in A_\alpha$, we also have that $y \in I_{<\alpha}$ hence $A_{<\alpha} \cap I_{<\alpha} \neq \emptyset$. Since we know that $A_\gamma \subseteq A_\beta$ and $I_\gamma \subseteq I_\beta$ whenever $\gamma < \beta$, this implies that there exists a $\beta < \alpha$ such that $A_\beta \cap I_\beta \neq \emptyset$, contradicting the induction hypothesis on β.
2. By induction over α. If $x \in I_\alpha$, then there exists $y \in A_{<\alpha}$ such that $y \curvearrowright x$; but this implies that $x \in \,]f(y), y[\,$, which contradicts $y \in A_{<\alpha} = RF$. On the other hand if $x \in A_\alpha$, then the induction hypothesis $I_{<\alpha} = \emptyset$ implies that $y \not\curvearrowright x$ for any y, that is $x \in A_0$ and $A_0 = RF$ by hypothesis, and therefore $A_\alpha \subseteq RF$ by the arbitrary choice of x: since it is always the case that $RF \subseteq A_\alpha$, we conclude.
3. Take $X = \mathbb{N}$ and let $f(1) = 0$ and $f(x) = x - 2$ for all $x \geq 2$. Then $RF^X = \{1\} = A_0^X$, so that $A^X = \{1\}, I^X = \emptyset$ by 2 of this lemma, and therefore $U^X = \mathbb{N} \setminus (\{0\} \cup A^X \cup I^X) = \mathbb{N} \setminus \{0, 1\} = \{x \in \mathbb{N} \mid x > 1\}$.

■

In fact, we can check that if $U^X \neq \emptyset$, then U^X itself is an infinite subset of X.

Lemma 3.3 *Let U be the set of unstable edges of (X, f). Then*

$$\forall x \in U\ \exists y \in U.\ y \curvearrowright x.$$

Therefore if $U \neq \emptyset$ there exists a chain of infinite cardinality $x_0 \curvearrowleft x_1 \curvearrowleft x_2 \cdots$ in U.

Proof. If $x \in U$, then $x \notin A \cup I$ by definition of U. By definition of A^X and I^X it follows that $\exists y \curvearrowright x.\ y \notin I$ and $\forall y \curvearrowright x.\ y \notin A$. Thus, $\exists y \curvearrowright x.\ y \notin I \cup A$. From $y \curvearrowright x$ we have $y > x$, hence $y \notin \{0\}$, so that, by definition of U, it must be the case that $y \in U$. By repeatedly applying this remark, if $U \neq \emptyset$ we construct an infinite sequence $x_0 \curvearrowleft x_1 \curvearrowleft x_2 \curvearrowleft x_3 \cdots$ all in U. By definition of $\curvearrowleft$, we have $x_0 < x_1 < x_2 < \ldots$, therefore the elements of this chain are pairwise distinct. ■

In the following, we aim at proving a kind of correctness theorem for the notion of limit, namely that (L^X, f) is always a pointing sequence. We have only to prove that L^X is f-closed, which is although not obvious because of the recursive definitions of active and inactive edges. We first establish two lemmas, in which we fix the sequence (X, f), writing just A, I and U for A^X, I^X, U^X.

Lemma 3.4 1. *If $x \in A$ and $x \curvearrowright y$ or $y \curvearrowright x$, then both $y \in I$ and $y \notin A \cup U$.*

2. *If $x, y \in A$, then the edges x, y are equal, or disjoint, or adjacent, or one strictly included in the other. Besides, $x \in]f(y), y[\Leftrightarrow]f(x), x[\subset]f(y), y[$.*
3. *If $x \in A$, then x is a maximal edge in $A \Leftrightarrow x$ is in the interior of no $y \in A$.*

Proof.

1. If $x \in A$ and $x \curvearrowright y$ then $y \in I$ because of the definition of I; if instead $y \curvearrowright x$ then $y \in I$ because of the definition of A. In both cases we have that $y \notin A$ by 1 of Lemma 3.2, and that $y \notin U$ by Definition 3.1.
2. By Lemma 2.3 on mutual positions, and the fact that the previous point forbids both $x \curvearrowleft y$ and $y \curvearrowleft x$. If $x \in]f(y), y[$ then the two edges are different, neither disjoint nor adjacent, and we have $y \notin]f(x), x[$. The only possibility left is $]f(x), x[\subset]f(y), y[$. Conversely, if $]f(x), x[\subset]f(y), y[$ then $f(y) \leq f(x) < x \leq y$, and $x \neq y$, therefore $x \in]f(y), y[$.
3. By the previous point, if $x, y \in A$, then the edge x is strictly included in the edge $y \Leftrightarrow$ the point x is in the interior of the edge y.

■

Lemma 3.5 *Inactive edges are definitely retracted: $I \subseteq R$; the limit always includes the first point: $0 \in L$.*

Proof. If $x \in I$ then $y \curvearrowright x$ for some $y \in A$, namely $x \in]f(y), y[$: hence $x \in R$ by Definition 3.1. On the other hand for no $x \in X$ it is the case that $0 \in]f(x), x[$, otherwise $f(x) < 0$. Hence $0 \in L$ by definition of L.

■

Theorem 3.6 *If (X, f) is a pointing sequence and L^X is its limit, then (L^X, f) is a pointing sequence as well.*

Proof. By Lemma 3.5, $L \neq \emptyset$. To see that it is f-closed, let $x \in L$ and suppose, toward a contradiction, that $f(x) \notin L$. By definition of L we know that $f(x) \in]f(y), y[$ for some $y \in A$; since $x \in L$ we have that $x \notin]f(y), y[$. We also know that $x \neq y$, since $f(x) \neq f(y)$. Therefore it must be the case that $f(y) < f(x) < y < x$, that is, $x \curvearrowright y$. By Lemma 3.4 and the fact that $y \in A$, it follows that $x \in I$, and hence $x \in R$ by Lemma 3.5, namely $x \notin L$, contradiction. ■

4 Pointing sequences and the learning process

The original motivation for developing the notion of limiting recursion in [17, 18] was to model the learning process. However the bare sequence of guesses considered by Gold, and representing the trace of all attempts by the learner, is too sketchy to describe sophisticated learning strategies, because it only includes the chronological ordering on guesses, and not also the "justification" (or "logical") ordering between guesses as we would like. In this section we propose learning sequences (pointing sequences with each point associated to a "guess"), in which the "justification" ordering between guesses is modelled by the backward pointer of the underlying pointing sequence. In a sense, pointers model the "memory" of the learner.

We move from the idea of Coquand's game theoretic interpretation of arithmetic [12, 13], in which pointing sequences are used to model the interaction between Myself and Nature. Hence we borrow some of intuitions and terminology from game theory, and see guesses as pairs of question/answers by the learner and the environment respectively, more than as putting forward a hypothesis.

Definition 4.1 (Learning Sequence) *A* learning sequence *is a triple* (X, f, γ) *where:*

1. (X, f) *is a pointing sequence, with* f *computable;*
2. $\gamma : X \to G$ *is a computable function associating points in* X *to* guesses *in* G.

We think of $X = \{x_0, x_1, x_2, \ldots\}$ as the time, which is discrete and linearly ordered, and of $\{\gamma(x_0), \ldots, \gamma(x_n)\}$, with the pointer structure described by f, as the state of memory at step x_n of the learning agent we are modelling. At each instant $x \in X$ the learner produces her guess $\gamma(x) \in G$, and a pointer $f(x)$ to the relative guess $\gamma(f(x))$; this extends the list of guesses $\gamma(0) = \gamma(f^n(x)), \ldots, \gamma(f^2(x)), \gamma(f(x)), \gamma(x)$ related to the thread of x, which is seen as a picture of the evidences motivating the current guess of the learner, and her subsequent acts.

No specific assumptions are made about the set G of guesses but the concreteness of its elements, as both f and γ are computable. We ask for the computability of f and γ to rule out non-realistic learners. In concrete examples f, γ are often defined by mutual recursion from their values on all $y < x$ with $y \in X$.

Assume x is in the thread of $y - 1$. Then the crossing relation $y \curvearrowright x$ can be seen as the retraction of the guess $\gamma(x)$ from the thread of y (in case it was included in it). However, since the guess $\gamma(y)$ can be itself retracted, for instance when the edge from y becomes inactive, it is still possible for the guess $\gamma(x)$ to be restored, provided that x is not inactivated elsewhere. This mechanism is intended to model the ability of the learner to suspend her judgment about some line of thoughts she had in the past, and possibly to resume it by changing her mind when this is suggested by some subsequent evidence.

The examples by which we illustrate our interpretation all come from applications of proof theory to computer science. Coquand [13] and Hayashi [19] discovered that non-constructive existence proofs of classical mathematics can be interpreted as (highly non-trivial) learning algorithms. Similar algorithms were already considered by Duval and others in [15]. These algorithms learn the object whose existence is stated in theorems, and their structure reflects the structure of the relative proofs. Learning algorithms from proof theory are, as we could expect, quite different from learning processes in other settings; nevertheless the underlying concepts are the same, up to a proper interpretation.

First in proofs we do not have guesses; rather their role is played by the hypotheses and their consequences. If a statement depends on certain hypothesis, it can be interpreted as a guess which is consistent with a chain of previous guesses, with the guess chain described by the justification function.

Were we about constructive reasoning only, each guess would depend on the one immediately before, and learning would be nothing else than direct computation of a retraction-free stream of guesses. The primary goal of our paper is, instead, to account for non constructive reasonings that use the excluded middle principle, or equivalently the *reductio ad absurdum*. When a proof uses reductio ad absurdum, once we arrive to a contradiction, this is viewed as a failure. The learner interpreting the proof backtracks to the point immediately before to the point in which she guessed that the absurd hypothesis was true, retracts all guesses depending on the absurd hypothesis, and guesses its negation. We say that the learner "has changed her mind".

To illustrate the concept of learning sequence and to see in which sense its limit might represent the object of learning, we go through some examples.

4.1 Simple Backtracking

According to [12], the act of backtracking to some previous guess is *simple* when the learner (there called "Myself") "never changes her mind about a value she has considered as wrong". As an example of simple backtracking we consider the winning strategy given in [12] for the formula $\exists x \forall y (A[x] \vee A^{\perp}[y])$, where $A^{\perp}$ is the dual of A and A is some decidable formula of arithmetic. We rephrase this strategy as a learning sequence $(\mathbb{N}, f, \gamma)$ as follows. We set $\gamma(0) = \perp$, meaning: "don't know". $\perp$ is some guess which is put in the step 0 only to make retractable the first meaningful guess (here the second one). For all $i > 0$, $\gamma(i)$ is a pair of natural numbers (n_i, m_i) against which we test the predicate $A[n_i] \vee A^{\perp}[m_i]$. We try to guess some n such that $\forall y. A[n] \vee A^{\perp}[y]$. We start by guessing $n = 0$, and we produce a flow $(n, m) = (0,0), (0,1), (0,2), \ldots$ of guesses to provide an evidence for $\forall y. A[0] \vee A^{\perp}[y]$. We proceed as long as $A[n] \vee A^{\perp}[m]$ is true, with each guess justified by the previous one. If $A[0]$ is true or $A[m]$ is false for all m, then $A[0] \vee A^{\perp}[m]$ is always true, therefore the learning sequence is $\perp, (0,0), (0,1), (0,2), \ldots$, with all arrows directed to the point immediately preceding and therefore all (interiors of) edges are empty. Otherwise, if $A[0], \ldots, A[m_0 - 1]$ are all false and $A[m_0]$ is true, then $A[0] \vee A^{\perp}[m_0]$ is the first false formula we found. In this case we backtrack to step 0 and we change n to 1, and we assume $\forall y. A[n] \vee A^{\perp}[y]$ for this new n. We then produce a new flow $(1,0), (1,1), (1,2), \ldots$ of guesses considered as an evidence for the truth of the formula $\forall y. A[1] \vee A^{\perp}[y]$. If $A[1] \vee A^{\perp}[m_0]$ is the next false formula we find, then we backtrack again to the step 0 and change n to 2, assuming this time $\forall y. A[1] \vee A^{\perp}[y]$. And so forth, until $n = m_0$, therefore $A[m_0] \vee A^{\perp}[m]$ is true for all m. In this case we eventually produce an infinite flow $(m_0, 0), (m_0, 1), (m_0, 2), \ldots$ of guesses as an evidence of $\forall y. A[m_0] \vee A^{\perp}[y]$.

Of course this is not the most clever way to learn the truth of the given formula: think of a different strategy such that, as soon as the m_0 above is discovered starts immediately to produce the flow $(m_0, 0), (m_0, 1), (m_0, 2), \ldots$ But this is not our point: we can represent both the former and the latter strategy, and although choosing among them would lead to the same limit, this is not the case in general.

Taking $m = m_0 - 1$, a picture of the first learning sequence is:

$$(0,0), (0,1), \ldots, (0,m), (1,0), (1,1), \ldots, (1,m), (2,0), \ldots$$

with a pointer from each guess $(0,0), (1,0), (2,0), \ldots$ to $\perp$, and all other arrows empty. Here is a picture of the learning sequence, where we put labels representing guesses above their indexes:

$$\begin{tikzcd}
\overset{\perp}{0} & \overset{(0,0)}{1} \arrow[l] & \cdots \arrow[l] & \overset{(0,m)}{m+1} \arrow[l] & \overset{(1,0)}{m+2} \arrow[llll, bend right] & \cdots \arrow[l] & \overset{(1,m)}{2m+2} \arrow[l] & \overset{(2,0)}{2m+3} \arrow[lllllll, bend right] & \cdots \arrow[l]
\end{tikzcd}$$

Note that there are no crossing edges here, since the only possible backtracking are to point 0. Therefore all edges are active and the limit is obtained by removing the interior of all edges. There exist only finitely many backtracking steps, though their number is in general not computable from A. There exists $i > 0$ such that for all $j > i$ we have $f(j) = j - 1$ and the pairs from point i up are $(m_0, 0), (m_0, 1), (m_0, 2), \ldots$: m_0 is the first x such that $\forall y (A[x] \vee A^{\perp}[y])$. Otherwise stated, the limit (in Gold sense) of the sequence obtained by taking the first coordinate of each pair exists and it is m_0. On the other hand if we take the limit of the learning sequence in our sense (see Def. 3.1), then we remove the interior of all edges, and we obtain the infinite sequence $(m_0, 0), (m_0, 1), (m_0, 2), \ldots$ whose first coordinate is m_0. Formally, for all $i > 0$ $\gamma(i) = (a_i, b_i)$ is defined together with $f(i)$ as follows:

1. $a_1 = b_1 = 0$, and $f(1) = 0$;
2. if $A[a_i] \vee A^{\perp}[b_i]$ is true then $a_{i+1} = a_i, b_{i+1} = b_i + 1$ and $f(i+1) = i$;
3. if $A[a_i] \vee A^{\perp}[b_i]$ is false then $a_{i+1} = a_i + 1, b_{i+1} = 0$ and $f(i+1) = 0$.

Since A is decidable, both f and γ are computable total functions.

4.2 The Constant Subsequence Principle

In the first example of this section the retraction of a guess can be only definitive. The second example exhibits a subtler and stronger use of backtracking than the first one, namely the ability of the learner of changing her mind about previously retracted guesses. In learning theory this ability is called *non-monotonic learning*.

By Proposition 2.7 there exists a measure α of how much involved backtracking can be. When $\alpha = 1$, the learner can decide that she was wrong but cannot retract such a decision, and we speak of 1-backtracking; when $\alpha = 2$, instead, the learner can decide she was wrong in believing she was wrong, and we will speak of 2-backtracking, and so forth (see [8] for more information about n-backtracking). We conjecture that if $U^X = \emptyset$ then α is a recursive ordinal, and the $\Sigma^0_{\alpha+1}$ sets are exactly the limits of learning sequences of complexity α and having $U^X = \emptyset$.

As a second example we propose an interpretation of a simple but relevant classical principle, the *constant subsequence principle*, which has been pointed out by Stolzenberg. This principle says that any infinite recursive sequence $s : \mathbb{N} \setminus \{0\} \to \{0, 1\}$ has some infinite constant subsequence, but we can restate it in many ways (see [6]). In combinatorics it is equivalent to König Lemma for r.e. trees, and to the fact that every recursive sequence: $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$ has some weakly increasing infinite subsequence. In Analysis, it is equivalent to the Subsequence Lemma for recursive sequences (every recursive sequence in $[a, b]$ has some convergent subsequence), to the Intermediate value Theorem for recursive continuous maps (if $f : R \to R$ is continuous and $f(a) < x < f(b)$, then $x = f(c)$ for some $c \in R$), and to many others famous theorems (see [28]). All these equivalences can be established by intuitionistic proofs though, clearly, none of these is a constructive principle.

Let us fix any infinite recursive sequence s; then we might express the constant subsequence principle as follows:

$$\forall x \exists y > x\ (s(y) = 0)\ \vee\ \forall x \exists y > x\ (s(y) = 1). \tag{1}$$

A proof by contradiction is then obtained by observing that, if we negate (1) then there exists an x which is not followed by 0's nor by 1's, which must be the last one of the infinite sequence s.

For the sake of constructing a sequence learning s, let us tentatively consider $(\mathbb{N}, f, \gamma)$, where γ is just s extended by $\gamma(0) = \perp$, and $f(x)$ is the last y such that $s(x) = s(y)$, and it is 0 if there is no such a y:

$$f(x) = \max\{y < x \mid y = 0 \vee s(x) = s(y)\}.$$

With this simple choice, however, it does not seem that a limit coinciding with s can be obtained in a similar manner as in the previous example. Indeed if $s(2n+1) = 1$ and $s(2n+2) = 0$ for all n then the learning sequence has the shape represented in Figure 5.

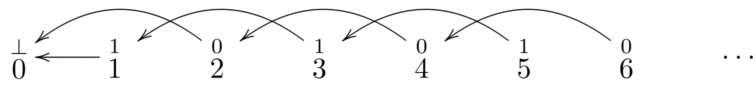

Figure 5: A tentative learning sequence when $s(2n+1) = 1$ and $s(2n+2) = 0$ for all n.

This is unfortunate since all non-empty edges in the sequence are inactivated and resumed infinitely many times, and in fact they are unstable according to Definition 3.1. Therefore the limit is no meaningful object (in fact, it is just the original sequence).

A proper solution takes a different attitude with respect to 0 and 1. We join two 0's no matter how far they are, and two 1's only if they are consecutive. Here is an example, in which no two 1's are consecutive:

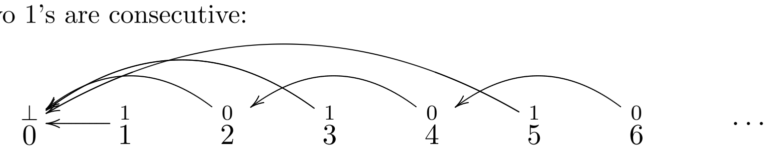

Figure 6: $\mathtt{CS}(s)$ when $s(2n+1) = 1$ and $s(2n+2) = 0$ for all n.

Of course the roles of 0 and 1 can be safely interchanged, but it seems that if we use sequential computations (as we do by choosing sequences, because the time line is totally ordered) then some asymmetry is unavoidable, as first remarked by Stolzenberg and later in [4].

The learning sequence $\mathtt{CS}(s)$ we will define is non monotonic, that is, it is an example of 2-backtracking. The learner starts with the hypothesis that there are infinitely many 0's. Whenever a 1 is met she retracts all previous guesses, and she starts a new thread made of consecutive 1's. But this is not definitive, since each time a new 0 is met she retracts *definitively* this thread of 1's, and resumes the hypothesis: "s has infinitely many 0's" , together with all previous 0's in the sequence. This reads as: "the learner thinks she was wrong in believing she was wrong about the existence of infinitely many 0's". Here is the formal definition:

Definition 4.2 *Let $s : \mathbb{N} \setminus \{0\} \to \{0,1\}$ be an infinite recursive sequence. Define $\mathtt{CS}(s) = (\mathbb{N}, f, \gamma)$ where γ is s extended by $\gamma(0) = \perp$, and*

$$f(x) = \begin{cases} \max\{y < x \mid y = 0 \vee \gamma(y) = 0\} & \textit{if } \gamma(x) = 0 \\ x - 1 & \textit{if } \gamma(x) = \gamma(x-1) = 1 \\ 0 & \textit{if } \gamma(x) = 1 \textit{ but } \gamma(x-1) \neq 1. \end{cases}$$

Figure 6 illustrates an initial segment of $\mathtt{CS}(s)$ in the case of $s(2n+1) = 1$ and $s(2n) = 0$ considered above.

Using learning sequences we get a meaningful limit even in the case of a non-monotonic learning. For instance, in the given example all edges are retracted, but only the edges labeled by 0 are resumed. More precisely for all n we have that $2n+2 \curvearrowright 2n+1$, while $2n+1$ crosses no edge, thus $2n+2$ is crossed by no edge. This means that $2n+2$ is active, while $2n+1$, being crossed by $2n+2$, is inactive. In other words we have $A = \{2n+2 \mid n \in \mathbb{N}\}$, and the interior of each $2n+2 \in A$ is $\{2n+1\}$. If we take as L the set $\mathbb{N} \setminus \bigcup_{x \in A}]f(x), x[$, namely eliminate the interiors of all active edges which are all $2n+1$ for $n \in \mathbb{N}$, we get $L = \{2n \mid n \in \mathbb{N}\}$. L is exactly the subsequence of all 0's (but for the first value which is $\perp$).

This learning sequence can be translated in term of limit iterated twice, but the result would be more involved and less natural.

For any recursive sequence s, the limit of the learning sequence is the subsequence of all 0's if there are infinitely many 0's, while if this is not the case, it is the longest suffix of s made only of 1's.

4.3 Testing for totality

As a further illustration of the notion of 2-backtracking let $\varphi : \{x \in \mathbb{N} \mid x > 0\} \to \mathbb{N}$ be some partial recursive function from positive to non-negative integers, of the form $\varphi(x) = \mu\, m.P(x, m)$ for some total recursive predicate P. We aim to define a sequence learning the graph of φ if φ is total; otherwise the sequence will be the learning of the first x such that $\varphi(x)$ is divergent. E.g.: $P(x, m) \Leftrightarrow h^m(x) = 1$ where $h(2x) = (2x)/2 = x$ and $h(2x + 1) = 3(2x + 1) + 1 = 6x + 4$. With the given P the function φ is total if and only if Collatz's conjecture about the $3x + 1$ problem is true, while the first x in which $\varphi(x)$ diverges would be the first counterexample to the conjecture, if any exists. To the time of writing, it is unknown whether Collatz conjecture holds, and therefore the limit of the learning sequence we construct exists in principle, but it is actually unknown.

We define a learning sequence $\mathtt{CL}(P) = (\mathbb{N}, f, \gamma)$ associated to φ (or more properly to its intentional definition via P) much in the same way people try to find some empirical evidence for or against Collatz conjecture by direct computation. At step 0 the learner makes the empty guess $\perp$ and the hypothesis that φ is total. At step 1 she assumes the incompatible hypothesis that $\varphi(1)$ is undefined. She maintains the latter as long as $\neg P(1, 0), \neg P(1, 1), \neg P(1, 2), \ldots$, producing the flow $(1, 0), (1, 1), (1, 2), \ldots$ of guesses considered as an evidence of the current hypothesis. If and when some m_1 is found such that $P(1, m_1)$ is true, she knows that $\varphi(1) = m_1$ converges. In such a case she retracts all guesses done before knowing that $\varphi(1) = m_1$, and restores the hypothesis that φ should be total. In the next step however, she retracts her hypothesis and the guess $(1, m_1)$ at the basis of its likelihood, this time assuming that $\varphi(2)$ is divergent, and keeping this as long as $\neg P(2, 0), \neg P(2, 1), \neg P(2, 2), \ldots$. Hence the flow of guesses $(2, 0), (2, 1), (2, 2), \ldots$ is generated to asses that $\varphi(2)$ diverges. If and when some m_2 is met such that $P(2, m_2)$ is true, the learner knows that $\varphi(2) = m_2$; she changes her mind once again reverting to the hypothesis that φ is total, and she "justifies" this by *resuming* the guess $(1, m_1)$ produced in advance. And so forth. This is 2-backtracking, because the learner is able to believe "to be wrong believing that she was wrong" about the fact that φ is total.

The learning sequence $\mathtt{CL}(P)$ is depicted in Figure 7, where we show the pointing structure by the arrows among the guesses.

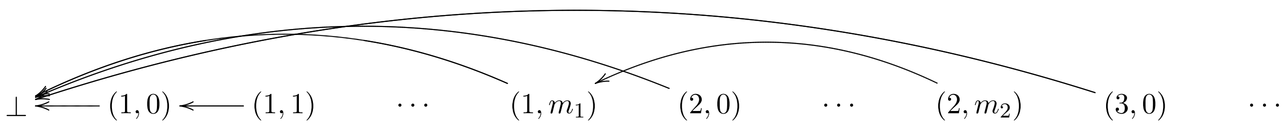

Figure 7: The learning sequence $\mathtt{CL}(P)$.

This is the formal definition of $\mathtt{CL}(P)$. We define $\gamma(0) = \perp$, and $\gamma(x) = (a(x), b(x))$ for all $x > 0$, with $a(1) = 1$ and $b(1) = 0$, and we put for all $x > 1$:

$$\gamma(x) = (a(x), b(x)) = \begin{cases} (a(x-1), b(x-1)+1) & \text{if } \neg P(a(x-1), b(x-1)) \\ (a(x-1)+1, 0) & \text{if } P(a(x-1), b(x-1)) \end{cases}$$

We define $f(1) = 0$ and for all $x > 1$:

$$f(x) = \begin{cases} x-1 & \text{if } \neg P(a(x), b(x)) \ \& \ \neg P(a(x-1), b(x-1)) \\ 0 & \text{if } \neg P(a(x), b(x)) \ \& \ P(a(x-1), b(x-1)) \\ \max(\{0\} \cup \{y \mid y < x \ \& \ P(a(y), b(y))\}) & \text{if } P(a(x), b(x)) \end{cases}$$

Let T be the largest thread $\perp, (1, m_1), (2, m_2), \ldots$ in $\mathtt{CL}(P)$ (we abuse terminology, identifying the sequence of guesses of the points in a thread with the thread itself), such that each (i, m_i) with $i > 0$ is followed by $(i+1, m_{i+1})$. If T is infinite then for all $n > 1$, each edge $(n, 0)$ such that $\neg P(n, 0)$ is crossed by some edge (n, m_n) such that $\neg P(n, m_n)$ and therefore is inactive. There is no other non-empty edge in the sequence, therefore by removing the interior of all edges of T we obtain T itself as the limit of the sequence. T gives (but for its first point $\perp$) the graph of φ, and it is a witness of the fact that φ is total and the Collatz conjecture is true.

Viceversa suppose that T is finite, ending in some (i, m_i). Then the next guess $(i+1, 0)$ is part of some infinite thread $D = \perp, (i+1, 0), (i+1, 1), (i+1, 2), \ldots$ such that $\neg P(i+1, 0), \neg P(i+1, 1), \neg P(i+1, 2), \ldots$. The edge from $(i+1, 0)$ to $\perp$ is the last non-empty edge, therefore it is active and it includes all non-empty edges, hence the limit of the sequence is obtained by removing the interior of $(i+1, 0)$, and it is D itself. D is an evidence of the fact that $\varphi(i+1)$ is divergent and $i+1$ is a counterexample to Collatz conjecture.

We do not know, today, which of the two threads T or D is the actual limit of $\mathtt{CL}(P)$. What we have is an effective construction, thought an implicit one, of the actual limit, which exists by a non constructive argument.

5 Limits and normalization

There is a jump from (X, f) to its limit (L^X, f), due to definition of A^X and I^X which requires the Fixed Point Theorem. To make this more concrete we introduce a normalization procedure, namely a process converging to L^X. As it should be expected, the complexity of the original X, which disappears in L^X, is mirrored by the complexity of the normalization procedure.

The basic idea is that we get L^X by progressively eliminating the interior of active edges x crossed by no edges, instead of removing all of the interior of all active edges in a single step. We will check that this process preserves the structure we have in X, that is, the f-closure of X and the original partition of edges of X into active, inactive and unstable. This process is already considered in [13], where, indeed, the process of cut elimination repeatedly removes the interior of edges not intersected by any other.

We formalize this procedure via a reduction relation $\twoheadrightarrow$ between learning sequences, which however should be understood as an equivalent definition rather than a computation of the limit. Indeed even the basic reduction step will be infinitary. This is first due to the infinitary nature of both X and L^X in general; but also, and more importantly, to the fact that the definitions of A^X and I^X are not even arithmetical in general. As a matter of fact it would not be sensible to take as basic reduction step the removal of the interior of an edge $x \in A^X$ because the logical complexity of the definition of the latter set is Π^1_1 in general. We start instead by removing the interiors of all non-empty edges $z \in A^X_0$.

Definition 5.1 (Removable Interiors) *The interior* $]f(x), x[_X$ *of an edge in the pointing sequence* (X, f) *is* removable *if it is non empty and* $x \in A_0^X$.

To explain the last definition, we say that the (non empty) interior of an edge x is removable if there exists no edge y such that $y \curvearrowright x$, that is, such that $f(y) \in\,]f(x), x[$ and $y \notin\,]f(x), x[$. The effect of the removal of the interior of x is to step from (X, f) to $(X \setminus\,]f(x), x[\ , f)$, which is a pointing sequence by construction. Figure 8 is an example in which we remove the interior of the edge 4.

$$\begin{tikzcd} 0 & 1 \arrow[l] & \cancel{2} \arrow[l] & \cancel{3} \arrow[l] & 4 \arrow[lll, bend right] & 5 \arrow[l] & 6 \arrow[l] & \cdots \end{tikzcd}$$

Figure 8: Removable edges.

We ask for the interior of a removable edge to be non empty since otherwise $X \setminus\,]f(x), x[$ is just X, in contrast to the idea of strict one-step reduction which is introduced below.

Definition 5.2 (Strict One Step Reduction) *Let* $\vec{x} = \{x_\gamma\}_{\gamma<\alpha} \subseteq \mathbb{N}$ *be any sequence, where* $\alpha > 0$ *is an arbitrary ordinal; then we say that a learning sequence* (X, f) *reduces in one step to* (Y, h) *w.r.t.* $\vec{x}$*, written* $(X, f) \rightarrow_{\vec{x}} (Y, h)$ *if:*

1. $h = f \restriction Y$,
2. $]f(x_\gamma), x_\gamma[$ *is removable in* X*, for all* $\gamma < \alpha$,
3. $Y = X \setminus \bigcup_{\gamma<\alpha}]f(x_\gamma), x_\gamma[$.

Lemma 5.3 *If* (X, f) *is a learning sequence and* $(X, f) \rightarrow_{\vec{x}} (Y, h)$ *for some sequence* $\vec{x}$*, then* (Y, h) *is a learning sequence.*

Proof. Immediate by definition: indeed if $y \in Y$ then $h(y) = f(y)$ and $f(y) \notin\,]f(x_\gamma), x_\gamma[$ for all $x_\gamma \in \vec{x}$ since the interior of x_γ is removable: so $h(y) \in Y = X \setminus \bigcup_{\gamma<\alpha}]f(x_\gamma), x_\gamma[$.

■

Since $\alpha > 0$, the sequence $\vec{x} = \{x_\gamma\}_{\gamma<\alpha}$ cannot be empty. Because of condition (1) of the above definition and Lemma 5.3, in $(X, f) \rightarrow_{\vec{x}} (Y, h)$ we can forget about the function h, since h is just the restriction of f to Y. Henceforth we fix a regressive function f and just write $X \rightarrow_{\vec{x}} Y$, assuming that X, Y are f-closed sets. Moreover we write $X \rightarrow Y$ when $X \rightarrow_{\vec{x}} Y$ for some sequence $\vec{x}$.

In all examples of Section 4, the relation $\rightarrow$ suffices to reach the limit. However, this is not true in general. There is a simple reason: like in a play with Mahjong tiles, the interiors of certain edges might become removable only after other interiors have been removed. For instance in the case of Figure 3 in Section 2 we have that the interior of 4 is removable, while that one of 2 is not, because of $3 \curvearrowright 2$. However once $]f(4), 4[$ has been removed, no edge is crossing 2 any more, so that the interior of 2 becomes removable in the next step.

The transitive closure of $\rightarrow$ solves the problem if X is finite, but not in general. In fact in the infinite case there is no upper bound to the number of edges inactivating an edge x only temporarily but not definitely. Therefore it could take infinitely many reductions to remove all edges temporarily inactivating x. This calls for the closure under transfinite chains of reductions.

Definition 5.4 (Reduction) *The reduction relation* $\twoheadrightarrow$ *among pointing sequences is the closure of* $\rightarrow$ *w.r.t.*

1. *transitivity,*
2. *denumerable intersections of chains: if* $X = X_0 \twoheadrightarrow X_1 \twoheadrightarrow \cdots$ *for denumerably many* X_i *then* $X \twoheadrightarrow \bigcap_{i\in\omega} X_i$.

We first check that reduction turns pointing sequences into pointing sequences.

Lemma 5.5 *If* X *is* f*-closed and* $X \twoheadrightarrow Y$ *then* $Y \subset X$*, and it is* f*-closed; hence* (Y, f) *is a pointing sequence.*

Proof. That $Y \subset X$ is an immediate consequence of the fact that if $X \to Y$ then $X \to_{\vec{x}} Y$ for some non empty $\vec{x}$. The proof that Y is f-closed is by induction over the definition of $\twoheadrightarrow$: the basis is Lemma 5.3; transitivity is obvious; for the case of denumerable intersections of chains just observe that any intersection of f-closed sets is f-closed.

■

In order to establish that reduction always attains the limit (Theorem 5.9), we prove the following claim: "if $X \twoheadrightarrow Y$, then active and inactive edges of Y are the active and inactive edges of X which belong to Y". We split this proof into two steps. In Lemma 5.6, we check that the claim holds if Y is obtained out of X by removing the interior of edges which are active in X (this proviso makes the statement much weaker). Then, in Lemma 5.7, we prove that, indeed, if $X \twoheadrightarrow Y$, then Y is obtained out of X by removing the interior of edges which are active in X. We will conclude in Theorem 5.9, showing that the normal form is the limit.

Lemma 5.6 *Let* $Z \subseteq A^X$ *and* $Z^* = \bigcup_{z\in Z}]f(z), z[\subseteq R^X$ *a subset of the retracted part of* X*. Assume* $Y = X \setminus Z^*$ *is a pointing sequence. Then:*

1. *If* $x \in X$ *is removed in* Y*, but the interior of* x *is not completely removed in* Y*, then the edge* x *is inactive: formally, if* $x \in Z^*$ *but* $]f(x), x[\not\subseteq Z^*$*, then* $x \in I^X$.
2. $A^Y_\alpha \subseteq A^X \cap Y$ *and* $I^Y_\alpha \subseteq I^X \cap Y$*, for all ordinal* α.
3. *If* $x \in X$ *is active and removed in* Y*, then the interior of* x *is removed in* Y*: formally, if* $x \in A^X$ *and* $x \in Z^*$ *then* $]f(x), x[\subseteq Z^*$.
4. $A^X_\alpha \cap Y \subseteq A^Y_\alpha$ *and* $I^X_\alpha \cap Y \subseteq I^Y_\alpha$*, for all ordinal* α.
5. $A^Y = A^X \cap Y$ and $I^Y = I^X \cap Y$.

Proof.

1. By definition if $x \in Z^*$ then $x \in]f(z), z[$ for some $z \in Z \subseteq A^X$. Either $f(x) \geq f(z)$, or $f(x) < f(z)$. In the first case $f(z) \leq f(x)$, therefore $]f(x), x[\subset]f(z), z[\subseteq Z^*$, contradicting the assumption $]f(x), x[\not\subset Z^*$. Thus, $f(x) < f(z)$. From $x \in]f(z), z[$ we conclude $z \curvearrowright x$. By assumption, $z \in A^X$. We conclude $x \in I^X$.
2. By induction on α. By induction hypothesis, for all $\beta < \alpha$ we have $A^Y_\beta \subseteq A^X \cap Y$ and $I^Y_\beta \subseteq I^X \cap Y$. By definition unfolding of $A^Y_{<\alpha}, I^Y_{<\alpha}$, we deduce $A^Y_{<\alpha} \subseteq A^X \cap Y$ and $I^Y_{<\alpha} \subseteq I^X \cap Y$.
   Let us first prove that $A^Y_\alpha \subseteq A^X \cap Y$, for which it suffices to show that $A^Y_\alpha \subseteq A^X$. Assume $y \in A^Y_\alpha \subseteq Y$, in order to prove, for all $x \in X$, $x \curvearrowright y$, that $x \in I^X$. If $x \in Y$, then, by definition unfolding of $y \in A^Y_\alpha$, we have $x \in I^Y_{<\alpha} \subseteq I^X \cap Y \subseteq I^X$. If $x \in X \setminus Y$, then $x \in I^X$ by $y \in Y$, $y \in]f(x), x[$ (hence $]f(x), x[\not\subset Z^*$), and part (1) of this lemma. Therefore $x \in I^X$ in both cases. We conclude $y \in A^X_\alpha$.
   It is also true that $I^Y_\alpha \subseteq I^X_\alpha \cap Y$, for which we just have to check that $I^Y_\alpha \subseteq I^X$. Indeed if $y \in I^Y_\alpha$ then, by definition unfolding, there exists $y' \in A^Y_{<\alpha}$ such that $y' \curvearrowright y$: now the fact that $A^Y_{<\alpha} \subseteq A^X \cap Y \subseteq A^X$, just proved, immediately yields that $y \in I^X$.

3. Assume for contradiction that $x \in A^X$, $x \in Z^*$ but $]f(x), x[\not\subseteq Z^*$. Then by point (1) above we have $x \in I^X$, contradicting (1) of Lemma 3.2.
4. By induction on α. By induction hypothesis, for all $\beta < \alpha$ we have $A^Y_\beta \supseteq A^X_\beta \cap Y$ and $I^Y_\beta \supseteq I^X_\beta \cap Y$, that is $A^Y_{<\alpha} \supseteq A^X_{<\alpha} \cap Y$ and $I^Y_{<\alpha} \supseteq I^X_{<\alpha} \cap Y$.

   We claim that $A^Y_\alpha \supseteq A^X_\alpha \cap Y$. Indeed, if $x \in A^X_\alpha \cap Y$, then for all $y \in Y$, $y \curvearrowright x$ we have $y \in I^X_{<\alpha}$ since $x \in A^X_\alpha$. By the fact that $y \in I^X_{<\alpha} \cap Y$ and $I^X_{<\alpha} \cap Y \subseteq I^Y_{<\alpha}$ we conclude $y \in I^Y_{<\alpha}$. Thus, $x \in A^Y_\alpha$.

   To finish the proof, we check $I^Y_\alpha \supseteq I^X_\alpha \cap Y$. If $y \in I^X_\alpha \cap Y$, then $x \curvearrowright y$ for some $x \in A^X_{<\alpha}$, hence $y \in]f(x), x[$. By $x \in A^X_{<\alpha}$, $y \in]f(x), x[$, $y \in Y$ (hence $]f(x), x[\not\subseteq Z^*$, and part (3) of this lemma, we deduce $x \notin Z^*$, that is, $x \in Y$. Thus, $x \in A^X_{<\alpha} \cap Y \subseteq A^Y_{<\alpha}$. We conclude $y \in I^Y_\alpha$.
5. By points (2) and (4), using the fact: $A^Y_\alpha = A^Y, A^X_\alpha = A^X, I^Y_\alpha = I^Y, I^X_\alpha = I^X$ from some α on.

■

We can now establish that active and inactive edges after a reduction are subsets of the active and inactive edges before the same reduction.

Lemma 5.7 *Assume $X \twoheadrightarrow Y$.*

1. $Y = X \setminus \bigcup_{z \in Z}]f(z), z[$ *for some* $\emptyset \neq Z \subseteq A^X$.
2. $A^Y = A^X \cap Y$ and $I^Y = I^X \cap Y$.

Proof.

1. By induction over the length of the reduction of $X \twoheadrightarrow Y$. In the basic case $X \rightarrow_{\vec{z}} Y$ for a sequence $\vec{z} \neq \emptyset$ of removable points, so that $Y = X - \bigcup_{z \in \vec{z}}]f(z), z[$. By definition of $\rightarrow_{\vec{z}}$, for any $z \in \vec{z}$ there exists no $y \in X$ such that $y \curvearrowright z$: hence $\vec{z} \subseteq A^X$ vacuously.

   Suppose that $X \twoheadrightarrow Y$ because $X \twoheadrightarrow V$ and $V \twoheadrightarrow Y$ for some V. By induction hypothesis we have:
$$\begin{array}{rcll} V & = & X \setminus \bigcup_{z \in Z_1}]f(z), z[& \text{for some } \emptyset \neq Z_1 \subseteq A^X \\ Y & = & V \setminus \bigcup_{z \in Z_2}]f(z), z[& \text{for some } \emptyset \neq Z_2 \subseteq A^V. \end{array}$$

   Hence
$$Y = X \setminus \bigcup_{z \in Z_1 \cup Z_2}]f(z), z[\; .$$

   By (2) of Lemma 5.6, $A^V \subseteq A^X$, therefore $\emptyset \neq Z_1 \cup Z_2 \subseteq A^X$ as desired.

   In the last case $X = X_0$ and $X_i \twoheadrightarrow X_{i+1}$ for all $i \in \omega$, and $Y = \bigcap_{i \in \omega} X_i$. Repeating the same reasoning as in the previous case, we know that $X_{i+1} = X_i \setminus \bigcup_{z \in Z_i}]f(z), z[$ for some $\emptyset \neq Z_i \subseteq A^{X_i} \subseteq A^X$ by induction. If $Z = \bigcup_{i \in \omega} Z_i$, then $\emptyset \neq Z \subseteq A^X$ and $\bigcap_{i \in \omega} X_i = X \setminus \bigcup_{z \in Z}]f(z), z[$.
2. By Lemma 5.6.5 and part (1) of this lemma.

■

We then prove a lemma saying that a pointing sequence is irreducible if and only if all its active edges are empty.

Lemma 5.8 *An f-closed set X is irreducible w.r.t. $\twoheadrightarrow$ if and only if $A^X = RF^X$.*

Proof. The if part is just a rephrasing of Lemma 5.7. Indeed by contraposition if $X \twoheadrightarrow Y$ then $Y = X \setminus \bigcup_{z \in Z}]f(z), z[$ for some $Z \subseteq A^X$; by Lemma 5.5 we know that $Y \subset X$, so that it must be the case that $]f(z), z[\neq \emptyset$ for some $z \in A^X$.

On the other hand if $X \not\twoheadrightarrow Y$ for any Y, then $]f(z), z[= \emptyset$ for all $z \in A_0^X$, since otherwise $X \rightarrow_z X \setminus]f(z), z[$. This means that $A_0^X = RF^X$, which implies, by (2) of Lemma 3.2, that $A^X = RF^X$, and we are done.

■

Theorem 5.9 *If $X \twoheadrightarrow Y$ and Y is irreducible, then $Y = L^X$.*

Proof. Recall that, by definition, $L^X = X \setminus \bigcup_{x \in A^X}]f(x), x[_X$. By Lemma 5.7, $Y = X \setminus \bigcup_{x \in Z}]f(x), x[_X$ for some $Z \subseteq A^X$. We deduce $\bigcup_{x \in Z}]f(x), x[_X \subseteq \bigcup_{x \in A^X}]f(x), x[_X$, hence $Y \supseteq L^X$.

To prove that $Y \subseteq L^X$, suppose toward a contradiction that there is some $y \in Y$ such that $y \in]f(x), x[_X$ for some $x \in A^X$. By part (3) of Lemma 5.6, we deduce $x \in Y$. But $A^Y = A^X \cap Y$ by (2) of the same lemma, so that $x \in A^Y$ and $y \in]f(x), x[_Y$, which contradicts Lemma 5.8 by the irreducibility of Y.

■

We eventually conclude that reduction is a way to achieve the limit. In fact on one hand an irreducible X coincides with its limit; on the other hand if X is reducible then the limit is the "normal form" of (X, f), which then exists and is unique.

Corollary 5.10 *For any learning sequence (X, f) either $X = L^X$ and X is irreducible, or $X \twoheadrightarrow L^X$.*

Proof. If $X = L^X$ then all active edges are empty, that is $A^X = RF^X$. By Lemma 5.8, X is irreducible and we are done. Assume $X \neq L^X$. Then by definition of limit there exists $x \in A^X$ such that $]f(x), x[\cap X \neq \emptyset$; by Lemma 5.8, this implies that X is reducible. We construct a sequence $\{X_i\}$ such that $X_0 = X$ and either $X_i \twoheadrightarrow X_{i+1}$ or X_i is irreducible. If there exists such an irreducible X_i (with $i > 0$) then $X \twoheadrightarrow X_i$ by transitivity and we conclude that $X_i = L^X$ by the Theorem 5.9. Otherwise $X \twoheadrightarrow X_\omega = \bigcap_{i<\omega} X_i$ by denumerable intersection of chains. We can continue with $X_\omega \twoheadrightarrow X_{\omega+1}$, $X_{\omega+1} \twoheadrightarrow X_{\omega+2}$, ..., and so forth. By definition of reduction, $\mathbb{N} \supset X \supset X_1 \supset X_2 \supset X_3 \ldots \supset X_\omega \supset X_{\omega+1} \ldots$ A cardinality reasoning shows that, for some $\alpha < \omega_1$, $X \twoheadrightarrow X_\alpha$ and X_α is irreducible, so that Theorem 5.9 applies, and we conclude $X_\alpha = L^X$.

■

6 A characterization of limits

In this section we associate an ordinal to pointing sequences that can be understood as a measure of their complexity and we characterize the pointing sequences having a retraction-free limit (a "meaningful" limit, in a sense) as those for which such a concept is well defined. Recall that a pointing sequence having retraction-free limit can be interpreted as a learning sequence reaching a stable form of knowledge in the limit.

The ordinal is 0 for retraction-free sequences, is 1 for monotonic learning; it is some ordinal $\alpha \geq 2$, for non-monotonic learning having a retraction-free limit. When this measure is not well defined, we obtain a limit with non-empty edges, representing some "unstable" state of belief.

The first step in our characterization is proving that if $X \twoheadrightarrow Y$, then $U^X = U^Y$, that is, U^X is invariant under reduction. We start with some simple remarks. As a consequence of the results in Section 5 we have that if $L = L^X$ is the limit of a learning sequence (X, f) then $A^L = A^X \cap L = RF^L$ and therefore $I^L = I^X \cap L = \emptyset$. What about U^L and U^X? It would be easy to prove that $U^L = U^X \cap L$, but we will in fact prove the stronger statement that $U^L = U^X$. We first check that the points of U^X are not in the retracted part of X.

Lemma 6.1 *If $x \in R^X$, that is, if $x \in]f(z), z[_X$ for some $z \in A^X$, then $x \notin U^X$.*

Proof. Toward a contradiction let $x \in U^X$. By Lemma 3.3 there exists an infinite chain $x = x_0 \curvearrowleft x_1 \curvearrowleft \cdots$ all in U^X. Since $]f(z), z[$ is finite there exists a minimum i such that $x_i \notin]f(z), z[$, which necessarily is greater than 0. By the choice of i, we have $x_{i-1} \in]f(z), z[$, and $x_i \geq z$. We necessarily have $x_i > z$, because $x_i = z$ contradicts $x_i \in U^X$ and $z \in A^X$. We claim that either $z \curvearrowright x_{i-1}$, or $x_i \curvearrowright z$: in both cases we have a contradiction with Lemma 3.4, because $x_i, x_{i-1} \in U^X$, while $z \in A^X$. If $f(x_{i-1}) < f(z)$, then by $x_{i-1} \in]f(z), z[$ we conclude $f(x_{i-1}) < f(z) < x_{i-1} < z$, that is, $z \curvearrowright x_{i-1}$. If $f(x_{i-1}) \geq f(z)$ then by $x_{i-1} \in]f(z), z[$ we deduce $]f(x_{i-1}), x_{i-1}[\subset]f(z), z[$. From $x_i \curvearrowright x_{i-1}$ we obtain $f(x_i) \in]f(x_{i-1}), x_{i-1}[\subset]f(z), z[$. From this latter and $x_i > z$ we conclude $x_i \curvearrowright z$, the required contradiction. ■

By definition unfolding, we are now in place to show that U^X is invariant under reduction.

Proposition 6.2 *If $X \twoheadrightarrow Y$ then $U^X = U^Y$.*

Proof. By Lemma 5.6 we know that $A^Y = A^X \cap Y$ and that $I^Y = I^X \cap Y$; therefore

$$\begin{array}{rcl} U^Y & = & Y \setminus (\{0\} \cup A^Y \cup I^Y) \\ & = & (X \cap Y) \setminus ((\{0\} \cap Y) \cup (A^X \cap Y) \cup (I^X \cap Y) \\ & = & (X \setminus (\{0\} \cup A^X \cup I^X)) \cap Y \\ & = & U^X \cap Y, \end{array}$$

hence $U^Y \subseteq U^X$. Now we prove $U^X \subseteq U^Y$. By Lemma 5.7, we have that $Y = X \setminus \bigcup_{z \in Z}]f(z), z[_X$ for some $Z \subseteq A_X$. If $U^X \not\subseteq U^Y$ then there exists $x \in U^X \setminus Y$ that is $x \in]f(z), z[_X$ for some $z \in A^X$, which is against Lemma 6.1. We conclude that $U^X \subseteq U^Y$, and the proposition follows. ■

We now formalize the idea of an edge inactivated by some another edge by a relation $\rhd_X$ over X we call the *undo* relation. The measure of complexity of a learning sequence will be the ordinal height of $\rhd_X$.

Definition 6.3 (Inactivation) *If $x, y \in X$, where (X, f) is a pointing sequence, then we say that x is* inactivated *by y, or that y undoes x, if $y \curvearrowright x$, and if x was active right before y:*

$$x \rhd_X y \Leftrightarrow x \in A^{[0,y[_X} \wedge y \curvearrowright x.$$

Note that, since y is active in $[0, y]_X$, as it is its last element, if $x \rhd_X y$ then $x \in I^{[0,y]_X}$. Also observe that if there is a chain of undo's starting from x, then each new undo changes the state of x from "active" to "inactive" and conversely. If there is an infinite chain of undo's ending in x, there is no obvious way of telling if x is active or inactive. Therefore it is reasonable to conjecture the following: the unstable edges (neither active nor inactive) are exactly the edges x such that there is an infinite chain of undo's starting from x; this will be the case for some x if the relation $\rhd_X$ is *not* well-founded.

We prove first that if X is a limit, then active edges are empty, inactive edges do not exist, and unstable edges, if any, are non-empty.

Lemma 6.4 *If $X = L^X$ then $A^X = RF^X$ and $I^X = \emptyset$.*

Proof. In general (that is for any X) if $]f(x), x[_X = \emptyset$ then there exist no $y \in X$ such that $y \curvearrowright x$: it follows that $x \in A^X$: therefore $RF^X \subseteq A^X$.

On the other in the particular case when $X = L^X$ we have that $A^X \subseteq RF^X = \{z \in X \mid\ ]f(z), z[_X = \emptyset\}$ since $X = L^X =$ (by definition)$X \setminus \bigcup_{z \in A^X}]f(z), z[_X$.

Now from the fact that $A^X = RF^X$ we conclude that for any $x \in X$ there exists no $y \in A^X$ such that $y \curvearrowright x$, since otherwise $x \in]f(y), y[_X$, so that $I^X = \emptyset$.

■

Lemma 6.5 *If $X = L^X$ then $U^X = \{x \in X \mid\ ]f(x), x[_X \neq \emptyset\}$.*

Proof. By Lemma 3.3 if $x \in U^X$ then there exists $y \in U^X \subseteq X$ such that $y \curvearrowright x$: hence $f(y) \in]f(x), x[_X$. Viceversa let $]f(x), x[_X \neq \emptyset$; the hypothesis that $X = L^X$ implies that $A^X = RF^X$ and $I^X = \emptyset$ by Lemma 6.4, hence $x \in X \setminus (\{0\} \cup A^X \cup I^X) = U^X$.

■

Next we see that the unstable edges are exactly those belonging to some infinite descending chain w.r.t. $\rhd_X$, so that U^X is empty if and only if the latter relation is well-founded. We first prove this when X is a limit.

Lemma 6.6 *If $X = L^X$ then U^X is the set of all $x \in X$ such that there exists an infinite descending chain $x = x_0 \rhd_X x_1 \rhd_X x_2 \rhd_X \cdots$.*

Proof. By Lemma 6.5 U^X consists of the edges with non empty interior. If $x \in U^X$ then by Lemma 3.3 there exists y such that $y \curvearrowright x$. Choose the first such y: then x is crossed by no edge in $[0, y[\, \cap X$, thus it is active in that interval and we have $x \rhd_X y$. On the other hand $y \in U^X$ since $x \in]f(y), y[_X \neq \emptyset$ so that we can iterate the same reasoning building an infinite descending chain $x = x_0 \rhd_X x_1 \rhd_X x_2 \rhd_X \cdots$.

Viceversa if there exists such descending chain, then $x \curvearrowleft x_1$, which implies that $f(x_1) \in]f(x), x[_X$. Then $x \in U^X$ by Lemma 6.5.

■

Then we aim to establish that Lemma 6.6 holds in the general case, namely also for non limit X. We first prove that reduction commutes with restriction to certain finite prefixes of the given sequence X; then that it commutes w.r.t. the undo relation, and eventually that it commutes with any infinite chain of undo's.

Lemma 6.7 *Let $y \in Y$ then:*

1. *if $X \to Y$ then either $X \cap [0, y] = Y \cap [0, y]$ or $X \cap [0, y] \to Y \cap [0, y]$, and $y \in X$,*
2. *if $X \twoheadrightarrow Y$ then either $X \cap [0, y] = Y \cap [0, y]$ or $X \cap [0, y] \twoheadrightarrow Y \cap [0, y]$, and $y \in X$.*

Proof. Let us first observe that $X \twoheadrightarrow Y$ implies $Y \subseteq X$, hence $y \in X$.

To prove part (1) recall that the set of removable edges in X is just A_0^X, hence we know that

$$Y = X - \bigcup_{z \in Z}]f(z), z[_X \quad \text{for some } Z \subseteq A_0^X.$$

The fact that $y \in Y$ gives $y \notin]f(z), z[_X$ for any $z \in Z$: hence either $y \leq f(z)$ or $z \leq y$ for all $z \in Z$. Let $Z' = \{z \in Z \mid z \leq y\}$, then if $X \cap [0, y] \neq Y \cap [0, y]$ we have $Z' \neq \emptyset$ and

$$Y \cap [0, y] = (X \cap [0, y]) \setminus \bigcup_{z \in Z'}]f(z), z[\,,$$

where we note that $X \cap [0, y]$ is f-closed (for $X = (X, f, v)$) since X is such and $[0, y]$ is an initial segment of $\mathbb{N}$, so that it is f-closed: the intersection of f-closed sets is f-closed. Now, because of $Z' \subseteq Z \subseteq A_0^X$, we conclude that $X \cap [0, y] \to Y \cap [0, y]$ as desired.

Part (2) is proved by induction over the definition of $\twoheadrightarrow$ using part 1 as the basic case. Transitivity is straightforward. Suppose that $X = X_0 \twoheadrightarrow X_1 \twoheadrightarrow \cdots$ for denumerably many X_i, and that $Y = \bigcap_{i \in \mathbb{N}} X_i$. By ind. hyp., for all i we have either $X_i \cap [0, y] \twoheadrightarrow X_{i+1} \cap [0, y]$, or $X_i \cap [0, y] = X_{i+1} \cap [0, y]$. Assume that for some i equality does not holds. Then by finite or denumerable intersection of chains we have:

$$X \cap [0, y] \twoheadrightarrow \bigcap_{i \in \mathbb{N}} (X_i \cap [0, y]) = Y \cap [0, y].$$

Assume equality holds for all i. Then we have: $X \cap [0, y] = \bigcap_{i \in \mathbb{N}} (X_i \cap [0, y]) = Y \cap [0, y]$. ■

We will now prove that reduction commutes with the undo relation: if $X \twoheadrightarrow Y$, then $\rhd_Y$ is the restriction to Y of $\rhd_X$.

Lemma 6.8 *If $X \twoheadrightarrow Y$ and $x, y \in Y$ then $x \rhd_X y$ if and only if $x \rhd_Y y$.*

Proof. Suppose that $x \rhd_X y$ and let $y' = \max(X \cap [0, y[)$ so that $X \cap [0, y'] = X \cap [0, y[$. We first claim that $y' \in Y$ (and therefore $Y \cap [0, y[= Y \cap [0, y']$). Indeed, by Lemma 5.7 there exists $Z \subseteq A^X$ such that $Y = X \setminus \bigcup_{z \in Z}]f(z), z[$. Assume for contradiction that $y' \notin Y$. Then $y' \in X \cap]f(z), z[$ for some $z \in Z$. Such a z cannot be greater than y since otherwise we had $f(z) < y' < y < z$ against the hypothesis that $y \in Y$. Such a z cannot be smaller than y, otherwise we had $z \leq y'$ by $z \in X$ and the choice of y' in X. This is in contradiction with $y' \in]f(z), z[$. Thus, $z = y$, and by $z \in Z$ we deduce $y \in Z$. We have $f(y) < x < y$ by $x \rhd_X y$. We conclude $x \notin Y$, contradiction.

Now suppose that $X \cap [0, y'] \neq Y \cap [0, y']$ (otherwise we conclude that $x \rhd_Y y$ trivially): since $y' \in Y$, by part (2) of Lemma 6.7 we have $X \cap [0, y'] \twoheadrightarrow Y \cap [0, y']$, so that by (2) of Lemma 5.6:

$$A^{Y \cap [0, y']} = A^{X \cap [0, y']} \cap Y \cap [0, y']. \tag{2}$$

It follows that $x \in A^{Y \cap [0, y']} = A^{Y \cap [0, y[}$, which together with $y \curvearrowright x$ implies $x \rhd_Y y$.

For the opposite implication, namely if $x \rhd_Y y$, we use again equation (2) to infer that $x \in A^{Y \cap [0, y[} \subseteq A^{X \cap [0, y[}$; we know that $y \curvearrowright x$, then $x \rhd_X y$ follows.

■

The last step before proving our characterization Theorem is to establish that any reduction $X \twoheadrightarrow Y$ preserves all infinite chains of the relation $\rhd_X$, if they exist.

Lemma 6.9 *If $X \twoheadrightarrow Y$ and $\mathcal{C} = \{x_0, x_1, \ldots\} \subseteq X$ is an infinite chain w.r.t. $\rhd_X$ then $\mathcal{C} \subseteq Y$, and $\mathcal{C}$ is also an infinite chain w.r.t. $\rhd_Y$.*

Proof. By induction over the definition of $\twoheadrightarrow$.

$X \to Y$: then $Y = X \setminus \bigcup_{z \in Z}]f(z), z[_X$ for some $Z \subseteq A_0^X$. We prove that $\mathcal{C} \cap \bigcup_{z \in Z}]f(z), z[= \emptyset$. For contradiction, let $x_i \in]f(z), z[$ for some $z \in Z \subseteq A_0^X$, with $i = \max\{j \mid x_j \in \mathcal{C} \cap]f(z), z[\}$. Because of the choice of x_i, we have $z \leq x_{i+1}$: were they equal, from $x_{i+2} \curvearrowright x_{i+1}$ and $x_{i+1} = z$, we immediately had $x_{i+2} \curvearrowright z$, contradicting $z \in A_0^X$ (no edge crosses z). Thus, $z < x_{i+1}$.

Claim: either $z \curvearrowright x_i$, or $x_{i+1} \curvearrowright z$.

Proof of the Claim Either $f(x_i) < f(z)$, or $f(x_i) \geq f(z)$. If $f(x_i) < f(z)$, then from $x_i \in]f(z), z[$ we deduce $z \curvearrowright x_i$. If $f(x_i) \geq f(z)$, then $]f(x_i), x_i[\subset]f(z), z[$, and from $x_{i+1} \curvearrowright x_i$ we deduce first $f(x_{i+1}) \in]f(x_i), x_i[$, then $f(x_{i+1}) \in]f(z), z[$ and eventually, from $z < x_{i+1}$, also $x_{i+1} \curvearrowright z$.

Proof of point 1 *from Claim* If $z \curvearrowright x_i$, then x_i is inactive in $X \cap [0, x_{i+1}[$, because it is crossed by z, and z is active in $X \cap [0, x_{i+1}[$. Indeed, no edge crosses z in X, hence, with more reason, crosses z in $X \cap [0, x_{i+1}[$. But the fact that x_i inactive in $X \cap [0, x_{i+1}[$ is in contradiction with the definition of $x_i \rhd_X x_{i+1}$. If $x_{i+1} \curvearrowright z$, we are instead in contradiction with $z \in A^X_0$ (no edge crosses z).

We conclude that $\mathcal{C}$ cannot intersect any interior $]f(z), z[_X$ with $z \in A^X_0$, and therefore $\mathcal{C} \subseteq Y$. From this and Lemma 6.8 it follows that $\mathcal{C}$ is also an infinite chain w.r.t. $\rhd_Y$.

$X \twoheadrightarrow Y$ because $X \twoheadrightarrow V \twoheadrightarrow Y$ for some V. By the first induction hypothesis $\mathcal{C} \subseteq V$ and it is an infinite chain w.r.t. $\rhd_V$; hence the thesis by the second induction hypothesis.

$X = X_0 \twoheadrightarrow X_1 \twoheadrightarrow \cdots$ for denumerably many X_i, and $Y = \bigcap_{i\in\mathbb{N}} X_i$. By a secondary induction over i and the principal induction hypothesis we know that $\mathcal{C} \subseteq X_i$ for all $i \in \mathbb{N}$, therefore $\mathcal{C} \subseteq Y$. That $\mathcal{C}$ is a chain w.r.t. $\rhd_Y$ is now a consequence of Lemma 6.8.

We can now characterize the unstable edges of any X as those for which the undo relation is not well-founded.

Theorem 6.10 *For any pointing sequence (X, f), U^X is the set of all $x \in X$ such that there exists an infinite descending chain $x = x_0 \rhd_X x_1 \rhd_X x_2 \rhd_X \cdots$.*

Proof. By Corollary 5.10 either $X = L^X$ or $X \twoheadrightarrow L^X$. In the first case the thesis is just Lemma 6.6. Suppose that $X \twoheadrightarrow L = L^X$. By Proposition 6.2 we have $U^X = U^L$, therefore U^X consists of all x such that there exists an infinite chain $x = x_0 \rhd_L x_1 \rhd_L x_2 \rhd_L \cdots$. By Lemma 6.8 this is the same as a chain w.r.t. $\rhd_X$, hence if $x \in U^X$ then there exists an infinite chain $x = x_0 \rhd_X x_1 \rhd_X x_2 \rhd_X \cdots$ out of it.

Viceversa, if $x = x_0 \rhd_X x_1 \rhd_X x_2 \rhd_X \cdots$ is an infinite chain then it belongs to L by Lemma 6.9, so that $x = x_0 \rhd_L x_1 \rhd_L x_2 \rhd_L \cdots$ is an infinite chain by Lemma 6.8, and we conclude that $x \in U^L = U^X$. ■

As immediate consequences of the theorem and of Lemma 3.4.3, we can characterize the sets A^L, I^L, U^L of the limit L of X:

Corollary 6.11 *For any pointing sequence (X, f), $A^X \cup I^X$ is the set of all $x \in X$ which do not belong to any infinite chain w.r.t. $\rhd_X$.*

Corollary 6.12 *Let (X, f) be a pointing sequence, and $L = L^X$ its limit. Then $A^L = A^X \cap L$ is the set of those edges x which are empty in L and maximal w.r.t. inclusion in A^X; $I^L = \emptyset$, and $U^L = U^X$ is the set of all non empty edges in L (if any), that is:*

$$L^X = RF^L \Leftrightarrow U^X = \emptyset.$$

7 Related works

A primary source of the present research is Coquand's semantics of evidence for classical arithmetic [13], where the notion of limit is implicit in the cut-elimination procedure on plays, which is essentially our reduction relation on learning sequences. More on the relation between cut-elimination on plays and cut-elimination on proofs has been investigated e.g. in [20].

Pointing sequences are possibly infinite plays in the sense of [2, 21, 23]. Of these, because of a careful study of the structure of the replies in a play and of the explicit definition of the concept of view, the paper [21] is especially relevant to us. A game, there called "computational arena", is a set of rules defining the justification relation among moves, which are divided into questions and answers. To define both a play and positions in a play, the notion of "well formed sequence" is used: it is a sequence of moves each equipped with a pointer to some previous move in the sequence, but in the case of the first one. To be well formed, the sequence has to satisfy certain conditions that impose that a move always replies to some previous move, and "justifies" this latter, and, more importantly, that Gandy's "no dangling-question-mark condition" is satisfied, namely that the last asked question is answered first.

The latter condition, also called the "bracketing condition" in [2], does not entail that in a well formed sequence crossing edges cannot exist. Consider the arena associated to the type $(N \to N) \to N$ (using the intuitionistic arrow in place of the linear one to avoid the use of exponentials, as e.g. in [25]) and the functional $F(f) = f(f(0))$. A play where Player defends F according to a call-by-value strategy does not involve any crossing edge; but if we switch to a call-by-name evaluation strategy, Player's reply to Opponent's question: "what is the value of $f(f(0))$?" is a question about the value of the outermost f; Opponent's reaction is now asking about its argument, that is the value of $f(0)$, which forces Player to suspend his answer, continuing with a new question about the value of the innermost f. As soon as the latter comes, the pointer of Player's reply to the suspended question crosses Player's own question about the value of the innermost f. As a matter of fact the crossed edge disappears in the Opponent view right after, together with all the interior to which it belongs.

There is a tight connection between the limit of a finite pointing sequence and the view of a player on turn in a position of a play of [21]. More, the concept of view remains well defined even for plays of transfinite length [7, 9], but the idea of using limits to extract the content of an infinite dialogue seems new, as well as the characterization results we present in Section 6.

In [12], a preliminary version of [13], the author explicitly hinted at an interpretation of plays and strategies in terms of learning. The notion of backtracking as a retraction of previous guesses is sketched there, and has been investigated in the case of 1-backtracking in [8]. In the simpler cases this matches with Gold's ideas of learning in the limit [17, 18] and with Hayashi Limit-Computable Mathematics [19], though the latter were deliberately limited to classical theorems of low logical complexity.

To go behind Δ^0_2 problems, in fact, one has to resort to Schubert's iterated limits [26], which have been proved to cover exactly the arithmetical hierarchy [14]. With respect to iterated limits and to the latter result we improve by giving a more direct construction of the limit itself, which seems to be suitable to model more refined strategies as are implicit in actual non constructive proofs. Moreover we know that the set of true arithmetical formulas is the limit of a suitable learning sequence [5], hence the class of learnable sets is not included into any level of the arithmetical hierarchy, so that limits in the sense of the present paper and iterated limits are non equivalent concepts: if our conjecture that iterated limits are a particular case of learning sequences is true, then we have a proper extension of the notion of Schubert. More precisely we conjecture that learnable sets are exactly the Δ^1_1 sets, and that the sets learnable by a learning sequence whose undo relation has height k are exactly the Σ^0_{k+1} sets.

Gold's work is considered as the mathematical foundation of algorithmic learning theory, a research area involving investigations by philosophers and computer scientists, who call it machine learning. Although there exists a large variety of approaches in the field, there are some common traits illustrated e.g. in [24]. The "guesses" of a learning sequence in our sense can be seen as a training set, where the instances are provided via an "incremental method". This consists of an "infinite sequence of evidence items" [27], but differently than in machine learning models, the exhibition of the evidences is caused by the interaction of the learner with the environment: in our setting she is indeed an active agent, posing questions and reacting to answers given by the "nature". On the other hand the environment is supposed to provide honest replies, playing the role of a supervisor in a training process. It is important to remark that we think of the environment as a reliable actor, who might though behave nondeterministically, say because is himself learning from learner guesses, and challenging her hypothesis in some clever way.

In the setting of the present work the "hypothesis" is that a certain claim is true, namely classically valid (in this sense we do not have a set of hypotheses nor a bias to restrict the "version space"). But at the same time the learner acquires some knowledge which is the limit in our sense: hence an infinite object is learned, which is actually a function, and which belongs to an infinite set.

The limit is an ideal entity, which is however effectively constructed by the learner, but it is not computable in general, and it remains implicit in the memory of the learner who in general is not aware that her knowledge has become stable (the "no bell tolls" for a scientific discovery of the pragmatist philosopher William James quoted in [27]). Hence in our model the learner is an "extrapolating machine", rather than an "inductive inference machine" in the sense of [11]. There is however an essential difference between our model and those usually considered in machine learning theory: it is the fact that we model the memory of the learner by recording how her attitude with respect to the hypothesis evolves via a pointing sequence and the related notion of thread. On the contrary in learning theory the learner is a black box, whose internal states are perfectly irrelevant. She can change her mind during the learning process, but there is no description of the logical dependance of her subsequent guesses. This enforces the adoption of a notion of learning in the limit which is just stabilization, at least in the discrete case. It follows that the "limit" considered in machine learning theory is essentially Gold's notion, and there is no room for considering the hierarchy of backtracking as it emerges from the main results of the present paper.

8 Conclusion

We have studied pointing sequences of countable length and defined a notion of limit which is essentially the main "thread" of the sequence. We motivate our study by the idea of modeling non constructive reasoning by means a concept of learning, namely learning sequences and their limits, extending the constructive interpretation from intuitionistic to classical arithmetic. We also investigate the structure of the limit and show that, when it has a definite content, there exists a measure of its logical complexity, which is an ordinal.

Much remains to be done. First a full characterization of the learnable functions, or sets, is in order: we conjecture that they are exactly Δ^1_1 concepts, namely that learning sequences are a model of predicative reasoning. Second we could relax the sequentiality assumption, namely that backtracking to some point implies the retraction of all intermediate points (the interior of the edge). This could be accomplished by considering partial orders instead of sequences. Learning sequences are our view of strategies in games, hence they are candidates for interpreting proofs: the limit is then the computational content of a proof. A formal study of this interpretation is indeed the final goal of our study. While seeking that, it is not secondary to clarify and to further investigate the relation between the proposed interpretation of classical arithmetic and the topological foundations of machine learning.

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