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TOWARDS A COMBINATORIAL REPRESENTATION THEORY FOR THE RATIONAL CHEREDNIK ALGEBRA OF TYPE $G(r, p, n)$

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Abstract This paper aims to lay the foundations for a combinatorial study, via orthogonal functions and intertwining operators, of category $\mathcal{O}$ for the rational Cherednik algebra of type $G(r, p, n)$. As a first application, a self-contained and elementary proof of the analogue for the groups $G(r, p, n)$, with $r > 1$, of Gordon's Theorem (previously Haiman's Conjecture) on the diagonal co-invariant ring is given. No restriction is imposed on p; the result for $p \neq r$ has been proved by Vale using a technique analogous to Gordon's. Because of the combinatorial application to Haiman's Conjecture, the paper is logically self-contained except for standard facts about complex reflection groups. The main results should be accessible to mathematicians working in algebraic combinatorics who are unfamiliar with the impressive range of ideas used in Gordon's proof of his theorem.

Keywords: complex reflection groups; Catalan numbers; diagonal co-invariant rings; Jack polynomials

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Secondary 33C52; 33C80

1. Introduction

The aim of this paper is twofold. First, we introduce intertwining operators for the rational Cherednik algebra (RCA) $\mathbb{H}$ of type $G(r, p, n)$ and carry out sufficient calculation of the relations they satisfy to be useful for a combinatorial study of the representation theory of $\mathbb{H}$. This work forms the basis for the sequels [**15**, **16**], where we study the combinatorics of the ordinary co-invariant ring and begin the combinatorial study of the lattice of submodules of each standard module for the rational Cherednik algebra. Our goal is the construction of canonical bases for the standard modules $M(V)$ and their composition factors, including the irreducible quotients $L(V)$.

Second, we use the intertwining operators to give a new and self-contained (modulo standard facts about complex reflection groups) proof of the analogue of Gordon's Theorem [**12**, **25**] on the diagonal co-invariant ring for $G(r, p, n)$. The proof here works only when $r > 1$ but does not place any restriction on p; Vale [**25**] establishes the result for $p \neq r$. We use neither the KZ functor and cyclotomic Hecke algebras (as in [**12**, **25**]) nor degeneration from the double affine Hecke algebra (as in [**5**]) as there is no double affine Hecke algebra available for the groups $G(r, p, n)$ with $r > 2$, although we found the ideas of those papers inspirational. The existence of this paper also owes much to the foundational papers [**9**, **11**].

One of our goals being self-containment, we begin in § 2 with definitions and a sketch of the proof of the Poincaré–Birkhoff–Witt Theorem for rational Cherednik algebras. In § 3 we explain the formalism that connects the rational Cherednik algebra to quotients of the diagonal co-invariant ring of the type conjectured by Haiman [**17**]. In § 4 we specialize to the case of the groups $G(r, p, n)$ and review the construction given in [**9**] of an important commutative subalgebra of the rational Cherednik algebra $\mathbb{H}$. We study the intertwining operators and their basic properties in § 5 and study their action on a particular basis of the polynomial representation in § 6. We determine the submodule structure of the polynomial representation of $G(r, p, n)$ in the cases that we will use for the study of the diagonal co-invariant ring in § 7. The results of § 7 are similar to, but more detailed than, those contained in [**6**]. The extra detail is crucial for the material of § 8, where we study the $\mathbb{H}$-modules fulfilling the requirements of § 3 and relevant to the diagonal co-invariant ring.

2. Definitions and the Poincaré–Birkhoff–Witt Theorem

Let V be a finite-dimensional vector space over a field k, and let $W \subseteq \mathrm{GL}(V)$ be a finite subgroup. Let TV be the tensor algebra of V and let kW be the group algebra of W over k, with basis t_w for $w \in W$ and multiplication $t_w t_v = t_{wv}$. The *semi-direct product* $TV \rtimes W$ is $TV \otimes_k kW$ with multiplication

$$(f \otimes t_w)(g \otimes t_v) = f(wg) \otimes t_{wv} \quad \text{for } f, g \in TV \text{ and } w, v \in W. \tag{2.1}$$

From now on we will drop the tensor signs when it will not cause confusion. Fix a collection of skew-symmetric forms indexed by the elements of W:

$$\langle \cdot\,, \cdot \rangle_w : V \times V \to k \quad \text{for } w \in W. \tag{2.2}$$

The *Drinfel′d–Hecke algebra* $\mathbb{H}$ corresponding to these data is the quotient of the algebra $TV \otimes_k kW$ by the relations

$$xy - yx = \sum_{w \in W} \langle x, y \rangle_w t_w \quad \text{for } x, y \in V. \tag{2.3}$$

Now let $\mathfrak{h}$ be a finite-dimensional k-vector space. A *reflection* is an element $s \in \mathrm{GL}(\mathfrak{h})$ such that $\mathrm{codim}(\mathrm{fix}(s)) = 1$. A *reflection group* is a finite subgroup $W \subseteq \mathrm{GL}(\mathfrak{h})$ that is generated by the set of reflections it contains. Assume now that W is a reflection group, let T be the set of reflections it contains, and put $V = \mathfrak{h}^* \oplus \mathfrak{h}$. Write $\langle x, y \rangle = x(y)$ for $x \in \mathfrak{h}^*$ and $y \in \mathfrak{h}$. For each $s \in T$, fix $\alpha_s \in \mathfrak{h}^*$ and $\alpha_s^\vee \in \mathfrak{h}$ with

$$sx = x - \langle x, \alpha_s^\vee \rangle \alpha_s \quad \text{for all } x \in \mathfrak{h}^*, \tag{2.4}$$

and define skew symmetric bilinear forms $\langle \cdot\,, \cdot \rangle_s$ on V by the requirements

$$\langle x, y \rangle_s = \begin{cases} 0 & \text{if } x, y \in \mathfrak{h} \text{ or } x, y \in \mathfrak{h}^*, \\ c_s \langle \alpha_s, y \rangle \langle x, \alpha_s^\vee \rangle & \text{if } x \in \mathfrak{h}^* \text{ and } y \in \mathfrak{h}, \end{cases} \tag{2.5}$$

where $c_s \in k$ satisfy $c_{wsw^{-1}} = c_s$ for all $s \in T$ and $w \in W$. It is straightforward to check that $\langle \cdot , \cdot \rangle_s$ does not depend on the choice of α_s and $\alpha_s^\vee$ satisfying (2.4). We extend the pairing $\langle \cdot , \cdot \rangle$ between $\mathfrak{h}$ and $\mathfrak{h}^*$ to a symplectic form on V by requiring that $\mathfrak{h}$ and $\mathfrak{h}^*$ be isotropic.

The *rational Cherednik algebra* corresponding to a reflection group W is the quotient of the semi-direct product $TV \rtimes W$ by the relations

$$yx - xy = \kappa \langle x, y \rangle - \sum_{s \in T} \langle x, y \rangle_s t_s \quad \text{for } x, y \in V, \tag{2.6}$$

where $\kappa \in k$. It is thus a special case of the Drinfel′d–Hecke algebra. Note that by the definitions of $\langle \cdot , \cdot \rangle$ and $\langle \cdot , \cdot \rangle_s$, we have

$$yx - xy = 0 \quad \text{if } x, y \in \mathfrak{h} \text{ or } x, y \in \mathfrak{h}^*, \tag{2.7}$$

and hence there are canonical maps $S(\mathfrak{h}) \to \mathbb{H}$ and $S(\mathfrak{h}^*) \to \mathbb{H}$.

Now assume that $\mathbb{H}$ is the Drinfel′d–Hecke algebra associated to a collection $\langle \cdot , \cdot \rangle_w$ of skew symmetric forms as above. We say that *the PBW Theorem holds for* $\mathbb{H}$ if, for any basis $x_1, \dots , x_n$ of V, the collection $\{x_{i_1} x_{i_2} \cdots x_{i_p} t_w \mid 1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_p \leqslant n,\ w \in W\}$ is a basis for $\mathbb{H}$. The following theorem was stated in [**8**] and many proofs have now appeared (see, for example, [**2**, **10**, **11**, **18**, **23**]). Our proof has the virtue of working in arbitrary characteristic; more importantly, it is conceptually extremely simple.

Theorem 2.1 (the Poincaré–Birkhoff–Witt Theorem for Drinfel′d–Hecke algebras). *The Poincaré–Birkhoff–Witt (PBW) Theorem holds for* $\mathbb{H}$ *if and only if the following two conditions hold:*

(a) $\langle vx, vy \rangle_{vwv^{-1}} = \langle x, y \rangle_w$ *for all* $x, y \in V$ *and* $v, w \in W$, *and*

(b) $\langle x, y \rangle_w (wz - z) + \langle y, z \rangle_w (wx - x) + \langle z, x \rangle_w (wy - y) = 0$ *for all* $x, y, z \in V$ *and* $w \in W$.

Proof. First assume that the PBW Theorem holds for $\mathbb{H}$. Let $x, y \in V$ and $v \in W$. Equating coefficients on both sides of

$$\sum_{w \in W} \langle vx, vy \rangle_w t_w = [vx, vy] = t_v [x, y] t_v^{-1} = \sum_{w \in W} \langle x, y \rangle_w t_{vwv^{-1}} \tag{2.8}$$

implies that (a) holds. Let $x, y, z \in V$. By the Jacobi identity,

$$\begin{aligned} 0 &= [[x, y], z] + [[y, z], x] + [[z, x], y] \\ &= \left[\sum_{w \in W} \langle x, y \rangle_w t_w, z \right] + \left[\sum_{w \in W} \langle y, z \rangle_w t_w, x \right] + \left[\sum_{w \in W} \langle z, x \rangle_w t_w, y \right] \\ &= \sum_{w \in W} (\langle x, y \rangle_w (wz - z) + \langle y, z \rangle_w (wx - x) + \langle z, x \rangle_w (wy - y)) t_w. \end{aligned}$$

Now, equating coefficients of t_w on both sides implies that (b) holds.

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Conversely, assume that (a) and (b) hold. The defining relations for $\mathbb{H}$ evidently imply that, given any basis $x_1, x_2, \ldots, x_n$ of V, the set $\{x_{i_1} x_{i_2} \cdots x_{i_p} t_w \mid 1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_p \leqslant n,\ w \in W\}$ spans $\mathbb{H}$. We will show that these elements are also linearly independent by mimicking the standard proof of the PBW Theorem for universal enveloping algebras of Lie algebras: we construct the module that ought to be the left regular representation of $\mathbb{H}$. Let M be the vector space with basis consisting of the words $\{x_{i_1} x_{i_2} \cdots x_{i_p} t_w \mid 1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_p \leqslant n,\ w \in W\}$. Define operators l_x and l_v on M for $x \in V$ and $v \in W$ inductively as follows:

$$l_x \cdot t_w = x t_w, \qquad l_v \cdot t_w = t_{vw}, \tag{2.9}$$

and, for $p \geqslant 1$,

$$l_{x_i} \cdot x_{i_1} \cdots x_{i_p} t_w = \begin{cases} x_i x_{i_1} \cdots x_{i_p} t_w & \text{if } i \leqslant i_1, \\ l_{x_{i_1}} \cdot l_{x_i} \cdot x_{i_2} \cdots x_{i_p} t_w + \sum_{v \in W} \langle x_i, x_{i_1} \rangle_v l_v \cdot x_{i_2} \cdots x_{i_p} t_w & \text{if } i > i_1, \end{cases} \tag{2.10}$$

and

$$l_v \cdot x_{i_1} \cdots x_{i_p} t_w = l_{v x_{i_1}} \cdot l_v \cdot x_{i_2} \cdots x_{i_p} t_w. \tag{2.11}$$

A straightforward but lengthy calculation shows that these operators satisfy the defining relations for $\mathbb{H}$. It follows that M is an $\mathbb{H}$-module, with x acting by l_x and t_w acting by l_w. Suppose that there is a relation in $\mathbb{H}$ of the form

$$\sum_{\substack{w \in W, \\ 1 \leqslant i_1 \leqslant \cdots \leqslant i_p \leqslant n}} a_{i_1 \cdots i_p, w} x_{i_1} \cdots x_{i_p} t_w = 0,$$

with $a_{i_1 \cdots i_p, w} \in k$. Applying both sides of this relation to the element $1 = t_1 \in M$ implies that all the coefficients $a_{i_1 \cdots i_p, w}$ are zero, and the proof is complete. □

Corollary 2.2. *Let $\mathbb{H}$ be the rational Cherednik algebra corresponding to the reflection group W. Then the multiplication map $S(\mathfrak{h}^*) \otimes_k S(\mathfrak{h}) \otimes_k kW \to \mathbb{H}$ is an isomorphism.*

Proof. The result follows from the previous theorem once we check that conditions (a) and (b) of that theorem hold. Condition (a) is straightforward to verify. Condition (b) holds trivially for $w \notin T$. If $s \in T$ and $z \in \mathrm{fix}_V(s)$, then the definitions (2.4) and (2.5) imply $\langle z, x \rangle_s = 0$ for all $x \in V$. Therefore, the radical of $\langle \cdot, \cdot \rangle_s$ has codimension at most 2. If $\langle \cdot, \cdot \rangle_s \neq 0$, then we may choose $x, y \in V$ with $\langle x, y \rangle_s = 1$, and for any $z \in V$ we have

$$z = \langle x, z \rangle_s y - \langle y, z \rangle_s x + f \quad \text{with } f \in \mathrm{fix}_V(s). \tag{2.12}$$

Applying $s - 1$ to both sides and rearranging terms shows that the identity (b) holds for x, y and arbitrary z. In general, identity (b) holds trivially if every two of x, y, z are linearly dependent modulo the radical of $\langle \cdot, \cdot \rangle_s$, so we are reduced to the case just treated. □

There is an important filtration of $\mathbb{H}$ defined by

$$\mathbb{H}^{\leqslant m} = k\text{-span}\{x_{i_1} \cdots x_{i_l} t_w \mid l \leqslant m,\ w \in W \text{ and } x_{i_j} \in \mathfrak{h}^* \oplus \mathfrak{h}\}. \tag{2.13}$$

By the PBW Theorem for $\mathbb{H}$, the associated graded algebra of $\mathbb{H}$ with respect to this filtration is the semidirect product $S(\mathfrak{h}^* \oplus \mathfrak{h}) \rtimes W$.

Our next proposition is a fundamental computation. It expresses some commutators in $\mathbb{H}$ as linear combinations of derivatives and divided differences of elements of $S(\mathfrak{h}^*)$ and $S(\mathfrak{h})$. For $y \in \mathfrak{h}$, we write ∂_y for the derivation of $S(\mathfrak{h}^*)$ determined by

$$\partial_y(x) = \langle x, y\rangle \quad \text{for } x \in \mathfrak{h}^*, \tag{2.14}$$

and we define a derivation ∂_x of $S(\mathfrak{h})$ analogously.

Proposition 2.3. *Let $y \in \mathfrak{h}$ and $f \in S(\mathfrak{h}^*)$. Then*

$$yf - fy = \kappa \partial_y f - \sum_{s \in T} c_s \langle \alpha_s, y\rangle \frac{f - sf}{\alpha_s} t_s. \tag{2.15}$$

Similarly, for $x \in \mathfrak{h}^*$ and $g \in S(\mathfrak{h})$, we have

$$gx - xg = \kappa \partial_x g - \sum_{s \in T} c_s \langle x, \alpha_s^\vee\rangle t_s \frac{g - s^{-1}g}{\alpha_s^\vee}. \tag{2.16}$$

Remark 2.4. Note the placement of t_s in the second formula. In practice, it is sometimes convenient to rewrite it as

$$gx - xg = \kappa \partial_x g - \sum_{s \in T} c_s \langle x, \alpha_s^\vee\rangle \frac{sg - g}{s\alpha_s^\vee} t_s. \tag{2.17}$$

Proof. Observe that if $f = x \in \mathfrak{h}^*$, the first formula to be proved is

$$yx - xy = \kappa\langle x, y\rangle - \sum_{s \in T} c_s \langle \alpha_s, y\rangle \frac{x - sx}{\alpha_s} t_s,$$

and the right-hand side may be rewritten as

$$\kappa\langle x, y\rangle - \sum_{s \in T} c_s \langle \alpha_s, y\rangle \langle x, \alpha_s^\vee\rangle t_s,$$

so that the formula to be proved is one of the defining relations for $\mathbb{H}$. We proceed by induction on the degree of f. Assume we have proved the result for $h \in S^d(\mathfrak{h}^*)$ and all $d \leqslant m$. For $f, g \in S^{\leqslant m}(\mathfrak{h}^*)$ and $y \in \mathfrak{h}$ we have

$$\begin{aligned}
[y, fg] &= [y, f]g + f[y, g] \\
&= \Bigl(\kappa\partial_y(f) - \sum_{s \in T} c_s \langle \alpha_s, y\rangle \frac{f - sf}{\alpha_s} t_s\Bigr) g + f\Bigl(\kappa\partial_y(g) - \sum_{s \in T} c_s \langle \alpha_s, y\rangle \frac{g - sg}{\alpha_s} t_s\Bigr) \\
&= \kappa(\partial_y(f)g + f\partial_y(g)) - \sum_{s \in T} c_s \langle \alpha_s, y\rangle \Bigl(\frac{f - sf}{\alpha_s} sg + f\frac{g - sg}{\alpha_s}\Bigr) t_s \\
&= \kappa\partial_y(fg) - \sum_{s \in T} c_s \langle \alpha_s, y\rangle \frac{fg - s(fg)}{\alpha_s} t_s,
\end{aligned}$$

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by using the inductive hypothesis in the second equality and the Leibniz rule for ∂_y and a *skew* Leibniz rule for the divided differences in the fourth equality. This proves the first commutator formula, and the proof of the second one is exactly analogous. □

Let V be a kW-module and define an $S(\mathfrak{h}) \rtimes kW$ action on V by

$$f \cdot v = f(0)v \quad \text{and} \quad t_w \cdot v = wv \quad \text{for } w \in W,\ f \in S(\mathfrak{h}). \tag{2.18}$$

The *standard module* corresponding to V is

$$M(V) = \operatorname{Ind}_{S(\mathfrak{h}) \rtimes kW}^{\mathbb{H}} V. \tag{2.19}$$

The PBW Theorem shows that $\mathbb{H}$ is a free $S(\mathfrak{h}) \rtimes kW$-module, so that the additive functor $V \mapsto M(V)$ is exact and, as a k-vector space,

$$M(V) \simeq S(\mathfrak{h}^*) \otimes_k V. \tag{2.20}$$

In particular, when $V = \mathbf{1}$ is the trivial kW-module, we obtain from Proposition 2.3 that

$$M(\mathbf{1}) \simeq S(\mathfrak{h}^*) \quad \text{with } y \cdot f = \kappa \partial_y f - \sum_{s \in T} c_s \langle \alpha_s, y \rangle \frac{f - sf}{\alpha_s} \tag{2.21}$$

for $y \in \mathfrak{h}$ and $f \in S(\mathfrak{h}^*)$. These are the famous *Dunkl operators*. From our point of view, the fact that they commute is a consequence of the PBW Theorem, though it is possible to prove the commutativity independently (see, for example, [**9**]) and then use it to establish the PBW Theorem.

The definition (2.19) implies that the module $M(V)$ has the following universal property: given an $\mathbb{H}$-module M and a W-stable subspace $U \subseteq M$ such that $V \cong U$ as W modules and $y \cdot U = 0$ for all $y \in \mathfrak{h}$, there is a unique $\mathbb{H}$-module homomorphism $M(V) \to M$ which restricts to the given isomorphism $V \cong U$.

Define the element $h \in \mathbb{H}$ by

$$h = \sum_{i=1}^{n} x_i y_i + \sum_{s \in T} c_s (1 - t_s), \tag{2.22}$$

where x_i is a basis of $\mathfrak{h}^*$ and y_i is the dual basis of $\mathfrak{h}$. Calculations using the defining relations for $\mathbb{H}$ show that

$$[h, x] = \kappa x, \quad [h, y] = -\kappa y \quad \text{and} \quad [h, t_w] = 0 \tag{2.23}$$

for $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$ and $w \in W$. Thus, if $\kappa = 1$ and V is an irreducible W-module, the h action on the Verma module $M(V)$ is given by

$$h \cdot fv = (\deg(f) + c_V) fv \quad \text{for } f \in S(\mathfrak{h}^*) \text{ homogeneous and } v \in V, \tag{2.24}$$

where c_V is the scalar by which $\sum_{s \in T} c_s (1 - t_s)$ acts on V.

When $\kappa = 1$ the formula (2.24) implies that each standard module $M(V)$ has a unique irreducible quotient $L(V)$. This paper is primarily concerned with the module $L(\mathbf{1})$ in those cases related to diagonal co-invariants, but the techniques developed will be applied in the sequel, [**16**], to obtain detailed information on the submodule structure of $M(V)$ for more general representations V.

3. Diagonal co-invariants

We now describe a situation in which we can relate $L(\mathbf{1})$ to the diagonal co-invariant ring

$$R = S(\mathfrak{h}^* \oplus \mathfrak{h})/I, \quad \text{where } I = S(\mathfrak{h}^* \oplus \mathfrak{h})^W_+ S(\mathfrak{h}^* \oplus \mathfrak{h}). \tag{3.1}$$

Recall the filtration (2.13) of $\mathbb{H}$ with $\mathfrak{h}^* \oplus \mathfrak{h}$ in degree 1 and $\mathbb{C}W$ in degree 0. By (2.6) and the PBW Theorem, the associated graded algebra of $\mathbb{H}$ with respect to this filtration is $S(\mathfrak{h}^* \oplus \mathfrak{h}) \rtimes W$. It is this fact that was used in [**12**] (and that we will use in Theorem 3.2) to make the connection to diagonal co-invariants.

In this section, we assume that we are working with an irreducible complex reflection group W of rank n. When A is a graded vector space, we write A_i for the ith graded piece.

Lemma 3.1. *Assume that there is an irreducible module V of dimension n such that there is an exact sequence $M(V) \to M(\mathbf{1}) \to L(\mathbf{1}) \to 0$ with $L(\mathbf{1})$ finite dimensional. Then the Koszul complex*

$$0 \to S(\mathfrak{h}^*) \otimes \Lambda^n V \to \cdots \to S(\mathfrak{h}^*) \otimes \Lambda^1 V \to S(\mathfrak{h}^*) \to L(\mathbf{1}) \to 0$$

is exact and the maps are maps of $\mathbb{H}$-modules. The graded W-character of $L(\mathbf{1})$ is

$$\sum_{i \geqslant 0} \operatorname{tr}(w, L(\mathbf{1})_i) t^i = \frac{\det(1 - t^k w_V)}{\det(1 - t w_{\mathfrak{h}^*})},$$

where the image of V in $M(\mathbf{1})$ lies in degree k and w_V and $w_{\mathfrak{h}^*}$ denote w regarded as an endomorphism of V and $\mathfrak{h}^*$, respectively.

Proof. Since V is n-dimensional and $L(\mathbf{1})$ is finite dimensional, the image of V under the map $M(V) \to M(\mathbf{1})$ is spanned by a regular sequence. Hence, the Koszul complex is exact. As a vector space $M(\Lambda^i V) \cong S(\mathfrak{h}^*) \otimes \Lambda^i V$ and, using this identification, the vector spaces in the Koszul complex are $\mathbb{H}$-modules. By assumption, the first map is a map of $\mathbb{H}$-modules; its kernel is therefore an $\mathbb{H}$-submodule. The kernel is generated as an $S(\mathfrak{h}^*)$-submodule by $\Lambda^2 V$ and it follows that the second map in the Koszul complex is a map of $\mathbb{H}$-modules. One proves in the same way, by induction on i, that the ith map in the Koszul complex is a map of $\mathbb{H}$-modules. That the graded W-character of $L(\mathbf{1})$ is as asserted is a routine calculation using the Koszul resolution. □

If V is an irreducible W-module of dimension l, its *exponents* are the integers $e_1 \leqslant e_2 \leqslant \cdots \leqslant e_l$ defined by the equation

$$\sum_{i \geqslant 0} [(S(\mathfrak{h}^*)/J)_i : V] t^i = \sum_{i=1}^{r} t^{e_i},$$

where J is the ideal generated by the positive degree elements of the invariant ring $S(\mathfrak{h}^*)^W$ and $(S(\mathfrak{h}^*)/J)_i$ denotes the ith graded piece. Since W is a complex reflection

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group, the invariant ring $S(\mathfrak{h}^*)^W$ is generated by n algebraically independent polynomials $f_1, \dots, f_n$ with degrees $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$.

An irreducible representation V is *free* if $(S(\mathfrak{h}^*) \otimes \Lambda^* V^*)^W$ is a free exterior algebra over $S(\mathfrak{h}^*)^W$. By [**21**, Theorem 3.1], V is free if $e_1 + \cdots + e_l$ is the (unique) exponent of $\Lambda^l V$. By the proof of [**21**, Theorem 3.3], the Galois conjugates of the reflection representation $\mathfrak{h}$ are all free. In Lemma 8.1 we will give some other examples of free representations of the group $G(r,p,n)$ that will be relevant to the diagonal co-invariant ring.

Theorem 3.2. *With assumptions as in Lemma 3.1, assume moreover that V is free and the image of V in $M(\mathbf{1})$ lies in degree k for an integer k such that the multisets $\{k - e_i\}_{i=1}^n$ and $\{d_i\}_{i=1}^n$ are equal. Then there is a unique occurrence of $\Lambda^n V$ in $L(\mathbf{1})$, lying in degree $e_1 + \cdots + e_n$. Let $v \in L(\mathbf{1})$ span this occurrence of $\Lambda^n V$, and filter $L(\mathbf{1})$ by*

$$L(\mathbf{1})^{\leqslant i} = \mathbb{H}^{\leqslant i} \cdot v.$$

Then the map $\operatorname{gr} \mathbb{H} \to \operatorname{gr} L(\mathbf{1})$ restricts to a surjection $R \to \operatorname{gr} L(\mathbf{1})$, which has W-character given by Lemma 3.1. Finally, the image in $\operatorname{gr} L(\mathbf{1})$ of $S(\mathfrak{h}^*)$ is isomorphic to the ordinary co-invariant ring $S(\mathfrak{h}^*)/J$.

Proof. In light of Lemma 3.1, the occurrences of $\Lambda^n V$ in $L(\mathbf{1})$ are given by the formula

$$\sum [L(1)_j : \Lambda^n V] t^j = \sum (-1)^i [S(\mathfrak{h}^*)_{j-ik} \otimes \Lambda^i V : \Lambda^n V] t^j,$$

and we compute the occurrences of $\Lambda^n V$ in $S(\mathfrak{h}^*) \otimes \Lambda^* V$ by use of the W-equivariant isomorphism $\Lambda^i V \otimes \Lambda^n V^* \cong \Lambda^{n-i} V^*$. Thus,

$$[S(\mathfrak{h}^*)_j \otimes \Lambda^i V : \Lambda^n V] = \dim_{\mathbb{C}} (S(\mathfrak{h}^*)_j \otimes \Lambda^{n-i} V^*)^W.$$

On the other hand, the assumption that $(S(\mathfrak{h}^*) \otimes \Lambda^* V^*)^W$ is a free exterior algebra over $S(\mathfrak{h}^*)^W$ implies that

$$\sum \dim_{\mathbb{C}} (S(\mathfrak{h}^*)_j \otimes \Lambda^i V^*)^W q^i t^j = \prod_{i=1}^n \frac{1 + q t^{e_i}}{1 - t^{d_i}}.$$

Thus,

$$\begin{aligned}
\sum [L(1)_j : \Lambda^n V] t^j &= \sum (-1)^i [S(\mathfrak{h}^*)_j \otimes \Lambda^i V : \Lambda^n V] t^{j+ik} \\
&= \sum (-1)^i \dim_{\mathbb{C}} (S(\mathfrak{h}^*)_j \otimes \Lambda^{n-i} V^*)^W t^{j+ik} \\
&= \sum (-1)^{n-i} \dim_{\mathbb{C}} (S(\mathfrak{h}^*)_j \otimes \Lambda^i V^*)^W t^{j+(n-i)k} \\
&= (-1)^n t^{nk} \Big[\sum \dim_{\mathbb{C}} (S(\mathfrak{h}^*)_j \otimes \Lambda^i V^*)^W q^i t^j \Big]_{q=-t^{-k}} \\
&= (-1)^n t^{nk} \prod_{i=1}^n \frac{1 - t^{e_i - k}}{1 - t^{d_i}} \\
&= t^{e_1 + \cdots + e_n} \prod_{i=1}^n \frac{1 - t^{k - e_i}}{1 - t^{d_i}} \\
&= t^{e_1 + \cdots + e_n}.
\end{aligned}$$

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Upon filtering $L(\mathbf{1})$ and taking the corresponding associated graded module as in the statement of the theorem, it follows that there is a unique copy of $\Lambda^n V$ in $\operatorname{gr} L(\mathbf{1})$. Since $L(\mathbf{1})$ is an irreducible $\mathbb{H}$-module, the map

$$\left.\begin{aligned} \operatorname{gr}\mathbb{H} &\to \operatorname{gr} L(\mathbf{1}), \\ f &\to f\cdot v, \end{aligned}\right\} \tag{3.2}$$

is surjective. By the PBW-theorem, $\operatorname{gr}\mathbb{H} = S(\mathfrak{h}^*\oplus\mathfrak{h})\otimes\mathbb{C}W$, and the above map remains surjective upon restriction to $S(\mathfrak{h}^*\oplus\mathfrak{h})$. Since v is the unique occurrence of $\Lambda^n V$ in $\operatorname{gr} L(\mathbf{1})$, we have $S(\mathfrak{h}^*\oplus\mathfrak{h})_+^W\cdot v = 0$ and it follows that $\operatorname{gr} L(\mathbf{1})$ is a quotient of the diagonal co-invariant ring.

Let $N = |T|$ be the number of reflections in W; it is a standard fact that the socle of the ordinary co-invariant ring $S(\mathfrak{h}^*)/J$ is spanned by $\prod_{s\in T}\alpha_s$ and hence lies in degree N. The equation $N = d_1 - 1 + d_2 - 1 + \cdots + d_n - 1$ is also well known. By Lemma 3.1 and the hypotheses of the theorem, the top degree piece of $L(\mathbf{1})$ lies in degree

$$n(k-1) = e_1 + \cdots + e_n + d_1 + \cdots + d_n - n = e_1 + \cdots + e_n + N.$$

Since $L(\mathbf{1})$ is irreducible the map

$$\left.\begin{aligned} S(\mathfrak{h}^*)\otimes S(\mathfrak{h}) &\to \operatorname{gr} L(\mathbf{1}), \\ f &\to f\cdot v, \end{aligned}\right\} \tag{3.3}$$

is surjective. It follows that the socle of $S(\mathfrak{h}^*)/J$ is not in the kernel of the induced map, and hence the co-invariant ring is the image of $S(\mathfrak{h}^*)$. □

If in addition to the hypotheses of Lemma 3.1 we assume that V^* is free, then an analogous calculation shows that

$$\sum \dim_{\mathbb{C}} L(\mathbf{1})_i^W t^i = \prod_{i=1}^n \frac{1-t^{k+e_i'}}{1-t^{d_i}}, \tag{3.4}$$

where $e_1',\dots,e_n'$ are the exponents of V^*. This fact establishes a connection to the conjectural t-analogue of the W-Catalan number discovered by Bessis and Reiner [**4**]: if, with the assumptions of Lemma 3.1, W is a complex reflection group that can be generated by n reflections, $k = h+1$, where $h = d_n$ is the largest degree (i.e. *Coxeter number*), and $V = \mathfrak{h}^*$, then

$$\sum \dim_{\mathbb{C}} L(\mathbf{1})_i^W t^i = \prod_{i=1}^n \frac{1-t^{h+d_i}}{1-t^{d_i}}. \tag{3.5}$$

It does not seem unreasonable to expect that, for most of the exceptional complex reflection groups W and appropriate values of the parameters in the definition of $\mathbb{H}$, the module $L(\mathbf{1})$ gives rise to both a nice quotient of the diagonal co-invariant ring and a t-analogue of the W-Catalan number.

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4. The rational Cherednik algebra for $G(r,p,n)$

Let $G(r,1,n)$ be the group of $n\times n$ monomial matrices whose entries are rth roots of 1. Let

$$\zeta = \mathrm{e}^{2\pi\mathrm{i}/r} \quad\text{and}\quad \zeta_i^l = \mathrm{diag}(1,\dots,\zeta^l,\dots,1) \text{ for } 1\leqslant i\leqslant n. \tag{4.1}$$

Let

$$s_i = s_{i,i+1}, \quad \text{where } s_{ij} = (ij) \text{ for } 1\leqslant i<j\leqslant n, \tag{4.2}$$

is the transposition interchanging i and j. There are r conjugacy classes of reflections in $G(r,1,n)$:

(a) the reflections of order 2,

$$\zeta_i^l s_{ij}\zeta_i^{-l} \quad \text{for } 1\leqslant i<j\leqslant n,\ 0\leqslant l\leqslant r-1, \tag{4.3}$$

and

(b) the remaining $r-1$ classes, consisting of diagonal matrices

$$\zeta_i^l \quad \text{for } 1\leqslant i\leqslant n,\ 1\leqslant l\leqslant r-1, \tag{4.4}$$

where ζ_i^l and ζ_j^k are conjugate if and only if $k=l$.

Let

$$y_i = (0,\dots,1,\dots,0)^t \quad\text{and}\quad x_i = (0,\dots,1,\dots,0)$$

have 1s in the ith position and 0s elsewhere, so that $y_1,\dots,y_n$ is the standard basis of $\mathfrak{h}=\mathbb{C}^n$ and $x_1,\dots,x_n$ is the dual basis in $\mathfrak{h}^*$. If

$$\alpha_s = \zeta^{-l-1}x_i, \qquad \alpha_s^\vee = (\zeta^{l+1}-\zeta)y_i \quad \text{for } s=\zeta_i^l,$$

and

$$\alpha_s = x_i - \zeta^l x_j, \quad \alpha_s^\vee = y_i - \zeta^{-l}y_j \qquad \text{for } s = \zeta_i^l s_{ij}\zeta_i^{-l}.$$

then, with $\langle\cdot\,,\cdot\rangle$ denoting the canonical pairing between $\mathfrak{h}^*$ and $\mathfrak{h}$,

$$sx = x - \langle x,\alpha_s^\vee\rangle\alpha_s \quad\text{and}\quad s^{-1}(y) = y - \langle\alpha_s,y\rangle\alpha_s^\vee,$$

for $s\in T$, $x\in\mathfrak{h}^*$ and $y\in\mathfrak{h}$. We relabel the parameters defining $\mathbb{H}$ by letting

$$c_0 = c_{s_1} \quad\text{and}\quad c_i = c_{\zeta_1^i} \text{ for } 1\leqslant i\leqslant r-1. \tag{4.5}$$

Proposition 4.1. *The rational Cherednik algebra for $W=G(r,1,n)$ with parameters κ, $c_0,c_1,\dots,c_{r-1}$, is the algebra generated by $\mathbb{C}[x_1,\dots,x_n]$, $\mathbb{C}[y_1,\dots,y_n]$ and t_w for $w\in W$ with relations*

$$t_wt_v = t_{wv}, \quad t_wx = (wx)t_w \quad\text{and}\quad t_wy = (wy)t_w,$$

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for $w, v \in W$, $x \in \mathfrak{h}^*$ *and* $y \in \mathfrak{h}$,

$$y_i x_j = x_j y_i + c_0 \sum_{l=0}^{r-1} \zeta^{-l} t_{\zeta_i^l s_{ij} \zeta_i^{-l}} \tag{4.6}$$

for $1 \leqslant i \neq j \leqslant n$, *and*

$$y_i x_i = x_i y_i + \kappa - \sum_{l=1}^{r-1} c_l (1 - \zeta^{-l}) t_{\zeta_i^l} - c_0 \sum_{j \neq i} \sum_{l=0}^{r-1} t_{\zeta_i^l s_{ij} \zeta_i^{-l}} \tag{4.7}$$

for $1 \leqslant i \leqslant n$.

Proof. This is just a matter of rewriting formula (2.6) using our $G(r, 1, n)$-specific notation. For $1 \leqslant i < j \leqslant n$,

$$\begin{aligned} y_i x_j = x_j y_i &+ \kappa \langle x_j, y_i \rangle \\ &- c_0 \sum_{1 \leqslant k < m \leqslant n} \sum_{l=0}^{r-1} \langle x_k - \zeta^l x_m, y_i \rangle \langle x_j, y_k - \zeta^{-l} y_m \rangle t_{\zeta_k^l s_{km} \zeta_k^{-l}} \\ &- \sum_{k=1}^{n} \sum_{l=1}^{r-1} c_l \langle \zeta^{-l-1} x_k, y_i \rangle \langle x_j, (\zeta^{l+1} - \zeta) y_k \rangle t_{\zeta_k^l} \\ = x_j y_i &+ \kappa \cdot 0 - c_0 \sum_{l=0}^{r-1} (-\zeta^{-l}) t_{\zeta_i^l s_{ij} \zeta_i^{-l}} - 0 \\ = x_j y_i &+ c_0 \sum_{l=0}^{r-1} \zeta^{-l} t_{\zeta_i^l s_{ij} \zeta_i^{-l}}. \end{aligned}$$

The calculation for $1 \leqslant j < i \leqslant n$ is similar. For $i = j$,

$$\begin{aligned} y_i x_i = x_i y_i &+ \kappa \langle x_i, y_i \rangle \\ &- c_0 \sum_{1 \leqslant k < m \leqslant n} \sum_{l=0}^{r-1} \langle x_k - \zeta^l x_m, y_i \rangle \langle x_i, y_k - \zeta^{-l} y_m \rangle t_{\zeta_k^l s_{km} \zeta_k^{-l}} \\ &- \sum_{k=1}^{n} \sum_{l=1}^{r-1} c_l \langle \zeta^{-l-1} x_k, y_i \rangle \langle x_i, (\zeta^{l+1} - \zeta) y_k \rangle t_{\zeta_k^l} \\ = x_i y_i &+ \kappa - c_0 \sum_{1 \leqslant i < m \leqslant n} \sum_{l=0}^{r-1} t_{\zeta_i^l s_{im} \zeta_i^{-l}} \\ &- c_0 \sum_{1 \leqslant k < i \leqslant n} \sum_{l=0}^{r-1} t_{\zeta_k^l s_{ik} \zeta_k^{-l}} - \sum_{l=1}^{r-1} c_l (1 - \zeta^{-l}) t_{\zeta_i^l}. \end{aligned}$$

□

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Most of the equations that occur later on are simpler in terms of a certain reparametrization. For $j \in \mathbb{Z}$ define

$$d_j = \sum_{l=1}^{r-1} \zeta^{lj} c_l. \tag{4.8}$$

It follows that $d_0 + d_1 + \cdots + d_{r-1} = 0$ and that, for $1 \leqslant l \leqslant r-1$,

$$c_l = \frac{1}{r} \sum_{j=0}^{r-1} \zeta^{-lj} d_j. \tag{4.9}$$

The defining relation (4.7) becomes

$$y_i x_i = x_i y_i + \kappa - \sum_{j=0}^{r-1} (d_j - d_{j-1}) \epsilon_{ij} - c_0 \sum_{j \neq i} \sum_{l=0}^{r-1} t_{\zeta_i^l s_{ij} \zeta_i^{-l}}, \tag{4.10}$$

where, for $0 \leqslant j \leqslant r-1$, the primitive idempotents for the cyclic reflection subgroup of W generated by ζ_i are

$$\epsilon_{ij} = \frac{1}{r} \sum_{l=0}^{r-1} \zeta^{-lj} t_{\zeta_i^l}. \tag{4.11}$$

The complex reflection group $G(r,p,n)$ is the subgroup of $G(r,1,n)$ consisting of those matrices so that the product of the non-zero entries is an r/pth root of 1. The reflections in $G(r,p,n)$ are

(a) $\zeta_i^l s_{ij} \zeta_i^{-l}$ for $1 \leqslant i < j \leqslant n$ and $0 \leqslant l \leqslant r-1$, and

(b) ζ_i^{lp} for $1 \leqslant i \leqslant n$ and $0 \leqslant l \leqslant r/p - 1$.

When $n \geqslant 3$, the rational Cherednik algebra for $G(r,p,n)$ is the subalgebra of the rational Cherednik algebra $\mathbb{H}$ for $G(r,1,n)$ with parameters

$$c_l = 0 \quad \text{if } p \text{ does not divide } l,$$

generated by $\mathbb{C}[x_1, \dots, x_n]$, $\mathbb{C}[y_1, \dots, y_n]$ and $\mathbb{C}G(r,p,n)$. Although this is not strictly speaking true for $n = 2$, our results on the diagonal co-invariant ring still hold in that case except when $p = r = 2$.

Although not strictly necessary for the results of this paper, it seems worthwhile to mention here that when $c_l = 0$ for l not divisible by p there is a cyclic group of automorphisms, generated by

$$x \mapsto x, \quad y \mapsto y, \quad s_i \mapsto s_i, \quad t_{\zeta_j} \mapsto \zeta^{r/p} t_{\zeta_j}, \tag{4.12}$$

for $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$, $1 \leqslant i \leqslant n-1$, and $1 \leqslant j \leqslant n$, of the rational Cherednik algebra $\mathbb{H}$ for $G(r,1,n)$ so that the rational Cherednik algebra for $G(r,p,n)$ is the fixed subalgebra. The version of Clifford theory given in [**22**] therefore applies to deduce representation theoretic results for the $G(r,p,n)$ RCA from those for the $G(r,1,n)$ RCA.

From now on, p dividing r will be fixed and we work with the rational Cherednik algebra $\mathbb{H}$ for $G(r,p,n)$. Note that, with the parameters $c_l = 0$ for l not divisible by p, we have $d_j = d_k$ if $j = k \bmod r/p$.

Our first goal is identify a certain commutative subalgebra $\mathfrak{t}$ of $\mathbb{H}$. Later on we will use the subalgebra $\mathfrak{t}$ to diagonalize the standard module $M(\mathbf{1})$ (as in [**9**]), a result which is generalized in the paper [**16**]. For $1 \leqslant i \leqslant n$ define

$$z_i = y_i x_i + c_0 \phi_i, \quad \text{where } \phi_i = \sum_{1\leqslant j<i} \sum_{l=0}^{r-1} t_{\zeta_i^l s_{ij} \zeta_i^{-l}}. \tag{4.13}$$

The following proposition is proved in [**9**].

Proposition 4.2. *The elements $z_1, \dots, z_n$ of $\mathbb{H}$ are pairwise commutative:*

$$z_i z_j = z_j z_i \quad \text{for } 1 \leqslant i, j \leqslant n.$$

Proof. We begin by computing

$$\begin{aligned}
[y_i x_i, y_j x_j] &= y_i x_i y_j x_j - y_j x_j y_i x_i \\
&= y_i (x_i y_j - y_j x_i) x_j + y_j (y_i x_j - x_j y_i) x_i \\
&= -y_i \left(c_0 \sum_{l=0}^{r-1} \zeta^{-l} t_{\zeta_j^l s_{ij} \zeta_j^{-l}} \right) x_j + y_j \left(c_0 \sum_{l=0}^{r-1} \zeta^{-l} t_{\zeta_i^l s_{ij} \zeta_i^{-l}} \right) x_i \\
&= -y_i x_i \left(c_0 \sum_{l=0}^{r-1} t_{\zeta_j^l s_{ij} \zeta_j^{-l}} \right) + \left(c_0 \sum_{l=0}^{r-1} t_{\zeta_i^l s_{ij} \zeta_i^{-l}} \right) y_i x_i \\
&= -\left[y_i x_i, c_0 \sum_{l=0}^{r-1} t_{\zeta_i^l s_{ij} \zeta_i^{-l}} \right].
\end{aligned}$$

Thus,

$$\left[y_i x_i, y_j x_j + c_0 \sum_{l=0}^{r-1} t_{\zeta_i^l s_{ij} \zeta_i^{-l}} \right] = 0 \tag{4.14}$$

Let

$$\psi_i = \phi_1 + \cdots + \phi_i = \sum_{1\leqslant j<k\leqslant i} \sum_{l=0}^{r-1} t_{\zeta_k^l s_{jk} \zeta_k^{-l}}.$$

Then ψ_i is a conjugacy class sum and therefore a central element of the group algebra of $G(r,1,i)$. It follows that ψ_i commutes with $\psi_1, \dots, \psi_i$. Therefore, $\psi_1, \psi_2, \dots, \psi_n$ are pairwise commutative and hence $\phi_1, \dots, \phi_n$ are pairwise commutative.

Using the commutativity of the ϕ_i, the commutator formula (4.14), and the fact that y_jx_j commutes with ϕ_i for $i<j$, we assume that $i<j$ and compute

$$\begin{aligned}[z_i,z_j] &= [y_ix_i + c_0\phi_i, y_jx_j + c_0\phi_j] \\ &= [y_ix_i, y_jx_j + c_0\phi_j] \\ &= \biggl[y_ix_i, y_jx_j + c_0\sum_{l=0}^{r-1} t_{\zeta_i^l s_{ij}\zeta_i^{-l}} + c_0 \sum_{1\leqslant k\neq i<j}\sum_{l=0}^{r-1} t_{\zeta_j^l s_{jk}\zeta_j^{-l}}\biggr] \\ &= 0.\end{aligned}$$

□

As observed in [**7**, Proposition 1.1], the relations in the following lemma imply that the subalgebra of $\mathbb{H}$ generated by $G(r,p,n)$ and $z_1,\dots,z_n$ is isomorphic to the graded Hecke algebra for $G(r,p,n)$ defined in [**23**, § 5] (the elements $z_1,\dots,z_n$ are algebraically independent over $\mathbb{C}$ by the PBW Theorem).

Proposition 4.3. *Working in the rational Cherednik algebra* $\mathbb{H}$ *for* $G(r,1,n)$, *we have*

$$z_it_{\zeta_j} = t_{\zeta_j}z_i \qquad \text{for } 1\leqslant i,j\leqslant n, \tag{4.15}$$

$$z_it_{s_i} = t_{s_i}z_{i+1} - c_0\sum_{l=0}^{r-1} t_{\zeta_i^l\zeta_{i+1}^{-l}} \qquad \text{for } 1\leqslant i\leqslant n, \tag{4.16}$$

$$z_it_{s_j} = t_{s_j}z_i \qquad \text{for } 1\leqslant i\leqslant n \text{ and } j\neq i,i+1. \tag{4.17}$$

Proof. First we observe that the elements t_{ζ_i} and ϕ_j commute for all $1\leqslant i,j\leqslant n$. This is clear if $i>j$; if $i=j$, then

$$t_{\zeta_j}\phi_jt_{\zeta_j}^{-1} = t_{\zeta_j}\sum_{1\leqslant k<j}\sum_{l=0}^{r-1} t_{\zeta_j^l s_{jk}\zeta_j^{-l}}t_{\zeta_j}^{-1} = \sum_{1\leqslant k<j}\sum_{l=0}^{r-1} t_{\zeta_j^{l+1}s_{jk}\zeta_j^{-l-1}} = \phi_j;$$

a similar computation handles the case $i<j$. Then (4.15) follows from

$$t_{\zeta_i}y_ix_i = \zeta y_it_{\zeta_i}x_i = \zeta\zeta^{-1}y_ix_it_{\zeta_i} = y_ix_it_{\zeta_i} \quad \text{for } 1\leqslant i,j\leqslant n.$$

For (4.16),

$$\begin{aligned} z_it_{s_i} &= \biggl(y_ix_i + c_0\sum_{1\leqslant j<i}\sum_{l=0}^{r-1} t_{\zeta_i^l s_{ij}\zeta_i^{-l}}\biggr)t_{s_i} \\ &= t_{s_i}\biggl(y_{i+1}x_{i+1} + c_0\sum_{1\leqslant j<i}\sum_{l=0}^{r-1} t_{\zeta_{i+1}^l s_{i+1,j}\zeta_{i+1}^{-l}}\biggr) \\ &= t_{s_i}z_{i+1} - t_{s_i}c_0\sum_{l=0}^{r-1} t_{\zeta_{i+1}^l s_{i+1,i}\zeta_{i+1}^{-l}} \\ &= t_{s_i}z_{i+1} - c_0\sum_{l=0}^{r-1} t_{\zeta_i^l\zeta_{i+1}^{-l}}.\end{aligned}$$

Finally, we observe that if $j \neq i, i+1$, then t_{s_j} commutes with ϕ_i and with $y_i x_i$, and hence with $z_i = y_i x_i + c_0 \phi_i$. □

Let

$$\mathfrak{t} = \mathbb{C}[z_1, \dots, z_n, t_{\zeta_1 \zeta_2^{-1}}, \dots, t_{\zeta_{n-1}\zeta_n^{-1}}, t_{\zeta_1^p}, \dots, t_{\zeta_n^p}]. \tag{4.18}$$

By Proposition 4.2 and Lemma 4.3, the subalgebra $\mathfrak{t}$ is a commutative subalgebra of $\mathbb{H}$. Our goal is to use $\mathfrak{t}$ in much the same way as a Cartan subalgebra of a semisimple Lie algebra.

5. Intertwiners

In this section we will prove many formulae using elements of $G(r,1,n)$ that are not in $G(r,p,n)$; since the rational Cherednik algebra for $G(r,p,n)$ is a subalgebra of a specialization of that for $G(r,1,n)$, these formulae have consequences in the rational Cherednik algebra for $G(r,p,n)$.

The following lemma is a generalization of (4.16) and (4.17). Let

$$\pi_i = \sum_{l=0}^{r-1} t_{\zeta_i^l \zeta_{i+1}^{-l}}. \tag{5.1}$$

Lemma 5.1. *Let f be a rational function of $z_1, \dots, z_n$. Then*

$$t_{s_i} f = (s_i f) t_{s_i} - c_0 \pi_i \frac{f - s_i f}{z_i - z_{i+1}} \quad \text{for } 1 \leqslant i \leqslant n-1. \tag{5.2}$$

Proof. Observe that if f is z_i, z_{i+1} or z_j for $j \neq i, i+1$, then the relation to be proved follows from (4.16) and (4.17). Assume that the relation (5.2) is true for rational functions f and g. Then it is evidently true for $f+g$ and af for all $a \in \mathbb{C}$, and we compute

$$\begin{aligned} t_{s_i} fg &= (t_{s_i} f - (s_i f) t_{s_i}) g + (s_i f)(t_{s_i} g - (s_i g) t_{s_i}) + (s_i f g) t_{s_i} \\ &= \left(-c_0 \pi_i \frac{f - s_i f}{z_i - z_{i+1}} \right) g + (s_i f) \left(-c_0 \pi_i \frac{g - s_i g}{z_i - z_{i+1}} \right) + (s_i f g) t_{s_i} \\ &= (s_i f g) t_{s_i} - c_0 \pi_i \frac{fg - s_i fg}{z_i - z_{i+1}}, \end{aligned}$$

so (5.2) is true for fg. Assuming it is true for the rational function f, we compute

$$\begin{aligned} (t_{s_i} 1/f - (1/s_i f) t_{s_i}) f (s_i f) &= t_{s_i} s_i f - 1/s_i f \left((s_i f) t_{s_i} - c_0 \pi_i \frac{f - s_i f}{z_i - z_{i+1}} \right) s_i f \\ &= c_0 \pi_i \frac{f - s_i f}{z_i - z_{i+1}}, \end{aligned}$$

and dividing by $f(s_i f)$ proves that the relation holds for $1/f$. Since it holds for $z_1, \dots, z_n$, it is true for all rational functions in $z_1, \dots, z_n$. □

The *intertwining operators* σ_i for $1 \leqslant i \leqslant n-1$ are

$$\sigma_i = t_{s_i} + \frac{c_0}{z_i - z_{i+1}}\pi_i, \quad \text{where } \pi_i = \sum_{l=0}^{r-1} t_{\zeta_i^l \zeta_{i+1}^{-l}}, \tag{5.3}$$

and we define intertwining operators Φ and Ψ by

$$\Phi = x_n t_{s_{n-1}s_{n-2}\cdots s_1} \quad \text{and} \quad \Psi = y_1 t_{s_1 s_2 \dots s_{n-1}}. \tag{5.4}$$

The intertwiner Φ was first defined in [**19**, § 4], where it is used for the symmetric group case. The other intertwiners were defined for the first time in the author's thesis [**14**], where some of the results that follow were also recorded.

The intertwiner σ_i is well defined when $\pi_i = 0$ or $z_i - z_{i+1} \neq 0$. The intertwiners are important because, as Lemma 5.3 shows, they permute the z_is from (4.13). Our first task is to compute the squares σ_i^2 of the intertwiners and the products $\Phi\Psi$ and $\Psi\Phi$. Since these compositions all lie in $\mathfrak{t}$, this calculation is useful for deciding when the intertwiners applied to a $\mathfrak{t}$-eigenvector (or generalized eigenvector) are non-zero.

Lemma 5.2.

(a) *For* $1 \leqslant i \leqslant n-1$,

$$\sigma_i^2 = 1 - \left(\frac{c_0 \pi_i}{z_i - z_{i+1}}\right)^2.$$

(b)

$$\Psi\Phi = z_1 \quad \textit{and} \quad \Phi\Psi = z_n - \kappa + \sum_{j=0}^{r-1}(d_j - d_{j-1})\epsilon_{nj}.$$

Proof. Using Lemma 5.1,

$$\begin{aligned}
\sigma_i^2 &= \left(t_{s_i} + \frac{c_0\pi_i}{z_i - z_{i+1}}\right)\left(t_{s_i} + \frac{c_0\pi_i}{z_i - z_{i+1}}\right) \\
&= 1 + t_{s_i}\frac{c_0\pi_i}{z_i - z_{i+1}} + \frac{c_0\pi_i}{z_i - z_{i+1}}t_{s_i} + \left(\frac{c_0\pi_i}{z_i - z_{i+1}}\right)^2 \\
&= 1 + \frac{c_0\pi_i}{z_{i+1} - z_i}t_{s_i} - \frac{c_0\pi_i}{z_i - z_{i+1}}\left(\frac{c_0\pi_i}{z_i - z_{i+1}} - \frac{c_0\pi_i}{z_{i+1} - z_i}\right) \\
&\qquad + \frac{c_0\pi_i}{z_i - z_{i+1}}t_{s_i} + \left(\frac{c_0\pi_i}{z_i - z_{i+1}}\right)^2 \\
&= 1 - \left(\frac{c_0\pi_i}{z_i - z_{i+1}}\right)^2.
\end{aligned}$$

This proves (a), and (b) follows from the definition (5.4) and the relation (4.10). □

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We define a symmetric group action on $\mathfrak{t}$ by letting S_n simultaneously permute $z_1, z_2, \dots, z_n$ and $t_{\zeta_1}, \dots, t_{\zeta_n}$. We also define an automorphism ϕ of $\mathfrak{t}$ by

$$\phi(t_{\zeta_i}) = t_{\zeta_{i+1}} \quad \text{for } 1 \leqslant i \leqslant n-1, \qquad \phi(t_{\zeta_n}) = \zeta^{-1} t_{\zeta_1} \tag{5.5}$$

and

$$\phi(z_i) = z_{i+1} \text{ for } 1 \leqslant i \leqslant n-1 \quad \text{and} \quad \phi(z_n) = z_1 + \kappa - \sum_{j=0}^{r-1} (d_{j-1} - d_{j-2}) \epsilon_{1j}, \tag{5.6}$$

where, as in (4.11), ϵ_{1j} are the primitive idempotents for the cyclic reflection subgroup generated by ζ_1. There are corresponding operators on the space of homomorphisms $\mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(\mathfrak{t}, \mathbb{C})$. Identify such an algebra homomorphism α with the sequence $(\alpha(z_1), \dots, \alpha(z_n), \alpha(\zeta_1), \dots, \alpha(\zeta_n))$. Then S_n acts by permutations, and the operators ϕ and ψ are as follows:

$$\begin{aligned} &\phi \cdot (\alpha_1, \dots, \alpha_n, \zeta^{\beta_1}, \dots, \zeta^{\beta_n}) \\ &\qquad = (\alpha_2, \dots, \alpha_n, \alpha_1 + \kappa - d_{\beta_1 - 1} + d_{\beta_1 - 2}, \zeta^{\beta_2}, \dots, \zeta^{\beta_n}, \zeta^{\beta_1 - 1}). \end{aligned} \tag{5.7}$$

Let $\psi = \phi^{-1}$, so that

$$\begin{aligned} &\psi \cdot (\alpha_1, \dots, \alpha_n, \zeta^{\beta_1}, \dots, \zeta^{\beta_n}) \\ &\qquad = (\alpha_n - \kappa + d_{\beta_n} - d_{\beta_n - 1}, \alpha_1, \dots, \alpha_{n-1}, \zeta^{\beta_n + 1}, \zeta^{\beta_1}, \dots, \zeta^{\beta_{n-1}}). \end{aligned} \tag{5.8}$$

Lemma 5.3.

(a) *For* $1 \leqslant i \leqslant n-1$ *and* $f \in \mathfrak{t}$,

$$\sigma_i f = (s_i \cdot f) \sigma_i.$$

(b) *For* $f \in \mathfrak{t}$,

$$f \Phi = \Phi(\phi \cdot f) \quad \textit{and} \quad f \Psi = \Psi(\phi^{-1} \cdot f).$$

Proof. The commutation relation (4.16) for z_i and t_{s_i} gives

$$\begin{aligned} z_i \sigma_i &= z_i \left(t_{s_i} + \frac{c_0 \pi_i}{z_i - z_{i+1}} \right) \\ &= t_{s_i} z_{i+1} - c_0 \sum_{l=0}^{r-1} t_{\zeta_i^l \zeta_{i+1}^{-l}} + \frac{c_0 \pi_i z_i}{z_i - z_{i+1}} \\ &= \sigma_i z_{i+1} - \frac{c_0 \pi_i z_{i+1}}{z_i - z_{i+1}} - c_0 \pi_i + \frac{c_0 \pi_i z_i}{z_i - z_{i+1}} \\ &= \sigma_i z_{i+1}. \end{aligned}$$

The proof that $z_{i+1} \sigma_i = \sigma_i z_i$ is exactly analogous, and the fact that σ_i and z_j commute if $j \neq i, i+1$ is obvious.

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Using the relation $t_{\zeta_i}\pi_i = \pi_i t_{\zeta_{i+1}}$,

$$t_{\zeta_i}\sigma_i = t_{\zeta_i}\left(t_{s_i} + \frac{c_0\pi_i}{z_i - z_{i+1}}\right) = t_{s_i}t_{\zeta_{i+1}} + \frac{c_0\pi_i}{z_i - z_{i+1}}t_{\zeta_{i+1}} = \sigma_i t_{\zeta_{i+1}}.$$

The proof that $t_{\zeta_{i+1}}\sigma_i = \sigma_i t_{\zeta_i}$ is the same, and the fact that σ_i and t_{ζ_j} commute if $j \neq i, i+1$ is obvious. This proves (a).

Using the commutation formula (4.10) for y_n and x_n,

$$\begin{aligned} y_n x_n \Phi &= \left(x_n y_n + \kappa - \sum_{j=0}^{r-1}(d_j - d_{j-1})\epsilon_{nj} - c_0\phi_n\right)x_n t_{s_{n-1}}\cdots t_{s_1} \\ &= \Phi y_1 x_1 + \kappa\Phi - \Phi\sum_{j=0}^{r-1}(d_j - d_{j-1})\epsilon_{1,j+1} - c_0\phi_n\Phi. \end{aligned}$$

Hence,

$$z_n\Phi = (y_n x_n + c_0\phi_n)\Phi = \Phi\left(z_1 + \kappa - \sum_{j=0}^{r-1}(d_{j-1} - d_{j-2})\epsilon_{1j}\right).$$

Let $1 \leqslant i < n$. Since

$$\begin{aligned} y_i x_i \Phi &= y_i x_i x_n t_{s_{n-1}\cdots s_1} \\ &= \left(x_n y_i + c_0\sum_{l=0}^{r-1}\zeta^{-l}t_{\zeta_i^l s_{in}\zeta_i^{-l}}\right)x_i t_{s_{n-1}\cdots s_1} \\ &= \Phi y_{i+1}x_{i+1} + \Phi c_0\sum_{l=0}^{r-1}t_{\zeta_{i+1}^l s_{i+1,1}\zeta_{i+1}^{-l}} \end{aligned}$$

and

$$\begin{aligned} \phi_i\Phi &= \sum_{1\leqslant j<i}\sum_{l=0}^{r-1}t_{\zeta_i^l s_{ij}\zeta_i^{-l}}x_n t_{s_{n-1}\cdots s_1} \\ &= x_n t_{s_{n-1}\cdots s_1}\sum_{1\leqslant j<i}\sum_{l=0}^{r-1}t_{\zeta_{i+1}^l s_{i+1,j+1}\zeta_{i+1}^{-l}} \\ &= \Phi\left(\phi_{i+1} - \sum_{l=0}^{r-1}t_{\zeta_{i+1}^l s_{i+1,1}\zeta_{i+1}^{-l}}\right), \end{aligned}$$

it follows that

$$\begin{aligned} z_i\Phi &= (y_i x_i + c_0\phi_i)\Phi \\ &= \Phi y_{i+1}x_{i+1} + \Phi c_0\sum_{l=0}^{r-1}t_{\zeta_{i+1}^l s_{i+1,1}\zeta_{i+1}^{-l}} + \Phi c_0\left(\phi_{i+1} - \sum_{l=0}^{r-1}t_{\zeta_{i+1}^l s_{i+1,1}\zeta_{i+1}^{-l}}\right) \\ &= \Phi z_{i+1}. \end{aligned}$$

Finally,

$$t_{\zeta_i}\Phi = t_{\zeta_i} x_n t_{s_{n-1}\cdots s_1} = x_n t_{s_{n-1}\cdots s_1} t_{\zeta_{i+1}} = \Phi t_{\zeta_{i+1}}$$

and

$$t_{\zeta_n}\Phi = t_{\zeta_n} x_n t_{s_{n-1}\cdots s_1} = x_n t_{s_{n-1}\cdots s_1}\zeta^{-1} t_{\zeta_1} = \Phi(\zeta^{-1} t_{\zeta_1})$$

for $1 \leqslant i < n$. This proves the formula involving Φ. The formula for Ψ follows from that for Φ by using the relations in part (b) of Lemma 5.2. □

6. An eigenbasis of $M(\mathbf{1})$

Let $\mathbb{C}[x_1, x_2, \dots, x_n] = M(\mathbf{1})$ be the polynomial representation of $\mathbb{H}$. We will show that, for generic choices of the parameters κ and c_i, the ring $\mathbb{C}[x_1, \dots, x_n]$ has an $\mathfrak{t}$-eigenbasis indexed by the set $\mathbb{Z}^n_{\geqslant 0}$ and we will describe how the intertwining operators act on this basis.

For $\mu \in \mathbb{Z}^n_{\geqslant 0}$, let w_μ be the maximal length permutation such that

$$w_\mu \cdot \mu = \mu_-, \quad \text{where } \mu_- \text{ is the non-decreasing (anti-partition) rearrangement of } \mu. \tag{6.1}$$

We write μ_+ for the partition rearrangement of μ, and define a partial order on $\mathbb{Z}^n_{\geqslant 0}$ by

$$\lambda < \mu \quad \Longleftrightarrow \quad \lambda_+ <_d \mu_+ \text{ or } \lambda_+ = \mu_+ \text{ and } w_\lambda < w_\mu, \tag{6.2}$$

where we use the Bruhat order on S_n, and $<_d$ denotes dominance order on $\mathbb{Z}^n_{\geqslant 0}$, given by

$$\lambda \leqslant_d \mu \quad \text{if } \mu - \lambda \in \sum_{i=1}^{n-1} \mathbb{Z}_{\geqslant 0}(\epsilon_i - \epsilon_{i+1}). \tag{6.3}$$

If $\mu_i > \mu_{i+1}$, then

$$\mu > s_i \cdot \mu + k(\epsilon_i - \epsilon_{i+1}) \quad \text{for } 0 \leqslant k < \mu_i - \mu_{i+1}. \tag{6.4}$$

The next theorem is the analogue of [**20**, Theorem 2.6] in our setting. It shows that the z_is are upper triangular as operators on $\mathbb{C}[x_1, \dots, x_n]$ with respect to the order on $\mathbb{Z}^n_{\geqslant 0}$ defined in (6.2). Equivalent results are proved [**9**] by reduction to the symmetric group case. The proof we give below is generalized to the modules $M(V)$ for all irreducible $\mathbb{C}W$-modules V in [**16**].

Theorem 6.1.

(a) *The actions of* $t^p_{\zeta_i}$, $t_{\zeta_i^{-1}\zeta_{i+1}}$, *and* z_i *on* $M(\mathbf{1})$ *are given by*

$$t^p_{\zeta_i} \cdot x^\mu = \zeta^{-p\mu_i} x^\mu, \qquad t_{\zeta_i^{-1}\zeta_{i+1}} \cdot x^\mu = \zeta^{\mu_i - \mu_{i+1}} x^\mu,$$

and, with d_j *as in (4.8),*

$$z_i \cdot x^\mu = (\kappa(\mu_i + 1) - (d_0 - d_{-\mu_i - 1}) - r(w_\mu(i) - 1)c_0)x^\mu + \sum_{\nu < \mu} c_\nu x^\nu.$$

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(b) *Assuming that the parameters are generic, so* $k = \mathbb{C}(\kappa, c_0, d_1, \dots, d_{r/p-1})$, *for each* $\mu \in \mathbb{Z}^n_{\geqslant 0}$ *there exists a unique* $\mathfrak{t}$*-eigenvector* $f_\mu \in M(\mathbf{1})$ *such that*

$$f_{\mu,T} = x^\mu + \text{lower terms.}$$

The $\mathfrak{t}$*-eigenvalue of* f_μ *is determined by the formulae in part (a).*

Proof. The statements about the action of $t_{\zeta_i^p}$ and $t_{\zeta_i\zeta_{i+1}^{-1}}$ follow from the commutation relation in the definition of the rational Cherednik algebra and the definition of the representation $M(\mathbf{1})$. Using the commutation formula in Proposition 2.3 for $f \in \mathbb{C}[x_1, \dots, x_n]$ and $y \in \mathfrak{h}$ and the geometric series formula to evaluate the divided differences, we obtain the following formula for the action of $y_i x_i$ on x^μ:

$$\begin{aligned}
y_i \cdot x^{\mu+\epsilon_i} &= \kappa(\mu_i+1)x^\mu - c_0 \sum_{1\leqslant j<k\leqslant n} \sum_{l=0}^{r-1} \langle x_j - \zeta^l x_k, y_i\rangle \frac{x^{\mu+\epsilon_i} - \zeta_j^l s_{jk} \zeta_j^{-l} x^{\mu+\epsilon_i}}{x_j - \zeta^l x_k} \\
&\quad - \sum_{1\leqslant j\leqslant n} \sum_{l=1}^{r/p-1} c_{lp} \langle x_j, y_i\rangle \frac{x^{\mu+\epsilon_i} - \zeta_j^{lp} x^{\mu+\epsilon_i}}{x_j} \\
&= \kappa(\mu_i+1)x^\mu - c_0 \sum_{j\neq i} \sum_{l=0}^{r-1} \frac{x^{\mu+\epsilon_i} - \zeta_i^l s_{ij} \zeta_i^{-l} x^{\mu+\epsilon_i}}{x_i - \zeta^l x_j} \\
&\quad - \sum_{l=1}^{r/p-1} c_{lp}(1 - \zeta^{-lp(\mu_i+1)}) x^\mu \\
&= \left(\kappa(\mu_i+1) - \sum_{l=1}^{r/p-1} c_{lp}(1 - \zeta^{-lp(\mu_i+1)})\right) x^\mu \\
&\quad - c_0 \sum_{\substack{j\neq i,\\ \mu_i \geqslant \mu_j}} (x^\mu + \zeta^l x^{\mu+(\epsilon_j-\epsilon_i)} + \dots + \zeta^{l(\mu_i-\mu_j)} x^{s_{ij}\mu}) \\
&\quad + c_0 \sum_{\substack{j\neq i,\\ \mu_j > \mu_i}} \sum_{k=1}^{\mu_j-\mu_i-1} \zeta^{-lk} x^{\mu+k(\epsilon_i-\epsilon_j)}.
\end{aligned}$$

Using this equation and (6.4) to identify lower terms,

$$\begin{aligned}
z_i \cdot x^\mu &= \left(y_i x_i + c_0 \sum_{1\leqslant j<i} \sum_{0\leqslant l\leqslant r-1} t_{\zeta_i^l s_{ij} \zeta_i^{-l}}\right) x^\mu \\
&= y_i \cdot x^{\mu+\epsilon_i} + c_0 \sum_{1\leqslant j<i} \sum_{l=0}^{r-1} t_{\zeta_i^l s_{ij} \zeta_i^{-l}} \cdot x^\mu
\end{aligned}$$

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$$= \left(\kappa(\mu_i+1) - \sum_{l=1}^{r/p-1} c_{lp}(1-\zeta^{-lp(\mu_i+1)})\right)x^{\mu} - c_0 \sum_{\substack{1\leqslant j<i,\\ \mu_j\leqslant\mu_i,\\ 0\leqslant l\leqslant r-1}} \zeta^{l(\mu_i-\mu_j)}x^{s_{ij}\mu}$$

$$- c_0 \sum_{\substack{1\leqslant j<i,\\ \mu_j<\mu_i,\\ 0\leqslant l\leqslant r-1}} x^{\mu} - c_0 \sum_{\substack{i<j\leqslant n,\\ \mu_j\leqslant\mu_i,\\ 0\leqslant l\leqslant r-1}} x^{\mu} + c_0 \sum_{\substack{1\leqslant j<i,\\ 0\leqslant l\leqslant r-1}} \zeta^{l(\mu_i-\mu_j)}x^{s_{ij}\mu} + \text{lower terms}$$

$$= \left(\kappa(\mu_i+1) - \sum_{l=1}^{r/p-1} c_{lp}(1-\zeta^{-lp(\mu_i+1)}) - c_0 r(w_{\mu}(i)-1)\right)x^{\mu} + \text{lower terms},$$

where to obtain the last line we used the formula

$$w_{\mu}(i) = |\{j<i \mid \mu_j<\mu_i\}| + |\{j>i \mid \mu_j\leqslant\mu_i\}| + 1.$$

Now, rewriting in terms of the d_js from (4.8) proves part (a) of the theorem.

For part (b), simply observe that the coefficient of κ in the formula for the action of z_i on x^{μ} is μ_{i+1}; it follows that the $\mathfrak{t}$-eigenspaces are all one dimensional, and hence a simultaneous eigenbasis exists. □

Define the *weight* $\mathrm{wt}(\mu)$ of $\mu\in\mathbb{Z}^n_{\geqslant 0}$ to be the $\mathfrak{t}$-homomorphism mapping z_i to

$$\kappa(\mu_i+1) - (d_0 - d_{-\mu_i-1}) - r(w_{\mu}(i)-1)c_0$$

and ζ_i to $\zeta^{-\mu_i}$. Let S_n act on $\mathbb{Z}^n_{\geqslant 0}$ by permuting coordinates, and define operators ϕ and ψ as follows:

$$\left.\begin{aligned} \phi\cdot(\mu_1,\dots,\mu_n) &= (\mu_2,\mu_3,\dots,\mu_n,\mu_1+1),\\ \psi\cdot(\mu_1,\dots,\mu_n) &= (\mu_n-1,\mu_1,\dots,\mu_{n-1}). \end{aligned}\right\} \tag{6.5}$$

Lemma 6.2. *The action of the intertwiners on the basis f_{μ} is given by*

$$\sigma_i\cdot f_{\mu} = \begin{cases} f_{s_i\cdot\mu} & \text{if } \mu_i<\mu_{i+1} \text{ or } \mu_i\neq\mu_{i+1} \bmod r,\\ 0 & \text{if } \mu_i=\mu_{i+1},\\ \dfrac{(\delta-rc_0)(\delta+rc_0)}{\delta^2}f_{s_i\cdot\mu} & \text{if } \mu_i=\mu_{i+1} \bmod r \text{ and } \mu_i>\mu_{i+1}, \end{cases} \tag{6.6}$$

where

$$\begin{gathered} \delta = \kappa(\mu_i-\mu_{i+1}) - c_0 r(w_{\mu}(i)-w_{\mu}(i+1)),\\ \Phi\cdot f_{\mu} = f_{\phi\cdot\mu}, \end{gathered} \tag{6.7}$$

and

$$\Psi\cdot f_{\mu} = \begin{cases} 0 & \text{if } \mu_n=0,\\ (\kappa\mu_n - (d_0-d_{-\mu_n}) - c_0 r(w_{\mu}(n)-1))f_{\psi\cdot\mu} & \text{if } \mu_n\neq 0. \end{cases} \tag{6.8}$$

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Proof. We will establish the formulae for σ_i; the formulae for Φ and Ψ are proved in an analogous fashion. If $\mu_i < \mu_{i+1}$, then for all $\nu \leqslant \mu$ one has $s_i \cdot \nu \leqslant s_i \cdot \mu$ and it follows that the leading term of $\sigma_i \cdot f_\mu$ is $x^{s_i \cdot \mu}$. Since $\sigma_i \cdot f_\mu$ is a $\mathfrak{t}$-eigenvector by Lemma 5.3, we have $\sigma_i \cdot f_\mu = f_{s_i \cdot \mu}$. If $\mu_i > \mu_{i+1}$ then by Lemma 5.2 one has

$$\begin{aligned}\sigma_i \cdot f_\mu &= \sigma_i^2 \cdot f_{s_i \cdot \mu} \\ &= \frac{(z_i - z_{i+1} - c_0 \pi_i)(z_i - z_{i+1} + c_0 \pi_i)}{(z_i - z_{i+1})^2} f_{s_i \cdot \mu} \\ &= \begin{cases} f_{s_i \cdot \mu} & \text{if } \mu_i \neq \mu_{i+1} \bmod r, \\ \dfrac{(\delta - r c_0)(\delta + r c_0)}{\delta^2} f_{s_i \cdot \mu} & \text{if } \mu_i = \mu_{i+1} \bmod r. \end{cases}\end{aligned}$$

□

Corollary 6.3. *Suppose that $\kappa = 1$. Then the $\mathfrak{t}$-eigenspaces of $M(\mathbf{1})$ are all one dimensional provided that $c_0 \notin \bigcup_{j=1}^{n} (1/j)\mathbb{Z}_{>0}$.*

Proof. We assume there is a two-dimensional $\mathfrak{t}$-eigenspace and prove that $c_0 \in \bigcup_{j=1}^{n} (1/j)\mathbb{Z}_{>0}$. Let $\mu, \nu \in \mathbb{Z}_{\geqslant 0}^n$ be distinct and assume that $\mathrm{wt}(\mu) = \mathrm{wt}(\nu)$; since $\mathrm{wt}(w \cdot \mu) = w \cdot \mathrm{wt}(\mu)$ for all $w \in S_n$, we may assume that $\mu = \mu_+$ is a partition. Write $w = w_\nu$. Thus, $\mu_i - \mu_{i+1} = \nu_i - \nu_{i+1} \bmod r$ for $1 \leqslant i \leqslant n-1$ and

$$\mu_i - \nu_i = r(n - i + 1 - w(i))c_0 \tag{6.9}$$

for $1 \leqslant i \leqslant n$. Since $\mu \neq \nu$, $c_0 \in \mathbb{Q}$ is a rational number. Let i be minimal with $w(i) \neq n - i + 1$. Then $w(i) < n - i + 1$ and there is some $k > i$ with $w(k) = n - i + 1$. Therefore, if $c_0 < 0$, then

$$\left.\begin{aligned} \mu_i - \nu_i &= r(n - i + 1 - w(i))c_0 < 0, \\ \mu_k - \nu_k &= r(n - k + 1 - (n - i + 1))c_0 > 0, \end{aligned}\right\} \tag{6.10}$$

whence $\mu_i < \nu_i$ and $\mu_k > \nu_k$. But $\mu_k > \nu_k \geqslant \nu_i > \mu_i$ contradicts $\mu = \mu_+$, and it follows that $c_0 > 0$. Now for $1 \leqslant i \leqslant n-1$ we have

$$\begin{aligned}\frac{\mu_i - \nu_i - (\mu_{i+1} - \nu_{i+1})}{r} &= (n - i + 1 - w(i) - (n - i - w(i+1)))c_0 \\ &= (1 + w(i+1) - w(i))c_0 \end{aligned} \tag{6.11}$$

and the corollary follows unless $w(i+1) - w(i) = -1$ for $1 \leqslant i \leqslant n-1$. But in that case $w = w_0$ and $\mu = \nu$, which is a contradiction. □

In fact, the preceding corollary can be sharpened somewhat: provided that either $p = 1$ or n does not divide r, the $\mathfrak{t}$-eigenspaces are one dimensional as long as

$$c_0 \notin \bigcup_{j=1}^{n-1} (1/j)\mathbb{Z}_{>0}.$$

We will not need this fact in this paper.

7. Koszul resolutions of some finite-dimensional $\mathbb{H}$-modules

We assume for the rest of the paper that $r > 1$. For $1 \leqslant k \leqslant n$ and $j \in \mathbb{Z}_{>0}$ with $j \neq 0 \bmod r$ define (affine) hyperplanes

$$H_{j,k} = \{(c_0, d_1, \ldots, d_{r/p-1}) \mid d_0 - d_{-j} + rc_0(n-k) = j\} \subseteq \mathbb{C}^{r/p} \tag{7.1}$$

and, for $x \in (1/n)\mathbb{Z}_{>0}$,

$$H_x = \{(c_0, d_1, \ldots, d_{r/p-1}) \mid c_0 = x\}. \tag{7.2}$$

The hyperplane $H_{j,1}$ was introduced (modulo different conventions for the parameters) in [**6**], where it was called E_j. Chmutova and Etingof [**6**, Theorems 4.2 and 4.3] have proved that there is a finite-dimensional quotient of $M(\mathbf{1})$ when the parameter lies on $H_{j,1}$ for $j \neq 0 \bmod r$, and that if $p = 1$, this quotient is irreducible for generic choices of the parameters. Also, Dunkl and Opdam have proved [**9**, § 3.4] that $M(\mathbf{1})$ is reducible exactly if the parameter is on some $H_{j,k}$ for some positive $j \neq 0$ and $1 \leqslant k \leqslant n \bmod r$ or H_x for some $x \in (1/j)\mathbb{Z}_{>0} - \mathbb{Z}$ with $2 \leqslant j \leqslant n$. The following theorem describes the structure of the module $M(\mathbf{1})$ in the case in which $\operatorname{gr} L(\mathbf{1})$ is the quotient of the diagonal co-invariant ring predicted by Haiman. It has the advantage of working for arbitrary divisors p or r: this is what makes our strengthening of Vale's result possible.

Theorem 7.1. *Suppose that $\kappa = 1$, that $k \in \mathbb{Z}_{>0}$ with $k \neq 0 \bmod r$, that $(c_0, d_1, \ldots, d_{r/p-1}) \in H_{k,1}$ and that the parameters do not lie on any other hyperplane $H_{l,j}$ or H_x for $1 \leqslant l \leqslant n$, $j \in \mathbb{Z}_{>0}$ and $x \in (1/n)\mathbb{Z}_{>0}$. Then the unique proper submodule of $M(\mathbf{1})$ is*

$$\mathbb{C}\{f_\lambda \mid \lambda \text{ has at least one part of size at least } k\}.$$

Proof. By Corollary 6.3 the $\mathfrak{t}$-eigenspaces of $M(\mathbf{1})$ are all one dimensional, and hence the Jack polynomials f_μ are all well defined. Suppose that M is a proper non-zero submodule of $M(\mathbf{1})$. Then M contains f_μ for some $\mu \in \mathbb{Z}^n_{\geqslant 0}$. By our assumption on the parameters and Lemma 6.2, $\sigma_i \cdot f_\mu$ is a non-zero multiple of $f_{s_i \cdot \mu}$ whenever $\mu_i \neq \mu_{i+1}$, and it follows that M also contains $f_{s_i \cdot \mu}$ for all $1 \leqslant i \leqslant n-1$. Hence, M contains f_{μ_-}, where μ_- is the non-decreasing rearrangement of μ. By Lemma 6.2, we have $\Psi \cdot f_\mu = 0$ exactly if $\mu_n = 0$ or

$$\mu_n = d_0 - d_{-\mu_n} + r(w_\mu(n) - 1)c_0. \tag{7.3}$$

The latter equation holds exactly if $\mu_n = k$ and $w_\mu(n) = n$ or, equivalently, exactly if $\mu_n = k$ is strictly larger than all other parts of μ. It follows that if all the parts of μ are of size less than k, then M contains $f_0 = 1$, contradicting the fact that M is a proper submodule. On the other hand, if μ has at least one part of size k, it follows from the preceding discussion that by applying an appropriate sequence of intertwiners to f_μ we may obtain a non-zero multiple of f_ν, where $\nu = (k, 0, \ldots, 0)$. Since f_ν generates $\mathbb{C}\{f_\lambda \mid \lambda \text{ has at least one part of size at least } k\}$ as an $\mathbb{H}$-module the result follows. $\square$

In [**16**, Theorem 7.5] the author generalizes Theorem 7.1 to the case of a Verma module $M(V)$ with one-dimensional $\mathfrak{t}$-eigenspaces, giving a combinatorial description of the submodule structure (which can be much more intricate than the situation we study in this paper).

8. Diagonal co-invariants for $G(r,p,n)$

We continue to assume that $r > 1$. The *Coxeter number* of $G(r,p,n)$ is

$$h = \begin{cases} r(n-1) + r/p & \text{if } p < r, \\ r(n-1) & \text{if } p = r. \end{cases} \tag{8.1}$$

This agrees with the usual definition of the Coxeter number (the largest degree of a basic invariant) when $r > 1$ and $p = 1$ or $p = r$. The following theorem constructs an analogue for the groups $G(r,p,n)$ of the quotient of the diagonal co-invariant ring discovered by Gordon in [**12**]. For $p < r$, a very similar theorem is proved in [**25**]. Our techniques (which are conceptually very similar to those of [**5**], but work directly in $\mathbb{H}$) allow us to handle the case $p = r$ in the same way as $p < r$.

Lemma 8.1. *Let m be a positive integer not divisible by r and let V be the representation $\mathbb{C}\{x_1^m, \dots, x_n^m\}$. Then V is free. If $m = h+1$ and $e_1, \dots, e_n$ are the exponents of V, then the multisets $\{h+1-e_i\}_{i=1}^n$ and $\{d_i\}_{i=1}^n$ are equal.*

Proof. Let $\bar{m}$ and m' be the integers determined by

$$0 \leqslant \bar{m} < r, \quad 0 \leqslant m' < r/p, \quad \bar{m} = m \bmod r \quad \text{and} \quad m' = m \bmod r/p. \tag{8.2}$$

Observe that the representation $\Lambda^n V$ is $\epsilon\delta^{-m}$, where ϵ and δ are the one-dimensional $G(r,1,n)$-representations determined by $\epsilon(\zeta_i^l s_i j \zeta_i^{-l}) = -1$, $\epsilon(\zeta_i^l) = 1$, $\delta(\zeta_i^l s_i j \zeta_i^{-l}) = 1$ and $\delta(\zeta_i^l) = \zeta^l$. This is carried by the non-zero element

$$(x_1 \cdots x_n)^{m'} \prod_{1 \leqslant i < j \leqslant n} (x_i^r - x_j^r)$$

of the ordinary co-invariant ring.

For $1 \leqslant i, j \leqslant n$, let

$$f_{i,j} = x_j^{(i-1)r+\bar{m}} \text{ and put } v_j = x_1^{r-\bar{m}+m'} \cdots x_j^{m'} \cdots x_n^{r-\bar{m}+m'}. \tag{8.3}$$

When $p = 1$, the functions $f_{i,j}$ for $1 \leqslant j \leqslant n$ span a copy of V, and, when $p > 1$, $v_1, \dots, v_n$ span a copy of V. One computes

$$\det(f_{i,j})_{i,j=1}^n = (x_1 \cdots x_n)^{\bar{m}} \prod_{1 \leqslant i < j \leqslant n} (x_i^r - x_j^r) \tag{8.4}$$

and if A is the matrix whose nth row is $v_1, v_2, \dots, v_n$ and whose ith row for $1 \leqslant i < n$ is $f_{i,1}, f_{i,2}, \dots, f_{i,n}$, then

$$\det(A) = (-1)^n (x_1 \cdots x_n)^{m'} \prod_{1 \leqslant i < j \leqslant n} (x_i^r - x_j^r). \tag{8.5}$$

It follows by [**21**, Theorem 3.1] that V is free and the exponents of V are

$$e_i(V) = \bar{m} + (i-1)r \text{ for } 1 \leqslant i \leqslant n \quad \text{if } p = 1, \tag{8.6}$$

and

$$\left.\begin{aligned} e_i(V) &= \bar{m} + (i-1)r \quad \text{for } 1 \leqslant i \leqslant n-1, \\ e_n(V) &= (n-1)(r-\bar{m}) + nm' \quad \text{if } p > 1. \end{aligned}\right\} \tag{8.7}$$

The degrees of $G(r,p,n)$ are $r, 2r, \dots, (n-1)r, n(r/p)$ if $p < r$ and $r, 2r, \dots, (n-1)r, n$ if $p = r$. When $m = h+1$ it is straightforward to verify the last claim. □

Theorem 8.2. *Suppose that $G(r,p,n)$ acts irreducibly on $\mathbb{C}^n$. With h as in (8.1) and $V = \mathbb{C}\{x_1^{h+1}, \dots, x_n^{h+1}\}$, there is a W-equivariant quotient L of the diagonal co-invariant ring R of $G(r,p,n)$ such that, for each $w \in W$,*

$$\sum \operatorname{tr}(w, (L \otimes \Lambda^n V)_i) t^i = \frac{\det(1 - t^{h+1} w_V)}{\det(1 - t w_{\mathfrak{h}^*})},$$

where w_V and $w_{\mathfrak{h}^*}$ denote w regarded as an endomorphism of V and $\mathfrak{h}^*$, respectively. The image of $S(\mathfrak{h}^*)$ in L is isomorphic to the ordinary co-invariant ring.

Proof. The theorem will follow from Theorem 3.2, with $L = \operatorname{gr} L(\mathbf{1}) \otimes \Lambda^n V^*$, once we verify its hypotheses. Let $\mu_i \in \mathbb{Z}_{\geqslant 0}^n$ have an $h+1$ in the ith position and 0s elsewhere. Then one checks that with $c_s = (h+1)/h$ for all reflections $s \in G(r,p,n)$ the hypotheses as in Theorem 7.1 are satisfied for $k = h+1$. Thus, the radical of $M(\mathbf{1})$ is generated by $\mathbb{C}\{f_{\mu_1}, \dots, f_{\mu_n}\}$. Lemma 6.2 shows that, as W-modules,

$$\mathbb{C}\{f_{\mu_1}, \dots, f_{\mu_n}\} \cong V = \mathbb{C}\{x_1^{h+1}, \dots, x_n^{h+1}\}. \tag{8.8}$$

Thus, the hypotheses of Lemma 3.1 hold, with $k = h+1$. Lemma 8.1 shows that the remaining hypotheses of Theorem 3.2 hold. □

For an arbitrary irreducible complex reflection group W we define the 'Coxeter' number of W to be

$$h = \frac{N + N^*}{n}, \tag{8.9}$$

where n is the dimension of the reflection representation of W, N is the number of reflections in W and N^* is the number of reflecting hyperplanes for W. This definition agrees with (8.1) for the groups $G(r,p,n)$ whenever they are irreducible. By a straightforward modification of [**3**, Proposition 2.3], when $c = 1/h$ there is a one-dimensional $\mathbb{H}$-module with a Bernstein–Gelfand–Gelfand (BGG) resolution.

Question 8.3. Is it possible that, for every complex reflection group W and every integer m coprime to the 'Coxeter' number h, when $c = (m/h)$ the $\mathbb{H}$-module $L(\mathbf{1})$ is m^n dimensional with BGG resolution

$$0 \to M(\Lambda^n V) \to \cdots \to M(\Lambda^1 V) \to M(\mathbf{1}) \to L(\mathbf{1}) \to 0, \tag{8.10}$$

where V is an irreducible $\mathbb{C}W$-module of dimension n?

This question is related to [**4**, Conjecture 4.3]: Bessis and Reiner conjecture that for an irreducible complex reflection group of dimension n that can be generated by n reflections, there is a homogeneous system of parameters in each degree $\pm 1 \bmod h$ that carries either the reflection representation or its dual. The existence of such a homogeneous system of parameters implies an interpretation of the q-Fuss/Catalan numbers as Hilbert series (see [**1**] for a survey of 'Catalan phenomena' and non-crossing partitions). We expect that, at parameters $c_s = 1 + 1/h$, if W can be generated by n reflections, then $L(\mathbf{1})$ gives rise to a nice quotient of the diagonal co-invariant ring; the results of [**24**, § 5] are sure to be relevant here.

Note added in proof

Question 8.3 is answered in [**13**], assuming that the Hecke algebra of W is free of rank $|W|$.

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