Towards a General Theory of Good Deal Bounds.

Tomas Björk,
Department of Finance,
Stockholm School of Economics,

Irina Slinko,
Department of Finance,
Stockholm School of Economics,

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Basic Framework

Exogenously Given:

- An underlying **incomplete** market.

- A contingent T-claim Z.

Recall: The arbitrage free price of Z is given by

$$\Pi(t, Z) = E^P \left[\frac{D_T}{D_t} \cdot Z \middle| \mathcal{F}_t \right] = E^Q \left[e^{-\int_t^T r_u du} \cdot Z \middle| \mathcal{F}_t \right]$$

where D is the stochastic discount factor (SDF)

$$D_t = e^{-\int_0^t r_u du} L_t, \quad L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t$$

However:

- Incomplete market $\Rightarrow$ D and Q are not unique.

- Thus no unique price process $\Pi(t, Z)$.

How can we price in this incomplete setting?

Sad Fact:
The no arbitrage bounds are far to wide to be useful.

Some standard techniques:

- Quadratic hedging.
- Utility indifference pricing.
- Minimize some distance between Q and P.

Our Goal:

- Find "reasonable" and **tight** no arbitrage bounds.
- Economic interpretation.
- Market data as input.

Cochrane and Saa-Requejo

- An arbitrage opportunity is a "ridiculously good deal".

- Thus, no arbitrage pricing is pricing subject to the constraint of ruling out ridiculously good deals.

The CSR Idea:
Find pricing bounds by ruling out, not only ridiculously good deals, but also "unreasonably good deals".

How is this formalized?:

- Impose restrictions on the volatility of the SDF (stochastic discount factor).

- Impose bounds on the Sharpe Ratio!

Sharpe Ratio

The Sharpe Ratio for an asset price S is defined by

$$SR = \text{risk premium per unit volatility}$$

i.e.

$$SR = \frac{\mu - r}{v}$$

where

$$\begin{aligned} \mu &= \text{mean rate of return} \\ r &= \text{short rate} \\ v &= \text{total volatility of } S \end{aligned}$$

i.e.

$$v_t^2 dt = Var^P \left[\left. \frac{dS_t}{S_{t-}} \right| \mathcal{F}_{t-} \right]$$

Moral:
High Sharpe Ratio = unreasonbly good deal.

Reasonable Values of the Sharp Ratio

- The market portfolio is not so dramatically inefficient $\Rightarrow$ we do not expect to see SR much higher then historical market SR, which is about 0,5.

- Using utility function approach, unless we make extreme assumptions about consumption volatility and risk aversion it is difficult to generate SR higher then 0,3.

- A hedge fund with a SR around 2 is doing extremely well.

CSR First Problem Formulation

Find upper and lower price bounds subject to a constraint of the Sharpe Ratio, i.e. find

$$\sup \; E^P\left[\frac{D_T}{D_t} \cdot Z \middle| \mathcal{F}_t\right]$$

subject to

$$|SR_t| \leq B. \quad \text{for all } t$$

However:

- Formulated this way, the problem is mathematically intractable.
- Even if we have a bound on the SR for the Z derivative, it may be possible to form portfolios (on underklying and derivative) with very high Sharpe ratios.

Reformulating the Constraint

Recall:
In a Wiener driven world we have the

Hansen-Jagannathan inequality:

$$|SR_t|^2 \leq \|h_t\|^2_{R^d}$$

where

$$-h_t = \text{market price vector of } W\text{-risk}$$

or in martingale language

$$dL_t = L_t h_t dW_t, \quad L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t$$

Idea:
Replace SR constraint with constraint on $\|h_t\|$

Second CSR Problem Formulation

Find

$$\sup_{h} \; E^P \left[\frac{D_T}{D_t} \cdot Z \middle| \mathcal{F}_t \right]$$

subject to

$$\|h_t\|_{R^d}^2 \leq B^2 \quad \forall t \in [0, T].$$

CSR Results:

- Main analysis done in one-period framework.
- In continuous time, CSR derive a PDE for upper and lower price bounds through (informal) dynamic programming argument.
- Obtains nice numerical results.
- Surprisingly tight bounds.

Limitations of CSR

$$\sup_h \; E^P \left[\frac{D_T}{D_t} \cdot Z \middle| \mathcal{F}_t \right]$$

subject to

$$\|h_t\|^2_{R^d} \leq B^2 \quad \forall t \in [0, T].$$

- Only Wiener driven asset price processes.
- Analysis carried out entirely in terms of SDFs.
- Connection to martingale measures not clarified.
- CSR derive a HJB equation, but the precise underlying control problem is never made precise.
- Some ad hoc assumptions on the upper an lower bounds processes.

Main Contributions of the Present Paper

- We focus on martingale measures rather than on SDF, which is mathematically equivalent but
  - allows to use the technical machinery of martingale theory
  - considerably streamlines the arguments - "good-deal" pricing problem can be formulated as a **standard stochastic control problem**
- We **do not** assume the existence, **nor do we** make assumptions about the explicit dynamics of the price bounds
- We introduce a driving general marked **point process**, thus allowing the possibility of jumps in the random processes describing the financial markets.

A Generic Example

The Merton model:

$$dS_t = S_t \alpha dt + S_t \sigma dW_t + S_{t-} \delta_t dN_t$$

Here N is Poisson and δ lognormal at jumps.

- To obtain a unique derivatives pricing formula Merton assumes **zero market price of jump risk**.

Can we do better?

The Model

- An n-dimensional traded asset price process $S = (S^1, \ldots, S^n)$

$$
\begin{aligned}
dS_t^i &= S_t^i \alpha_i\,(S_t, Y_t)\,dt + S_t^i \sigma_i(S_t, Y_t) dW_t \\
&\quad + S_{t-}^i \int_X \delta_i(S_{t-}, Y_{t-}, x) \mu(dt, dx), \quad i = 1, \ldots, n
\end{aligned}
$$

- A k-dimensional factor process $Y = (Y^1, \ldots, Y^n)$

$$
\begin{aligned}
dY_t^j &= a_j\,(S_t, Y_t)\,dt + b_j(S_t, Y_t) dW_t \\
&\quad + \int_X c_j(S_{t-}, Y_{t-}, x) \mu(dt, dx). \quad j = 1, \ldots, k
\end{aligned}
$$

Recap on Marked Point Processes

- $\mu(dt, dx)$ - number of events in $(dt, dx) \in R_+ \times X$

- Typically we assume that $\mu(dt, dx)$ has predictable P-intensity measure process λ This essentially means that

$$\lambda_t(dx)dt = E^P\left[\mu(dt, dx)\right| F_{t-}]$$

- $\lambda_t(dx)$- expected rate of events at time t with marks in dx.

- For each x, the differential $\mu(dt, dx) - \lambda_t(dx)dt$ is a P-martingale differential.

- $\lambda_t(X)$=global intensity (regardless of mark)

- The probability distribution of marks, given that there is a jump at t is

$$\frac{1}{\lambda_t(X)} \cdot \lambda_t(dx)$$

Assumptions

- The point process μ has a predictable P-intensity measure λ, of the form

$$\lambda_t(dx) = \lambda(S_{t-}, Y_{t-}, dx)dt.$$

- We assume the existence of a short rate r of the form

$$r_t = r(S_t, Y_t).$$

- We assume that the model is free of arbitrage in the sense that there exists a (not necessarily unique) risk neutral martingale measure Q.

- $\delta_i(s, y, x) \geq -1 \quad \forall i$ and $\forall (s, y, x)$

- We consider claims of the form

$$Z = \Phi(S_T, Y_T)$$

Girsanov for MPP and Wiener

Assume that $\mu(dt,dx)$ has predictable P-intensity $\lambda_t(dx)$ and that W is d-dimensional P-Wiener

- Choose predictable processes h_t and $\varphi_t(x) \geq -1$
- Define likelihood process L by

$$\begin{cases} dL_t &= L_t h_t dW_t + L_{t-} \int_X \varphi_t(x) \tilde{\mu}(dt,dx) \\ L_0 &= 1 \end{cases}$$

$$\tilde{\mu}(dt,dx) = \mu(dt,dx) - \lambda_t(dx)dt$$

Then:

- $\mu(dt,dx)$ has Q-intensity

$$\lambda_t^Q(dx) = \{1 + \varphi_t(x)\} \lambda_t(dx)$$

- We have

$$dW = h_t^\star + dW_t^Q$$

Extended Hansen-Jagannathan Bounds

Proposition:
For all arbitrage free price processes S and for all Girsanov kernels $h_t, \varphi_t(x)$, defining a martingale measure, the following inequality holds

$$|SR_t|^2 \leq \|h_t\|^2_{R^d} + \int_X \varphi_t^2(x)\lambda_t(dx)$$

or

$$|SR_t|^2 \leq \|h_t\|^2_{R^d} + \|\varphi_t\|^2_{\lambda_t},$$

where $\|\cdot\|_{\lambda_t}$ denotes the norm in the Hilbert space $L^2\left[X, \lambda_t(dx)\right]$.

Good Deal Bounds

The upper good deal price bound process is defined as the optimal value process for the following optimal control problem.

$$V(t,s,y) = \sup_{h,\varphi} \quad E^Q\left[e^{-\int_t^T r_u du} \Phi\left(S_T, Y_T\right) \middle| \mathcal{F}_t \right]$$

Q dynamics:

$$\begin{aligned} dS_t^i &= S_t^i \left\{ r_t - \int_X \delta_i(x) \left\{1 + \varphi_t(x)\right\} \lambda_t(dx) \right\} dt \\ &\quad + S_t^i \sigma_i dW_t^Q + S_{t-}^i \int_X \delta_i(x) \mu(dt, dx), \\ &\quad\; i = 1, \ldots, n \end{aligned}$$

$$\begin{aligned} dY_t^j &= \left\{a_j + b_j h_t\right\} dt + b_j dW_t^Q \\ &\quad + \int_X c_j(x) \mu(dt, dx). \quad j = 1, \ldots, k \end{aligned}$$

Standard stochastic control problem

Constraints on h and φ

- (Guarantees that Q is a martingale measure)

$$\alpha_i + \sigma_i h_t + \int_X \delta_i(x)\left\{1 + \varphi_t(x)\right\}\lambda_t(dx) = r_t, \quad \forall i$$

- (Rules out "good deals")

$$\|h_t\|^2_{R^d} + \int_X \varphi_t^2(x)\lambda_t(dx) \le B^2,$$

- (Ensures that Q is a positive measure)

$$\varphi_t(x) \ge -1, \quad \forall t, x.$$

HJB Equation

Theorem The upper good deal bound function is the solution V to the following boundary value problem

$$\begin{aligned}\frac{\partial V}{\partial t}(t,s,y) + \sup_{h,\varphi} A^{h,\varphi} V(t,s,y) - r(s,y)V(t,s,y) &= 0, \\ V(T,s,y) &= \Phi(s,y)\end{aligned}$$

NB:
The embedded static problem

$$\sup_{h,\varphi} \left\{ A^{h,\varphi} V(t,s,y) \right\}$$

is a full fledged variational problem. For each (t,s,y) we have to determine $\varphi(t,s,y,\cdot)$ as a function of x.

$$
\begin{aligned}
& A^{h,\varphi} V(t,s,y) \\
= \quad & \sum_{i=1}^{n} \frac{\partial V}{\partial s_i} s_i \left\{ r - \int_X \delta_i(x) \left\{1+\varphi(x)\right\} \lambda_t(dx) \right\} \\
+ \quad & \sum_{j=1}^{k} \frac{\partial V}{\partial y_j} \left\{ a_j + b_j h \right\} + \int_X \Delta V(x) \left\{1+\varphi(x)\right\} \lambda_t(dx) \\
+ \quad & \frac{1}{2} \sum_{i,l=1}^{n} \frac{\partial^2 V}{\partial s_i \partial s_l} s_i s_l \sigma_i^\star \sigma_l + \frac{1}{2} \sum_{j,l=1}^{k} \frac{\partial^2 V}{\partial y_j \partial y_l} b_j^\star b_l + \sum_{i,j=1}^{k} \frac{\partial^2 V}{\partial s_i \partial y_j} s_i \sigma_i^\star b_j
\end{aligned}
$$

Here

$$
\Delta V(x) = V\left(t, s(1+\delta(x)), y + c(x)\right) - V(t,s,y)
$$

Examples. Purely Wiener-driven Model

$$\begin{aligned} dS_t^i &= S_t^i \alpha_i\,(S_t, Y_t)\,dt + S_t^i \sigma_i(S_t, Y_t) dW_t, \quad \forall i \\ dY_t^j &= a_j\,(S_t, Y_t)\,dt + b_j(S_t, Y_t) dW_t, \quad \forall j \end{aligned}$$

The static problem takes the form

$$\max_h \sum_{j=1}^k \frac{\partial V}{\partial y_j}(t, s, y) b_j(s, y) h(t, s, y)$$

subject to the constraints

$$\begin{aligned} \alpha_i + \sigma_i h &= r, \quad i = 1, \ldots, n \\ \|h\|_{R^d}^2 &\leq A^2. \end{aligned}$$

- Maximize **linear** function subject to **linear** and **quadratic** constraints.
- Piece of cake.
- Includes the Cochrane Saa-Requejo theory.

Point Process Examples

Consider a financial market and a scalar price process S satisfying the SDE

$$dS_t = S_t \alpha dt + S_t \sigma dW_t + S_{t-} \int_X \delta(x) \mu(dt, dx).$$

The point process μ has a P-compensator of the form

$$\nu^P(dt, dx) = \lambda(dx)dt$$

λ is a finite nonnegative measure on $(X, \mathcal{X})$.

I. The Poisson-Wiener Model
$X = \{x_0\}$, the measure $\lambda(dx)$ is a point mass $\lambda(x_0)$, the jump function is a real number $\delta = \delta(x_0)$

$$dS_t = S_t \alpha dt + S_t \sigma dW_t + S_{t-} \delta dN_t$$

1. The infinitesimal generator is given now as

$$A^{h,\varphi}V(t,s) = \frac{\partial V}{\partial s}s\left\{r - \delta\lambda(1+\varphi)\right\} + \frac{1}{2}s^2\sigma^2\frac{\partial^2 V}{\partial s^2} + \left\{V(t,s(1+\delta)) - V(t,s)\right\}\lambda(1+\varphi).$$

2. The static optimization problem becomes

$$\max_{h,\varphi} \quad \lambda\left\{V(t,s(1+\delta)) - V(t,s) - V_s(t,s)s\delta\right\}\varphi$$

3. subject to the constraints

$$\begin{aligned} \alpha + \sigma h + \delta\lambda\left\{1+\varphi\right\} &= r, \\ h^2 + \varphi^2\lambda &\leq B^2, \\ \varphi &\geq -1. \end{aligned}$$

The structure of the solution

- In general the optimal kernels have "bang-bang" structure depending on the sign of

$$V(t, s(1+\delta)) - V(t, s) - V_s(t, s)s\delta$$

- In case contract funcion Φ is convex
  - The optimal upper bound value function is convex
  - $V(t, s(1+\delta)) - V(t, s) - V_s(t, s)s\delta \geq 0$
  - The optimal kernels are constant

Solution to the Poisson-Wiener Model

The optimal upper bound value function satisfies the following PIDE

$$\frac{\partial V}{\partial t}(t,s) + \frac{\partial V}{\partial s} s\left\{r - \delta\lambda(1+\hat{\varphi})\right\} + \frac{1}{2}s^2\sigma^2\frac{\partial^2 V}{\partial s^2}$$

$$+\left\{V(t, s(1+\delta)) - V(t,s)\right\}\lambda(1+\hat{\varphi}) - rV(t,s) = 0,$$

$$V(T,s) = \Phi(s)$$

where $\hat{h}, \hat{\varphi}$ are defined by as follows

$$\begin{aligned}
h_{\max} &= -\frac{\sigma R}{(\sigma^2+\delta^2\lambda)\lambda} - \frac{\delta\sqrt{B^2\left(\sigma^2+\delta^2\lambda\right) - R^2)}}{\left(\sigma^2+\delta^2\lambda\right)\sqrt{\lambda}} \\
\varphi_{\max} &= -\frac{\delta R}{\sigma^2+\delta^2\lambda} + \frac{\sigma\sqrt{B^2\left(\sigma^2+\delta^2\lambda\right) - R^2)}}{\left(\sigma^2+\delta^2\lambda\right)\sqrt{\lambda}}
\end{aligned}$$

II. The Compound Poisson-Wiener Model

In this case the static problem has the following form

$$\max_{h,\varphi} \quad \int_X \Delta V(t,s,x)\varphi(t,s,x)\lambda(dx)$$
$$-sV_s(t,s)\int_X \delta(x)\varphi(t,s,x)\lambda(dx),$$

subject to

$$\begin{aligned}
\alpha + \sigma h + \int_X \delta(x)\lambda(dx) + \int_X \delta(x)\varphi(x)\lambda(dx) &= r, \\
h^2 + \int_X \varphi^2(x)\lambda(dx) &\leq B^2, \\
\varphi(x) &\geq -1,
\end{aligned}$$

where, as before,

$$\Delta V(t,s,x) = V(t,s(1+\delta(x))) - V(t,s).$$

- The static problem has to be solved for every fixed choice of (t, s, y) and the control variables are h and φ

- For fixed (t, s, y) h is d-dimensional vector

- However, φ is a function of x and thus infinite-dimensional control variable

- We are faced thus not a standard finite dimensional programming problem, but variational problem

Numerical Aspects of Static Problem

- Linear objective with:
  - Linear constraints.
  - Quadratic constraints.
  - A positivity constraint!
- The positivity constraint makes it messy.

Present situation:

- Without the postivity constraint, the static problem can easily be solved using Hilbert space techniques. This may lead to a signed "martingale measure" and to bounds which are to wide.
- Including the positivity constraint, we have used an interior point method.

The Minimal Martingale Measure

Assume price dynamics

$$dS_t = S_t \alpha dt + S_t \sigma dW_t + S_{t-} \int_X \delta(x) \mu(dt, dx).$$

The **minimal martingale measure** is defined as the (possibly signed) martingale measure with minimum norm for the price of risk, i.e. by the problem

$$\min_{h,\varphi} \quad \|h_t\|^2_{R^d} + \int_X \varphi_t^2(x) \lambda_t(dx)$$

s.t.

$$\alpha + \sigma h_t + \int_X \delta(x) \left\{1 + \varphi_t(x)\right\} \lambda_t(dx) = r_t,$$

NB: No positivity constraint!

The good deal constraint is

$$\|h_t\|^2_{R^d} + \int_X \varphi_t^2(x) \lambda_t(dx) \leq B^2$$

If the MMM is a positive measure then the MMM price is always within the good deal bounds.

The Relaxed Pricing Bounds

Definition:

$$\begin{aligned} V^s &= \text{optimal upper price bound} \\ V^i &= \text{optimal lower price bound} \\ \bar{V}^s &= \text{upper relaxed price bound} \\ \bar{V}^i &= \text{lower relaxed price bound} \\ \bar{V}^m &= \text{MMM price} \end{aligned}$$

The relaxed prices bounds $\bar{V}^s$, $\bar{V}^i$, and the MMM price $\bar{V}^m$ are **very** easy to compute.

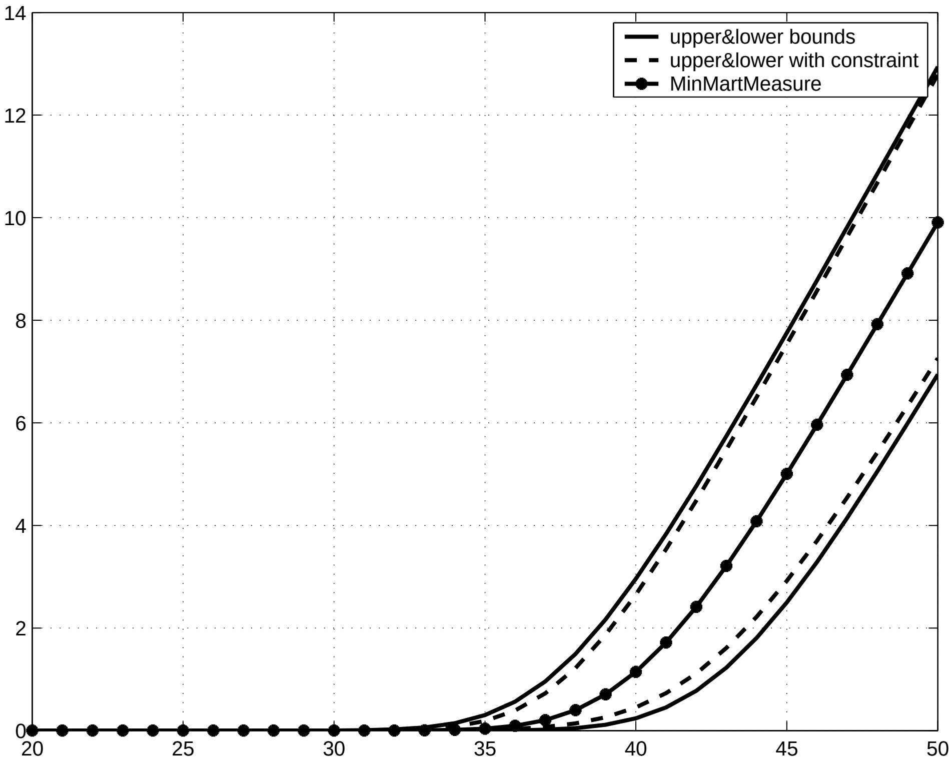
In general we have

$$\bar{V}^i \leq V^i \leq V^s \leq \bar{V}^s$$

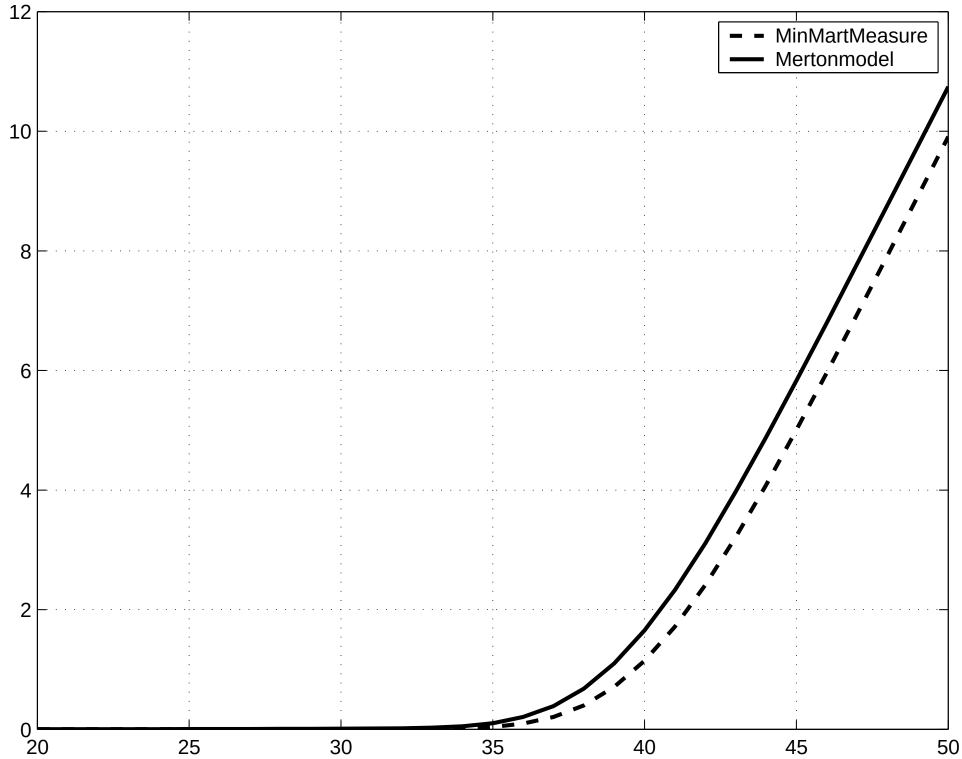
and

$$\bar{V}^i \leq \bar{V}^m \leq \bar{V}^s$$

Good deal pricing bounds



The minimal martingale measure and the Merton model



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