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Towards a Physical Understanding of the Earthquake Frequency Distribution

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Summary

In this paper the cumulative frequency–magnitude relationship is replaced by the cumulative frequency–moment equation in which the B-value takes the place of the b-value

$$N(M_0) = \alpha \cdot M_0{}^{-B}$$

where the constants α and B can be observed or derived from the magnitude–moment and frequency–magnitude relationships. The average and maximal moments of a given set of earthquakes are found to be directly related to B and α respectively:

$$\overline{M}_0 = M_0(\min)\cdot e^{1/B} \qquad \text{and} \qquad M_0(\max) = (\alpha/N_{\max})^{1/B}$$

The total cumulative moment of an earthquake sequence can be expressed as

$$M_0(\text{tot}) = \frac{B}{1-B}\cdot M_0(\max)$$

which is approximately equal to twice $M_0(\max)$ for most observed sets of earthquakes.

Using the definition of the moment we then derive the general area frequency relationship

$$N(A, D) = \alpha_2 \cdot (A \cdot D)^{-B}$$

which shows that B gives the distribution of the product of average displacement D and fault area A. We may reformulate this in terms of the product of stress-drop $\Delta\sigma$ and fault area

$$N(A, \Delta\sigma) = \alpha_4 \cdot (\Delta\sigma \cdot A^{1.5})^{-B}$$

If we make the assumption that the stress-drop is a known function of the source dimension,

$$\Delta\sigma(r) = \beta \cdot r^{\gamma}$$

which can be verified for a given sequence, we may express the frequency of earthquake occurrence as a function of one source parameter

$$N(A) = \alpha_6' \cdot A^{-(3+\gamma)B/2}$$

from which we can obtain the mean rupture area of a set as a function of

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B, γ and the smallest area in the set

$$\bar{A} = A_{\min}\cdot\exp\left[\frac{2}{B(\gamma+3)}\right]$$

Alternatively we may eliminate the area and obtain

$$N(\Delta\sigma) = \alpha_7\cdot\Delta\sigma^{-((3+\gamma)/\gamma)B}$$

$$\bar{\Delta}\sigma = \Delta\sigma(\min)\cdot\exp\left[\frac{\gamma}{B(\gamma+3)}\right].$$

From this we may calculate the ratio of the average stress-drops of two sets with different B. From the different b-values of Denver earthquakes during low and high injection pressure the stress-drop is computed as 30 per cent higher at low-pore pressure. This difference is in good agreement with the difference of the respectively necessary failure stresses, which is 14 per cent.

In addition high apparent stresses $\eta\cdot\bar{\sigma}$ and high stress-drop $\Delta\sigma$ were found to correlate with low b- and low B-values, as they do in microfracture experiments. The combination of high average $\eta\bar{\sigma}$, high mean $\Delta\sigma$, large mean A and low b- or low B-value can be explained by high regional shear stress.

Introduction

The magnitude–frequency relations of earthquakes satisfy the empirical relation

$$\log N = a - b\cdot M \qquad (1)$$

where N is the number of shocks of magnitude M or greater (e.g. Ishimoto & Iida 1939; Gutenberg & Richter 1949). The fact that equation (1) holds for earthquakes and microfractures (e.g. Mogi 1962; Scholz 1968) indicates that a very fundamental physical property of the fracture process would be discovered if (1) could be explained completely. The purpose of this paper is to make some progress towards the physical understanding of (1).

It is clear that the constant a in this relation is variable since it measures the number of events in a sample with given b and $M_{\min}$. Many seismicity studies have led to the conclusion that the b-value is approximately constant (e.g. Riznichenko 1959; Allen *et al.* 1965). However, more recently, convincing evidence was accumulated that shows b to vary in time (Ikegami 1967; Healy *et al.* 1968) and space (e.g. Miyamura 1962; Karnik 1965; Francis 1968a, b; Eaton, O'Neill & Murdock 1970; Utsu 1971; Butovskaya & Kuznetsova 1971).

The statistical properties and implications of (1) have been studied in detail (e.g. Utsu 1969) and can be considered well understood. Lomnitz (1966) and Hamilton (1967) showed that b determines the average magnitude $\bar{M}$.

On the basis of fracture experiments in the laboratory, Mogi (1962) and Scholz (1968) proposed two different interpretations of the physical meaning of the b-value for microfractures. Mogi observed that the degree of the heterogeneity of the samples increased the b-value. From this he concluded that b measured the size distribution of fault areas. Scholz, however, showed that the b-value was primarily a function of applied stress, and he interpreted Mogi's results in this way also. Scholz offered a theoretical explanation of his observations in which the number of events was a function of the rupture area distribution, which in turn was governed by the applied stress.

In this paper we will replace the magnitude by the seismic moment as a scale of earthquake strength. In this way one can show that the frequency distribution of earthquakes is a function of the fault area distribution or alternatively of the stress-drop distribution given by b. It is also demonstrated that low b-values of earthquakes correlate with high stresses as in the microfracture experiments by Scholz (1968).

The moment–frequency relation

The seismic moment M_0 and the magnitude M are both measures of the strength of an earthquake. These two parameters can function as scales by which earthquakes may be classified. The relationship between the two scales can be estimated by a theoretical model (Aki 1967) or determined empirically (Brune & King 1967; Wyss & Brune 1968). For certain sizes of earthquakes both the moment and magnitude are obtained from the amplitude of waves of the same period. In these cases the relationship between M_0 and M should be one to one. For instance, the surface wave magnitude for an $M_s = 6$ event directly corresponds to its moment (Brune & King 1967). Also, the local magnitude (measured at 1 cps) should correspond directly to the moment when $M_L \leqslant 1$. In the magnitude–frequency relation (1) we will substitute M_0 for M using

$$\log M_0 = c + d \cdot M \tag{2}$$

and thus we obtain the moment–frequency relation

$$N(M_0) = \alpha \cdot M_0^{-B} \tag{3}$$

where $\alpha = \exp\,[2.3(a+bc/d)]$ and $B = b/d$ is the exponent that describes the moment–frequency relation. It is preferable to observe B directly rather than to infer it from the b-value. M_0 is measured in dyne-cm.

From (3) the moment of the largest earthquake or earthquakes included in the sequence for which (1) is valid can be obtained. The number of events with the largest moment M_0(max) will be called N_{max}. This is the smallest number for any M_0 in the set and for aftershock sequences it is usually 1. M_0(max) is a statistical quantity not necessarily equal to be observed maximal moment, and is not to be confused with the largest earthquake possible in a region. From (3) we get

$$M_0(\text{max}) = (\alpha/N_{max})^{1/B} \tag{4}$$

The value of d in (2) changes for different magnitude ranges depending on the definition of magnitude Wyss & Brune (1968). This does not imply that B should change in these intervals but rather it implies that b cannot be constant for magnitudes based on different definitions. To be useful, the magnitude had to be based on the amplitude at a defined frequency. Without knowledge of the radiated spectrum one cannot say whether and how much the amplitude at the defined frequency was decreased owing to the finiteness of the source. For this reason it would be greatly preferable to use observed moments in future studies rather than rely on the moment–magnitude relation as we are forced to do here.

In order to convince the reader that approximately the same frequency relation is obtained on the basis of magnitude as well as moment as a scale, Fig. 1 shows a population of Californian earthquakes for which both M_L and M_0 were observed by Wyss & Brune (1971). This figure can be considered a graphic check on equation (2) for this particular earthquake set. In the statistical presentation of Fig. 1 the scatter in the data of Wyss and Brune (1971) is greatly reduced, adding credibility to the constants chosen for (2) by Wyss & Brune. The sequence is not good for b-determinations because it was selected at random and is incomplete at either end due to instrument limitations.

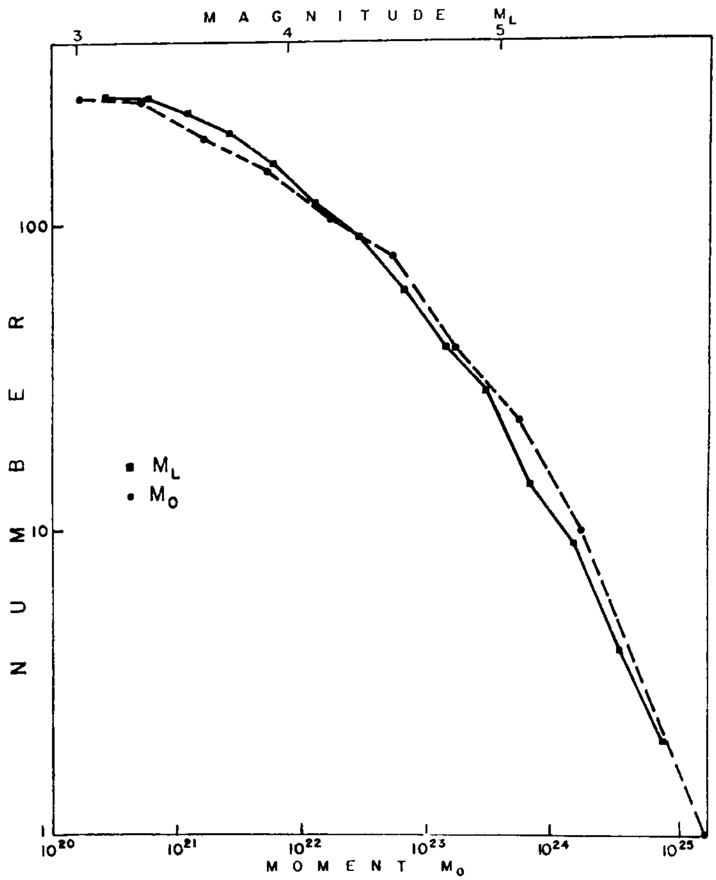

FIG. 1. Cumulative frequency-moment relation compared to cumulative frequency-magnitude relation. Scale of moment and magnitude adjusted after Wyss & Brune (1968).

The moment–frequency distribution

Analogous to the magnitude–frequency distribution we define the moment probability density function as the derivative of the moment–frequency relation (3)

$$\frac{dN(M_0)}{dM_0} = f(M_0) = -\alpha \cdot B \cdot M_0^{-(B+1)}. \tag{5}$$

The cumulative moment of all earthquakes of a given moment is

$$-dN \cdot M_0 = -M_0 \cdot dM_0 \cdot f(M_0) = \alpha \cdot B \cdot M_0^{-B}$$

and the sum of all moments in a given earthquake population is the integral of the above expression. For $B \neq 1$ we get

$$M_0(\text{tot}) = \int_{M_0(\min)}^{M_0(\max)} M_0 \cdot f(M_0)\, dM_0 = \frac{\alpha \cdot B}{1-B} \cdot [M_0^{1-B}]_{M_0(\min)}^{M_0(\max)}$$

Since $M_0(\text{min}) \ll M_0(\text{max})$ and $B < 1$ we use (4) to obtain $N_{\text{max}} = 1$

$$M_0(\text{tot}) = \frac{B}{1-B} \cdot \alpha^{1/B} = \frac{B}{1-B} \cdot M_0(\text{max}). \tag{6}$$

Which can be expressed in terms of b as

$$M_0(\text{tot}) = \frac{b}{d-b} \cdot \exp\left[2.3\left(a\frac{d}{b}+c\right)\right]. \tag{6a}$$

Since $M_0(\text{tot})$ in a defined moment interval must remain finite we see that

$$b < d$$

For a typical value of $b = 1$ and $d = 1.5$ equation (6) gives $M_0(\text{tot}) \cong 2M_0(\text{max})$. This means that for the consideration of the total moment of a sequence of earthquakes only the very largest earthquakes count. Brune (1968) pointed out that this is true for most aftershock sequences except for earthquakes swarms, and it applies to the energy as well, as can be deduced in the same manner as the above result (Gutenberg & Richter 1949).

We also may consider the geometric mean moment, $\overline{M}_0$, of a given sequence. Lomnitz (1966) and Hamilton (1967) give the algebraic average $\overline{M}$ of an earthquake set which obeys equation (1) as

$$\overline{M} = M_{\text{min}} + (2.3b)^{-1}$$

together with (2) we obtain the geometric mean of the corresponding moments

$$\overline{M}_0 = M_0(\text{min}) \cdot e^{1/B}. \tag{7}$$

Analogous to the statistical interpretation of b by Lomnitz we find that B determines the mean moment for a given earthquake population.

The general area frequency relation

The advantage of M_0 as a scale is that we know what it means in terms of source parameters, such as source dimensions r, average dislocation at the source D, and stress-drop $\Delta\sigma$. Aki (1966) derived from the results of Burridge & Knopoff (1964).

$$M_0 = \mu DA = \mu D\pi r^2 \tag{8}$$

where μ is the shear modulus. We will assume a circular source. For this Keilis-Borok (1959) gives

$$\Delta\sigma = (7\pi/16)\mu\frac{D}{r} \tag{9}$$

and we obtain

$$M_0 = \frac{16}{7} r^3 \Delta\sigma \tag{10}$$

From (3) and (8) we can express the frequency of earthquake occurrence as a function of the source dimension and the dislocation

$$N(D, r) = \alpha_1 \cdot (D \cdot r^2)^{-B} = \alpha_2 \cdot (D \cdot A)^{-B} \tag{11}$$

or alternatively using (3) and (10) we get a frequency dependence on source dimension and stress-drop

$$N(\Delta\sigma, r) = \alpha_3 \cdot (\Delta\sigma \cdot r^3)^{-B} = \alpha_4 \cdot (\Delta\sigma \cdot A^{1.5})^{-B} \tag{12}$$

where N is the number of earthquakes exceeding a given product of $(D \cdot r^2)$ or $(\Delta\sigma \cdot r^3)$ respectively. r is measured in centimetres, $\Delta\sigma$ in dyne cm^{-2} and α_i are constants given explicitly in Table 1.

Equations (11) and (12) can be called the general radius frequency relation because no simplifying assumptions that would restrict the applicability of (11) and (12) have been made at this point. The only simplification made was that the geometry of the fault area be circular. This affects the constants α_3 and α_4 only in a minor way, since the constant in (9) does not vary more than a factor of 2 for different geometries of the rupture area.

Table 1

Constants

$$\alpha = \exp\left[2.3\left(a + \frac{bc}{d}\right)\right]$$

$$\alpha_1 = \alpha/(\mu\pi)^B$$

$$\alpha_2 = \alpha/\mu^B$$

$$\alpha_3 = \alpha(\tfrac{7}{16})^B$$

$$\alpha_4 = \alpha(\tfrac{7}{16}\pi^{1.5})^B$$

$$\alpha_5 = \alpha\left(\frac{7}{16\beta}\right)^B$$

$$\alpha_6 = \alpha\left(\frac{7\pi^{(3+\gamma)/2}}{16\beta}\right)^B$$

$$\alpha_7 = \alpha(\tfrac{7}{16}\beta^{3/\gamma})^B$$

From the above equations one sees that the number of earthquakes is mainly a function of the source dimension r as one would expect from equation (1). The number decreases with increasing r by some power of the radius modified by a possible variation of D or $\Delta\sigma$ from region to region or as a function of r.

The mean product $(\overline{\Delta\sigma \cdot r^3})$ follows from (7) as

$$\overline{\Delta\sigma \cdot r^3} = (\Delta\sigma \cdot r^3)^{\min} e^{1/B}. \tag{13}$$

Reduction of the frequency distribution to a one-parameter function

If we can justify an assumption of a relation between D and r, or equivalently between $\Delta\sigma$ and r, we can reduce the general radius frequency relation (11) or (12) to a one-parameter equation.

The simplest assumption is the one of similarity (Aki 1967) which postulates that D scales directly with r, i.e. $\Delta\sigma = \text{const}$. However, empirical evidence shows that this assumption is not true in general (King & Knopoff 1968; Wyss & Brune 1971; Thatcher 1972; Molnar & Wyss 1972). It is possible that Aki's assumption holds for some particular earthquake sequence.

A less restricting, if only slightly different assumption is that $\Delta\sigma$ and D are known functions of r to be determined by observation for each earthquake sequence in question. For some sets of earthquakes such a function may not be found and in such cases the idea of a one-parameter frequency distribution function has to be abandoned. Heterogeneity of the source region may cause great scatter in the dependence of $\Delta\sigma(r)$ because equal volumes of high strength and low strength material will produce large and low stress-drops respectively.

In order to make the assumption

$$\Delta\sigma(r) = \beta \cdot r^\gamma \tag{14}$$

where β and γ are constants to be determined, we therefore postulate that the strength of the source region of a given earthquake sequence be relatively homogeneous. This could be approximately true for a well-established fault zone if we consider only events in this zone and none outside of it, and if we imagine that the fault area distribution is mainly governed by stress heterogeneities along the fault. It is for such earthquake populations that the increase of stress-drop with magnitude or moment holds, as observed by King & Knopoff (1968) and Molnar & Wyss (1972).

From the LD^2 versus magnitude plot of King & Knopoff (1968) one can derive the stress-drop versus moment (or magnitude) dependence directly without any assumptions in the following way: From (8) and (9) we get

$$M_0 \cdot \Delta\sigma = \frac{7\pi}{16}\mu^2 \frac{\pi}{2} LD^2$$

where $L = 2r$. This means the LD^2 versus magnitude plot is equivalent to a $M_0 \cdot \Delta\sigma$ versus magnitude relationship. Using (2) for a given set of earthquakes we then can eliminate the magnitude from this relationship obtaining

$$M_0 = 10^{23} \cdot \Delta\sigma^{2.86} \tag{15}$$

where $\Delta\sigma$ is in bars. Equation (15) fits the data by King and Knopoff as well as some other data of small earthquakes in major fault zones (Wyss 1970a). From (10) and (15) we get

$$\Delta\sigma = 0.13 \cdot r^{1.6} \tag{14a}$$

where $\Delta\sigma$ is measured in bars and r in kilometres. Substituting (14) in (11b) we obtain the cumulative frequency as a function of only one source parameter, the source dimensions.

$$N(r) = \alpha_5 \cdot r^{-(\gamma+3)B} = \alpha_6 \cdot A^{-((\gamma+3)/2)\,B}. \tag{16}$$

In order to obtain this result one has to know the function (14) for the earthquake sequence in question. It is possible to determine (14) for sets of earthquakes since stress-drop determinations can now be done in large numbers.

Alternatively r may be eliminated in (11b) and we obtain the stress–drop–frequency relation

$$N(\Delta\sigma) = \alpha_7 \cdot \Delta\sigma^{-((\gamma+3)/\gamma)\,B} \tag{17}$$

In (16) and (17) N measures the number of earthquakes the radius or stress-drop of which exceeds a given value. We note that for $\gamma = 1$ the dependence of N on r and $\Delta\sigma$ will be identical except for a constant factor. The same is the case for N as a function of A and $\Delta\sigma$ when $\gamma = 2$. The reason for this is the definition of γ in (14). For $\gamma=0$, i.e. $\Delta\sigma=$const., (17) does not exist. In this case the number of events can only be given as a function of source dimension as in (16). γ is expected to have a value between 0 and 2.

From (16) and (17) follows that r and $\Delta\sigma$ of the largest event for which (1) or (3) is valid are given by

$$r_{max} = \alpha_6^{\,1/((\gamma+3)\,B)}$$

$$\Delta\sigma_{max} = \alpha_7^{\,1/((1+(3/\gamma))\,B)}.$$

These equations assume $N_{max} = 1$ but can easily be generalized analogous to (4).

From (7), (10) and (14) we obtain the mean source dimension and the mean stress-drop of a set

$$\bar{r} = r_{\min} \cdot \exp\left[\frac{1}{B(\gamma+3)}\right]$$

$$\overline{\Delta\sigma} = \Delta\sigma_{\min} \cdot \exp\left[\frac{\gamma}{B(\gamma+3)}\right]. \tag{18}$$

From (18) we conclude that the B-value (or b) measures directly the mean dimension or stress-drop of an earthquake sequence, if a relation between $\Delta\sigma$ and r exists. Earthquake sets with large B-values (or large b) will have comparatively small mean stress-drops, which is in agreement with Scholz's (1968) microfracturing results. Large B-values also imply relatively small mean source dimensions.

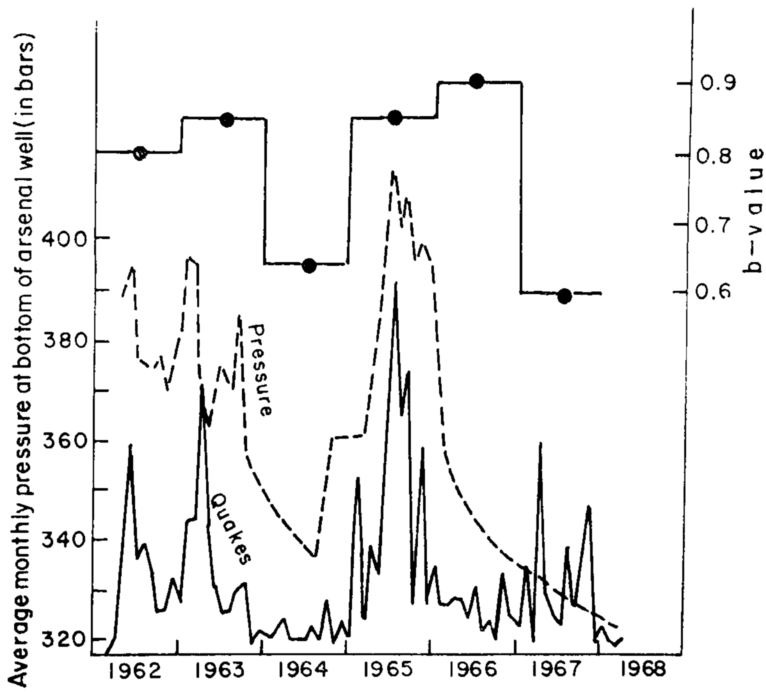

FIG. 2. Number of earthquakes near Denver and pressure in the well compared to b-values (From Healy *et al.* 1968).

Numerical examples

Now we wish to inspect some numerical examples of earthquake sequences for which some of the parameters considered in this paper are known. The purpose of this is to check whether the derivations above lead to reasonable results when numbers are used and to compare some aftershock sequences with each other.

(*a*) *The Parkfield, California aftershock sequence* was studied in great detail by Eaton *et al.* (1970). They found $a = 3.4$ and $b = 0.85$. Wyss & Brune (1968) give for the same sequence $c = 17.0$, $d = 1.4$ and $\gamma \cong 1.5$. The smallest magnitude for which (1) holds is $M_L = 1$ which corresponds to $M_0(\min) = 2.5 \cdot 10^{18}$. It follows that

$$N(M_0) = 10^{13.8}\, M_0^{-0.61}.$$

Further, the inequality $b < d$ is fulfilled and from (6) we get the total moment of that part of the sequence which Eaton *et al.* (1970) used

$$M_0(\text{tot}) = 1.56 \cdot M_0(\text{max}) = 6.25 \cdot 10^{22} \text{ erg}.$$

From (7) the mean moment is estimated as

$$\overline{M}_0 = M_0(\text{min}) \cdot 5.15 = 1.3 \cdot 10^{19}.$$

(*b*) *The Borrego Mt California aftershock sequence.* A detailed study of this sequence was done by Hamilton (1972). The *b*-value was obtained in this paper from Hamilton's data and will be discussed as a function of depth later (Fig. 3). The averages for *b* are given in Table 2. From 29 earthquakes Wyss (1970a) found *c* and *d*. Approximate $\Delta\sigma$ determinations give an estimate for γ (Wyss 1970a) and the resulting parameters of the sequence are given in Table 2.

The relation of stress to the *b*-value

Mogi (1962), Vinogradov (1962) and Scholz (1968) found that equation (1) holds for microfracturing experiments. Scholz observed that in his experiments the *b*-value depended strongly on the applied stress: *b* decreased with increasing stress. Scholz interpreted Mogi's result as due to the same phenomenon.

Scholz also proposed a theoretical model explaining his observation. In his model he came to the conclusion that the number of events was a function of fault area, the distribution of which was governed by the applied stress $\bar{\sigma}$ and the strength S by

$$N(A) = A^{-[1-F(s;\, \bar{\sigma})]} \qquad \text{(Scholz 1968, equation (8))}$$

where $F(S;\, \bar{\sigma})$ is the probability that locally the stress exceeds the strength. Because $F(S;\, \bar{\sigma})$ is a distribution function the exponent of A can only vary between 0 and -1. Equation (S8) and (16) are equivalent since $\Delta\sigma$ depends directly on $\bar{\sigma}$ in laboratory experiments (Scholz *et al*, in preparation). Comparing these two equations we get

$$(1.5+\gamma/2)\, B = 1 - F(S;\, \bar{\sigma}). \tag{21}$$

If Scholz's theory is correct one can calculate from (21) what the probability $F(S;\, \bar{\sigma})$ is for a given B and γ. Further, since $0 \leqslant F(S;\, \bar{\sigma}) \leqslant 1$

Table 2

Numerical examples

Region	b	d	B	γ	$\bar{\Delta}\bar{\sigma}(b)$
Set 1	0·85	1·7*	0·5	1·5*	D
Denver					
Set 2	0·62	1·7*	0·36	1·5*	1·3 D
6 < km	0·94	1·4	0·67	1·5	P_1
Parkfield					
6 ⩾ km	0·73	1·4	0·52	1·5	1·15P_1
average	0·85	1·4	0·61	1·5	P
off fault	0·64	1·4*	0·46	1·5*	1·2P
6 < km	0·92	1·7	0·54	1·5	B_1
Borrego Mt					
6 ⩾ km	0·75	1·7	0·44	1·5	1·15B_1
average	0·92	1·7	0·54	1·5	1·07P
Danville					
6 < km	0·7	1·7*	0·41	1·5*	1·3P
Baja California	0·85	1·7	0·5	1·5*	1·3G
Gulf of California	1·2	1·4	0·86	1·5*	G
Fracture zones	0·63	1·5*	0·42	1·5*	R
Rifts	1·08	1·5*	0·72	1·5*	1·4 R

* assumed

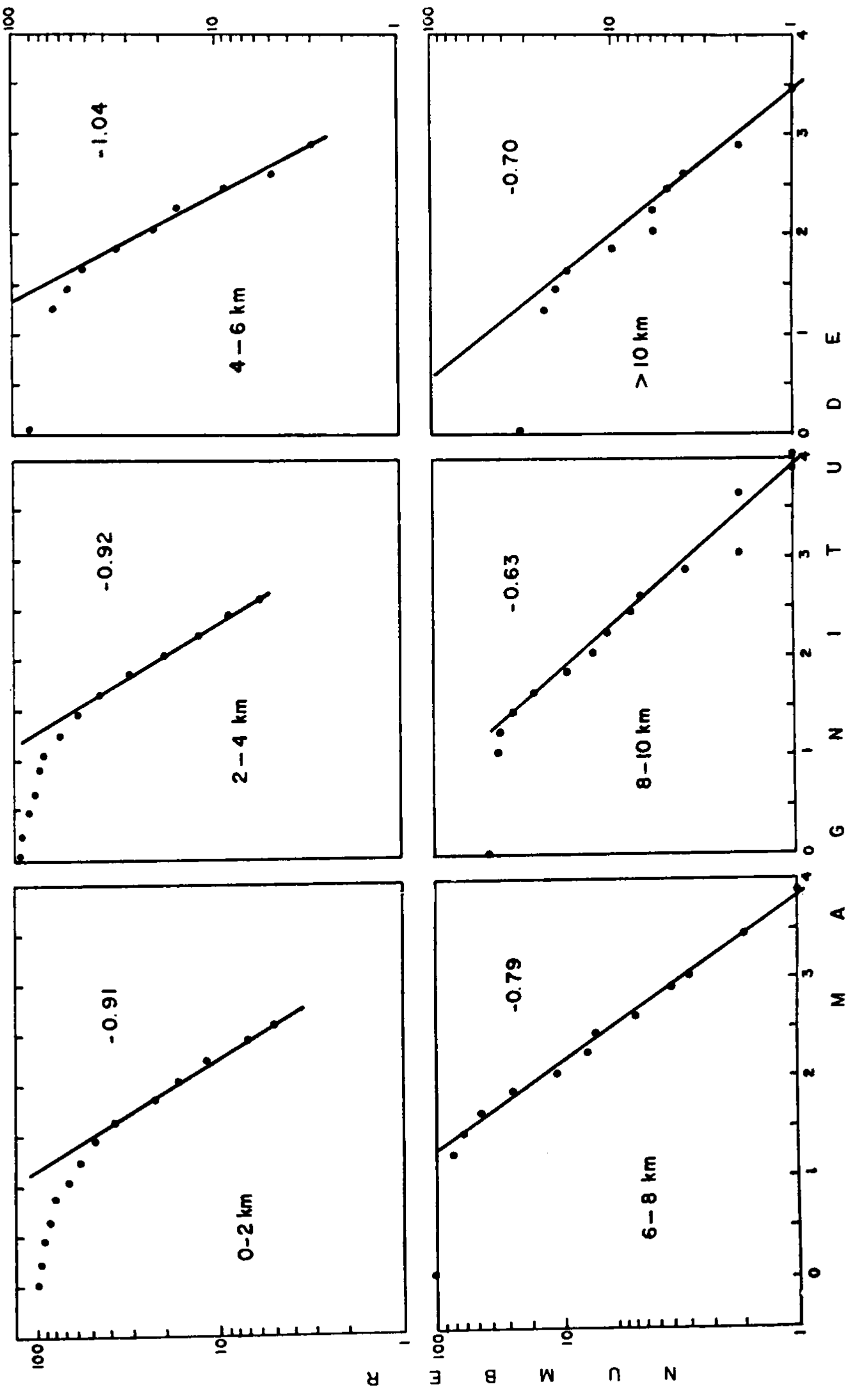

FIG. 3. Cumulative frequency magnitude distribution of the Borrego Mt, California, aftershock sequence. Data from Hamilton (1972).

Scholz's theory leads to

$$\gamma \leqslant \left(\frac{1}{B} - 1.5\right)$$

which puts a restriction on the function $\Delta\sigma(r)$.

For the California sets where B is in the order of 0.5 we get $\gamma \leqslant 1$, which implies

$$\Delta\sigma = \text{const.} \cdot r.$$

This is quite possible for the California sets and we can consider ourselves in approximate agreement with Scholz.

The ratio of the mean stress-drops of two earthquake sets is related to the inverse difference of their respective B-values. If (18) is valid for two sequences we get

$$\frac{\overline{\Delta\sigma}_1}{\overline{\Delta\sigma}_2} = \frac{\Delta\sigma_1(\text{min})}{\Delta\sigma_2(\text{min})} \cdot \exp\left[\frac{1}{B_1(1+3/\gamma_1)} - \frac{1}{B_2(1+3/\gamma_2)}\right]$$

where the indices 1 and 2 refer to set 1 and set 2 respectively. For a comparison of two sets to have meaning we need $M_{0_1}(\text{min}) = M_{0_2}(\text{min})$, and we obtain for the ratio of the mean stress-drops

$$\frac{\overline{\Delta\sigma}_1}{\overline{\Delta\sigma}_2} = \left(\frac{\beta_1{}^{1/\gamma_1}}{\beta_2{}^{1/\gamma_2}}\right)^{(3/\gamma_2)/(\gamma_2+3)} \cdot \Delta\sigma_1(\text{min})^{[3/\gamma_1((\gamma_1-\gamma_2)/(3+\gamma_2))]} \times$$

$$\times \exp\left[\frac{1}{B_1(1+(3/\gamma_1))} - \frac{1}{B_2(1+(3/\gamma_2))}\right]. \qquad (19a)$$

From this equation it becomes evident that differences in B-values may indicate differences in stress-drop. On the other hand such differences may be offset by variations in β and γ. Also B-values could be the same but $\overline{\Delta\sigma}$ would vary because of different β and γ. Combined studies of $\Delta\sigma$, γ and β may furnish some laws by which these parameters are interconnected.

In practice it may well be that sets of earthquakes have the same slope of $\Delta\sigma(r)$, i.e. $\gamma = \text{const}$ in (14), with different β. In this case (19a) reduces to

$$\frac{\overline{\Delta\sigma}_1}{\overline{\Delta\sigma}_2} = \left(\frac{\beta_1}{\beta_2}\right)^{[3/\gamma+(3/\gamma)^2]} \cdot \exp\left[\frac{(B_2-B_1)\cdot\gamma}{B_1\cdot B_2\cdot(\gamma+3)}\right]. \qquad (19b)$$

If we deal with earthquake sets which have the same equations (14) but different B-values, then (19b) further reduces to

$$\frac{\overline{\Delta\sigma}_1}{\overline{\Delta\sigma}_2} = \exp\left[\frac{B_2-B_1)\cdot\gamma}{B_1\cdot B_2\cdot(\gamma+3)}\right] \qquad \begin{matrix}\gamma = \text{const.}\\ \beta = \text{const.}\end{matrix} \qquad (19c)$$

in which case the difference of the B-values together with γ measure directly the natural logarithm of the inverse ratio of the stress-drops.

Alternatively we may consider the ratio of the mean source-dimensions. Using the same derivation we get

$$\frac{\bar{r}_1}{\bar{r}_2} = \left(\frac{\beta_1}{\beta_2}\right)^{1/(3+\gamma_2)} \cdot r_1(\text{min})^{((\gamma_2-\gamma_1)/(\gamma_2+3))} \times$$

$$\times \exp\left[\frac{1}{B_1(\gamma_1+3)} - \frac{1}{B_2(\gamma_2+3)}\right]. \qquad (20a)$$

If (14) is identical for the two sets, (20a) reduces to

$$\frac{\bar{r}_1}{\bar{r}_2} = \exp\left[\frac{B_2 - B_1}{B_1 \cdot B_2 \cdot (\gamma+3)}\right] \qquad \begin{array}{l}\gamma = \text{const.}\\ \beta = \text{const.}\end{array} \tag{20b}$$

a result which can also be obtained from (14) and (19c).

With equations (16), (17) and (19) we have shown that the same relation between b and stress must hold for earthquakes as for microfractures. Scholz (1968) showed that by increasing the shear stress or by decreasing the confining pressure the b-value will be decreased. Either of these two stress changes will produce larger stress-drop and it was shown above, that larger stress-drop in earthquakes will decrease the B (or b) value.

The Denver earthquakes provide an opportunity to prove from observations that the above derivations are correct. These earthquakes have been triggered by the injection of waste fluid (Evans 1966). Healy *et al.* (1968) showed that over a six-year period there was a strong correlation of earthquake activity near the well with the pressure in the well. They concluded that the earthquakes are triggered because at high pumping pressure, i.e. at high-pore pressure p, the frictional resistance to fracture is decreased according to

$$\tau = \tau_0 + (S_n - p)\tan\phi$$

where τ is the shear stress on the fault plane at failure, τ_0 is the cohesive strength, S_n is the normal stress across the fault plane and $\tan\phi$ is the coefficient of friction. Healy *et al.* (1968) also calculated $\tau_1 = 203$ bars when $p_1 = 389$ bars (the subscript refers to set 1) and they found $\tau_0 = 150$ bars. This situation corresponds to the years 1962, 1963, 1965 (set 1). In Fig. 2 we see that in these years the b-values (Healy *et al.* 1968) are high, typically $b_1 = 0.85$. In the years of low pumping pressure (1964, 1967; set 2) $p_2 = 340$ bars and $b_2 = 0.62$. Fig. 2 shows that there is a very good correlation of pumping pressure, number of earthquakes and b-values. Following Healy *et al.* we obtain $\tau_2 = 231$ bars for 1964, 1967 and we see that high stresses produce low b-values in agreement with Scholz's (1968) microfracture results.

We can test our equation (19) quantitatively by considering set 1 and set 2 of the Denver earthquakes as defined above. Since the two sets have the same source region we will assume $\beta_1 = \beta_2$ and $\gamma_1 = \gamma_2$. In this case equation (19c) applies, and assuming $d = 1.7$ and $\gamma = 1.5$ we obtain $\overline{\Delta\sigma_2}/\overline{\Delta\sigma_1} = 1.3$. This predicted ratio of the stress-drops is in good agreement with the ratio of the yielding stresses $\tau_2/\tau_1 = 1.14$. Increase of stress-drop with yielding stress is also observed in the laboratory (Scholz *et al.* 1972).

It is felt that the data from the Denver earthquakes are a very strong support for the connection of the b-value with stress-drop and applied stress, developed in this paper and by Scholz (1968), even though in the year 1966 b does not well correlate with p (Fig. 2). Possibly the sequence of 1966 was still dominated by the high-pore pressure regime during the early part of the year. It should also be pointed out that, for optimal correlation, b-values should be obtained for sets selected on the basis of the pore pressure regime and not on an annual basis.

A sequence chosen on the pressure regime basis could be considered a homogeneous set. In order to obtain the best results for B-value analysis one should observe $N(M_0)$ and $\Delta\sigma(r)$ directly eliminating assumptions regarding d, β and γ and one should only consider homogeneous sets.

Correlation of *b*-values with source parameters in homogeneous earthquake sequences

If the b-value of an earthquake sequence is to make sense in terms of physical parameters the sequence must be homogeneous with regard to the involved earthquake mechanism and conditions at the source. For future studies of the relation of B to source parameters, one should be cautious with the choice of the earthquake set to be studied. Some sources of heterogeneity other than varying pore pressure could be the following:

(a) Depth of focus

In a detailed study of an aftershock sequence near Parkfield, California, Eaton *et al.* (1970) recently found that the b-value changes with depth in the uppermost 12 km of the crust. Fig. 3 shows the b-values of different subsets of another earthquake sequence, near Borrego Mt, which was studied by Hamilton (1972). It is clear that for $h \geqslant 6$ km there are more large earthquakes and lower b-values than for $h < 6$ km. Butovskaya & Kuznetsova (1971) and Papazachos *et al.* (1967) also found b to vary as a function of depth in the crust.

Table 3

b-values as a function of depth

Depth (km)	Parkfield* (b)	Borrego (b)
0–2	—	0·91
2–4	1·03	0·92
4–6	0·97	1·04
6–8	0·84	0·79
8–10	0·61	0·63
10–12	0·87	0·70
12–14	0·95	—

* Eaton *et al.* 1970.

From both Parkfield and Borrego data (Table 3) it appears that there is a discontinuity of the b-value at a depth of 6 km in parts of California. From (19) we can estimate the ratio of the stress-drops for the two depth ranges. Assuming that $\gamma = 1.5$ and $\beta =$ const at all depths we get approximately

$$\overline{\Delta\sigma}(h \geqslant 6)/\overline{\Delta\sigma}(h < 6) = 1.15 \qquad \text{Parkfield } (d = 1.4)$$

$$\overline{\Delta\sigma}(h \geqslant 6)/\overline{\Delta\sigma}(h < 6) = 1.15 \qquad \text{Borego Mt } (d = 1.7).$$

These ratios of mean stress-drop are calculated on the assumption that $\beta =$ const. Since the apparent stress at Borrego Mt increases by approximately a factor of 10 with depth (Wyss & Brune 1971) we are led to suspect that β increases also with depth. The observation from the b-value, that the stress-drop increases with depth, makes qualitative sense, since frictional resistance to failure increases with depth. Note in Table 3 that the smallest b-value occurs at 8–10 km depth; below that b increases again. This may indicate that the stresses resisting fracture decrease between 10 and 14 km, owing to the thermal gradient, until at 16 km (deepest foci in California) resistance to shear is so low that all deformation occurs as creep.

(b) Location on or off major faults

It appears that earthquakes located off major faults have higher apparent stresses than earthquakes on them (Wyss & Brune 1971). This difference is presumably due to the weakness of fault zones compared to the surrounding rock volume. It might be that b-values of these two types of earthquakes also differ, and that for homogeneity one should consider these groups separately.

The Danville, California, sequence occurred off the nearby Calaveras fault. This set of events was studied in great detail by Lee, Eaton & Brabb (1971), who found that $b = 0.7$ considering the whole set. This value is lower than that found at Parkfield by Eaton *et al.* (1970), which was 0.85, or that at Borrego 0.92. The Parkfield and Borrego sequences both occurred on major mapped faults. The Danville sequence, however, contains relatively fewer shallow earthquakes than the other sequences. We therefore determined the *b*-value at Danville for $h < 6$ km and found 0.7 (Fig. 4) which contrasts with 0.94 and 0.92 values found for the Parkfield and Borrego Mt sequences respectively at $h < 6$ km. From (19c) the stress-drop ratios between Parkfield and Danville shallow events can be estimated as approximately

$$\overline{\Delta\sigma}(D)/\overline{\Delta\sigma}(P) = 1.3.$$

This is in agreement with the generally higher apparent stress off major fault zones.

Another example of lower *b*-value off the main fault can be found at Parkfield. This aftershock sequence defined very closely a smooth rupture along the San Andreas fault (Eaton *et al.* 1970). The few earthquakes located off the fault are plotted in Fig. 5. The *b*-value of these earthquakes is about 0.64 implying that their mean stress-drops have to be at least 20 per cent higher in the mean than those on the fault.

At Borrego Mt, however, earthquakes with epicentres distant from the surface rupture did not differ in *b*-value from the others. This may be because this aftershock sequence did not define a smooth fault area, but scattered in a band several kilometres wide (Hamilton 1972).

It is interesting to note that the high average *b*-value at Borrego Mt compared to Parkfield is offset by a difference in *d* between the two regions. The resulting *B*-values indicate that the stress-drops at Borrego Mt are slightly higher in the mean than at Parkfield (Table 2).

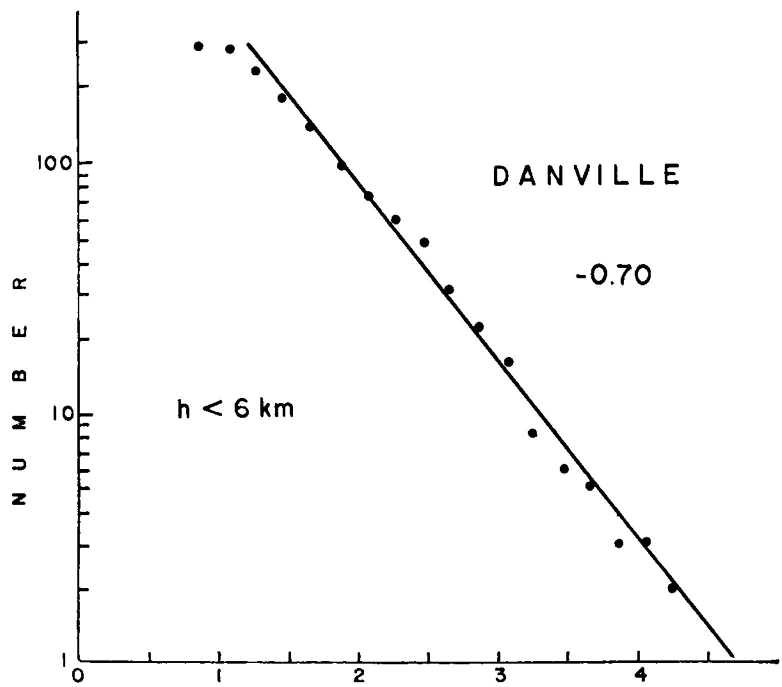

FIG. 4. Cumulative frequency magnitude distribution at Danville, California, for $h < 6$ km. Data from Lee *et al.* (1971).

(c) Ridge and fracture-zone

Earthquakes have considerably different b-values (Francis 1968a). Francis (1968b) gives b (rift) $= 1.33$ and b (fracture) $= 0.65$ when the surface-wave magnitude is considered. If we make the assumption of $\beta =$ const., $\gamma = 1.5$ and $d = 1.5$, the fracture zone earthquakes would be expected to have approximately 30 per cent higher stress-drops in the mean or 20 per cent larger source dimensions according to (19) and (20).

The lower $\overline{\Delta\sigma}$ on the ridge crest could be caused by the higher temperatures and weaker crust there. Or alternatively, we may understand the larger mean source dimensions away from the central rift as a function of the thickening of the brittle crust away from the ridge. If the crust thickens due to cooling as it travels away from the ridge, larger fracture areas become possible, and the mean source dimension is increased.

(d) The Gulf of California and Baja California regions

Earthquakes in the northern parts of the Gulf of California and of Baja California show great differences in the spectra of the emitted waves. This was first noticed by Brune, Epinosa & Oliver (1963) and the difference was quantitatively measured by Wyss & Brune (1971) who found that $(\eta\bar{\sigma})$ in Northern Baja California was on the average 100 times larger than in the Northern Gulf region. Thatcher (1972) studied the spectra of earthquakes in the two regions in detail and found that $\Delta\sigma$ tended to be larger in Baja California.

On the basis of Pasadena local magnitudes and USCGS magnitudes Mota (private communication) found $b = 0.85$ in Baja California and $b = 1.2$ in the Gulf. Using

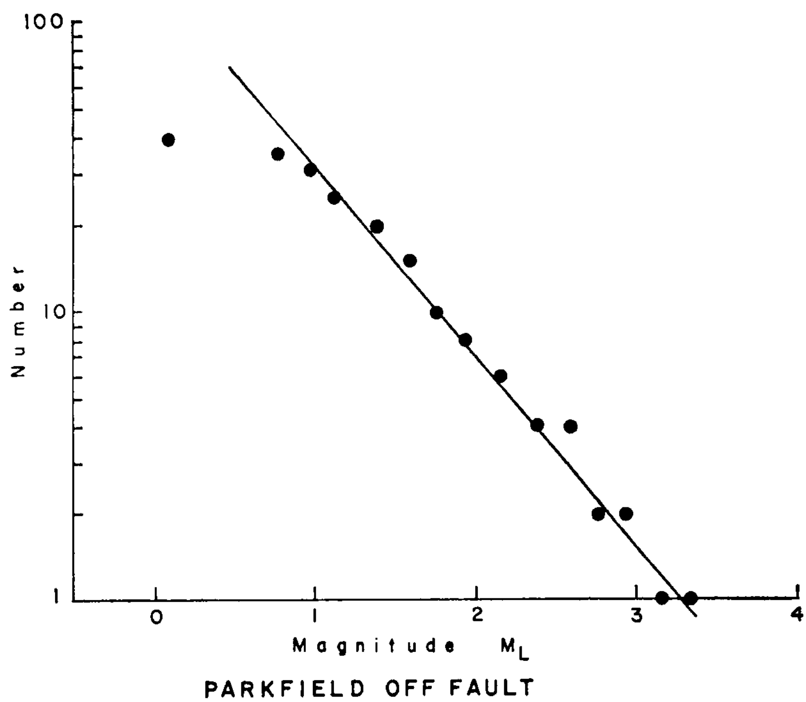

FIG. 5. Cumulative frequency magnitude distribution of Parkfield, California, earthquakes which occurred off the San Andreas fault. Data from Eaton *et al.* 1970.

(19c) with $\gamma = 1.5$, we find that, due to b-variations alone, Baja California earthquakes should have 30 per cent higher stress-drops than those in the Gulf of California. Further differences could arise from variations of β and γ.

Again we see that a low b-value correlated with high stress ($\Delta\sigma$ and $\eta\bar{\sigma}$) in Baja California, whereas the high b-value in the Gulf correlated with low stresses.

(e) *Deep earthquakes in the South American subduction zone*

Acharya (1971) determined that the b-value of deep earthquakes in this zone was very small. Wyss (1970b) on the other hand found that ($\eta\bar{\sigma}$) at great depth was smaller than at intermediate depth. Thus low b and high stress do not correlate in this example. Lomnitz (1972) has discussed a few possible tectonic explanations for this observation.

(f) *Difference of b-values with time*

In addition to the man-caused differences of b with time at the pumping operation in Denver, Suyehiro (1966, 1969) observed lower b-values for foreshock-sequences compared to the b-values of the respective aftershock-sequences. According to our results this indicates that foreshocks have higher mean stress-drops than aftershocks, implying higher tectonic stress for foreshocks. This is to be expected since the main shock will decrease the local tectonic stress to a considerable degree.

In order to estimate the stress-drop change from (19) we will have to assume β = const and γ = const. Since all earthquakes considered occurred in the same source region, these assumptions might be reasonably good. The problem here lies with the magnitude definition used by Suyehiro. Since we do not exactly know how his magnitudes correspond to moment we will have to assume $d = 1.5$. We also assume $\gamma = 1.5$. With these assumptions we estimate that the mean stress-drop was lowered by the main shocks in the order of 10–40 per cent (Table 4) in the three data sets.

Table 4

The mean stress-drops of foreshocks and aftershocks

Location	Year		Fore	After
Kaphallenia	1953	b†	0·65	0·85
		$\overline{\Delta\sigma}$	K	0·89K
Volos	1955	b†	0·43	0·63
		$\overline{\Delta\sigma}$	V	0·78V
NE Crete	1956	b†	0·55	0·65
		$\overline{\Delta\sigma}$	N	0·91N
Japan	1964	b*	0·35	0·76
		$\overline{\Delta\sigma}$	J_1	0·6J_1
Japan	1967	b*	0·59	0·89
		$\overline{\Delta\sigma}$	J_2	0·83J_2
Chile	1960	b*	0·55	1·13
		$\overline{\Delta\sigma}$	C	0·73C

† Papazachos *et al.* (1967).
* Suyehiro (1966 1969).

These estimates seem reasonable; however, they are very uncertain because of the necessary assumptions. It would be of great interest to observe B, and possibly β and γ, directly for fore- and aftershock-sequences. It is expected that from such a study the difference in tectonic stress before and after the main shock could be estimated quite reliably.

(*g*) *Small versus large earthquakes*

Caution is necessary when small and large earthquakes are jointly used in *b*-value studies, since the source dimension versus magnitude (or moment) curve contains a strong discontinuity near magnitude 5 1/2. On either side of this magnitude the slope and level of the curve is different (Wyss & Brune 1968). This may imply that the details of the source mechanism on either side of $M_s = 5\ 1/2$ differ. Earthquakes with $M > 5\ 1/2$ should be considered separately from earthquakes with $M_s < 5\ 1/2$.

Conclusions

The replacement of the magnitude M by the moment M_0 as a scale for earthquakes has great advantages and leads to a number of corollaries which throw light on the physical meaning of the *b*-value. It was shown that low *b*-values indicate high stress in the source region. Higher stresses explain the relatively low *b*-values of foreshocks and the decrease of *b*-values with focal depth observed for crustal earthquakes. Low *b*-values may be used as a guide to find earthquakes with unusually high stress-drops; the type of earthquakes indicating high local tectonic stresses and the type most likely to be confused with an explosive disturbance. Moment and stress-drop determinations for complete sets of earthquakes could consolidate and extend the conclusions of this paper, leading to a more complete physical understanding of the frequency-moment distribution of earthquakes.

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