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TOWARD A THEORY OF FUZZY SYSTEMS

by L. A. Zadeh

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ABSTRACT

          Many of the systems encountered in the real world are
too complex and/or too ill-defined to be susceptible of exact analysis.
The concept of a fuzzy set, that is, a class which admits of intermediate
grades of membership in it, opens the possibility of analyzing such
systems both qualitatively and quantitatively by allowing the input and/or
the output and/or the state of the system to range over fuzzy sets. In
this paper, several basic concepts relating to the characterization of
discrete-time fuzzy systems are introduced and input-output-state
equations for such systems are developed.

CONTENTS

INTRODUCTION

Many of the advances in network theory and system theory
during the past three decades are traceable to the influence and contri-
butions of Ernst Guillemin, Norbert Wiener, Richard Bellman, Rudolph
Kalman, and their students.   In sum, we now possess an impressive arma-
mentarium of techniques for the analysis and synthesis of linear and non-
linear systems of various types - techniques which are particularly ef-
fective in dealing with systems characterized by ordinary differential
or difference equations of moderately high order such as are encountered
in network theory, control theory and related fields.

What we still lack, and lack rather acutely, are methods
for dealing with systems which are too complex or too ill-defined to ad-
mit of precise analysis.   Such systems pervade life sciences, social
sciences, philosophy, economics, psychology and many other "soft" fields.
Furthermore, they are encountered in what are normally regarded as "non-
soft" fields when the complexity of a system rules out the possibility
of analyzing it by conventional mathematical means, whether with or with-
out the aid of computers.   Many examples of such systems are found among
large-scale traffic control systems, pattern recognition systems, machine
translators, large-scale information processing systems, large-scale
power distribution networks, neural networks and games such as chess,
checkers, etc.

Perhaps the major reason for the ineffectiveness of classi-
cal mathematical techniques in dealing with systems of high order of com-
plexity lies in their failure to come to grips with the issue of fuzzi-
ness, that is, with imprecision which stems not from randomness but from
a lack of sharp transition from membership in a class to non-membership
in it.   It is this type of imprecision which arises when one speaks, for
example, of the class of real numbers which are much larger than 10,
since the real numbers cannot be divided dichotomously into those that
are much larger than 10 and those that are not.   The same applies to
classes such as "tall men," "good strategies for playing chess," "pairs
of numbers which are approximately equal to one another," "systems which

are approximately linear," etc. Actually, most of the classes encounter-
ed in the real world are of this fuzzy, imprecisely defined, kind. What
sets such classes apart from classes which are well defined in the con-
ventional mathematical sense is the fuzziness of their boundaries. In
effect, in the case of a class with a fuzzy boundary, an object may have
a grade of membership in it which lies somewhere between full membership
and non-membership.

A class which admits of the possibility of partial member-
ship in it is called a fuzzy set.[1] In this sense, the class of tall men,
for example, is a fuzzy set, as is the class of real numbers which are
much larger than 10. We make a _fuzzy_ _statement_ or assertion when some
of the words appearing in the statement or assertion in question are
names for fuzzy sets. This is true, for example, of statements such as
"John is _tall_," "x is _approximately_ equal to 5," "y is _much_ _larger_ _than_
10," etc. In these statements, the sources of fuzziness are the under-
lined words, which, in effect, are labels for fuzzy sets.

Why is fuzziness so relevant to complexity? Because no
matter what the nature of a system is, when its complexity exceeds a cer-
tain threshold it becomes impractical or computationally infeasible to
make precise assertions about it. For example, in the case of chess the
size of the decision tree is so large that it is impossible, in general,
to find a precise algorithmic solution to the following problem: Given
the position of pieces on the board, determine an optimal next move.
Similarly, in the case of a large-scale traffic control system, the com-
plexity of the system precludes the possibility of precise evaluation of
its performance. Thus, any significant assertion about the performance
of such a system must necessarily be fuzzy in nature, with the degree of
fuzziness increasing with the complexity of the system.

How can fuzziness be made a part of system theory? A tenta-
tive step in this direction was taken in recent papers[2,3] in which the
notions of a fuzzy system* and fuzzy algorithm were introduced. In what

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* The maximin automata of Wee and Santos (Refs. 4 and 5) may be regarded
as instances of fuzzy systems.

follows, we shall proceed somewhat further in this direction, focusing our attention on the definition of a fuzzy system and its state. It should be emphasized, however, that the task of constructing a complete theory of fuzzy systems is one of very considerable magnitude, and that what we shall have to say about fuzzy systems in the sequel is merely a first step toward devising a conceptual framework for dealing with such systems in both qualitative and quantitative ways.

ELEMENTARY PROPERTIES OF FUZZY SETS

The concept of a fuzzy system is intimately related to that of a fuzzy set. In order to make our discussion self-contained, it will be helpful to begin with a brief summary of some of the basic definitions pertaining to such sets.*

Definition of a fuzzy set. Let X = {x} denote a space of points (objects), with x denoting a generic element of X. Then a fuzzy set A in X is a set of ordered pairs

$$A = \left\{ \left(x, \mu_A(x)\right) \right\} \quad , \quad x \in X \qquad\qquad 1.1$$

where $\mu_A(x)$ is termed the grade of membership of x in A. Thus, if $\mu_A(x)$ takes values in a space M – termed the membership space – then A is essentially a function from X to M. The function $\mu_A:X \to M$ which defines A is called the membership function of A. When M contains only two points 0 and 1, A is non-fuzzy and its membership function reduces to the conventional characteristic function of a non-fuzzy set.

Intuitively, a fuzzy set A in X is a class without sharply defined boundaries, that is, a class in which a point (object) x may have a grade of membership intermediate between full membership and non-membership. The important point to note is that such a fuzzy set can be defined precisely by associating with each x its grade of membership in A. In what follows, we shall assume for simplicity that M is the interval [0,1], with the grades 0 and 1 representing, respectively, non-membership and

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* More detailed discussions of fuzzy sets and their properties may be found in the references listed in the Bibliography.

full membership in a fuzzy set. (More generally, M can be a partially ordered set or, more particularly, a lattice.[6]) Thus, our basic assumption will be that a fuzzy set A in X, even though lacking in sharply defined boundaries, can be precisely characterized by a membership function which associates with each x in X a number in the interval [0,1] representing the grade of membership of x in A.

Example. Let A = {x | x >> 1} (i.e., A is the fuzzy set of real numbers which are much larger than 1). Such a set may be defined subjectively by a membership function such as:

$$\mu_A(x) = 0 \qquad \text{for} \quad x \le 1$$
$$= \left(1 + (x - 1)^{-2}\right)^{-1} \qquad \text{for} \quad x > 1 \qquad\qquad 1.2$$

It is important to note that in the case of a fuzzy set it is not meaningful to speak of an object as belonging or not belonging to that set, except for objects whose grade of membership in the set is unity or zero. Thus, if A is the fuzzy set of tall men, then the statement "John is tall" should not be interpreted as meaning that John belongs to A. Rather, such a statement should be interpreted as an association of John with fuzzy set A - an association which will be denoted by John $\underset{\sim}{\in}$ A to distinguish it from an assertion of belonging in the usual non-fuzzy sense, that is, John $\in$ A, which is meaningful only when A is non-fuzzy.*

Containment. Let A and B be fuzzy sets in X. Then A is contained in B (or A is a subset of B) written as A ⊂ B, if and only if $\mu_A(x) \le \mu_B(x)$

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* Here and elsewhere in this paper we shall employ the convention of underscoring a symbol with a wavy bar to represent a fuzzified version of the meaning of that symbol. For example, x $\underset{\sim}{=}$ y will denote a fuzzy equality of x and y; x $\underset{\sim}{\rightarrow}$ y will denote fuzzy implication, etc.

5

for all x in X. (In the sequel, to simplify the notation we shall omit x when an equality or inequality holds for all values of x in X.)

Example. If $\mu_A = \mu_B^2$, then $A \subset B$.

Equality. Two fuzzy sets are _equal_, written as A = B, if and only if $\mu_A = \mu_B$.

Complementation. A fuzzy set A' is the _complement_ of a fuzzy set A if and only if $\mu_{A'} = 1 - \mu_A$.

Example. The fuzzy sets A = {x | x >> 1} and A' = {x | x not >> 1} are complements of one another.

Union. The _union_ of A and B is denoted by A ∪ B and is defined as the smallest fuzzy set containing both A and B. The membership function of A ∪ B is given by $\mu_{A \cup B} = \text{Max}[\mu_A, \mu_B]$. Thus, if at a point x, $\mu_A(x) = 0.9$, say, and $\mu_B(x) = 0.4$, then at that point $\mu_{A \cup B}(x) = 0.9$.

As in the case of non-fuzzy sets, the notion of the union is closely related to that of the connective "or". Thus, if A is a class of tall men, B is a class of fat men and "John is tall" or "John is fat," then John is associated with the union of A and B. More generally, expressed in symbols, we have

$$x \in A \qquad \text{or} \qquad x \in B \;\rightarrow\; x \in A \cup B \qquad\qquad 1.3$$

Intersection. The _intersection_ of A and B is denoted by A ∩ B and is defined as the largest fuzzy set contained in both A and B. The membership function of A ∩ B is given by $\mu_{A \cap B} = \text{Min}[\mu_A, \mu_B]$. It is easy to verify that A ∩ B = (A' ∩ B')'. The relation between the connective "and" and ∩ is expressed by

Algebraic product. The algebraic product of A and B is denoted by AB and is defined by $\mu_{AB} = \mu_A \mu_B$.

Algebraic sum. The algebraic sum of A and B is denoted by A $\oplus$ B and is defined by $\mu_{A \oplus B} = \mu_A + \mu_B - \mu_{AB}$. It is trivial to verify that A $\oplus$ B = (A'B')'.

Relation. A fuzzy relation, R, in the product space X × Y = {(x,y)}, x ∈ X, y ∈ Y, is a fuzzy set in X × Y characterized by a membership function μ_R which associates with each ordered pair (x,y) a grade of membership $\mu_R(x,y)$ in R. More generally, an n-ary fuzzy relation in a product space X = X^1 × X^2 × .... × X^n is a fuzzy set in X characterized by an n-variate membership function $\mu_R(x_1, \ldots, x_n)$, $x_i \in X^1$, i = 1, ..., n.

Example. Let X = Y × R, where R is the real line (−∞,∞). Then x >> y is a fuzzy relation in R^2. A subjective expression for μ_R in this case might be: $\mu_R(x,y) = 0$ for x ≤ y; $\mu_R(x,y) = \left(1 + (x - y)^{-2}\right)^{-1}$ for x > y.

Composition of relations. If R_1 and R_2 are two fuzzy relations in X^2, then by the composition of R_1 and R_2 is meant a fuzzy relation in X^2 which is denoted by $R_1 \circ R_2$ and is defined by

$$\mu_{R_1 \circ R_2}(x,y) = \underset{\nu}{\text{Sup Min}}\left(R_1(x,\nu), R_2(\nu,y)\right) \qquad 1.5$$

where the supremum is taken over all ν in X.

Fuzzy sets induced by mappings. Let f:X→Y be a mapping from X to Y, with the image of x under f denoted by y = f(x). Let A be a fuzzy set in X. Then, the mapping f induces a fuzzy set B in Y whose membership function

is given by

$$\mu_B(y) = \underset{x \in f^{-1}(y)}{\text{Sup}} \mu_A(x) \qquad \qquad 1.6$$

where $f^{-1}(y)$ denotes the set of points in X which are mapped by f into y.

Shadow of a fuzzy set. Let A be a fuzzy set in X × Y, and let f denote the mapping which takes (x,y) into x. The fuzzy set in X which is induced by this mapping is called the shadow (projection) of A on X and is denoted by $S_X(A)$. In consequence of (1.6), the membership function of $S_X(A)$ is given by

$$\mu_{S_X(A)}(x) = \underset{y}{\text{Sup}} \ \mu_A(x,y) \qquad \qquad 1.7$$

where $\mu_A(x,y)$ is the membership function of A.

Conditioned fuzzy sets. A fuzzy set B(x) in Y will be said to be conditioned on x if its membership function depends on x as a parameter. To place this dependence in evidence, we shall denote the membership function of B(x) as $\mu_B(y \mid x)$ or - when B can be omitted with no risk of confusion - as $\mu(y \mid x)$.

Now suppose that the parameter x ranges over a space X. Then, the function $\mu_B(y \mid x)$ defines a mapping from X to the space of fuzzy sets defined on Y. Through this mapping, a fuzzy set A in X induces a fuzzy set B in Y which is defined by

$$\mu_B(y) = \underset{x \in X}{\text{Sup}} \ \text{Min}\left(\mu_A(x), \mu_B(y \mid x)\right) \qquad \qquad 1.8$$

where μ_A and μ_B denote the membership functions of A and B, respectively.

In effect, (1.8) is a special case of the composition of relations (1.5).

The notion of a conditioned fuzzy set bears some resemblance to the notion of a conditional probability distribution. Thus, (1.8) is the counterpart of the familiar identity

$$p_B(y) = \int_X p_B(y \mid x) p_A(x) dx \qquad 1.9$$

where for simplicity x and y are assumed to be real-valued, $p_A(x)$ denotes the probability density of x, $p_B(y \mid x)$ denotes the conditional probability density of y given x and $p_B(y)$ denotes the probability density of y.* It is worthy of note that, in this as well as many other instances involving fuzziness on the one hand and probability on the other, the corresponding formulae differ from one another in that to the operations of summation and integration involving probabilities corresponds the operation of taking the supremum (or maximum) of membership functions, and to the operation of multiplication of probabilities corresponds the operation of taking the infimum (or minimum) of membership functions. To make this correspondence more evident, it is convenient to use the symbols ∨ and ∧ for the supremum and infimum, respectively. Then, (1.9) becomes

$$\mu_B(y) = \bigvee_x \left(\mu_A(x) \wedge \mu_B(y)x \right) \qquad 1.10$$

Similarly, (1.7) becomes

$$\mu_B(y) = \bigvee_{x \in f^{-1}(y)} \mu_A(x) \qquad 1.11$$

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* To simplify the notation, we use the same symbol to denote a random variable and a generic value of that variable.

for which its probabilistic counterpart reads

$$p_B(y) = \sum_{x \in f^{-1}(y)} p_A(x) \qquad\qquad 1.12$$

where x and y are assumed to range over finite sets and $p_A(x)$ and $p_B(y)$ denote probabilities rather than probability densities as in (1.9).

This concludes our brief summary of some of the basic concepts relating to fuzzy sets. In what follows, we shall employ these concepts in defining a fuzzy system and explore some of the elementary properties of such systems.

CHAPTER II

SYSTEM, AGGREGATE AND STATE

For simplicity, we shall restrict our attention to time-invariant discrete-time systems in which t, time, ranges over integers, and the input and output at time t are real-valued.

In the theory of non-fuzzy discrete-time systems, it is customary to introduce the notion of state at the very outset by defining a system $\mathcal{Q}$ through its state equations:

$$x_{t+1} = f(x_t, u_t) \qquad t = \ldots, -1, 0, 1, \ldots$$

$$y_t = g(x_t, u_t)$$

2.1

where u_t denotes the <u>input</u> at time t, y_t is the <u>output</u> at time t and x_t is the <u>state</u> at time t, with the ranges of u_t, y_t and x_t denoted by U, Y and X, respectively. In this way, $\mathcal{Q}$ is characterized by two mappings, $f: X \times U \to \Sigma$ and $g: X \times U \to Y$. The space X is called the <u>state space</u> of $\mathcal{Q}$, and a point α in X is called a <u>state</u> of $\mathcal{Q}$.

Let u denote an input sequence starting at, say, t = 0. Thus, $u = u_0 u_1 \ldots u$ where $u_t \in U$, t = 0, 1, $\ldots$, ℓ, ℓ = non-negative integer. The set of all sequences whose elements are drawn from U will be denoted by U^*.

Now, to each state α in X and each input sequence $u = u_0 u_1 \ldots$ u_ℓ in U^* will correspond an output sequence $y = y_0 y_1 \ldots y_\ell$ in Y^*. The pair of sequences (u,y) is called an <u>input-output</u> pair of length $\ell+1$. The totality of input-output pairs, (u,y), of varying lengths which correspond to a particular state α in X will be referred to as an <u>aggregate</u> of input-

output pairs, or simply an aggregate, $\mathcal{Q}(\alpha)$, with α playing the role of a label for this aggregate. The union

$$\mathcal{Q} \;\; = \;\; \underset{\alpha \in X}{U} \; \mathcal{Q}(\alpha)$$

represents the totality of input-output pairs which correspond to all the states of $\mathcal{Q}$. It is this totality of input-output pairs that we shall equate with $\mathcal{Q}$.

The fact that a state is merely a label for an aggregate suggests that the concept of an aggregate be accorded a central place among the basic concepts of system theory. This is done implicitly in [8] and [9], and explicitly in [10]. The point of departure in the theory developed in [8] is the definition of a system as a collection of input-output pairs. An aggregate, then, may be defined as a subset of input-output pairs which satisfy certain consistency conditions, with a state playing the role of a name for an aggregate.

In what follows, we shall first generalize to fuzzy systems the conventional approach in which a system is described through its state equations. Then we shall indicate a connection between the notion of a fuzzy algorithm and a fuzzy system. Finally, we shall present in a summary form some of the basic definitions relating to the notion of an aggregate and briefly touch upon their generalization to fuzzy systems.

CHAPTER III

STATE EQUATIONS FOR FUZZY SYSTEMS

Let u_t, y_t and x_t denote, respectively, the input, output and state of a system $\mathcal{A}$ at time t. Such a system is said to be deterministic if it is characterized by state equations of the form

$$x_{t+1} = f(x_t, u_t) \qquad t = -1, 0, 1, 2, \ldots, \qquad 3.1$$

$$y_t = g(x_t, u_t) \qquad\qquad 3.2$$

in which f and g are mappings from X×U to X and Y, respectively.

$\mathcal{A}$ is said to be non-deterministic if x_{t+1} and/or y_t are not uniquely determined by x_t and u_t. Let $X^{t+1}(x_t, u_t)$ and $Y^t(x_t, u_t)$ or X^{t+1} and Y^t, for short, denote, respectively, the sets of possible values of x_{t+1} and y_t, given x_t and u_t. Then (3.1) and (3.2) can be replaced by equations of the form

$$X^{t+1} = F(x_t, u_t) \qquad\qquad 3.3$$

$$Y^t = G(x_t, u_t) \qquad\qquad 3.4$$

where F and G are mappings from X×U into the space of subsets of Σ and Y, respectively. Thus, a non-deterministic system is characterized by equations of the form (3.3) and (3.4), in which X^{t+1} and Y^t are subsets of X and Y, respectively.

The next step in the direction of further generalization is to assume that X^{t+1} and Y^t are fuzzy rather than non-fuzzy sets in X and

Y, respectively. In this case, we shall say that $\mathcal{A}$ is a _fuzzy_ discrete -
time system. Clearly, such a system reduces to a non-deterministic sys-
tem when X^{t+1} and Y^t are non-fuzzy sets. In turn, a non-deterministic
system reduces to a deterministic system when X^{t+1} and Y^t are single
points (singletons) in their respective spaces.

Let $\mu_X(x_{t+1} \mid x_t, u_t)$ and $\mu_Y(y_t \mid x_t, u_t)$ denote the member-
ship functions of X^{t+1} and Y^t, respectively, given x_t and u_t. Then we
can say that $\mathcal{A}$ is _characterized_ by the two membership functions $\mu_X(x_{t+1} \mid x_t, u_t)$ and $\mu_Y(y_t \mid x_t, u_t)$, which define conditioned fuzzy sets in X and
Y, respectively, involving x_t and u_t as parameters.

To illustrate, suppose that $X = R^3$. Then $\mathcal{A}$ is a fuzzy sys-
tem if its characterization contains statements such as: "If an input
$u_t = 5$ is applied to $\mathcal{A}$ in state $x_t = (3,5,1)$ at time t, then the state
of $\mathcal{A}$ at time t = 1 will be in the _vicinity_ of the point (7,3,5)." Here
the set of points in X which lie in the vicinity of a given point α is a
fuzzy set in X. Such a set may be characterized by a membership function
such as

$$\mu(x) \;=\; \exp - \frac{1}{k} \, \|x - \alpha\| \qquad\qquad 3.5$$

where x is a point in X, $\|x - \alpha\|$ denotes a norm of the vector $x - \alpha$, and
k is a positive constant.

By analogy with non-fuzzy systems, a fuzzy system $\mathcal{A}$ will be
said to be _memoryless_ if the fuzzy set Y^t is independent of x_t, that is,
if its membership function is of the form $\mu_Y(y_t \mid u_t)$. Just as a non-
fuzzy memoryless system is characterized by a graph $y_t = g(u_t)$, $u_t \in U$,
so a fuzzy memoryless system is characterized by a fuzzy graph which is
a family of fuzzy sets $\{Y^t(u_t), u_t \in U\}$.

In the case of a memoryless system, to each point u_t in U corresponds a fuzzy set $Y^t(u_t)$, or Y^t for short, in Y. Thus, we can write

$$Y^t = G(u_t) \qquad t = \ldots, -1, 0, 1, 2, \ldots, \qquad\qquad 3.6$$

where G is a function from R^1 to the space of fuzzy sets in Y. Now in consequence of (1.8), this implies that if U^t is a fuzzy set in U characterized by a membership function $\mu_U(u_t)$, then to U^t will correspond the fuzzy set Y^t defined by the membership function

$$\mu_Y(y_t) = \bigvee_{u_t} \left(\mu_U(u_t) \wedge \mu_Y(y_t \mid u_t) \right) \qquad\qquad 3.7$$

where $\vee$ and $\wedge$ denote the supremum and minimum, respectively. Thus, (3.7) establishes a relation between U^t and Y^t which can be expressed as

$$Y^t = G_o(U^t) \qquad t = \ldots, -1, 0, 1, 2, \ldots, \qquad\qquad 3.8$$

where G_o is a function from the space of fuzzy sets in U to the space of fuzzy sets in Y.

The important point to be noted here is that equation (3.6), which expresses Y^t as a function of u_t, induces equation (3.8), which expresses Y^t as a function of U^t. As should be expected, (3.8) reduces to (3.6) when U^t is taken to be the singleton $\{u_t\}$.

Intuitively, (3.6) and (3.8) may be interpreted as follows. If $\mathcal{A}$ is a fuzzy memoryless system, then to every non-fuzzy input u_t at time t corresponds a unique fuzzy output which is represented by a

conditioned fuzzy set Y^t in Y. The membership function of this fuzzy set is given by $\mu_Y(y_t \mid u_t)$.

If the input to $\mathcal{A}$ is fuzzy, i.e., is a fuzzy set U^t in U, then the corresponding fuzzy output Y^t is given uniquely by (3.8). The membership function for Y^t is expressed by (3.7).

As a very simple example, suppose that U and Y are finite sets: $U = \{1,2,3\}$ and $Y = \{1,2,3\}$. Furthermore, suppose that if the input u_t is 1, then the output is a fuzzy set described verbally as "y_t is approximately equal to 1." Similarly, if $u_t = 2$, then y_t is approximately equal to 2, and if $u_t = 3$, then y_t is approximately equal to 3. More concretely, we assume that $\mu_Y(y_t \mid u_t)$ is defined by the table:

$$\mu_Y(1 \mid 1) = 1 \quad ; \quad \mu_Y(2 \mid 1) = 0.3 \quad ; \quad \mu_Y(3 \mid 1) = 0.1 \quad ;$$

$$\mu_Y(1 \mid 2) = 0.2 \quad ; \quad \mu_T(2 \mid 2) = 1 \quad ; \quad \mu_Y(3 \mid 2) = 0.2 \quad ;$$

$$\mu_Y(1 \mid 3) = 0.1 \quad ; \quad \mu_Y(2 \mid 3) = 0.2 \quad ; \quad \mu_Y(3 \mid 3) = 1 \quad ;$$

Now assume that the input is a fuzzy set described verbally as "u_t is close to 1," and characterized by the membership function

$$\mu_U(1) = 1 \quad ; \quad \mu_U(2) = 0.2 \quad ; \quad \mu_U(3) = 0.1 \quad ;$$

Then, by using (3.7) the response to this fuzzy input is found to be a fuzzy set defined by the membership function

$$\mu_Y(1) = 1 \quad ; \quad \mu_Y(2) = 0.3 \quad ; \quad \mu_Y(3) = 0.2 \quad ;$$

16

It is convenient to regard (3.8) as a mapping from names of
fuzzy sets in U to names of fuzzy sets in Y. In many cases of practical
interest such a mapping can be adequately characterized by a finite, and
perhaps even fairly small, number of points (ordered paris (U,Y)) on the
graph of G_o. For example, G_o might be characterized approximately by a
table such as shown below. (For simplicity we suppress the subscript t
in u_t and y_t.)

U^t	Y^t
$\underset{\sim}{1}$	$\underset{\sim}{1}$
$\underset{\sim}{1.1}$	$\underset{\sim}{1.3}$
$\underset{\sim}{1.2}$	$\underset{\sim}{1.6}$
$\underset{\sim}{1.3}$	$\underset{\sim}{2}$
$\underset{\sim}{1.4}$	$\underset{\sim}{2.5}$
$\underset{\sim}{1.5}$	$\underset{\sim}{2.9}$
$\underset{\sim}{1.6}$	$\underset{\sim}{2.5}$
$\underset{\sim}{1.7}$	$\underset{\sim}{2.1}$
$\underset{\sim}{1.8}$	$\underset{\sim}{1.8}$
$\underset{\sim}{1.9}$	$\underset{\sim}{1.6}$
$\underset{\sim}{2}$	$\underset{\sim}{1.5}$
$\underset{\sim}{2.1}$	$\underset{\sim}{1.5}$
$\vdots$	$\vdots$
$\underset{\sim}{3}$	$\underset{\sim}{1.5}$

where $\underset{\sim}{x}$, $x \in R$, is the name for the fuzzy set of real numbers which are
approximately equal to x. Such a set may be characterized quantitatively
by a membership function. In many practical situations a very approxi-
mate description of this membership function would be sufficient. In
this way, equation (3.8) can serve the purpose of an approximate charac-
terization of a fuzzy memoryless system.

Turning to non-memoryless fuzzy systems, consider a system $\mathcal{A}$ which is characterized by state equations of the form

$$X^{t+1} = F(x_t, u_t) \qquad\qquad 3.9$$

$$Y^t = G(x_t, u_t) \qquad\qquad 3.10$$

where F is a function from the product space X×U to the space of fuzzy sets in X, G is a function from X×U to the space of fuzzy sets in Y, X^{t+1} denotes a fuzzy set in X which is conditioned on x_t and u_t, and Y^t denotes a fuzzy set in Y which, like X^{t+1}, is conditioned on x_t and u_t. X^{t+1} and Y^t represent, respectively, the fuzzy state and output of $\mathcal{A}$ at time t and are defined by the membership functions $\mu_X(x_{t+1} \mid x_t, u_t)$ and $\mu_Y(y_t \mid x_t, u_t)$.

Equations (3.9) and (3.10) relate the fuzzy state at time t+1 and the fuzzy output at time t to the non-fuzzy state and non-fuzzy input at time t. As in the case of a memoryless system, we can deduce from these equations - by repeated application of (1.8) - the state equations for $\mathcal{A}$ for the case where the state at time t or the input at time or both are fuzzy.

Specifically, let us assume that the state at time t is a fuzzy set characterized by a membership function $\mu_X(x_t)$. Then, by applying (1.8), we deduce from (3.9) and (3.10).

$$\mu_X(x_{t+1}) = \bigvee_{x_t} \left(\mu_X(x_t) \wedge \mu_X(x_{t+1} \mid x_t, u_t) \right) \qquad\qquad 3.11$$

$$\mu_Y(y_t) = \bigvee_{x_t} \left(\mu_X(x_t) \wedge \mu_Y(y_t \mid x_t, u_t) \right) \qquad\qquad 3.12$$

18

which in symbolic form may be expressed as[*]

$$X^{t+1} = F_o(X^t, u_t) \qquad\qquad 3.13$$

$$Y^t = G_o(X^t, u_t) \qquad\qquad 3.14$$

In what follows, to simplify the appearance of equations such as (3.11) and (3.12), we shall omit the subscripts X and Y in membership functions.

By n-fold iteration of (3.13) and (3.14), we can obtain expressions for X^{t+n+1} and Y^{t+n}, $n = 1, 2, 3, \ldots$, in terms of X^t and u_t, $\ldots$, u_{t+n}. For example, for $n = 1$, we have

$$X^{t+2} = F_o\big(F_o(X^t, u_t),\ u_{t+1}\big) \qquad\qquad 3.15$$

$$Y^{t+1} = G_o\big(F_o(X^t, u_t),\ u_{t+1}\big) \qquad\qquad 3.16$$

or more compactly

$$X^{t+2} = F_1\big(X^t,\ u_t,\ u_{t+1}\big) \qquad\qquad 3.17$$

$$Y^{t+1} = G_1\big(X^t,\ u_t,\ u_{t+1}\big) \qquad\qquad 3.18$$

To express (3.17) and (3.18) in terms of membership functions, we note that on replacing t with t+1 in (3.11) and (3.12), we obtain

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[*] Note that $\mu_X(x_{t+1})$ and $\mu_X(x_t)$ represent different membership functions. Strictly speaking, we should write them as $\mu_{X^{t+1}}(x_{t+1})$ and $\mu_{X^t}(x_t)$, respectively.

$$\mu(x_{t+2}) = \bigvee_{x_{t+1}} \left(\mu(x_{t+1}) \wedge \mu(x_{t+2} \mid x_{t+1}, u_{t+1}) \right) \qquad 3.19$$

$$\mu(y_{t+1}) = \bigvee_{x_{t+1}} \left(\mu(x_{t+1}) \wedge \mu(y_{t+1} \mid x_{t+1}, u_{t+1}) \right) \qquad 3.20$$

Then, on substituting $\mu(x_{t+1})$ from (3.11) into (3.19) and (3.20), we get

$$\mu(x_{t+2}) = \bigvee_{x_{t+1}} \left(\bigvee_{x_t} \left(\mu(x_t) \wedge \mu(x_{t+1} \mid x_t, u_t) \right) \wedge \mu(x_{t+2} \mid x_{t+1}, u_{t+1}) \right)$$

$$3.21$$

and

$$\mu(y_{t+1}) = \bigvee_{x_{t+1}} \left(\bigvee_{x_t} \left(\mu(x_t) \wedge \mu(x_{t+1} \mid x_t, u_t) \right) \wedge \mu(y_{t+1} \mid x_{t+1}, u_{t+1}) \right)$$

$$3.22$$

which by virtue of the distributivity of $\vee$ and $\wedge$ may be expressed as

$$\mu(x_{t+2}) = \bigvee_{x_{t+1}} \bigvee_{x_t} \left(\mu(x_t) \wedge \mu(x_{t+1} \mid x_t, u_t) \wedge \mu(x_{t+2} \mid x_{t+1}, u_{t+1}) \right)$$

$$3.23$$

$$\mu(y_{t+1}) = \bigvee_{x_{t+1}} \bigvee_{x_t} \left(\mu(x_t) \wedge \mu(x_{t+1} \mid x_t, u_t) \wedge \mu(y_{t+1} \mid x_{t+1}, u_{t+1}) \right)$$

$$3.24$$

and likewise for larger values of n. It should be noted that these relations are fuzzy counterparts of the corresponding expressions for stochastic systems,[11] with $\wedge$ and $\vee$ replacing product and sum, respectively, and membership functions replacing probability functions (see (3.6) et seq.).

In the above analysis, we have assumed that the successive inputs $u_t, \ldots, u_{t+n}$ are non-fuzzy. On this basis, we can obtain expressions

for X^{t+1}, ..., X^{t+n+1} and Y^t, ..., Y^{t+n} in terms of X^t and u_t, ..., u_{t+n}. It is natural to raise the question of what are the corresponding expressions for X^{t+1}, ..., X^{t+n+1} and Y^t, ..., Y^{t+n} when the successive inputs are fuzzy.

First, let us focus our attention on the state equations (3.3) and (3.4), in which F and G are functions from $X \times U$ to fuzzy sets in X and Y, respectively. Suppose that both the input at time t and the state at time t are fuzzy. What would be the expressions for the membership functions of X^{t+1} and Y^t in this simple case?

Let $\mu(x_t, u_t)$ denote the membership function of the fuzzy set whose elements are ordered pairs (x_t, u_t).[*] Then, using (1.8) we can express the membership functions of X^{t+1} and Y^t as follows:

$$\mu(x_{t+1}) = \bigvee_{x_t} \bigvee_{u_t} \left(\mu(x_t, u_t) \wedge \mu(x_{t+1} \mid x_t, u_t) \right) \qquad 3.25$$

$$\mu(y_t) = \bigvee_{x_t} \bigvee_{u_t} \left(\mu(x_t, u_t) \wedge \mu(y_t \mid x_t, u_t) \right) \qquad 3.26$$

These formulae assume a simpler form when $\mu(x_t, u_t)$ can be expressed as

$$\mu(x_t, u_t) = \mu(x_t) \wedge \mu(u_t) \qquad 3.27$$

where $\mu(x_t)$ and $\mu(u_t)$ denote, respectively, the membership function of the fuzzy state and the fuzzy input at time t. In this case, we shall say that the fuzzy sets X^t and U^t are __non-interacting__. Essentially, the

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[*] The probabilistic counterpart of this membership function is the joint probability of x and u.

notion of non-intersection of fuzzy sets corresponds to the notion of independence of random variables.

The assumption that X^t and U^t are non-interacting fuzzy sets is a reasonable one to make in many cases of practical interest. Under this assumption, (3.25) and (3.26) reduce to

$$\mu(x_{t+1}) \;=\; \bigvee_{x_t} \bigvee_{u_t} \left(\mu(x_t) \wedge \mu(u_t) \wedge \mu(x_{t+1} \mid x_t, u_t) \right) \qquad 3.28$$

$$\mu(y_t) \;=\; \bigvee_{x_t} \bigvee_{u_t} \left((x_t) \wedge \mu(u_t) \wedge \mu(y_t \mid x_t, u_t) \right) \qquad 3.29$$

It should be noted that the same expressions can be obtained by applying (1.8) to (3.13) and (3.14), with the input at time t assumed to be a fuzzy set characterized by $\mu(u_t)$.

In symbolic form, (3.28) and (3.29) can be expressed as

$$X^{t+1} \;=\; F_{oo}(X^t, U^t) \qquad 3.30$$

$$Y^t \;=\; G_{oo}(X^t, U^t) \qquad 3.31$$

where F_{oo} and G_{oo} are, respectively, functions from the product space of fuzzy sets in X and fuzzy sets in U to the space of fuzzy sets in X and fuzzy sets in Y. Thus, (3.30) expresses the fuzzy state at time t+1 as a function of the fuzzy state at time t and the fuzzy input at time t. Similarly, (3.31) expresses the fuzzy output at time t as a function of the fuzzy state at time t and the fuzzy input at time t. Note that (3.30) is induced via (1.8) by (3.9), which expresses the fuzzy state at time t+1 as a function of the non-fuzzy state at time t and the non-fuzzy input at time t. The same is true of (3.31) and (3.10).

A simple numerical example will serve to illustrate the use of the formulae derived above. Specifically, let us consider a fuzzy system with binary input and output, $U = Y = \{0,1\}$, and finite state space $X = \{\alpha, \beta, \gamma\}$. Suppose that the membership functions $\mu(x_{t+1} \mid x_t, u_t)$ and $\mu(y_t \mid x_t, u_t)$ for this system are characterized by the following tables

x_t \ x_{t+1}	$u_t = 0$			$u_t = 1$		
	α	β	γ	α	β	γ
α	1	0.8	0.6	0.8	0.5	1
β	0.7	0.2	1	0.2	1	0.6
γ	0.3	0.3	0.4	0.9	0.7	1

x_t \ y_t	$u_t = 0$		$u_t = 1$	
	0	1	0	1
α	0.8	0.3	0.6	0.3
β	1	0.1	0.5	1
γ	0.8	0.7	0.3	0.2

Further, assume that X^t and U^t are characterized by the membership functions tabulated below.

$$\mu(\alpha) = 1 \; ; \quad \mu(\beta) = 0.8 \; ; \quad \mu(\gamma) = 0.4 \; ;$$

$$\mu(0) = 1 \; ; \quad \mu(1) = 0.3 \; .$$

It should be noted that, as in the case of a memoryless fuzzy system, (3.30) and (3.31) can be used to provide an approximate characterization of a non-memoryless fuzzy system. To illustrate, let us employ the convention introduced earlier, namely, using the symbol $\underset{\sim}{x}$ to denote the name of a fuzzy set or real numbers which are approximately equal to x. Then, viewed as relations between names of fuzzy sets, (3.30) and (3.31) may take the appearance of tables such as shown below:

X^{t+1} :

U^t \ X^t	$\underset{\sim}{1}$	$\underset{\sim}{2}$	$\underset{\sim}{3}$	$\underset{\sim}{4}$
$\underset{\sim}{0}$ $\underset{\sim}{1}$	$\underset{\sim}{2}$ $\underset{\sim}{3}$	$\underset{\sim}{1}$ $\underset{\sim}{4}$	$\underset{\sim}{4}$ $\underset{\sim}{1}$	$\underset{\sim}{3}$ $\underset{\sim}{3}$

Y^t :

U^t \ X^t	$\underset{\sim}{1}$	$\underset{\sim}{2}$	$\underset{\sim}{3}$	$\underset{\sim}{4}$
$\underset{\sim}{0}$ $\underset{\sim}{1}$	$\underset{\sim}{0}$ $\underset{\sim}{1}$	$\underset{\sim}{1}$ $\underset{\sim}{0}$	$\underset{\sim}{1}$ $\underset{\sim}{1}$	$\underset{\sim}{0}$ $\underset{\sim}{1}$

where for simplicity we restricted x and u to integral values. More generally, the entries in these tables would be names for fuzzy sets in X, U and Y, and only a finite number of such names would be used as representative samples (paradigms) of the fuzzy sets in their respective spaces.

So far, we have restricted our attention to the case where a single fuzzy input U^t is applied to $\mathcal{Q}$ in state X^t. For this case, we found expressions for X^{t+1} and Y^t in terms of X^t and U^t. The same approach can readily be extended, however, to the case where the input is a sequence of non-interacting fuzzy inputs $U^t U^{t+1} \ldots U^{t+n}$, $n \geq 1$. The assumption of

non-interaction implies that

$$\mu(u_t, \ldots, u_{t+n}) = \mu(u_t) \wedge \mu(u_{t+1}) \wedge \ldots \wedge \mu(u_{t+n}) \qquad 3.32$$

To illustrate, let n = 1. Then by applying (1.8) to (3.23) and (3.24), we obtain

$$\mu(x_{t+2}) = \bigvee_{x_{t+1}} \bigvee_{x_t} \bigvee_{u_t} \bigvee_{u_{t+1}} \left(\mu(x_t) \wedge \mu(x_{t+1} \mid x_t, u_t) \wedge \right.$$

$$\left. \mu(x_{t+2} \mid x_{t+1}, u_{t+1}) \wedge \mu(u_t) \wedge \mu(u_{t+1}) \right) \qquad 3.33$$

$$\mu(y_{t+1}) = \bigvee_{x_{t+1}} \bigvee_{x_t} \bigvee_{u_t} \bigvee_{u_{t+1}} \left(\mu(x_t) \wedge \mu(x_{t+1} \mid x_t, u_t) \wedge \right.$$

$$\left. \mu(y_{t+1} \mid x_{t+1}, u_{t+1}) \wedge \mu(u_t) \wedge \mu(u_{t+1}) \right) \qquad 3.34$$

As in the case of stochastic systems, for higher values of n such relations can be expressed more compactly through the use of vector and matrix notation. For our purposes, the simple case of n = 1 considered above suffices to illustrate the main features of the method which can be used to compute the fuzzy state and fuzzy output of a system at the end of a finite sequence of non-interacting fuzzy inputs.

CHAPTER IV

FUZZY SYSTEMS AND FUZZY ALGORITHMS

As was shown in a recent note,[3] the notion of a fuzzy system bears a close relation to that of a fuzzy algorithm.

Roughly speaking, a fuzzy algorithm is an algorithm in which some of the instructions are fuzzy in nature. Examples of such instructions are: (a) Increase x _slightly_ if y is _slightly_ larger than 10; (b) Decrease u until it becomes _much_ _smaller_ than v; (c) _Reduce_ speed if the road is _slippery_. The sources of fuzziness in these instructions are the underlined words.

More generally, we may view a fuzzy algorithm as a fuzzy system A characterized by equations of the form:

$$X^{t+1} = F(X^t, U^t) \qquad 4.1$$

$$U^t = H(X^t) \qquad 4.2$$

where X^t is a fuzzy state of A at time t, U^t is a fuzzy input (representing a fuzzy instruction) at time t, and X^{t+1} is the fuzzy state at time t+1 resulting from the execution of the fuzzy instruction represented by U^t. As seen from (4.1) and (4.2), the function F defines the dependence of the fuzzy state at time t+1 on the fuzzy state at time t and the fuzzy input at time t, while the function H describes the dependence of the fuzzy input at time t on the fuzzy state at time t.

To illustrate (4.1) and (4.2), we shall consider a very simple example. Suppose that X is a fuzzy subset of a finite set X = $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and U^t is a fuzzy subset of a finite set U = $\{\beta_1, \beta_2\}$.

Since the membership functions of X^t and U^t are mappings from, respectively, X and U to the unit interval, these functions can be represented as points in unit hypercubes in R^4 and R^2, which we shall denote for convenience by C^4 and C^2. Thus, F may be defined by a mapping from $C^4 \times C^2$ to C^4 and H by a mapping from C^4 to C^2. For example, if the membership function of X^t is represented by the vector (0.5, 0.8, 1, 0.6) and that of U^t by the vector (1, 0.2), then the membership function of X^{t+1} would be defined by F as a vector, say, (0.2, 1, 0.8, 0.4), while that of U^t would be defined by H as a vector, say (1, 0.2).

It is clear that even in the very simple case where X and U are small finite sets, it is impracticable to attempt to characterize F with any degree of precision as a mapping from a product of unit hypercubes to a unit hypercube. Thus, in general, it would be necessary to resort to an approximate definition of F and H through the process of exemplification, as was done in the case of the relation between Y^t and U^t in Chapter 3 (Eq. (3.8) et seq.). This amounts to selecting a finite number of sample fuzzy sets in X and U, and tabulating finite approximations to F and H as mappings from and to the names of these fuzzy sets. In this light, an instruction such as "Reduce speed if the road is slippery" may be viewed as an ordered pair in H involving the names of fuzzy sets: "Reduce speed" and "Road is slippery."

Consider now the following situation. One is given an instruction of the form: "If x is much larger than 1 make y equal to 2. Otherwise make y equal to 1." Furthermore, the membership function of the class of numbers which are much larger than 1 is specified to be

$$\mu_E(x) = 0 \qquad\qquad \text{for } x < 1$$

$$= \left(1 + (x - 1)^{-2}\right)^{-1} \quad \text{for} \quad x \geq 1 \qquad\qquad 4.3$$

where E denotes the class in question and μ_E is its membership function.

Now suppose that x = 3. How should the above instruction be executed? Note that $\mu_E(3) = 0.8$.

The answer to this question is that the given instruction does not cover this contingency or, for that matter, any situation in which x is a number such that $\mu_E(x) > 0$. Specifically, the instruction in question tells us only that if the input is a fuzzy set characterized by the membership function (4.3), then y = 2; and if the input is characterized by the membership function $1 - \mu_E(x)$, then y = 1. Now when x is specified to be equal to 3, the input may be regarded as a fuzzy set whose membership function is equal to 1 for x = 3 and vanishes elsewhere. This fuzzy set is not in the domain of the instruction - viewing instruction as a function defined on a collection of fuzzy sets.

In some cases, it may be permissible to extend the domain of definition of a fuzzy instruction by an appropriate interpretation of its intent. For example, in the case considered above it may be reasonable to assume that y = 2 not just for the fuzzy set of numbers which are much larger than 1, but also for all fuzzy subsets of this set whose maximal grade of membership exceeds or is equal to a prescribed threshold; or, it may be reasonable to assume that y = 2 for all x whose grade of membership in E is greater than or equal to a threshold α. Alternatively, the domain of the instruction can be extended by employing randomized execution, that is, by choosing y = 2 and y = 1 for a given x with probabilities $\mu_E(x)$ and $1 - \mu_E(x)$, respectively. These and other ways of extending the domain of fuzzy instructions make the specification of F and H a problem which, though nontrivial, is well within the range of computational feasibility in many cases of practical interest.

Actually, crude forms of fuzzy algorithms are employed quite extensively in everyday practice. A cooking recipe is an example of an algorithm of this type. So is the set of instructions for parking a car or repairing a TV set. The effectiveness of such algorithms depends in

large measure on the existence of a fuzzy feedback which makes it possible to observe the output and apply a corrective input. Indeed, this is implicit in equation (4.3), except that in practice the H function is itself quite ill-defined.

The foregoing discussion of the notion of a fuzzy algorithm was intended primarily to point to a close connection between this notion and that of a fuzzy system. It may well turn out, however, that many of the complex problems - such as machine translation of languages - which so far have eluded all attempts to solve them by conventional techniques - cannot be properly formulated, much less solved, without the use, in one form or another, of a broader conceptual framework in which the notion of a fuzzy algorithm plays a basic role.

CHAPTER V

THE CONCEPT OF AGGREGATE

As was pointed out in Chapter 2, the state of a system may be viewed as a name for an aggregate of input-output pairs. In what follows, we shall summarize some of the principal notions relating to the concept of an aggregate, but will leave open the question of how these notions can be extended to fuzzy systems.

As in Chapter 2, let u and y denote a pair of sequences $u = u_0 u_1 \ldots u_t$ and $y = y_0 y_1 \ldots y_t$ of length t+1, where, for simplicity, t is assumed to range over non-negative integers. If $u = u_0 u_1 \ldots u_\tau$ and $v = v_{\tau+1} \ldots v_t$, then the concatenation of u and v is denoted by uv and is defined by $uv = u_0 u_1 \ldots u_\tau v_{\tau+1} \ldots v_t$.

Definition of a system. A system (discrete-time system) $\mathcal{A}$ is defined as a collection of ordered pairs of time-functions (u,y) satisfying the condition of closure under segmentation or CUS for short. Thus,

$$\mathcal{A} = \{(u,y)\} , \quad u \in U^* , \quad y \in Y^*$$

where u and y are, respectively, the input and output of $\mathcal{A}$, and (u,y) is an input-output pair belonging to $\mathcal{A}$. The expression for the CUS condition is:

If u = vv' and y = ww' (that is, u is a concatenation of time functions v and v' and y is a concatenation of w and w') and (u,y) ∈ $\mathcal{A}$, then (v,w) ∈ $\mathcal{A}$ and (v',w') ∈ $\mathcal{A}$. In effect, this condition requires that every segment of an input-output pair of $\mathcal{A}$ be an input-output pair of $\mathcal{A}$.

Comment. When we define a system as a collection of input-output pairs, we are in effect identifying a physical system or a mathematical model of

30

it with the totality of observations which can be made of its input and output time-functions. Furthermore, we tacitly assume that we have as many copies of the system as there are different initial states, and that each u is applied to all these copies, so that to each u correspond as many y's as there are copies of the system.

To characterize $\mathcal{A}$ as a collection of input-output pairs it is usually more expedient to employ an algorithm for generating input-output pairs belonging to $\mathcal{A}$ than to list them. From this point of view, a differential or difference equation relating the output of a system to its input may be viewed as a compact way of specifying the collection of input-output pairs which defines $\mathcal{A}$. An algorithm or an equation which serves this purpose is called an input-output relation.

Definition of an aggregate. Let $\mathcal{A}(t_0)$ denote a subset of $\mathcal{A}$ comprising those input-output pairs which start at time t_0. Now suppose we group together those input-output pairs in $\mathcal{A}(t_0)$ which have some property in common and call such groups bundles of input-output paris. As we shall see presently, the aggregates of $\mathcal{A}$ are bundles of input-output pairs with certain special properties which are defined in such a way as to make a state of $\mathcal{A}$ merely a name or a label for an aggregate of $\mathcal{A}$.

It is convenient to state the properties in question as a set of four conditions defining aggregates of $\mathcal{A}$. These conditions are as follows:

1. Covering condition. Let a generic bundle of input-output pairs in $\mathcal{A}(t_0)$ be denoted by $\mathcal{A}_{\alpha_0}(t_0)$, with α_0 serving as an identifying tag for a bundle. A collection of such bundles will be denoted by $\{\mathcal{A}_{\alpha_0}(t_0)\}$, $\alpha_0 \in \Sigma_{t_0}$, where Σ_{t_0} is the range of values which can be assumed by α_0 at t_0. Anticipating that α_0 will play the role of a state of $\mathcal{A}$, Σ_{t_0} will be referred to as the state space of $\mathcal{A}$ at time t_0. Note that t_0 is a variable ranging over the integers $0,1,2, \ldots$.

The covering condition requires that the collection $\{ \mathcal{A}_{\alpha_o}(t_o) \}$, $\alpha_o \in \Sigma_{t_o}$, be a covering for $\mathcal{A}(t_o)$, that is,

$$\bigcup_{\alpha_o} \mathcal{A}_{\alpha_o}(t_o) = \mathcal{A}(t_o) \quad , \qquad \text{for all } t_o \text{ in } \{0, 1, \ldots\} \quad . \qquad 4.4$$

In effect, this condition requires that every input-output pair in $\mathcal{A}(t_o)$ be included in some bundle in the collection $\{ \mathcal{A}_{\alpha_o}(t_o) \}$, $\alpha_o \in \Sigma_{t_o}$.

2. <u>Uniqueness condition.</u> The uniqueness condition is expressed by

$$(u,y) \in \mathcal{A}_{\alpha_o}(t_o) \qquad \text{and} \qquad (u,y') \in \mathcal{A}_{\alpha_o}(t_o) \; \rightarrow \; y = y'. \qquad 4.5$$

In other words, to each input u in the domain of the relation $\mathcal{A}_{\alpha_o}(t_o)$ corresponds a unique output y. (Note that the sequences u and y are assumed to be of the same length.)

3. <u>Prefix condition.</u> Consider an input-output pair (uu',yy') in $\mathcal{A}_{\alpha_o}(t_o)$ which is a concatenation of the input-output pairs (u,y) and (u',y'). The expression for the condition is

$$(uu',yy') \in \mathcal{A}_{\alpha_o}(t_o) \; \rightarrow \; (u,y) \in \mathcal{A}_{\alpha_o}(t_o) \quad . \qquad 4.6$$

Thus, this condition requires that any prefix (i.e., (u,y)) of an input-output pair in $\mathcal{A}_{\alpha_o}(t_o)$ also be an input-output pair in $\mathcal{A}_{\alpha_o}(t_o)$.

4. <u>Continuation condition.</u> As in the preceding condition, let (uu',yy') be an input-output pair in $\mathcal{A}_{\alpha_o}(t_o)$, with (u',y') starting at, say, t_1. The continuation condition may be expressed as

$$\{(u',y') \mid (uu',yy') \in \mathcal{A}_{\alpha_o}(t_o)\} = \mathcal{A}_{\alpha_1}(t_1) \qquad 4.7$$

32

where $\mathcal{A}_{\alpha_1}(t_1)$ denotes a bundle of input-output pairs starting at t_1, with the understanding that $\mathcal{A}_{\alpha_1}(t_1)$ is a member of the collection of bundles $\{\mathcal{A}_{\alpha_0}(t_0)\}$, $\alpha_0 \in \Sigma_{t_0}$, $t_0 = 0,1,2, \ldots$, and that α_1 ranges over Σ_{t_1}.

Informally, the continuation condition merely asserts that a state α_0 at time t_0 is transferred by input u into a state α_1 at time t_1.

In terms of the four conditions stated above the aggregates and states of a system can be defined as follows.

Definition. The aggregates of $\mathcal{A}$ are bundles of input-output pairs of $\mathcal{A}$ satisfying the covering, uniqueness, prefix and continuation conditions. The states of $\mathcal{A}$ are names (or tags) of the aggregates of $\mathcal{A}$. The set of names of the aggregates of input-output pairs starting at t_0 is the state-space of $\mathcal{A}$ at time t_0. Usually, the state space Σ_{t_0} is assumed to be independent of t_0.

With the above definitions as the point of departure, one can deduce all of the properties of the states and state equations of a system which, in the classical approach, are assumed at the outset. The way in which this can be done is described in [8] and, more explicitly though in lesser detail, in [9].

In Chapter 3, we have shown how the conventional approach in which the point of departure is the definition of a system through its state equations (3.1) and (3.2), can be generalized to fuzzy systems. This naturally gives rise to the question: How can the approach sketched above in which the starting point is (a) the definition of a system as a collection of input-output pairs; (b) the definition of an aggregate as a bundle of input-output pairs satisfying certain conditions; and (c) the definition of a state as a name for an aggregate, be similarly generalized to fuzzy systems?

If we could find an answer to this basic question, we might, perhaps, be able to develop effective techniques for the approximate analysis of complex systems for which state equations cannot be postulated at the outset. We state this question as an open problem because its solution can be perceived only dimly at this rudimentary stage of the development of the theory of fuzzy systems.

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