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Towards an acoustoelastic theory for measurement of residual stress

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TOWARDS AN ACOUSTOELASTIC THEORY FOR MEASUREMENT OF RESIDUAL STRESS

BY

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Towards an Acoustoelastic Theory for Measurement of Residual Stress[#]

Chi-Sing Man,* and W.Y. Lu**

Abstract

The rudiments of an acoustoelastic theory is developed within the framework of linear elasticity with initial stress. Since no assumption is made about the origin of the initial stress, our acoustoelastic theory will be applicable to evaluation of stress in plastically deformed bodies, provided that the superimposed ultrasonic waves be hyperelastic. New universal relations are deduced. An approach to evaluation of stress which does not use calibration specimens and makes full use of universal relations in our acoustoelastic theory is advocated. Examples are given which illustrate application of our theory to evaluate residual stress in plates. Preliminary corroboration of our theory are provided by the recent experiments of King & Fortunko and Thompson et al.

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§1. Introduction

It has been known for some time that the presence of stress in solids causes changes in the speeds of ultrasonic waves (the acoustoelastic effect), which raises the possibility of using ultrasonics as a nondestructive technique for measurement of stress. [1] Progress in ultrasonic measurement of stress, however, has been hampered by two outstanding difficulties:

(I) Current ultrasonic techniques for measurement of stress are founded on a theory which presumes that the body in question be hyperelastic and the stress in question be the result of elastic deformation from an unstressed "natural state". Residual stress in bodies, however, usually arises from processes that are not elastic. Consider, for example, the residual stress that results in a structure from welding. The residual stress in the heat-affected zone is due to the thermomechanical history, which certainly cannot be taken as an elastic deformation. Application of current ultrasonic techniques to evaluate residual stress in plastically deformed bodies has been known to be unreliable. [2]

(II) Structural materials such as aluminium and steel often acquire slight anisotropy from fabrication processes such as rolling, forging, and extrusion; the resulting anisotropy causes shifts in speeds of ultrasonic waves which are of the same order as those due to the presence of stress. It is commonly held that the effects of texture-induced anisotropy must be separated from the total velocity shifts before the existing acoustoelastic theory can be applied to infer the stress in question from the stress-induced changes in the speeds of ultrasonic waves. Since this problem was identified in the sixties [3], separating the effects of texture and stress has been commonly regarded as a major open problem in ultrasonic evaluation of stress.

Because of the preceding difficulties, ultrasonic measurement of stress has not yet borne out its early promise. Indeed a comparison of Crecraft's paper [3] of 1967, his review [4] of 1982, and the 1984 review of Pao et al. [5] will convince the reader that progress has been slow.

Technical advances in electromagnetic acoustic transducers have recently led to experiments which demonstrate that difficulty (II) can be resolved for some simple situations. In 1983 King & Fortunko [6] and Thompson et al. [7] independently reported success in overcoming difficulty (II) for the evaluation of in-plane principal shear stress in aluminium plates when the principal in-plane stress directions coincide with the rolling and the transverse directions of the plate. Although their methods were different, both groups exploited the capability of new electromagnetic acoustic transducers (EMATs) that can generate and detect horizontally polarized shear waves (SH-waves) from a wide range of oblique propagation directions. Both their methods were based on the usual acoustoelastic theory, so no headway was made on difficulty (I). The reported successes of King & Fortunko and Thompson et al. have immediately led to an explosion of print. (Here we refrain from listing all the follow-up papers we know; the interested reader can find a fair sampling of those in the British journal Ultrasonics.) All the follow-up papers, like the original works, have the usual acoustoelastic theory as their starting point; in other words, difficulty (I) survives unscathed. Reliance of an experimental method on the existing acoustoelastic theory could mean that the method in question has a severely limited range of applicability. When the method of Thompson et al. was first outlined [8], Pao & Gamer, for instance, were skeptical about its applicability to "a body with texture", because the proposed method of Thompson et al. was founded on a relation "derived on the basis of hyperelastic deformation at the initial state". [9]

Another recent development is the work of Hoger [10], in which she has proposed a statical approach to the nondestructive determination of residual stress. Behind her approach is a point of departure[1] which can be traced back to Cauchy [13], Rayleigh [14], Love [15], Biot [17], and others but has never been seriously taken up by practitioners of acoustoelasticity. In this paper we shall follow Hoger's lead; but, instead of pursuing further her statical approach, we shall develop the rudiments of an acoustoelastic theory for ultrasonic measurement of stress. Our theory will be applicable to both "applied stress" and "residual stress".[2] Plastically deformed bodies will cause no particular problem. In our theory we shall not attempt to separate the effects of "stress-induced" and "texture-induced" anisotropy per se. Instead, we shall seek relations, preferably universal relations, from which residual stress can be determined without the use of calibration specimens. Our theory will not only explain the successes of King & Fortunko [6] and Thompson et al. [7]; it will also reveal the weaknesses of their work, suggest improvements and new experiments.

To motivate our point of departure,[3] let us consider the following examples:

[1]Here we mean the classical theory of linear elasticity in its most general setting, which considers bodies with initial stress of arbitrary origin. As pointed out by Truesdell ([11], §55) and Truesdell & Noll ([12], §68, p. 246, Footnote 5; p. 250, paragraph in small print), Cauchy [13] was the first to derive the correct general equations, but "Cauchy's results were not understood and were reported obscurely or even incorrectly by nineteenth century expositors." ([12], p. 246) Among authors who obscured the work of Cauchy, Truesdell ([11], p. 209) mentioned Pearson and Love. After Cauchy "[t]here have been many subsequent treatments in various notations and subject to various restrictive assumptions." ([12], p. 246) For instance, when Rayleigh [14] proposed to consider the earth as a body with initial stress, Cauchy's results had long been forgotten. At that time the "usual elastic theory" "proceeds upon the assumption that the body is initially in a state of ease, free from stress and strain". After much deliberation Rayleigh came to the "conclusion" that "the usual equations ['for bodies in a state of ease'] may be applied to matter in a state of [initial] stress, provided we allow for altered values of the elasticities". Rayleigh's "conclusion" was vague and generally incorrect, but he apparently had a specific instance in mind, for which his suggestion would be sound (cf. Truesdell & Noll [12], §68, the paragraph that contains Eqs. (68.24) and (68.25)). According to Love ([15], p. 89), Rayleigh's "method" was really as follows: "The earth ought to be regarded as a body in a state of <u>initial stress</u>; this initial stress may be regarded as a hydrostatic pressure

(Footnote 1, continued)

balancing the self-gravitation of the body in the initial state; the stress in the body, when disturbed, may be taken to consist of the initial stress compounded with an additional stress; the additional stress may be taken to be connected with the strain, measured from the initial state as unstrained state, by the same formulae as hold in an isotropic elastic solid body slightly strained from a state of zero stress." Since "[t]he theory, as here described, is [still] ambiguous", Love proceeded to remove the ambiguity and apply Rayleigh's "method" in modelling "a gravitating compressible planet" ([15], Ch. VIII, Ch. X, and §§165-170 of Ch. XI). Among Love's results are those that concern "transmission of waves through a gravitating compressible body" ([15], §§165-170). Love ([16], §75) was clear about the fact that initial stress need not arise from elastic deformation. Biot (see [17] and references therein) rederived the general equations in the thirties and published a paper [18] on "the influence of initial stress on elastic waves" in 1940. While there are valuable historical comments by Truesdell ([11], §55 and annotations of §55 in pp. 208-209) and Truesdell & Noll ([12], §68; see in particular p. 246, Footnote 5, and p. 250, paragraph in small print) and there is a "correlation study of formulations of incremental deformation and stability of continuous bodies" by Bažant [19], a comprehensive historical analysis of the subject awaits to be written.

[2] In the literature the terms "applied stress" and "residual stress" may carry meanings different than what is intended here; we should make precise what we mean by them. Consider a body $\mathcal{B}$ in equilibrium at a configuration $\underset{\sim}{\kappa}$ with (Cauchy) prestress $\overset{\circ}{\underset{\sim}{T}}$. We say that the prestress $\overset{\circ}{\underset{\sim}{T}}$ is "residual" if the body $\mathcal{B}$ is subject to no external force at the configuration $\underset{\sim}{\kappa}$. In other words, $\overset{\circ}{\underset{\sim}{T}}$ is residual if it is divergence-free in $\underset{\sim}{\kappa}(\mathcal{B})$ and satisfies the zero-traction condition at the boundary $\partial\underset{\sim}{\kappa}(\mathcal{B})$. If the body force is not negligible or the traction at the boundary is not null, we refer to the prestress $\overset{\circ}{\underset{\sim}{T}}$ as "applied". In this paper we shall consider infinitesimal elastic motions superimposed on the given configuration $\underset{\sim}{\kappa}$; $\overset{\circ}{\underset{\sim}{T}}$ will be the only prestress that appears. Our acoustoelastic theory studies the effect of the prestress $\overset{\circ}{\underset{\sim}{T}}$ on various wave speeds; in this regard the conditions that determine whether $\overset{\circ}{\underset{\sim}{T}}$ is "residual" or "applied", namely the equation of equilibrium and the boundary condition for $\overset{\circ}{\underset{\sim}{T}}$, have no bearing whatsoever. Our theory does not distinguish what we call "residual stress" and "applied stress". Of course the situation can be completely different for other theories in which these terms carry meaning different from ours. For instance Bonilla & Keller [20], in a recent theoretical analysis of the acoustoelastic effect, in effect divided what we call here the prestress $\overset{\circ}{\underset{\sim}{T}}$ into a sum of "residual" and "applied" parts; they showed that in their theory the acoustoelastic effect of "residual stress" is different from that of "applied stress".

[3] As mentioned in Footnote 1 above, our point of departure is nothing novel. The reader, however, might still find it unfamiliar. Although linear elasticity with initial stress is the classical theory in its most general setting, generations of students have been trained to know only the special instance of zero prestress and regard this special instance as synonymous with the classical theory of linear elasticity. The general theory with initial stress, being forgotten, took revenge by coming back from time to time as a new research topic. For instance, seventy-seven years after Cauchy [13] obtained the correct general equations, Rayleigh [14] still took great pains trying to modify "the usual elastic theory for bodies in a state of ease" to make it applicable to "matter in state of stress". Similarly, while the reviews of Crecraft [4] and Pao et al. [5] together give a rather comprehensive picture of research activities in acoustoelasticity upto 1983, Biot is not mentioned in both reviews, albeit his pioneering work [18] on "the influence of initial stress on elastic waves" in 1940.

(1) A metal specimen $\mathcal{B}$ is loaded quasi-statically under a uniaxial tensile stress until the stress is well in the plastic range and the specimen comes to a configuration $\underset{\sim}{\kappa}$. While the specimen $\mathcal{B}$ is kept at the configuration $\underset{\sim}{\kappa}$ (i.e., without unloading), an ultrasonic wave is sent through $\underset{\sim}{\kappa}$. Will the small motion superimposed on $\underset{\sim}{\kappa}$ be elastic or plastic? The experiments of Bell and others in the early fifties (see [21], pp. 611-618 and references therein) and the more recent experiment of Lu [22] all indicated that the superimposed small motion would be elastic. In this example the specimen $\mathcal{B}$ is given at a state of plastic deformation and the stress in question is "applied stress".

(2) A body $\mathcal{B}$ has inherited residual stress from its forming process. It has subsequently experienced a complex loading history. At its present configuration $\underset{\sim}{\kappa}$, the traction at the boundary $\partial\underset{\sim}{\kappa}(\mathcal{B})$ is null. We have scant knowledge about the history of the body $\mathcal{B}$, and we are concerned only with the residual stress in $\mathcal{B}$ at the present configuration $\underset{\sim}{\kappa}$. The only thing we are sure about the body $\mathcal{B}$ is that a small-amplitude ultrasonic wave sent through $\mathcal{B}$ at its present configuration is elastic. Can we determine the residual stress in question by ultrasonic techniques?

The two examples above suggest that we should study the mechanics of infinitesimal elastic motions superimposed on some given prestressed reference-configuration of a body. The body in question is given as it is, at the reference configuration; the prestress may be "applied" or "residual". We are not really interested in nor have much knowledge about the history of the body. We want to study the effects of small incremental dynamic loadings (in particular, those pertaining to excitation of small amplitude ultrasonic waves) on the given body at the given prestressed configuration, whose response to such loadings we shall presume to be hyperelastic. From the hyperelastic responses to such loadings we want to infer the prestress in the original given configuration. We make no further constitutive assumptions on the body in question, except, when appropriate, those pertaining to material symmetry. Whether the responses of the given body to other more general loadings are elastic, viscoelastic or elasto-plastic, whether the body has a "natural state" or a placement at ease (i.e., a configuration with zero stress) are irrelevant to our present discussion. We make no assumption on the origin of the prestress. In fact we adopt an attitude identical to that in the classical theory of linear elasticity with initial stress. Our problem at hand is to find out how we can determine the initial stress in the given body by ultrasonic techniques.

§2. Two Forms of the General Constitutive Relation

Consider a body $\mathcal{B}$ in a given configuration κ. For a material point X in $\kappa(\mathcal{B})$, the following constitutive relation is valid for small elastic deformations and motions superimposed on the configuration κ: (see Hoger [10], §2.2)

$$S(H) = \mathring{T} + \mathbb{C}[H] = \mathring{T} + \mathbb{L}[E] + W\mathring{T} + \tfrac{1}{2}(E\mathring{T} - \mathring{T}E); \tag{1}$$

here the local configuration of X in $\kappa(\mathcal{B})$ is the reference configuration for the infinitesimal deformation at X; S is the first Piola-Kirchhoff stress tensor, E is the (incremental) infinitesimal strain, W is the (incremental) infinitesimal rotation, and $H = E + W$ is the (incremental) displacement gradient; $\mathring{T}$ is the residual or applied (Cauchy) prestress; $\mathbb{L}[\cdot]$ is a linear tensor-valued mapping defined on the set of displacement gradients and it has the minor symmetries, i.e., it depends only on the symmetric part E of the displacement gradient H and takes values in the space of symmetric tensors; $\mathbb{C}[H] = \mathbb{L}[E] + W\mathring{T} + \tfrac{1}{2}(E\mathring{T} - \mathring{T}E)$ is the elasticity tensor. Following Hoger [10], we call $\mathbb{L}$ the incremental elasticity tensor.

In general the incremental elasticity tensor $\mathbb{L}$ of a material point X will depend on the histories of loading, heating, etc., experienced by the entire body $\mathcal{B}$. Even in situations where it is meaningful to say that $\mathbb{L}(X)$ depends only on $\mathring{T}(X)$, this dependence will generally be nonlinear.

In Hoger's work [10], Eq. (1) is derived without recourse to any presumption on the origin of the residual or applied prestress $\mathring{\mathbf{T}}$ and without assuming the existence of a stored energy function for the incremental elastic deformations. For a given material point $\mathbf{X}$, the incremental elasticity tensor $\mathbf{L}$ will be symmetric (i.e., $\mathbf{E}_2 \cdot \mathbf{L}[\mathbf{E}_1] = \mathbf{L}[\mathbf{E}_2] \cdot \mathbf{E}_1$ for any two symmetric tensors $\mathbf{E}_1$ and $\mathbf{E}_2$) if there is a stored energy function for the incremental elastic deformations.

The purpose of this paper is to develop an acoustoelastic theory for ultrasonic measurement of stress. To this end we find it particularly convenient if we recast Eq. (1) in the form

$$\mathbf{S}(\mathbf{H}) = \mathring{\mathbf{T}} + \mathcal{L}[\mathbf{E}] + \mathbf{H}\mathring{\mathbf{T}}, \tag{2}$$

where

$$\mathcal{L}[\mathbf{E}] \equiv \mathbf{L}[\mathbf{E}] - \tfrac{1}{2}(\mathbf{E}\mathring{\mathbf{T}} + \mathring{\mathbf{T}}\mathbf{E}) = \mathbf{C}[\mathbf{H}] - \mathbf{H}\mathring{\mathbf{T}}. \tag{3}$$

Under a chosen Cartesian coordinate system we can express both $\mathbf{L}$ and $\mathcal{L}$ as 6×6 matrices, namely, (L_{ij}) and $(\mathcal{L}_{ij})$; here i and j, which run from 1 to 6, are the usual "abbreviated subscripts" (see Auld [23], §§1.F, 2.D, and 3.C). The difference $\mathcal{L} - \mathbf{L}$ is then represented by the matrix

$$
-\begin{bmatrix}
\hat{T}_{11} & 0 & 0 & 0 & \frac{1}{2}\hat{T}_{13} & \frac{1}{2}\hat{T}_{12} \\
0 & \hat{T}_{22} & 0 & \frac{1}{2}\hat{T}_{23} & 0 & \frac{1}{2}\hat{T}_{12} \\
0 & 0 & \hat{T}_{33} & \frac{1}{2}\hat{T}_{23} & \frac{1}{2}\hat{T}_{13} & 0 \\
0 & \frac{1}{2}\hat{T}_{23} & \frac{1}{2}\hat{T}_{23} & \frac{1}{4}(\hat{T}_{22}+\hat{T}_{33}) & \frac{1}{4}\hat{T}_{12} & \frac{1}{4}\hat{T}_{13} \\
\frac{1}{2}\hat{T}_{13} & 0 & \frac{1}{2}\hat{T}_{13} & \frac{1}{4}\hat{T}_{12} & \frac{1}{4}(\hat{T}_{11}+\hat{T}_{33}) & \frac{1}{4}\hat{T}_{23} \\
\frac{1}{2}\hat{T}_{12} & \frac{1}{2}\hat{T}_{12} & 0 & \frac{1}{4}\hat{T}_{13} & \frac{1}{4}\hat{T}_{23} & \frac{1}{4}(\hat{T}_{11}+\hat{T}_{22})
\end{bmatrix}. \qquad (4)
$$

Eq. (2) and Eq. (1) are equivalent forms of the same constitutive relation. It is easy to see that $\underline{\underline{\mathcal{L}}}$ is symmetric if and only if $\underline{\underline{L}}$ is symmetric.

In this paper we treat $\underset{\sim}{\hat{T}}$ as a constitutive quantity. Since the difference of $\underline{\underline{\mathcal{L}}}$ and $\underline{\underline{L}}$ depends only on $\underset{\sim}{\hat{T}}$, $\underline{\underline{\mathcal{L}}}$ is homogeneous if and only if $\underline{\underline{L}}$ is homogeneous, provided that the prestress $\underset{\sim}{\hat{T}}$ is homogeneous. Henceforth we shall say that a body $\mathcal{B}$ with prestress at the configuration $\underset{\sim}{\kappa}(\mathcal{B})$ is <u>homogeneous</u> if and only if both $\underline{\underline{L}}$ and $\underset{\sim}{\hat{T}}$ are homogeneous over $\underset{\sim}{\kappa}(\mathcal{B})$.

<u>Remark 2.1</u>. The reader should <u>not</u> interpret Eqs. (1), (2) and (3) as saying that $\underline{\underline{L}}$ is more basic than $\underline{\underline{\mathcal{L}}}$ and the effect of the prestress on the elasticity tensor $\underline{\underline{C}}$ in Eq. (1) is embodied solely in the term $\underset{\sim}{W}\underset{\sim}{\hat{T}} + \frac{1}{2}(\underset{\sim}{E}\underset{\sim}{\hat{T}} - \underset{\sim}{\hat{T}}\underset{\sim}{E})$. Consider the special instance in which the body $\mathcal{B}$ in question is hyperelastic and has a placement at ease $\underset{\sim}{\kappa}_o$. For a material

point $\mathbf{X}$, let $\mathbf{C}$ be the right Cauchy-Green tensor pertaining to the deformation from κ_0 to the given configuration κ. Both the incremental elasticity tensor $\mathbb{L}$ and the prestress $\mathring{\mathbf{T}}$ at $\mathbf{X}$ in $\kappa(\mathcal{B})$ will depend on $\mathbf{C}$. If the correspondence $\mathbf{C} \to \mathring{\mathbf{T}}(\mathbf{C})$ is locally invertible, then the dependence of $\mathbb{L}$ on $\mathbf{C}$ can be replaced by the dependence on $\mathring{\mathbf{T}}$. We should interpret Eqs. (1) and (2) as two equivalent forms of the same constitutive equation; indeed Eq. $(3)_1$, which relates $\mathbb{L}$ and $\mathcal{L}$, puts them on equal footing.

Remark 2.2. Hoger's derivation ([10], §2.2) of Eq. (1) is clear and elegant, but as she noted the constitutive equation is not new (cf. Footnote 1 above). For instance, it appeared in the work of Biot (see [17], Ch. 2, Eqs. (2.23) and (5.20); [24], Eq. (4.11)). While Biot's treatment of the subject may not be above reproach, he was fully aware of the generality of his equations, which includes Eq. (1): "They are applicable to non-elastic media undergoing an incremental deformation in the vicinity of a prestressed condition ... [W]e consider deformations which are elastic for the incremental deformations alone, irrespective of the manner by which the state of initial stress has been generated." (Biot [17], p. 56) One of his favorite example is possible application of his theory to "rapid deformations in the earth where the initial stress is associated with a slow process of creep due to viscous and plastic deformations." ([17], p. 6) While the quotations above are taken from his book, it is clear that Biot understood all these when he wrote his paper [18] of 1940 on "the influence of initial stress on elastic waves". The classical theory of linear elasticity with initial stress has a long and tortuous history (cf. Footnotes 1 and 3 above). A full historical appraisal of Biot's contributions to the subject is wanting.

Henceforth we shall use the general constitutive relation only in the form Eq. (2). Our basic assumption is that Eq. (2) be valid for infinitesimal progressive waves of ultrasonic frequencies superimposed on $\mathcal{B}$ at the given configuration $\underset{\sim}{\kappa}$. Since no confusion should arise, we shall also call $\underline{\underline{L}}$ by the name "incremental elasticity tensor". We shall assume that the incremental elasticity tensor $\underline{\underline{L}}(\underset{\sim}{X})$ be symmetric for each material point $\underset{\sim}{X}$. A sufficient condition for this assumption to be valid is that the superimposed motions be hyperelastic. We emphasize that the assumptions above pertain only to the response of the body to loadings which excite ultrasonic waves. The body in question need not behave elastically for other loadings.

§ 3. Material Symmetry. "Stress-Induced" and "Texture-Induced" Anisotropy

Material frame-indifference, Noll's definition of material symmetry, and the constitutive relation Eq. (2) dictate that an orthogonal tensor $\underset{\sim}{Q}$ belongs to the symmetry group $\mathcal{G}_{\underset{\sim}{\kappa}}$ of $\underset{\sim}{X}$ at the local configuration induced by the placement $\underset{\sim}{\kappa}$ if and only if

$$\underset{\sim}{S}(\underset{\sim}{Q}\underset{\sim}{H}\underset{\sim}{Q}^T) = \underset{\sim}{Q}\underset{\sim}{S}(\underset{\sim}{H})\underset{\sim}{Q}^T \tag{5}$$

for any displacement gradient $\underset{\sim}{H}$ (cf. Coleman & Noll [25]). It follows immediately that $\underset{\sim}{Q}$ belongs to $\mathcal{G}_{\underset{\sim}{\kappa}}$ if and only if

$$\underset{\sim}{Q}\overset{\circ}{\underset{\sim}{T}}\underset{\sim}{Q}^T = \overset{\circ}{\underset{\sim}{T}}, \tag{6}$$

and for every symmetric tensor $\underset{\sim}{E}$

$$\underset{\approx}{L}[\underset{\sim}{Q}\underset{\sim}{E}\underset{\sim}{Q}^T] = \underset{\sim}{Q}\underset{\approx}{L}[\underset{\sim}{E}]\underset{\sim}{Q}^T. \tag{7}$$

It is easy to see that we shall still obtain Eqs. (6) and (7) if we use Eq. (1) instead of Eq. (2). Cf. Hoger [10], Eqs. (2.1.6) and (2.2.18).

Remark 3.1. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the groups of orthogonal tensors which satisfy Eq. (6) and Eq. (7), respectively. How will $\mathcal{G}_1$ and $\mathcal{G}_2$ be related to $\mathcal{G}_{\underset{\sim}{\kappa}}$? After a moment's reflection on the special instance of hyperelastic bodies (discussed in Remark 2.1 above), the reader will convince himself that the only natural assumption to make is $\mathcal{G}_1 \supset \mathcal{G}_2 = \mathcal{G}_{\underset{\sim}{\kappa}}$.

In the special instance when the body in question is hyperelastic and has a "natural state" (see Remark 2.1 above), it is customary to refer to the anisotropy of a material point X at the natural state κ_o as "texture-induced". The anisotropy of X at the given configuration would be entirely "stress-induced" should the material point be isotropic at its natural state; otherwise it would be both "stress-induced" and "texture-induced". Since the sixties (cf. Crecraft [3]) the separation of "texture-induced" and "stress-induced" anisotropy has been the Gordian knot in ultrasonic measurement of residual stress. Years passed and progress was slow,[4] but people kept asking the same questions (cf. Crecraft [4], §11.4; Pao et al. [5], §§5.5, 6.1, 8.3 and 9). Indeed recent attempts have been made to extend the distinction of "texture-induced" and "stress-induced" anisotropy to more general situations, e.g., for plastically deformed bodies (see Pao et al. [5] and references therein).

With Eq. (2) in our hands, it is our conviction that the time is ripe to explore another approach. Instead of trying to identify and separate out the "stress-induced" anisotropy (we doubt whether the expression itself makes sense in general), we believe that to start with we should seek relations in which the effects of the incremental elasticity tensor $\mathbb{L}$ are eliminated altogether: more precisely, we shall first of all try to obtain relations which concern only $\mathring{T}$ and various wave speeds.

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[4]As we shall explain in detail in §§5-6 below, although King & Fortunko [6] and Thompson et al. [7] talked about separation of "texture-induced" and "stress-induced" velocity-shifts, it would not be unfitting to interpret these belated successes as the first conquests of the approach proposed in the present paper.

In §5 we shall show that such relations can indeed be obtained for instances that have practical engineering applications. Before we proceed, however, we hasten to point out the following advantages of our present approach:

(i) The relations to be obtained in §5 are results of material symmetry; they will be valid for any given homogeneous configuration of any continuous body which satisfies the specific conditions of material symmetry, provided that the superimposed ultrasonic waves in question be hyperelastic. In particular, those relations will be applicable to plastically deformed bodies.

(ii) All previous work on acoustoelastic determination of stress requires calibration specimens with known stress states.[5] For measurement of residual stress, whether we can generally have calibration specimens with known stress states is open to question. This difficulty will disappear completely in our approach. The relations to be obtained in §5 are universal in the sense that all the entries of the matrix (L_{ij}) do not appear; thence calibration specimens with known stress states will be superfluous in the present approach.

Remark 3.2. The idea to distinguish "stress-induced" and "texture-induced" anisotropy, as a rough but suggestive idea, is not completely worthless. Consider a material point $\underset{\sim}{X}$ given in a local configuration where micro-texture suggests that it would be orthotropic. Let us choose a Cartesian coordinate system such that the coordinate planes coincide with the planes of symmetry of $\underset{\sim}{X}$. Suppose $\underset{\sim}{\hat{T}}$ is symmetrical with respect to reflection about the 1-3 plane, i.e., $\underset{\sim}{Q}\hat{\underset{\sim}{T}}\underset{\sim}{Q}^T = \hat{\underset{\sim}{T}}$ for $\underset{\sim}{Q}$ with $Q_{11} = Q_{33} = 1$, $Q_{22} = -1$ and all other $Q_{ij} = 0$. Then $(L_{ij})(\underset{\sim}{X})$ will involve 13 elastic constants. When "stress-induced" anisotropy is weak, it is reasonable to assume that the elastic constants L_{15}, L_{25}, L_{35}, and L_{46} be small. We shall come back to this example in §7 below.

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[5]The only exception that we know of is the work of Thompson et al. ([7], [26], [27]). As we shall discuss in Remark 5.1 below, Thompson and coworkers were in effect following the approach that we advocate here.

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§4. Acoustoelasticity: General Considerations

Consider a body $\mathcal{B}$ in equilibrium at a given configuration κ with prestress $\mathring{T}$. The prestress $\mathring{T}$ satisfies the equation of equilibrium

$$\text{Div}\,\mathring{T} + \rho b = 0 \tag{8}$$

in $\kappa(\mathcal{B})$; here b is the body force per unit mass and ρ is the density at the configuration κ. When $b = 0$ and $\mathring{T}$ satisfies also the zero-traction boundary condition

$$\mathring{T}n\big|_{\partial\kappa(\mathcal{B})} = 0, \tag{9}$$

we call $\mathring{T}$ the residual stress; here n is the unit outward normal field on $\partial\kappa(\mathcal{B})$. In this section we shall assume that the body $\mathcal{B}$ in question is homogeneous over $\kappa(\mathcal{B})$.[6]

For small elastic motions superimposed on the given configuration

[6] Since a nonzero residual stress field must be inhomogeneous (see Hoger [10], §1), the assumption here would seem to preclude any possible application of the results below to determination of residual stress. As we shall explain in §6, there are situations involving nonzero residual stress for which the results in this and in the next section can be taken as approximately valid.

$\underset{\sim}{K}$ of body $\mathcal{B}$, by using Eqs. (2), $(3)_2$ and (8) we can write the equation of motion as follows:

$$\operatorname{Div} \underset{\approx}{C}[\underset{\approx}{H}] = \rho \partial^2 \underset{\sim}{u} / \partial t^2 ; \tag{10}$$

here $\underset{\sim}{u}$ is the displacement, t is the time, and $\underset{\approx}{C}$ is the elasticity tensor given by Eq. $(3)_2$.

Consider a plane sinusoidal progressive wave of the form

$$\underset{\sim}{u} = \underset{\sim}{a} \cos (\omega t - \underset{\sim}{k} \cdot \underset{\sim}{r}), \tag{11}$$

where ω is the angular frequency, $\underset{\sim}{k}$ is the propagation vector, $\underset{\sim}{a}$ is the amplitude and $\underset{\sim}{r}$ is the position vector; $\underset{\sim}{k}$ and $\underset{\sim}{a}$ are constant vectors, and ω is a constant scalar. Eq. (11) will be a solution of Eq. (10) if and only if $\underset{\sim}{a}$ satisfies the equation

$$k^2 \underset{\approx}{\Gamma} \underset{\sim}{a} = \rho \omega^2 \underset{\sim}{a} ; \tag{12}$$

here $k \equiv \|\underset{\sim}{k}\|$ and $\underset{\approx}{\Gamma}$, the generalized Christoffel tensor, is given in Cartesian coordinates by the components

$$\begin{aligned} &\Gamma_{11} = \alpha + \underset{\sim}{\ell} \cdot \overset{\circ}{\underset{\approx}{T}} \underset{\sim}{\ell}, \quad \Gamma_{22} = \beta + \underset{\sim}{\ell} \cdot \overset{\circ}{\underset{\approx}{T}} \underset{\sim}{\ell}, \quad \Gamma_{33} = \gamma + \underset{\sim}{\ell} \cdot \overset{\circ}{\underset{\approx}{T}} \underset{\sim}{\ell}, \\ &\Gamma_{12} = \Gamma_{21} = \delta, \quad \Gamma_{13} = \Gamma_{31} = \varepsilon, \quad \Gamma_{23} = \Gamma_{32} = \zeta, \end{aligned} \tag{13}$$

where $\underset{\sim}{\ell} \equiv \underset{\sim}{k}/k = (k_1, k_2, k_3)/k = (\ell_1, \ell_2, \ell_3)$ is the direction of propagation, and

$$
\begin{aligned}
\alpha &= L_{11}\ell_1^2 + L_{66}\ell_2^2 + L_{55}\ell_3^2 + 2L_{56}\ell_2\ell_3 + 2L_{15}\ell_3\ell_1 + 2L_{16}\ell_1\ell_2, \\
\beta &= L_{66}\ell_1^2 + L_{22}\ell_2^2 + L_{44}\ell_3^2 + 2L_{24}\ell_2\ell_3 + 2L_{46}\ell_3\ell_1 + 2L_{26}\ell_1\ell_2, \\
\gamma &= L_{55}\ell_1^2 + L_{44}\ell_2^2 + L_{33}\ell_3^2 + 2L_{34}\ell_2\ell_3 + 2L_{35}\ell_3\ell_1 + 2L_{45}\ell_1\ell_2, \\
\delta &= L_{16}\ell_1^2 + L_{26}\ell_2^2 + L_{45}\ell_3^2 + (L_{46}+L_{25})\ell_2\ell_3 + (L_{14}+L_{56})\ell_3\ell_1 + (L_{12}+L_{66})\ell_1\ell_2, \\
\varepsilon &= L_{15}\ell_1^2 + L_{46}\ell_2^2 + L_{35}\ell_3^2 + (L_{45}+L_{36})\ell_2\ell_3 + (L_{13}+L_{55})\ell_3\ell_1 + (L_{14}+L_{56})\ell_1\ell_2, \\
\zeta &= L_{56}\ell_1^2 + L_{24}\ell_2^2 + L_{34}\ell_3^2 + (L_{44}+L_{23})\ell_2\ell_3 + (L_{36}+L_{45})\ell_3\ell_1 + (L_{25}+L_{46})\ell_1\ell_2.
\end{aligned}
\tag{14}
$$

For a given direction of propagation $\underset{\sim}{\ell}$, $\underset{\sim}{\Gamma}$ is symmetric; thence the eigenvalue problem Eq. (12) has three real eigenvalues. If all the eigenvalues are positive, there will be three orthogonal directions of motion and three associated speeds of propagation for plane sinusoidal progressive waves. This result is completely analogous to that in the familiar special theory in which the body in question is given at a "natural state". Indeed, it is clear from the structure of the generalized Christoffel tensor $\underset{\sim}{\Gamma}$ that most theorems in the classical special context will have a counterpart in the present theory.

Remark 4.1. Like the constitutive relation Eq. (2), formally Eqs. (12), (13) and (14) are not new. For the special instance where the body in question is hyperelastic, they appeared already in Thurston's paper of 1965 ([28], §IX). As we have discussed in the previous sections, these same equations are in fact valid for infinitesimal hyperelastic waves of the form Eq. (11) superimposed on any given homogeneous configuration of any continuous body; here the adjective "any" refers to full generality in classical continuum mechanics.

Remark 4.2. A glance at Eqs. (13) and (14) reveals that Eq. (12) will not distinguish a medium with hydrostatic prestress $\mathring{\underset{\sim}{T}} = -p\underset{\sim}{I}$ and incremental elasticity tensor $\underset{\approx}{L}$ from one whose prestress is null and whose incremental elasticity tensor is equal to $\underset{\approx}{L}$ except that L_{jj} is replaced by $L_{jj} - p$ $(j = 1, \ldots, 6)$ and L_{12}, L_{13}, L_{23} are replaced by $L_{12} + p$, $L_{13} + p$, $L_{23} + p$, respectively. As mentioned in §3 above, we advocate an approach to ultrasonic measurement of stress which makes full use of universal relations that do not involve any coefficient of $\underset{\approx}{L}$. When restricted to using universal relations alone, this approach will not deliver mean normal prestress as a directly measured quantity; generally speaking, some auxilliary means must also be called upon to evaluate the complete prestress tensor. Cf. Example 6.1 below.

When Eq. (11) is a solution of Eq. (10), the quantity ω/k is the phase velocity of the plane wave in question. If a pulse of acoustic energy is radiated by a plane wave transducer, the wave packet is limited in two dimensions by the size of the transducer and in the third dimension by the pulse length. The wave fronts travel in the direction $\underset{\sim}{\ell}$, which is normal to the transducer surface; the modulation envelop of the wave packet, however, travels in the direction of the group velocity

$$\underset{\sim}{v}_g = -(\partial\Omega/\partial k_1,\ \partial\Omega/\partial k_2,\ \partial\Omega/\partial k_3)/(\partial\Omega/\partial\omega), \tag{15}$$

where $\Omega = \Omega(\omega, k_1, k_2, k_3) \equiv \det(k^2 \underset{\sim}{\Gamma} - \rho\omega^2 \underset{\sim}{I})$. Cf. Auld [23], §7.H. For later use let us calculate $\underset{\sim}{v}_g$ for a special instance: when $k_1 = 0$, $k_2 = k\cos\theta$, $k_3 = k\sin\theta$, and $k^2\Gamma_{11} - \rho\omega^2 = 0$,

$$\underset{\sim}{v}_g = (1/2\rho\omega)(0,\ \partial(k^2\Gamma_{11})/\partial k_2,\ \partial(k^2\Gamma_{11})/\partial k_3). \qquad (16)$$

§5. A Family of Universal Relations for Orthotropic Media

In this section we consider the special instance that the body $\mathcal{B}$ in question is homogeneous and orthotropic at the given configuration $\underset{\sim}{\kappa}$. We choose a Cartesian coordinate system whose coordinate planes are parallel to the planes of symmetry. By Eq. (6), the prestress must be of the form

$$\underset{\sim}{\mathring{T}} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} . \tag{17}$$

The components of the generalized Christoffel tensor $\underset{\sim}{\Gamma}$ are specialized as follows:

$$\begin{aligned}
\Gamma_{11} &= (L_{11} + \sigma_1)\ell_1^2 + (L_{66} + \sigma_2)\ell_2^2 + (L_{55} + \sigma_3)\ell_3^2, \\
\Gamma_{22} &= (L_{66} + \sigma_1)\ell_1^2 + (L_{22} + \sigma_2)\ell_2^2 + (L_{44} + \sigma_3)\ell_3^2, \\
\Gamma_{33} &= (L_{55} + \sigma_1)\ell_1^2 + (L_{44} + \sigma_2)\ell_2^2 + (L_{33} + \sigma_3)\ell_3^2, \\
\Gamma_{12} &= \Gamma_{21} = (L_{12} + L_{66})\ell_1\ell_2, \\
\Gamma_{13} &= \Gamma_{31} = (L_{13} + L_{55})\ell_3\ell_1, \\
\Gamma_{23} &= \Gamma_{32} = (L_{44} + L_{23})\ell_2\ell_3.
\end{aligned} \tag{18}$$

Consider a plane sinusoidal progressive wave with propagation direction $\underset{\sim}{\ell} = (0, \cos\theta, \sin\theta)$, for which Eq. (18) is further simplified as follows:

$$
\begin{aligned}
\Gamma_{11} &= (L_{66} + \sigma_2)\cos^2\theta + (L_{55} + \sigma_3)\sin^2\theta, \\
\Gamma_{22} &= (L_{22} + \sigma_2)\cos^2\theta + (L_{44} + \sigma_3)\sin^2\theta, \\
\Gamma_{33} &= (L_{44} + \sigma_2)\cos^2\theta + (L_{33} + \sigma_3)\sin^2\theta, \\
\Gamma_{12} &= \Gamma_{21} = \Gamma_{13} = \Gamma_{31} = 0, \\
\Gamma_{23} &= \Gamma_{32} = (L_{44} + L_{23})\cos\theta\sin\theta.
\end{aligned}
\tag{19}
$$

It is easy to see that by Eq. (19) the direction of motion (1, 0, 0) is an eigenvector of the eigenvalue problem Eq. (12). Let $\rho v_a^2(\theta)$ be the corresponding eigenvalue; thence

$$
\rho v_a^2(\theta) = (L_{66} + \sigma_2)\cos^2\theta + (L_{55} + \sigma_3)\sin^2\theta. \tag{20}
$$

Physically $v_a(\theta)$ is the phase velocity of shear wave with propagation direction $(0, \cos\theta, \sin\theta)$ and direction of motion (1, 0, 0). Similarly, for $\underset{\sim}{\ell} = (\sin\theta, \cos\theta, 0)$, the direction of motion (0, 0, 1) is an eigenvector of the eigenvalue problem Eq. (12). Let $\rho v_b^2(\theta)$ be the corresponding eigenvalue. Then

$$
\rho v_b^2(\theta) = (L_{44} + \sigma_2)\cos^2\theta + (L_{55} + \sigma_1)\sin^2\theta, \tag{21}
$$

and $v_b(\theta)$ is the phase velocity of shear wave with propagation direction $(\sin\theta, \cos\theta, 0)$ and direction of motion (0, 0, 1). Subtracting Eq. (21) from Eq. (20), we obtain the equation

$$(L_{66} - L_{44})\cos^2\theta + (\sigma_3 - \sigma_1)\sin^2\theta = \rho(v_a^2(\theta) - v_b^2(\theta)). \tag{22}$$

Suppose we can determine ρ, $v_a(\theta)$ and $v_b(\theta)$ experimentally. Suppose θ is given two suitable values θ_1 and θ_2; we obtain from Eq. (22) two linear equations in the unknowns $L_{66} - L_{44}$ and $\sigma_3 - \sigma_1$. Solving these equations, we obtain a family of universal relations:

$$\sigma_3 - \sigma_1 = \rho\,\frac{(v_a^2(\theta_2) - v_b^2(\theta_2))\cos^2\theta_1 - (v_a^2(\theta_1) - v_b^2(\theta_1))\cos^2\theta_2}{\cos^2\theta_1\sin^2\theta_2 - \cos^2\theta_2\sin^2\theta_1}. \tag{23}$$

In particular, for $\theta_1 = 0$ and $\theta_2 = \frac{1}{4}\pi$, Eq. (23) becomes simply

$$\sigma_3 - \sigma_1 = \rho\,(2(v_a^2(\tfrac{1}{4}\pi) - v_b^2(\tfrac{1}{4}\pi)) - (v_a^2(0) - v_b^2(0))). \tag{24}$$

In practice it is important to ascertain also the group velocities. Specializing Eq. (16) to the present context, we immediately deduce what follows: For a plane wave-packet with propagation direction $\underset{\sim}{\ell} = (0, \cos\theta, \sin\theta)$ and direction of motion $(1, 0, 0)$,

$$\underset{\sim}{v}_g = (\rho v_a(\theta))^{-1}(0,\ (L_{66} + \sigma_2)\cos\theta,\ (L_{55} + \sigma_3)\sin\theta). \tag{25}$$

Similarly, for a plane wave-packet with $\underset{\sim}{\ell} = (\sin\theta, \cos\theta, 0)$ and direction of propagation $(0, 0, 1)$,

$$\underset{\sim}{v}_g = (\rho v_b(\theta))^{-1}((L_{55} + \sigma_1)\sin\theta,\ (L_{44} + \sigma_2)\cos\theta,\ 0). \tag{26}$$

Remark 5.1. Eq. (23) seems to be new, although a special instance of it was already between the lines in Biot's paper [18] of 1940 and appeared explicitly as Eq. (8.3) in a later paper [29] (cf. also [17], Ch. 5, §4). We can easily obtain Biot's formula from Eq. (23). Let $v_{31} \equiv v_a(\frac{1}{2}\pi)$, $v_{13} \equiv v_b(\frac{1}{2}\pi)$; v_{31} and v_{13} are phase velocities of shear waves; v_{31} corresponds to shear wave with propagation direction (0, 0, 1) and direction of motion (1, 0, 0), and v_{13} corresponds to that for which the preceding directions are interchanged. On putting $\theta_2 = \frac{1}{2}\pi$ in Eq. (23) the terms with θ_1 drop out and there results Biot's formula

$$\sigma_3 - \sigma_1 = \rho(v_{31}^2 - v_{13}^2). \tag{27}$$

Biot took Eq. (27) as proof that "acoustic propagation under initial stress is fundamentally different from the stress-free case and cannot be represented by simply introducing into the classical theory stress-dependent elastic coefficients" ([17], p. 283; he made similar comments in his earlier papers [18] and [29]). Thurston ([15], §V), who apparently was unaware of Biot's papers [18] and [29], rederived Eq. (27) for the special instance of uniaxial prestress (in the 3-direction) in an otherwise isotropic hyperelastic medium; he also made observations similar to those of Biot quoted above. More recently MacDonald ([30], p. 78) suggested that Thurston's specialization of Biot's formula "can be used to determine the [uniaxial] stress"; Thompson et al. [8] singled out Eq. (27) as having "the potential of being a key element in overcoming" the problem "of differentiating stress induced velocity shifts from velocity shifts induced by microstructural variations", since by using Eq. (27) "the difference in principal stresses can be absolutely determined from measurements of density and velocity" — "no independent determination of texture is required and ... no microstructurally dependent acoustoelastic constant must be known". Of course Biot understood very well the meaning of his formula; he pointed out that his theory might lead to "possible development of new methods of measuring stresses in a solid" ([17], p. 291). While Biot did not go any further than making this suggestion, MacDonald and Thompson et al. were in fact still a step behind him in theory because they restricted their discussions to hyperelastic bodies and they alluded to second and third-order elastic coefficients in their respective paper. Commenting on the above proposal of Thompson et al. [8] to use Eq. (27) for measuring initial stress, Pao & Gamer [9] indeed questioned whether a relation "derived on the basis of hyperelastic deformation at the initial state ... can be applied to determine absolutely the difference of two principal stresses in a body with texture". Application of Biot's formula requires measurement of speeds of shear waves that propagate along two principal axes of stress. This requirement raises considerable practical difficulties should we want to put Biot's formula to an experimental test. Thompson et al. ([7], [26], [27]) overcame the aforementioned requirement by using electromagnetic acoustic transducers (EMATs) to excite and receive horizontally polarised shear waves that propagate in the plane of a thin plate. Altogether two experiments were reported in their papers.

The results of one experiment were "believed to represent a good preliminary confirmation" of Eq. (27). The other experiment "were intended to assess the influence of plastic deformation" on Eq. (27). Their results clearly showed that plastic deformation had no effect whatsoever on Biot's formula. Recalling that Biot's formula is a universal relation and is a special instance of our Eq. (23), we can interpret the experiments of Thompson et al. as corroborating our theory and as supportive evidence for the approach we advocate here regarding measurement of initial stress.

In the experiments of Thompson et al. the stress in question is homogeneous. Let us now turn to a possible application of Eq. (23) in the evaluation of residual stress, which is necessarily inhomogeneous.

§6. Determination of In-Plane Residual Stress in Orthotropic Plates

The results of §5 are useful in the determination of in-plane residual stress in orthotropic plates. At first sight the preceding assertion might appear paradoxical, because a nonzero residual stress field must be inhomogeneous (see Hoger [10], §1) and the results of §5 are derived under the assumption that the body $\mathcal{B}$ in question is homogeneous, by which we mean both $\mathbf{L}$ and $\mathring{\mathbf{T}}$ are homogeneous. Indeed it is the force of circumstances that render those results approximately valid. In most applications the residual stress $\mathring{\mathbf{T}}$ in a thin plate can be taken as homogeneous through its thickness. When a wave packet is sent through the plate in question by a transducer, only those points in the plate which are within the domain of influence of the wave packet during the time of transit will have any effect on velocity measurement. For a thin plate, the domain of influence will be approximately that small piece of the plate in contact with the transducer. We expect the results of §5 in effect to be valid, provided that the incremental elasticity tensor $\mathbf{L}$ and the residual stress $\mathring{\mathbf{T}}$ can be taken as effectively homogeneous within the domain of influence in question. Of course the discussion above should be made precise and be substantiated by mathematical theorems, which are as yet wanting. Nevertheless there are experiments (see, e.g., Hsu [31], King & Fortunko [6]) which support our contention; these experiments are based on the assumption that an inhomogeneous stress field in a thin plate can be taken as locally (i.e., for a region of the size of a transducer) homogeneous. Let us now consider a specific example.

Example 6.1. Consider a circular plate (with residual stress) which occupies the region $0 \leq r \leq R$, $0 \leq \phi < 2\pi$, $0 \leq z \leq h$; here (r, ϕ, z) are cylindrical coordinates, h is the thickness and R is the radius of the plate. We make the following assumptions:

(i) h is small when compared with R.

(ii) Each material point $\underset{\sim}{X}$ in the plate is orthotropic. For each $\underset{\sim}{X}$, the planes of orthotropic symmetry are those determined by the unit local base vectors $\underset{\sim}{e}_r$, $\underset{\sim}{e}_\phi$, and $\underset{\sim}{e}_z$ when they are grouped in pairs.

(iii) Under the given cylindrical coordinate system the residual stress has the form

$$\underset{\sim}{\overset{\circ}{T}} = \underset{\sim}{\overset{\circ}{T}}(r) = \begin{bmatrix} \overset{\circ}{T}_{rr}(r) & 0 & 0 \\ 0 & \overset{\circ}{T}_{\phi\phi}(r) & 0 \\ 0 & 0 & 0 \end{bmatrix} ; \qquad (28)$$

here $\overset{\circ}{T}_{rr}$ and $\overset{\circ}{T}_{\phi\phi}$ are physical components.

(iv) The rate of change of $\underset{\sim}{\overset{\circ}{T}}$ with respect to r is sufficiently small that $\underset{\sim}{\overset{\circ}{T}}$ can be taken as effectively homogeneous in any region of diameter $d + 2h$; here d is a length that characterizes the size of the transducers to be used in wave-speed measurements. (Without exact analysis the quantity $d + 2h$ and in particular the term $2h$ are somewhat arbitrary.)

Under the assumptions above we can determine, at least in principle, the function $\mathring{T}_{rr}(r) - \mathring{T}_{\phi\phi}(r)$ by appealing to Eq. (23) or Eq. (24) and by doing wave-speed measurements at various locations of the plate (cf. [6], [31]). Once $\mathring{T}_{rr} - \mathring{T}_{\phi\phi}$ is known, $\mathring{T}_{rr}$ (and thence also $\mathring{T}_{\phi\phi}$) can be calculated from the equation of equilibrium

$$d\mathring{T}_{rr}/dr + (\mathring{T}_{rr} - \mathring{T}_{\phi\phi})/r = 0 \tag{29}$$

and the boundary condition $\mathring{T}_{rr}(R) = 0$.

The foregoing example illustrates how the results of §5 can be applied in evaluation of residual stress. Of course, that something can be done in principle does not imply it can be put into practice. Let us devote the rest of this section to discuss the work of King & Fortunko [6], which shows indirectly that there are situations for which what we propose in the example above is indeed empirically feasible.

King & Fortunko considered an hyperelastic plate with inhomogeneous texture and inhomogeneous prestress. They assumed that (1.) the material points of the plate are orthotropic and almost isotropic in their unstressed "natural state"; (2.) at the given configuration of the plate, for each material point $\underset{\sim}{X}$, the principal axes of the prestress coincide with the axes which define the symmetry planes of the texture orthotropy. They developed an experimental procedure to evaluate the difference of the in-plane principal stresses. Their procedure is based on an equation which can be taken as a descendent of our Eq. (22). By appealing to approximations, they in effect replaced our $L_{66} - L_{44}$ in Eq. (22) by $(L^{o}_{66} - L^{o}_{44}) + K(\sigma_3 - \sigma_1)$; here K is a material constant, and L^{o}_{66} and L^{o}_{44} are elastic constants pertaining to the unstressed "natural state". In our notation their equation reads as follows:

$$(L^{o}_{66} - L^{o}_{44})\cos^2\theta + A(\theta)(\sigma_3 - \sigma_1) = \rho(v_a^2(\theta) - v_b^2(\theta)), \tag{30}$$

where $A(\theta) \equiv K\cos^2\theta + \sin^2\theta$. They used calibration specimens to determine $A(\theta)$ for two values of θ so that $\sigma_3 - \sigma_1$ could be evaluated by an equation similar to our Eq. (23). They tested their procedure by using an aluminium specimen with a known stress state. The values of $\sigma_3 - \sigma_1$ calculated through Eq. (30) from their data of wave-speed measurements agreed fairly well with the known stress values.

Remark 6.1. Let $a(\theta) = A(\theta)/2\mu$, where μ is the usual shear modulus of aluminium when we treat the metal as if it were isotropic. The calibration values of $a(\theta)$ determined by King & Fortunko were $a(33.8^\circ) = 3.7 \times 10^{-5}$ MPa^{-1} and $a(12.6^\circ) = 4.4 \times 10^{-5}$ MPa^{-1}. Since $A(\theta) \equiv K\cos^2\theta + \sin^2\theta$, we obtain from the two calibration values of $a(\theta)$ the following numerical values: $K = 2.44$, and $\mu = 0.27 \times 10^5$ MPa. The calculated value of μ agrees with that given in engineering handbooks, which confirms our interpretation of their work. Let us recast Eq. (30) as follows: $((L^{o}_{66} - L^{o}_{44}) + K(\sigma_3 - \sigma_1))\cos^2\theta + (\sigma_3 - \sigma_1)\sin^2\theta = \rho(v_a^2(\theta) - v_b^2(\theta))$. By comparing the preceding equation with Eq. (22), it is apparent that should King & Fortunko have used our Eq. (22) instead of Eq. (30) plus their calibration value of $a(\theta)$, they would have obtained the same values of $\sigma_3 - \sigma_1$ from their data of speed measurements. Thus we can interpret the experiment of King & Fortunko [6] as another corroboration of our theory.

Remark 6.2. The experimental techniques of King & Fortunko can be easily adapted in evaluation of residual stress for situations such as Example 6.1, provided that EMATs can be applied to generate and receive SH-waves.

Remark 6.3. A comparison of Eq. (22) and Eq. (30) will reveal the difference in philosophy between our approach and that of King & Fortunko [6], whose guiding idea was to separate the effects of "stress-induced" and "texture-induced" anisotropy. In our approach we strive to eliminate as a whole the effects of the incremental elasticity tensor $\underline{\underline{L}}$, which pertains to the given configuration. We do not attempt to separate the anisotropy of $\underline{\underline{L}}$ into "texture-induced" and "stress-induced" components; indeed we deem such a separation generally impossible and meaningless. Even for the special instance of hyperelastic body with "natural state", where such a separation makes sense, little will be gained by the separation except that the mean normal part of the prestress can appear explicitly in the formulae of the usual acoustoelastic theory (cf. Remark 4.2 above). As a disadvantage, material coefficients are introduced (e.g., the constant K in Eq. (30)), the elimination of which will generally require the use of calibration specimens. As compared with Eq. (30), Eq. (22) is simple, direct, exact, and not restricted to hyperelastic material with an unstressed "natural state". With Eq. (22) in hand, should we want to repeat the experiment of King & Fortunko, we could drop the calibration specimens and follow an otherwise identical experimental procedure.

§7. In-Plane Prestress in an Almost Orthotropic Plate [7]

Consider a homogeneous plate $\mathcal{B}$ of thickness h, every material point of which is monoclinic and has prestress $\mathring{\underset{\sim}{T}}$ in the given configuration $\underset{\sim}{\kappa}$. Let us choose a Cartesian coordinate system such that the plane of monoclinic symmetry at each material point $\underset{\sim}{X}$ is parallel to the plane $X_2 = 0$. At the given configuration $\underset{\sim}{\kappa}$, let the two faces of the plate $\mathcal{B}$ lie in the plane $X_2 = 0$ and $X_2 = h$, respectively, and let the prestress $\mathring{\underset{\sim}{T}}$ have Cartesian components given by the matrix

$$\mathring{\underset{\sim}{T}} = \begin{bmatrix} \mathring{T}_{11} & 0 & \mathring{T}_{13} \\ 0 & 0 & 0 \\ \mathring{T}_{13} & 0 & \mathring{T}_{33} \end{bmatrix}. \tag{31}$$

We assume that the given plate is almost orthotropic in the sense below: We can choose the 1- and 3-axis such that $\|\mathring{\underset{\sim}{T}}\|$, L_{15}, L_{25}, L_{35}, and L_{46} are small when compared with the other non-zero components of the "incremental elasticity tensor" $\underset{\approx}{L}$. Under the chosen coordinate system we can decompose the generalized Christoffel tensor $\underset{\sim}{\Gamma}$ of a material point in the plate as follows (cf. Eqs. (13) and (14) in §4 above):

$$\underset{\sim}{\Gamma} = \underset{\sim}{\Gamma}_o + \underset{\sim}{\Gamma}'; \tag{32}$$

here $\underset{\sim}{\Gamma}_o$ is that (orthotropic) part of $\underset{\sim}{\Gamma}$ whose components are given by Eq. (18) with σ_i (i = 1, 2, 3) set equal to zero; the components of $\underset{\sim}{\Gamma}'$

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[7] Cf. King & Fortunko [32].

are given by the equations

$$
\begin{aligned}
\Gamma'_{11} &= 2L_{15}\ell_3\ell_1 + \boldsymbol{\ell}\cdot\mathring{\mathbf{T}}\boldsymbol{\ell}, \\
\Gamma'_{22} &= 2L_{46}\ell_3\ell_1 + \boldsymbol{\ell}\cdot\mathring{\mathbf{T}}\boldsymbol{\ell}, \\
\Gamma'_{33} &= 2L_{35}\ell_3\ell_1 + \boldsymbol{\ell}\cdot\mathring{\mathbf{T}}\boldsymbol{\ell}, \\
\Gamma'_{12} &= \Gamma'_{21} = (L_{46} + L_{25})\ell_2\ell_3, \\
\Gamma'_{13} &= \Gamma'_{31} = L_{15}\ell_1^2 + L_{46}\ell_2^2 + L_{35}\ell_3^2, \\
\Gamma'_{23} &= \Gamma'_{32} = (L_{35} + L_{46})\ell_1\ell_2,
\end{aligned}
\tag{33}
$$

where

$$\boldsymbol{\ell}\cdot\mathring{\mathbf{T}}\boldsymbol{\ell} = \mathring{T}_{11}\ell_1^2 + 2\mathring{T}_{13}\ell_1\ell_3 + \mathring{T}_{33}\ell_3^2. \tag{34}$$

We shall regard $\boldsymbol{\Gamma}'$ as a small perturbation added onto $\boldsymbol{\Gamma}_o$. For each propagation direction $\boldsymbol{\ell}$ to be chosen below, we shall assume that $\boldsymbol{\Gamma}_o$ has three distinct positive eigenvalues and $\|\boldsymbol{\Gamma}'\|$ is small as compared with the absolute value of the difference of any two eigenvalues of $\boldsymbol{\Gamma}_o$. Here $\|\boldsymbol{\Gamma}'\|$ should be sufficiently small that we can apply perturbation theory (see Kato [33], Ch. II, Theorem 3.9 for an exact quantitative description of the smallness required).

For $\boldsymbol{\ell} = (0, \cos\theta, \sin\theta)$, $\mathbf{e}_1 = (1, 0, 0)$ is an eigenvector of $\boldsymbol{\Gamma}_o$ with eigenvalue $L_{66}\cos^2\theta + L_{55}\sin^2\theta$. For $\boldsymbol{\Gamma}$, the first-order correction to the corresponding eigenvalue is

$$\mathbf{e}_1\cdot\boldsymbol{\Gamma}'\mathbf{e}_1 = \mathring{T}_{33}\sin^2\theta. \tag{35}$$

Thence the speed $v_a(\theta)$ of the quasishear wave in question is, to first order, given by the equation

$$\rho v_a^2(\theta) = L_{66}\cos^2\theta + L_{55}\sin^2\theta + \mathring{T}_{33}\sin^2\theta. \tag{36}$$

Similarly, for $\underset{\sim}{\ell} = (\sin\theta, \cos\theta, 0)$, the speed $v_b(\theta)$ of quasishear wave whose displacement is to zeroth order in the direction of $\underset{\sim}{e}_3 = (0, 0, 1)$ is given by the equation

$$\rho v_b^2(\theta) = L_{44}\cos^2\theta + L_{55}\sin^2\theta + \mathring{T}_{11}\sin^2\theta. \tag{37}$$

Comparing Eqs. (36) and (37) with Eqs. (20) and (21) in §5, we see that by giving θ two suitable values θ_1 and θ_2, we can obtain the equation

$$\mathring{T}_{33} - \mathring{T}_{11} = \rho\frac{(v_a^2(\theta_2) - v_b^2(\theta_2))\cos^2\theta_1 - (v_a^2(\theta_1) - v_b^2(\theta_1))\cos^2\theta_2}{\cos^2\theta_1\sin^2\theta_2 - \cos^2\theta_2\sin^2\theta_1}. \tag{38}$$

Eq. (38) is the analog of Eq. (23) in the present context. It puts Eq. (23), our main result in §5, in a broader perspective.

Let ψ be the smallest positive angle of rotation about the 2-axis which will bring the 1- and 3-axes to the principal stress directions of $\underset{\sim}{\mathring{T}}$ in the 1-3 plane. Let σ_1 and σ_3 be the corresponding principal stresses. It is easy to deduce that

$$\mathring{T}_{33} - \mathring{T}_{11} = (\sigma_3 - \sigma_1)\cos 2\psi. \tag{39}$$

Let us give an example which illustrates a possible application of

Eqs. (38) and (39).

Example 7.1. Consider a plate with residual stress whose material points are orthotropic (cf. Example 6.1 for instance). Let the plate occupy the region $\mathcal{D} \times [0, h]$; here $\mathcal{D}$ is a domain in the 1-3 plane, and $0 \leq X_2 \leq h$, where h is the thickness of the plate. A piece of the plate, which originally occupies the region $\mathcal{D}_1 \times [0, h]$, is cut out from the plate so that a hole is formed. See Fig. 1. Suppose we are given the resulting plate with hole, which will no longer be orthotropic; for instance the principal stress directions at a point on the boundary $\partial\mathcal{D}_1 \times [0, h]$ generally will not coincide with the axes of original orthotropic symmetry. But it is still reasonable to assume that each material point of the plate remains monoclinic and almost orthotropic with respect to the axes of original orthotropic symmetry; the plane of monoclinic symmetry is parallel to the 1-3 plane. Suppose we want to determine the in-plane residual stress at the boundary of the hole.[8] Eqs. (38) and (39) will be useful in this regard. At a point $\underset{\sim}{X}$ on $\partial\mathcal{D}_1 \times [0, h]$, the residual stress $\overset{\circ}{\underset{\sim}{T}}(\underset{\sim}{X})$ will be of the form given by Eq. (31) under the Cartesian coordinate system defined by the axes of original orthotropic symmetry at $\underset{\sim}{X}$. The in-plane principal stress directions will be tangent and normal to $\partial\mathcal{D}_1$, respectively. The in-plane principal stress normal to $\partial\mathcal{D}_1$ will be null. The remaining in-plane principal stress can be determined through Eqs. (38) and (39) by wave-speed measurements. Cf. the opening paragraph of §6 and Example 6.1.

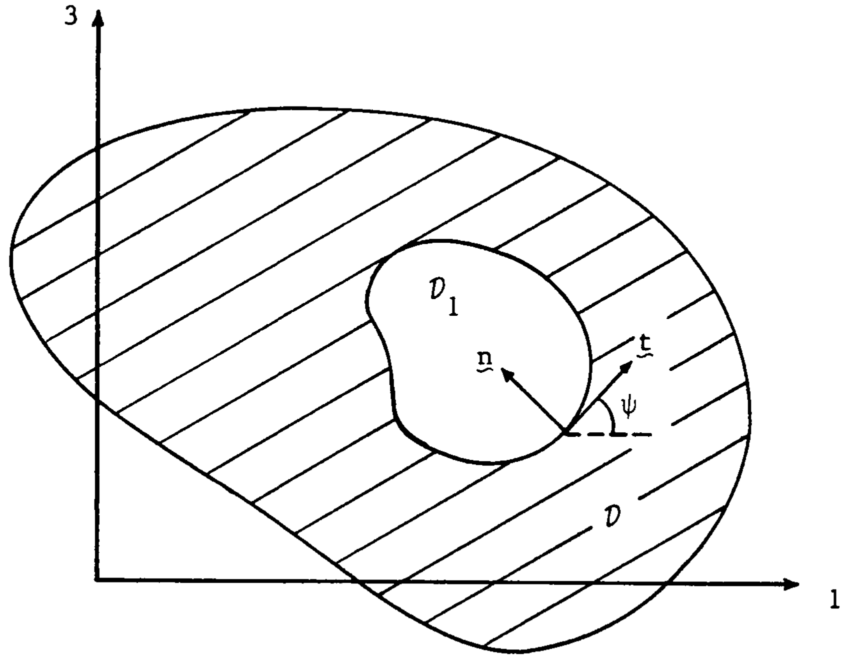

Fig. 1. Determination of in-plane residual stress around a hole in an almost orthotropic plate. $\underset{\sim}{t}$ and $\underset{\sim}{n}$ are the in-plane principal-stress directions tangent and normal to $\partial\mathcal{D}_1$, respectively; ψ is the angle between $\underset{\sim}{t}$ and the 1-direction.

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[8]In view of the hole-drilling method of residual stress determination, we want to emphasize that in this example we are given a plate with hole and we want to determine the residual stress after the hole is formed.

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§8. Love Waves and In-Plane Prestress in an Orthotropic Layer

Consider a homogeneous layer of thickness h deposited over a half-space with different acoustic properties. We choose a Cartesian coordinate system under which the layer and the half-space are defined by the conditions $-h \leqq X_2 \leqq 0$ and $X_2 \geqq 0$, respectively. Let the prestress in the layer be homogeneous and have the form

$$\overset{\circ}{T} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} . \qquad (40)$$

We assume that every material point of the given layer is orthotropic, the planes of symmetry being parallel to the coordinate planes of the chosen coordinate system. For simplicity we assume that the half-space is isotropic and is unstressed at the given configuration.

In this section we investigate the existence of Love waves that propagate along the free surface $X_2 = -h$ in the 1- and 3-directions, respectively. Since we shall go through essentially the same calculations as the simpler case where $\overset{\circ}{T} = 0$ (cf. Tournois & Lardat [34]), which are almost identical to what Love ([15], §177) did when the layer is also isotropic, we shall be brief below.

Consider a displacement field of the form

$$\underset{\sim}{u} = (0,\ 0,\ f(X_2)\cos(\omega t - kX_1)) \tag{41}$$

in the layer; here f is a smooth function of X_2. In order that Eq. (41) satisfies the equation of motion Eq. (10), f must observe the condition

$$\frac{d^2 f}{dX_2^2} + \left(\frac{\rho v^2 - (L_{55} + \sigma_1)}{L_{44}}\right)k^2 f = 0; \tag{42}$$

here the density ρ and the coefficients L_{44}, L_{55} all pertain to the layer; $v \equiv \omega/k$ is the phase velocity of the Love wave in question. We assume that $L_{44} > 0$ and $\rho v^2 > L_{55} + \sigma_1$; we set $a \equiv [(\rho v^2 - (L_{55} + \sigma_1))/L_{44}]^{\frac{1}{2}}$.

Similarly, consider a displacement of the form

$$\underset{\sim}{u}_\star = (0,\ 0,\ f_\star(X_2)\cos(\omega t - kX_1)) \tag{43}$$

in the half-space. In order that $\underset{\sim}{u}_\star$ observes the equation of motion, $f_\star$ must satisfy the differential equation

$$\frac{d^2 f_\star}{dX_2^2} - \left(1 - \frac{\rho_\star v^2}{\mu_\star}\right)k^2 f_\star = 0; \tag{44}$$

here $\rho_\star$ and $\mu_\star$ are the density and shear modulus of the half-space, respectively. We assume that $\rho_\star v^2 < \mu_\star$, and we put $b \equiv [1 - (\rho_\star v^2/\mu_\star)]^{\frac{1}{2}}$.

The general solution of Eq. (42) is

$$f(X_2) = A\sin(kaX_2) + B\cos(kaX_2); \tag{45}$$

that of Eq. (44) is

$$f_*(X_2) = C\exp(-kbX_2) + D\exp(kbX_2); \qquad (46)$$

here A, B, C and D are constants. We assume that $f_*(X_2) \to 0$ as $X_2 \to \infty$; it follows that $D = 0$. Since $\underset{\sim}{u} = \underset{\sim}{u}_*$ when $X_2 = 0$, we deduce that $B = C$. The plane $X_2 = -h$ is a free surface, on which the traction is null; as a result, $f'(-h) = 0$ or $\tan(kah) = -A/B$. By the continuity of the traction across the plane $X_2 = 0$, we deduce that $-A/C = b\mu_*/aL_{44}$. Combining all the results in this paragraph, we obtain the dispersion equation

$$\tan(kah) = \frac{b\mu_*}{aL_{44}}, \qquad (47)$$

which is analogous to the equation found by Love for the instance where $\hat{\underset{\sim}{T}} = \underset{\sim}{0}$ and both the layer and the half-space are isotropic. Let us regard the parameters ρ, L_{44}, L_{55}, ρ_*, μ_*, σ_1 and h as given. For each phase velocity v that satisfies the condition $(L_{55}+\sigma_1)/\rho < v^2 < \mu_*/\rho_*$, by the periodicity of the tangent function Eq. (47) has an infinite number of roots for k, each of which corresponds to a mode of Love wave with phase velocity v. The group velocity $\underset{\sim}{v}_g = (d\omega/dk, 0, 0)$ of the Love wave in question can be obtained from Eq. (47) by differentiation.

For Love waves that propagate in the 3-direction, it is apparent that we should also obtain a dispersion equation analogous to Eq. (47), but with σ_1 and L_{44} replaced by σ_3 and L_{66}, respectively. The quantity $L_{55} + \sigma_1$ appears in Eq. (47), while $L_{55} + \sigma_3$ appears in the other dispersion equation. Thence, inversion of Love-wave dispersion curves may provide an alternate way for us to ascertain the difference of in-plane principal stresses $\sigma_3 - \sigma_1$ in the layer.

§9. Conclusion

We have described above the rudiments of an acoustoelastic theory. We were motivated to develop this theory after we learned about Hoger's statical approach to measurement of residual stress [10] and read her clear and elegant rederivation of the constitutive relation Eq. (1). Our theory can be regarded as a generalization of and a natural sequel to Biot's work [18] and Thurston's study [28] of "effective elastic coefficients" in crystal acoustics. Our universal relation Eq. (23) is a direct generalization of Biot's formula, which is given as Eq. (27) above. Many of our basic equations in §4 already appeared in Thurston's paper. All these equations, however, have acquired new meaning in the theoretical setting we adopt, which is none other than the classical theory of linear elasticity with initial stress. The elusive concept of a "natural state", which has been both the basis of acoustoelasticity and the source of its difficulties, is once and for all eliminated from the entire picture. Complete emphasis is now put on the currently given configuration; previous history of loading and deformation has become irrelevant. In principle our equations and formulae could be applicable, for instance, to bodies given in a state of plastic deformation.

The main theme of this paper is to advocate an approach to ultrasonic measurement of initial stress which makes full use of universal relations in our acoustoelastic theory. We believe that the separation of "texture-induced" and "stress-induced" anisotropy, as a guiding idea in acousto-elastic research, is ultimately misleading. Although that idea has stimulated many efforts and activities, it should be counted on balance as among the negative factors that have impeded progress in ultrasonic measurement of stress during the last two decades. We can count the experiments of King & Fortunko [6] and Thompson et al. ([7], [26], [27]) as providing supportive evidence to the approach we advocate here. (See Remarks 5.1 and 6.1 above.)

In this paper we treat residual stress as a constitutive quantity. At first sight the task to determine residual stress would seem to be similar to that of measuring "effective elastic coefficients" in acoustics for crystals under stress. At the level of general theory the equations of wave propagation involved will be identical. The specific tasks at hand, however, make the difference. In crystal acoustics the (applied) prestress $\mathring{T}$ is usually taken as known and homogeneous; the problem is to measure the (homogeneous) "effective elastic coefficients". When we want to determine residual stress in a body, $\mathring{T}$ will be unknown and inhomogeneous; the coefficients of the (inhomogeneous) incremental elasticity tensor are not of prime concern. Inhomogeneity of $\mathring{T}$ will generally pose a real problem for stress determination by wave propagation methods. On the other hand, the fact that residual stress should be divergence-free and satisfy the zero-traction boundary condition will be helpful for its determination. Our examples in §§6-7 illustrate these points. There we discussed application to plates to evade the difficulty of inhomogeneity. We made use of the equation of equilibrium in Example 6.1 and the zero-traction boundary condition both in Example 6.1 and Example 7.1.

A physical theory will be untenable unless it can really model the class of phenomena it is meant for. While our theory has preliminary corroboration in the work of King & Fortunko and Thompson et al., a total number of three experiments is far too few; in particular, only one of the experiments enters the plastic regime. Moreover in all of the three experiments the quality of agreement between theoretical prediction and experimental data can only be described as between fair and good. More experimental testing of our theory is necessary. Currently we have an experimental program under way at the University of Kentucky. The first stage of the program is focused on the universal relation Eq. (23) and measurement of stress in plastically deformed bodies. The results obtained thus far are encouraging, and we shall report our experimental findings elsewhere. [35]

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