Towards an effective action for relativistic dissipative hydrodynamics

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Abstract: We propose an effective action for first order relativistic dissipative hydrodynamics that can be used to evaluate n-point symmetrized correlation functions, taking into account thermal fluctuations of the hydrodynamic variables.

Contents

1 Introduction

The study of fluid dynamics is a centuries-old discipline that still seems nowhere near closure. The "classical" relativistic fluid dynamics can be derived by requiring conservation of the energy-momentum tensor and global symmetry currents [1]. In modern understanding, these conservation equations should be constructed order by order in the derivative expansion of the hydrodynamic variables, similar to the derivative expansion in effective field theory. Truncating the derivative expansion at first order, second order and higher order one obtains ideal fluid dynamics (Euler equations), viscous fluid dynamics (Navier-Stokes equations), and higher-order dissipative hydrodynamics [2, 3]. The classical hydrodynamic correlation functions can then be obtained by varying with respect to external sources, see e.g. [4].

In non-relativistic fluids, it is well known that there are correlation functions of the hydrodynamic variables which can not be reproduced by such classical hydrodynamic equations, even when the frequency and momentum are arbitrarily small [5]. The reason is that while the classical equations describe flows generated by external sources, they neglect hydrodynamic excitations generated by thermal fluctuations within the fluid. These effects may be taken into account by supplementing the classical hydrodynamic equations with stochastic noise terms whose correlation functions are taken to be Gaussian white noise [5]. It is natural to expect that a similar stochastic modification is required for relativistic fluids in order to correctly reproduce physical observables.

In linear non-relativistic hydrodynamics, such noise terms were introduced long ago by Landau and Lifshitz [6]. When considering the full non-linear theory of stochastic hydrodynamics, one finds that the interactions lead to changes in the basic parameters of the classical theory, such as the shear viscosity coefficient [7]. More generally, correlation

functions evaluated in stochastic hydrodynamics will differ from their classical counterparts by fluctuation corrections involving loops of the hydrodynamic modes. Stochastic equations for hydrodynamic variables can be readily converted to a functional integral form [8], providing one with an effective field theory. The purpose of this note is to write down an effective action for dissipative relativistic fluids.

We emphasize that our interest is not in an action that will give rise to the classical hydrodynamic equations upon using a variational procedure. Rather, we are interested in an action which can be used in a standard way in the functional integral to evaluate hydrodynamic correlation functions. While it is straightforward to derive such an effective action for the linearized viscous relativistic hydrodynamics [4], the full non-linear hydrodynamics and the derivative expansion require more work. The fields in the effective theory include the hydrodynamic variables (fluid velocity, temperature etc), and we will refer to this effective theory as "statistical hydrodynamics", to distinguish it from classical hydrodynamics which ignores fluctuations. The 1PI effective action of statistical hydrodynamics should give rise to the classical hydrodynamic equations at tree level, but will contain corrections to classical hydrodynamics once the loops are taken into account. The loops here are not the quantum loops (as one is not quantizing the classical hydrodynamics), but rather reflect statistical fluctuations of the hydrodynamic variables.

We pause to comment on previous work addressing related questions. A variational formulation of classical ideal relativistic hydrodynamics (neglecting the derivative expansion, fluctuations, and dissipation) is an old subject discussed by many authors in various forms, see e.g. [9–11]. As mentioned above, we don't expect such classical constructions to be helpful for statistical hydrodynamics. Refs. [12, 13] studied effective actions for relativistic fluids, taking into account the derivative expansion, however the resulting effective action only captured non-dissipative information. Similarly, refs. [14, 15] derived generating functionals of relativistic fluids coupled to external sources in equilibrium. Again, this allowed a systematic construction to any order in the derivative expansion, but only captured static non-dissipative physics. For variational approaches aiming to incorporate dissipation in classical hydrodynamics, see e.g. [16–18].

Recently, there have also been efforts to understand dissipation in relativistic statistical hydrodynamics (with fluctuation corrections), partly motivated by the experimental study of the quark-gluon plasma in heavy-ion collisions. Refs. [4, 19–21] looked at statistical one-loop corrections to the shear viscosity, but lacked a systematic field-theoretic framework. See [22–25] for other recent work on relativistic fluctuating hydrodynamics, including the Israel-Stewart formulation. It is worth pointing out that the fluctuation corrections render the derivative expansion in purely classical relativistic hydrodynamics ill-defined [21]. Clearly, one needs a unified calculational framework that takes into account the full non-linearity of relativistic hydrodynamics, the derivative expansion, and fluctuations of the hydrodynamic variables. The present paper is a step in this direction.

2 Noisy hydrodynamics

2.1 Setup

Classical relativistic hydrodynamics [1] is a set of partial differential equations for the hydrodynamic fields $u^\mu(x)$, $T(x)$, and (for fluids with a global U(1) charge) $\mu(x)$. Collectively

denoting these hydrodynamic fields as ϕ, we will write the classical hydrodynamic equations in the form $E^a(\phi) = 0$, where $E^\mu = \partial_\nu T_{\rm cl}^{\nu\mu}$, $E^{d+1} = u^2 + 1$, $E^{d+2} = \partial_\mu J_{\rm cl}^\mu$, and d is the number of spatial dimensions. Here $T_{\rm cl}^{\mu\nu}$ and $J_{\rm cl}^\mu$ are the (symmetric) energy-momentum tensor and the U(1) global symmetry current, given in terms of ϕ. The fluid velocity is normalized[1] as $u^2 = -1$, T is the temperature, and μ is the chemical potential. The constitutive relations expressing $T_{\rm cl}^{\mu\nu}$ and $J_{\rm cl}^\mu$ in terms of ϕ are normally written in a given "frame" (a particular out-of equilibrium definition of ϕ), to a given order in the derivatives of ϕ. The starting point for the stochastic hydrodynamics is the modification of the classical hydrodynamic equations by "noise" terms which are interpreted as microscopic stresses and currents [6], so that the hydrodynamic equations take the form $\partial_\mu T^{\mu\nu} = 0$, and $\partial_\mu J^\mu = 0$, where $T^{\mu\nu} = T_{\rm cl}^{\mu\nu} + \tau^{\mu\nu}$, and $J^\mu = J_{\rm cl}^\mu + r^\mu$. The microscopic contributions $\tau^{\mu\nu}(\phi,\xi)$ and $r^\mu(\phi,\xi)$ are functionals of both the hydrodynamic fields ϕ and the noise fields collectively denoted as ξ, so that the hydrodynamic equations become stochastic equations

$$E^a(\phi) + f^a(\phi,\xi) = 0\,, \tag{2.1}$$

where $f^\mu = \partial_\nu \tau^{\nu\mu}$ and $f^{d+2} = \partial_\mu r^\mu$. The form of the force f^a and the dynamics of the noise fields need to be determined by the problem at hand. In particular, they must be such that the fluctuation-dissipation theorem is satisfied in equilibrium.

One can convert eq. (2.1) to a functional integral form. Let us denote the solution to eq. (2.1) as ϕ_ξ. Upon solving eq. (2.1), the energy-momentum tensor and the current will become functionals of the noise, $T^{\mu\nu}[\phi_\xi, \xi]$, $J^\mu[\phi_\xi, \xi]$. For a general function $O(\phi_\xi)$ we have

$$O(\phi_\xi) = \int D\phi\, \delta(E^a(\phi) + f^a(\phi,\xi))\, J(\phi,\xi) O(\phi)\,,$$

where the Jacobian is $J = \det \frac{\delta(E^a + f^a)}{\delta \phi_b}$. If the dynamics of ξ is independent of ϕ, so that the noise average is performed with some ϕ-independent action $S_{\rm n}[\xi]$, the correlation functions can be written as

$$\langle T^{\mu\nu} T^{\alpha\beta} \dots \rangle = \int D\xi\, D\phi\, D\tilde\phi\, e^{i\int \tilde\phi_a [E^a(\phi) + f^a(\phi,\xi)]}\, J(\phi,\xi)\, e^{-S_{\rm n}[\xi]}\, T^{\mu\nu}[\phi,\xi]\, T^{\alpha\beta}[\phi,\xi]\dots. \tag{2.2}$$

The corresponding partition function is

$$Z = \int D\xi\, D\phi\, D\tilde\phi\, e^{i\int \tilde\phi_a [E^a(\phi) + f^a(\phi,\xi)]}\, J(\phi,\xi)\, e^{-S_{\rm n}[\xi]}\,. \tag{2.3}$$

Alternatively, one can define stochastic hydrodynamics by the functional integral representation,

$$Z = \int D\xi\, D\phi\, D\tilde\phi\, e^{i\int \tilde\phi_a [E^a(\phi) + f^a(\phi,\xi)]}\, e^{-S_\xi[\phi,\xi]}\,, \tag{2.4}$$

where the auxiliary fields $\tilde\phi_a$ ensure that eq. (2.1) is satisfied, and the noise action S_ξ needs to be specified. Normally, the central limit theorem is invoked to argue that the noise is Gaussian, hence the noise action is quadratic in ξ. In this case ξ can be integrated

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[1]Our metric signature is $[-+++]$, e.g., space-positive.

out, leaving one with the effective action $S_{\text{eff}}(\phi, \tilde{\phi})$. A proposal for the effective action in stochastic hydrodynamics amounts to a choice of f^a and S_ξ.

The functional integral eq. (2.4) can in principle be used to compute correlation functions of the hydrodynamic fields, and hence of $T^{\mu\nu}$ and J^μ. As the order of the fields does not matter inside the functional integral, these are unordered (or symmetrized) correlation functions.[2] The effective action given by eq. (2.4) contains extra fields, in addition to the hydrodynamic fields ϕ. The extra fields can be thought of as "degrees of freedom" giving rise to dissipation. As the effective theory eq. (2.4) describes dissipative physics, the effective action need not be real.

In what follows we will apply the formulation eq. (2.4) to the first-order hydrodynamics in the Landau-Lifshitz "frame" [1]. The classical constitutive relations can be taken as

$$T_{\text{cl}}^{\mu\nu} = \epsilon u^\mu u^\nu + p\Delta^{\mu\nu} - G^{\mu\nu\alpha\beta}\partial_\alpha u_\beta \,, \tag{2.5a}$$

$$J_{\text{cl}}^\mu = n u^\mu - \sigma T \Delta^{\mu\nu}\partial_\nu(\mu/T) \,, \tag{2.5b}$$

with ϵ, p, and n the equilibrium energy density, pressure, and charge density, and with the last terms describing the dissipative part of the dynamics. Here $\Delta^{\mu\nu} \equiv \eta^{\mu\nu} + u^\mu u^\nu$ is the projector to the space components of the local rest frame, and $G^{\mu\nu\alpha\beta} \equiv 2\eta S_{\text{T}}^{\mu\nu\alpha\beta} + d\zeta S_{\text{L}}^{\mu\nu\alpha\beta}$, where $S_{\text{T}}^{\mu\nu\alpha\beta} \equiv \frac{1}{2}(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\alpha}\Delta^{\nu\beta} - \frac{2}{d}\Delta^{\mu\nu}\Delta^{\alpha\beta})$ and $S_{\text{L}}^{\mu\nu\alpha\beta} \equiv \frac{1}{d}\Delta^{\mu\nu}\Delta^{\alpha\beta}$ are transverse and longitudinal spatial projectors. $\eta(T,\mu)$ is the shear viscosity, $\zeta(T,\mu)$ is the bulk viscosity, and $\sigma(T,\mu)$ is the charge conductivity. Working in the Landau-Lifshitz frame, we will impose $u_\mu\tau^{\mu\nu} = 0$ and $u_\mu r^\mu = 0$.

2.2 Linear fluctuations in equilibrium

To illustrate the general procedure, let us look at small fluctuations in thermal equilibrium with constant $\bar{T}$, constant $\bar{\mu} = 0$, and constant $\bar{u}^\mu = (1, \mathbf{0})$. To linear order in fluctuations in the Landau-Lifshitz frame $\tau^{0\mu} = 0$, $r^0 = 0$, and the constitutive relations become

$$T_{ij} = \delta_{ij}(\bar{p} + \bar{s}\,\delta T) - \bar{\eta}\left(\partial_i v_j + \partial_j v_i - \frac{2}{d}\delta_{ij}\partial_k v^k\right) - \bar{\zeta}\delta_{ij}\partial_k v^k + \tau_{ij} \,,$$

$$J_i = -\bar{\sigma}\partial_i\mu + r_i \,.$$

To linear order in fluctuations, τ_{ij} and r_i do not depend on the hydrodynamic fields and can be treated as external sources. For the Fourier components it is then straightforward to find

$$\delta T(\omega, \mathbf{k}) = \frac{1}{\partial\bar{\epsilon}/\partial\bar{T}}\frac{k_i k_j \tau^{ij}}{\omega^2 - v_{\text{s}}^2 \mathbf{k}^2 + i\gamma_s\omega\mathbf{k}^2} \,,$$

$$v^i(\omega, \mathbf{k}) = \left(\delta^{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right)\frac{k_m\tau^{jm}}{\bar{w}(\omega + i\gamma_\eta\mathbf{k}^2)} + \frac{k^i}{\mathbf{k}^2}\frac{\omega}{\bar{w}}\frac{k_m k_n\tau^{mn}}{\omega^2 - v_{\text{s}}^2\mathbf{k}^2 + i\gamma_s\omega\mathbf{k}^2} \,,$$

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[2] The effective theory discussed here is supposed to be valid in the hydrodynamic limit $\omega \to 0$. In equilibrium, the difference between unordered and symmetrized functions is $O(\omega/T)$ for $\omega \ll T$. Out of equilibrium, we assume that there is a scale ω_0 such that the difference between unordered and symmetrized functions is negligible for $\omega \ll \omega_0$.

$$\mu(\omega, \mathbf{k}) = \frac{k_m r^m}{\bar{\chi}(\omega + iD\mathbf{k}^2)}\,,$$

where $v_{\rm s}^2 = s/(\partial\bar{\epsilon}/\partial\bar{T})$ is the speed of sound squared, $\gamma_\eta \equiv \bar{\eta}/\bar{w}$, $\gamma_\zeta \equiv \bar{\zeta}/\bar{w}$, $\gamma_s \equiv \gamma_\zeta + \frac{2d-2}{d}\gamma_\eta$, $\bar{w} \equiv \bar{\epsilon} + \bar{p}$ is the equilibrium enthalpy density, $\bar{\chi} \equiv (\partial\bar{n}/\partial\mu)_{\mu=0}$ is the equilibrium charge susceptibility, and $D = \bar{\sigma}/\bar{\chi}$ is the charge diffusion constant. The symmetrized two-point functions can be evaluated provided one specifies the dynamics of τ^{ij} and r^i. In our case they are taken as Gaussian fields with [26]

$$\langle r_i(x) r_j(y)\rangle = 2\bar{T}\bar{\sigma}\delta_{ij}\delta(x-y)\,, \tag{2.6a}$$

$$\langle \tau_{ij}(x) \tau_{kl}(y)\rangle = 2\bar{T}G_{ijkl}\delta(x-y)\,, \tag{2.6b}$$

where $G_{ijkl} = 2\bar{\eta}S_{{\rm T}\,ijkl} + d\bar{\zeta}S_{{\rm L}\,ijkl}$ as above. Choosing $\mathbf{k}$ along $\mathbf{z}$ gives the usual Kubo formula in terms of the symmetrized function of T_{xy},

$$G_{xy,xy}^S(\omega, \mathbf{k}) = 2\bar{T}\bar{\eta}\,.$$

The above noise average can be represented with a Gaussian functional integral,

$$\langle \dots \rangle = \int_{i\leqslant j} D\tau_{ij}\, Dr_k\, e^{-S_\xi[r,\tau]} \dots$$

where

$$S_\xi[r,\tau] = \frac{1}{2}\int dt\, d^dx \left(\frac{r_i\bar{\sigma}^{-1}r_i}{2\bar{T}} + \frac{\tau_{ij}G_{ijkl}^{-1}\tau_{kl}}{2\bar{T}}\right)\,,$$

and $G_{ijkl}^{-1} = \frac{1}{2\bar{\eta}}S_{{\rm T}\,ijkl} + \frac{1}{d\bar{\zeta}}S_{{\rm L}\,ijkl}$. The action S_ξ is positive definite (except for the trivial configuration $r_i = 0$, $\tau_{ij} = 0$), and in the case of linear fluctuations does not depend on the hydrodynamic fields ϕ. Integrating out the noise fields r_i and τ_{ij} in eq. (2.4) gives

$$Z = \int D\phi\, D\tilde{\phi}\, e^{-S_{\rm eff}[\phi,\tilde{\phi}]}\,,$$

where

$$S_{\rm eff}[\phi,\tilde{\phi}] = \int dt\, d^dx \left[i\,\partial_\mu\tilde{\phi}_\nu\, T_{\rm cl}^{\mu\nu} + \bar{T}\partial_i\tilde{\phi}_j G_{ijkl}\,\partial_k\tilde{\phi}_l + i\,\partial_\mu\tilde{\phi}_{d+2}\, J_{\rm cl}^\mu + \bar{T}\sigma\delta_{ij}\partial_i\tilde{\phi}_{d+2}\,\partial_j\tilde{\phi}_{d+2}\right] \tag{2.7}$$

is the effective action for linear viscous hydrodynamics [4]. The hydrodynamic fields are $\phi = (v_i, \delta T, \mu)$, and the stress tensor and the current are given by the classical linear constitutive relations, e.g. $T_{\rm cl}^{ij} = \delta_{ij}(\bar{p} + \bar{s}\,\delta T) - G^{ijkl}\partial_k v_l$. The equilibrium correlation functions of $T^{\mu\nu}$ and J^μ are straightforwardly evaluated using the effective action eq. (2.7). As expected, the effective action is local, but not real. Integrating out auxiliary fields $\tilde{\phi}_a$ will give rise to an action which is real, but non-local. If the auxiliary fields are rescaled as $\tilde{\phi} \to \bar{T}\tilde{\phi}$, the action can be written as $S_{\rm eff} = (1/\bar{T})\int \dots$, signifying that $\bar{T}$ determines the strength of thermal fluctuations.

2.3 The covariant form

Beyond the linear approximation, we use $\phi = (u^\lambda, T, \mu)$ as the hydrodynamic fields. The quadratic effective action eq. (2.7) naïvely generalizes to

$$S_{\text{eff}}[\phi, \tilde\phi] = \int dt\, d^d x \left[i\, \partial_\mu \tilde\phi_\nu\, T_{\text{cl}}^{\mu\nu} + T \partial_\mu \tilde\phi_\nu G^{\mu\nu\alpha\beta}\, \partial_\alpha \tilde\phi_\beta + i\, \partial_\mu \tilde\phi_5\, J_{\text{cl}}^\mu + T\sigma \Delta^{\mu\nu} \partial_\mu \tilde\phi_5 \partial_\nu \tilde\phi_5 \right]. \tag{2.8}$$

Here $T_{\text{cl}}^{\mu\nu}$ and J_{cl}^μ are given by eq. (2.5), $\tilde\phi_5 \equiv \tilde\phi_{d+2}$, and the constraint $u^2 = -1$ is implied. The effective action is not real, nor should it be. The action is invariant under complex conjugation combined with $\tilde\phi_\mu \to -\tilde\phi_\mu$, $\tilde\phi_5 \to -\tilde\phi_5$.

The auxiliary fields $\tilde\phi_\mu$ and $\tilde\phi_5$ are derivatively coupled, suggesting a Goldstone-boson interpretation, similar to ref. [17]. The Noether currents of the shift symmetry are

$$T^{\mu\nu} = T_{\text{cl}}^{\mu\nu} - 2iTG^{\mu\nu\alpha\beta}\partial_\alpha \tilde\phi_\beta\,, \tag{2.9a}$$

$$J^\mu = J_{\text{cl}}^\mu - 2iT\sigma \Delta^{\mu\nu} \partial_\nu \tilde\phi_5\,, \tag{2.9b}$$

and correspond to the full energy-momentum tensor and the current.

While this development is suggestive, we have not derived it, and in fact there are some problems, which we now enumerate:

- The relativistic version of eq. (2.6) is

$$\langle r_\mu(x) r_\nu(y) \rangle = 2T\sigma \Delta_{\mu\nu}\delta(x-y)\,, \tag{2.10a}$$

$$\langle \tau_{\mu\nu}(x)\tau_{\alpha\beta}(y) \rangle = 2TG_{\mu\nu\alpha\beta}\delta(x-y)\,, \tag{2.10b}$$

  indicating that the noise is not independent of ϕ, contrary to the assumptions made before eq. (2.2). This can be fixed by rescaling the noise fields r_μ and $\tau_{\mu\nu}$ by the "square root" of the coefficients appearing in the right-hand side of eq. (2.10). The conservation equations $\partial_\mu(T_{\text{cl}}^{\mu\nu} + \tau^{\mu\nu}) = 0$ and $\partial_\mu(J_{\text{cl}}^\mu + r^\mu) = 0$ become stochastic differential equations with multiplicative noise. Ambiguities in defining such stochastic differential equations must then be resolved in establishing the form of the functional integral.

- If an effective action is to be derived from a stochastic differential equation, there has to be the corresponding Jacobian, as indicated in eq. (2.3). In linear hydrodynamics, the Jacobian can be dropped because it is field-independent, which is not the case in non-linear hydrodynamics. Such a Jacobian is ignored in eq. (2.8).

We now turn to the above points.

3 The effective action

3.1 The noise

In order to arrive at the effective action, one can start from classical hydrodynamics augmented with noise terms. Rather than using the correlations eq. (2.10), one has to redefine

the noise so that the noise action does not depend on the hydrodynamic variables. To do so, we introduce a noise field ξ_μ with $\langle \xi_\alpha(x)\xi_\beta(x')\rangle = \eta_{\alpha\beta}\delta(x-x')$, and a symmetric noise field $\xi_{\mu\nu}$ with $\langle \xi_{\mu\nu}(x)\xi_{\alpha\beta}(x')\rangle = \frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\nu\alpha}\eta_{\mu\beta})\delta(x-x')$. The corresponding contributions to the energy-momentum tensor and the current are

$$\tau_{\mu\nu} = \sqrt{2T}\,G^{1/2}_{\mu\nu\alpha\beta}\,\xi^{\alpha\beta}\,, \qquad r_\mu = \sqrt{2T\sigma}\,\Delta_{\mu\nu}\,\xi^\nu\,, \tag{3.1}$$

where $G^{1/2}_{\mu\nu\alpha\beta} = \sqrt{2\eta}\,S_{\mathrm{T}\,\mu\nu\alpha\beta} + \sqrt{d\zeta}\,S_{\mathrm{L}\,\mu\nu\alpha\beta}$ satisfies $G^{1/2}_{\mu\nu\alpha\beta}G^{1/2\,\alpha\beta\rho\sigma} = G_{\mu\nu}{}^{\rho\sigma}$, and $G_{\mu\nu\rho\sigma}$ is defined below eq. (2.5). The construction of the effective action can now proceed as described in section 2.1, with[3]

$$S_{\mathrm{n}}[\xi] = \frac{1}{2}\int dt\,d^d x\,(\xi_\mu \xi^\mu + \xi_{\mu\nu}\xi^{\mu\nu})\,.$$

Ignoring the Jacobian in eq. (2.3) and integrating out ξ_μ and $\xi_{\mu\nu}$ gives the effective action eq. (2.8).

3.2 The discretization

Equation (2.1) as written is ambiguous because Langevin equations must generally be written as discrete-time equations, and the continuous time limit can depend on the manner of the discretization. This is particularly true for the case of multiplicative noise (see for instance ref. [27]). The essential feature of hydrodynamic equations is that current and stress conservation must be exact statements. For a discretization with time spacing Δ_t and spatial spacing Δ_x, and with ϕ defined on sites, we believe this should be achieved by defining J^μ and $T^{\mu\nu}$ on the μ-link, so that

$$\partial_\mu J^\mu(x) \equiv \sum_\mu \frac{J^\mu(x+\hat\mu\Delta_\mu/2) - J^\mu(x-\hat\mu\Delta_\mu/2)}{\Delta_\mu}\,, \tag{3.2}$$

and similarly for $\partial_\mu T^{\mu\nu}$.

Current conservation at a site is the vanishing of the signed sum of currents onto and off of that site, which implements conservation exactly. This is illustrated in figure 1. The current on a link should be defined using the average of ϕ at each end of the link, e.g.,[4]

$$J^0(x+\hat t\Delta_t/2) = \left(\frac{n(x)+n(x+\hat t\Delta_t)}{2}\right)\left(\frac{u^0(x)+u^0(x+\hat t\Delta_t)}{2}\right) - \dots .$$

We will assume that this is the prescription for defining the discrete equations of motion. The subsequent steps we describe should then be performed on these spacetime-discretized equations, with the continuum limit taken at the end (if at all). We expect this procedure to resolve discretization ambiguities, and we implicitly assume that it has been used in the following.

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[3]Note that ξ_μ and $\xi_{\mu\nu}$ contain negative-norm components. We implicitly assume that this is handled through continuation of some noise components to imaginary values, so the functional weight bounds the integrals. The projection operators ensure that wrong-norm components are never physically relevant.

[4]One feature of this definition is that particle number and 4-momentum density are not defined on sites, but on the temporal links between sites. Charge conservation means that J^0 summed over temporal links at time $t-\Delta_t/2$ equals J^0 summed over temporal links at time $t+\Delta_t/2$.
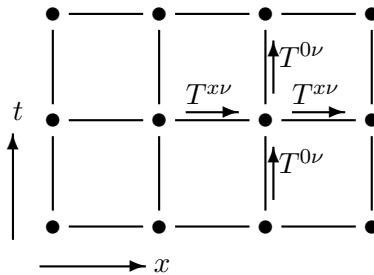
Figure 1. How $T^{\mu\nu}$ should be discretized. $T^{\mu\nu}$ is the flux of P^ν from one site to the neighboring site in the μ-direction. Stress-energy conservation at a site is the equality of the sum of all incoming P^ν contributions and the sum of all outgoing P^ν contributions.

3.3 The Jacobian

We next turn to the incorporation of the Jacobian of eq. (2.3) into the effective action. The Jacobian is $J = \det J_{ab}(\phi, \xi)$, where $J_{ab} = \delta(E^a + f^a)/\delta\phi_b$ is a differential operator linear in ξ. The Jacobian can be exponentiated by using ghost fields $\bar\psi_a$, ψ_a as

$$J = \int D\bar\psi \, D\psi \; e^{-S_{\det}} \, ,$$

where $S_{\det} = \int dt \, d^d x \, \bar\psi_a J_{ab}\psi_b$. As J_{ab} is linear in ξ, this action can be written as

$$S_{\det} = \int dt \, d^d x \left(\xi_\mu F^\mu(\phi, \bar\psi, \psi) + \xi_{\alpha\beta} F^{\alpha\beta}(\phi, \bar\psi, \psi) + \bar\psi_a \frac{\delta E^a}{\delta\phi_b} \psi_b \right) ,$$

where $F^\mu(\phi, \bar\psi, \psi)$ and $F^{\alpha\beta}(\phi, \bar\psi, \psi)$ are straightforward to evaluate, for given a choice of the hydro variables ϕ_a. Choosing $\phi = (u, T, \mu)$, we have

$$F^\mu = -\partial_\lambda \bar\psi_{d+2} \left[\frac{\delta C^{\lambda\mu}}{\delta u_\nu} \psi_\nu + \frac{\delta C^{\lambda\mu}}{\delta T} \psi_{d+1} + \frac{\delta C^{\lambda\mu}}{\delta\mu} \psi_{d+2} \right] ,$$

$$F^{\alpha\beta} = -\partial_\lambda \bar\psi_\mu \left[\frac{\delta C^{\lambda\mu\alpha\beta}}{\delta u_\nu} \psi_\nu + \frac{\delta C^{\lambda\mu\alpha\beta}}{\delta T} \psi_{d+1} + \frac{\delta C^{\lambda\mu\alpha\beta}}{\delta\mu} \psi_{d+2} \right] ,$$

with $C^{\mu\nu} \equiv \sqrt{2T\sigma}\Delta^{\mu\nu}$, $C_{\mu\nu\alpha\beta} \equiv \sqrt{2T}G^{1/2}_{\mu\nu\alpha\beta}$. Integrating out the noise ξ_μ and $\xi_{\mu\nu}$ gives the effective Lagrangian

$$\mathcal{L}_{\text{eff}} = i\,\partial_\mu\tilde\phi_\nu \, T^{\mu\nu}_{\text{cl}} - \frac{1}{2}(F_{\alpha\beta} + iP_{\alpha\beta})(F^{\alpha\beta} + iP^{\alpha\beta})$$
$$+ i\,\partial_\mu\tilde\phi_5 \, J^\mu_{\text{cl}} - \frac{1}{2}(F_\mu + iP_\mu)(F^\mu + iP^\mu) + \bar\psi_a \frac{\delta E^a}{\delta\phi_b}\psi_b \, , \tag{3.3}$$

where $P^\mu \equiv \sqrt{2T\sigma}\,\Delta^{\mu\lambda}\partial_\lambda\tilde\phi_5$, and $P^{\mu\nu} \equiv \sqrt{2T}\,G^{1/2\,\mu\nu\lambda\sigma}\partial_\lambda\tilde\phi_\sigma$.

4 Conclusions

Our proposal for the effective action in stochastic relativistic hydrodynamics is

$$S_{\text{eff}}[\phi, \tilde\phi, \dots] = \int dt \, d^d x \left[i\partial_\mu\tilde\phi_\nu \, T^{\mu\nu}_{\text{cl}} + TG^{\mu\nu\lambda\sigma}\partial_\mu\tilde\phi_\nu \, \partial_\lambda\tilde\phi_\sigma + i\partial_\mu\tilde\phi_5 \, J^\mu_{\text{cl}} + T\sigma\Delta^{\mu\nu}\partial_\mu\tilde\phi_5 \, \partial_\nu\tilde\phi_5 + \dots \right]$$
$$\tag{4.1}$$

where the classical energy-momentum tensor and the current are given by eq. (2.5). The hydrodynamic variables are $\phi^a = (u^\lambda, T, \mu)$, and the $\tilde{\phi}$'s are the corresponding auxiliary fields. The constraint $u^2 = -1$ is implied. The dots in eq. (4.1) denote ghost terms. The derivation of eq. (4.1) parallels existing work in simpler hydrodynamic systems. Several comments are in order.

- Evaluating the 1PI effective action in a given background in the theory eq. (4.1) should give rise to classical hydrodynamic equations plus loop corrections. Among other things, the loop corrections will renormalize the transport coefficients η, ζ, and σ, similar to what happens in simpler systems [7, 28].

- In the simplest case of a scale-invariant uncharged fluid in thermal equilibrium, the effective action has only three parameters: equilibrium temperature T, equilibrium entropy density s, and equilibrium shear viscosity η. There is only one combination, $\lambda \equiv (T/s^{1/d})\,(\eta/s)^{-1}$, which is dimensionless in the natural units $c = 1$ (the effective theory is classical, so there is no $\hbar$). In a large-N gauge theory, $\lambda \to 0$ as $N \to \infty$. One expects that fluctuation corrections will be suppressed by a positive power of λ, as happens in simpler models.

- We have only taken into account first-order gradient terms in the hydrodynamic equations of motion eq. (2.5). In principle, one can add second-order terms to the constitutive relations and repeat the derivation. Even without doing so, it is natural to expect that higher-order terms will be "generated" by the hydrodynamic loop corrections.

- We have assumed a particular convention for an off-equilibrium definition of hydrodynamic variables, the Landau-Lifshitz "frame". The correlation functions of the energy-momentum tensor and the current are independent of our choosing one or another frame, and it is desirable to have an effective action formulation where frame-invariance is manifest.

- We have checked that, for smooth fields, the discrete-space expression reproduces the continuum expressions we have written up to terms with 2 extra derivatives. But the loop expansion under this effective action will likely encounter UV divergences for which the discretization will matter. It is not clear to us how to deal with UV divergences in the effective theory. Also, the discretization we propose does not manifestly preserve the symmetry of $T^{\mu\nu} = T^{\nu\mu}$. It is not clear to us whether this could cause any problems.

- The ghost part of the action has unusual properties; in particular, the F^2 terms in eq. (3.3) are quartic in the ghosts, that is, there are nonlinear ghost interactions. The ghost part of the action could presumably be made quadratic by introducing more auxiliary fields, similar to what is done in other interacting fermion models.

- We have neglected the coupling of the hydrodynamic degrees of freedom to external sources. While it is straightforward to couple the action eq. (4.1) to the external

gauge field and the metric, one would like to have an effective action which provides us with the full set of n-point real-time correlation functions. This means that the action needs to be coupled to *two* sets of external sources, corresponding to the two branches of the Schwinger-Keldysh contour.

We plan to return to the above points in the future.

Acknowledgments

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