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Towards Field Theory Amplitudes From the Cohomology of Pure Spinor Superspace

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A simple BRST-closed expression for the color-ordered super-Yang-Mills 5-point amplitude at tree-level is proposed in pure spinor superspace and shown to be BRST-equivalent to the field theory limit of the open superstring 5-pt amplitude. It is manifestly cyclic invariant and each one of its five terms can be associated to the five Feynman diagrams which use only cubic vertices. Its form also suggests an empirical method to find superspace expressions in the cohomology of the pure spinor BRST operator for higher-point amplitudes based on their kinematic pole structure. Using this method, Ansätze for the 6- and 7-point 10D super-Yang-Mills amplitudes which map to their 14 and 42 color-ordered diagrams are conjectured and their 6- and 7-gluon expansions are explicitly computed.

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1. Introduction

As Parke and Taylor have shown for MHV amplitudes [1], it is sometimes possible to obtain simple expressions for seemingly complicated Yang-Mills amplitudes in four space-time dimensions. Using the pure spinor formalism [2] and its pure spinor superspace [3] (see also [4]) it will be proved that the tree-level color-ordered five-point super-Yang-Mills amplitude in ten dimensions can be written simply as

$$\mathcal{A}_5(1,2,3,4,5) = \frac{\langle L_{45} L_{12} V^3 \rangle}{s_{45} s_{12}} + \text{cyclic}(12345), \tag{1.1}$$

where V^j is the unintegrated massless vertex operator and L_{ij} is related to the OPE of a unintegrated and an integrated vertex operator in a way to be defined below.

It will also be suggested that higher-point amplitudes might have simple forms like the above, as there seems to be a direct correspondence between superspace expressions and Feynman diagrams which use only cubic vertices as in the arguments of [5]. Using the empirical method described in subsection 3.1, it will be argued that the super-Yang-Mills 6- and 7-point color-ordered amplitudes are given by

$$\mathcal{A}_6(1,2,3,4,5,6) = \frac{\langle L_{12} L_{34} L_{56} \rangle}{3 s_1 s_3 s_5} + \frac{\langle L_{23} L_{45} L_{61} \rangle}{3 s_2 s_4 s_6} \tag{1.2}$$

$$+\frac{1}{2}\frac{\langle T_{123}}{s_1 t_1}\Big(\frac{V^4 L_{56}}{s_5} + \frac{L_{45} V^6 \rangle}{s_4}\Big) - \frac{1}{2}\frac{\langle T_{126}}{s_1 t_3}\Big(\frac{V^3 L_{45}}{s_4} + \frac{L_{34} V^5 \rangle}{s_3}\Big) + \text{cyclic}(1\ldots 6)$$

and

$$\mathcal{A}_7(1,2,3,4,5,6,7) = +\frac{\langle T_{231} L_{45} L_{67} \rangle}{s_2 t_1 s_4 s_6} + \frac{\langle T_{123} T_{564} V_7 \rangle}{s_1 t_1 s_5 t_4} + \frac{\langle T_{127} T_{345} V_6 \rangle}{s_1 t_7 s_3 t_3} \tag{1.3}$$

$$-\frac{\langle T_{123} T_{456} V_7 \rangle}{s_1 t_1 s_4 t_4} - \frac{\langle T_{127} T_{453} V_6 \rangle}{s_1 t_7 s_4 t_3} - \frac{\langle T_{123} L_{45} L_{67} \rangle}{s_1 t_1 s_4 s_6} + \text{cyclic}(1\ldots 7)$$

where T_{ijk} is related to the OPE of one unintegrated and two integrated vertices in a way to be defined below and $s_1, \ldots, s_6$ and $t_1, \ldots, t_3$ ($s_1, \ldots, s_7$ and $t_1, \ldots, t_7$) are the 6-point (7-point) generalized Mandelstam variables of [6,7]. Using a computer program [8], the 6- and 7-gluon expansions of (1.2) and (1.3) are computed in the Appendix B and their form lend support to their correctness[2].

[2] In the amplitude computations of [6,7] the results were written in the 4D helicity formalism language, so a 10D comparison of results is not straightforward. However a comparison to the result [9] should be made [10].

Furthermore, given that the tree-level SYM 4-point amplitude can be written as [11]

$$\mathcal{A}_4(1,2,3,4) = \frac{1}{s_{12}}\langle L_{12}V^3V^4\rangle + \frac{1}{s_{41}}\langle L_{41}V^2V^3\rangle, \tag{1.4}$$

it is pointed out that the four-point Jacobi-like Bern-Carrasco-Johansson kinematic identity [5] becomes

$$\langle L_{\{12}V_{3\}}V_4\rangle = 0, \tag{1.5}$$

where $\{ijk\}$ means a sum over cyclic permutations of (ijk). For the five-point amplitude (1.1), the generalized BCJ identities of [12,13] hold in the form of

$$-\frac{L_{45}}{s_{45}}L_{\{12}V_{3\}} + \frac{L_{42}}{s_{24}}L_{\{13}V_{5\}} - \frac{L_{12}}{s_{12}}L_{\{34}V_{5\}} + \frac{L_{51}}{s_{51}}L_{\{23}V_{4\}} = 0, \tag{1.6}$$

etc. It is well-known that there are powerful four-dimensional methods to compute scattering amplitudes recursively (see [14] and references therein). The hints of a simplified ten-dimensional parametrization of field theory tree-level amplitudes using pure spinors[3] seem to suggest that there might be similar methods in a ten-dimensional pure spinor superspace setup – which is desirable since there is no need to differentiate between MHV and NMHV contributions as in the four-dimensional methods.

This paper is organized as follows. In section 2 an ansatz will be given for the tree-level five-point SYM amplitude by analogy with the structure of the known four-point amplitude. In section 3 the five-point ansatz will be derived from the field theory limit of a BRST-equivalent expression of the superstring amplitude computed in [11]. In subsection 3.1 an empirical method to write down similar Ansätze for higher-point amplitudes is presented, and expressions for the 6- and 7-point super-Yang-Mills amplitudes in ten-dimensional space-time are conjectured. In Appendix A the BCJ kinematic relations and its generalization [12,13] are written down using the pure spinor representations of the previous sections. Finally, in Appendix B the first few terms of the (rather long) 5-, 6- and 7-gluon expansions from (1.1), (1.2) and (1.3) are written down (the full expansions can be easily generated with a computer using [8] or other methods).

[3] It was suggested a long time ago that pure spinors simplify the description of super-Yang-Mills and supergravity theories [15]. The superspace results obtained with the pure spinor formalism seem to realize those expectations.

2. The field theory 5-pt tree-level amplitude from pure spinor cohomology

The color-ordered tree-level n-point amplitudes are denoted by $\mathcal{A}(1,\ldots,n) = \langle A(1,\ldots,n)\rangle$. The OPE between the unintegrated $V^i(z) = (\lambda A^i)$ and the integrated vertex operator[4] $U^j(z) = \partial\theta^\alpha A^j_\alpha + \Pi^m A^j_m + (dW^j) + \frac{1}{2}\mathcal{F}^j_{mn}N^{mn}$ is given by $V^i(z)U^j(w) \to \frac{\tilde{L}_{ij}}{z-w}$, with [18]

$$\tilde{L}_{ij}(\theta) = A^i_m(\lambda\gamma^m W^j) + (\lambda A^i)(k^i \cdot A^j), \tag{2.1}$$

where $[A_\alpha, A_m, W^\alpha, \mathcal{F}_{mn}]$ are the super-Yang-Mills superfields in ten dimensions satisfying the equations of motion [4,19,17],

$$Q\mathcal{F}_{mn} = 2k_{[m}(\lambda\gamma_{n]}W),\ QW^\alpha = \frac{1}{4}(\lambda\gamma^{mn})^\alpha\mathcal{F}_{mn},\ QA_m = (\lambda\gamma_m W) + k_m(\lambda A),\ QV = 0, \tag{2.2}$$

where $Q = \lambda^\alpha D_\alpha$ is the pure spinor BRST operator. Using (2.2) it follows that

$$Q\tilde{L}_{ij} = -s_{ij}(\lambda A^i)(\lambda A^j), \quad Q(A^i \cdot A^j) = \tilde{L}_{ij} + \tilde{L}_{ji} \equiv 2\tilde{L}_{(ij)} \tag{2.3}$$

where $s_{ij} = (k^i \cdot k^j)$. Using (2.3) and defining $L_{ij} = 1/2(\tilde{L}_{ij} - \tilde{L}_{ji})$ the superfield $\tilde{L}_{ij}$ can be written as[5]

$$\tilde{L}_{ij} = L_{ij} + \frac{1}{2}Q(A^i \cdot A^j). \tag{2.4}$$

The massless 4-point super-Yang-Mills amplitude obtained from the field theory limit of the open string amplitude is given by [11]

$$\mathcal{A}(1,2,3,4) = \frac{1}{s_{12}}\langle\tilde{L}_{12}V^3V^4\rangle + \frac{1}{s_{41}}\langle\tilde{L}_{41}V^2V^3\rangle = \frac{1}{s_{12}}\langle L_{12}V^3V^4\rangle + \frac{1}{s_{41}}\langle L_{41}V^2V^3\rangle \tag{2.5}$$

where we used that $\langle Q(A^i \cdot A^j)V^kV^l\rangle = 0$, as can be checked by integrating the BRST charge by parts. The other sub-amplitudes are obtained from (2.5) by relabeling,

$$\begin{aligned}
\mathcal{A}(1,2,3,4) &= \frac{1}{s_{12}}\langle L_{12}V^3V^4\rangle + \frac{1}{s_{41}}\langle L_{41}V^2V^3\rangle \\
\mathcal{A}(1,3,4,2) &= -\frac{1}{s_{13}}\langle L_{13}V^2V^4\rangle - \frac{1}{s_{12}}\langle L_{12}V^3V^4\rangle \\
\mathcal{A}(1,4,2,3) &= -\frac{1}{s_{14}}\langle L_{41}V^2V^3\rangle + \frac{1}{s_{13}}\langle L_{13}V^2V^4\rangle.
\end{aligned} \tag{2.6}$$

It is easy to check that the amplitudes in (2.6) satisfy $QA(i,j,k,l) = 0$.

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[4] For background material in the pure spinor formalism, see [16,17].

[5] I thank Dimitrios Tsimpis for suggesting the separation of the BRST-trivial part of $\tilde{L}_{ij}$.

As emphasized in [5], a color-ordered 5-point tree-level amplitude consists of five diagrams with purely cubic vertices specifying the poles,

$$\mathcal{A}(1,2,3,4,5) = \frac{n_1}{s_{45}s_{12}} + \frac{n_2}{s_{51}s_{23}} + \frac{n_3}{s_{12}s_{34}} + \frac{n_4}{s_{23}s_{45}} + \frac{n_5}{s_{34}s_{51}}. \tag{2.7}$$

As the BRST variation of L_{ij} is proportional to s_{ij}, the idea now is to construct a pure spinor superspace expression using L_{ij} and L_{kl} in the numerators of the terms containing poles in s_{ij} and s_{kl}, in such a way as to obtain a BRST-closed expression. It is straightforward to see that the amplitudes

$$\begin{aligned}
\mathcal{A}(1,2,3,4,5) &= \frac{\langle L_{45}L_{12}V^3\rangle}{s_{45}s_{12}} + \frac{\langle L_{51}L_{23}V^4\rangle}{s_{51}s_{23}} + \frac{\langle L_{12}L_{34}V^5\rangle}{s_{12}s_{34}} + \frac{\langle L_{23}L_{45}V^1\rangle}{s_{23}s_{45}} + \frac{\langle L_{34}L_{51}V^2\rangle}{s_{34}s_{51}}\\
\mathcal{A}(1,3,2,4,5) &= \frac{\langle L_{45}L_{13}V^2\rangle}{s_{45}s_{13}} - \frac{\langle L_{51}L_{23}V^4\rangle}{s_{51}s_{23}} - \frac{\langle L_{13}L_{42}V^5\rangle}{s_{13}s_{24}} - \frac{\langle L_{23}L_{45}V^1\rangle}{s_{23}s_{45}} - \frac{\langle L_{42}L_{51}V^3\rangle}{s_{24}s_{51}}\\
\mathcal{A}(1,4,3,2,5) &= \frac{\langle L_{25}L_{14}V^3\rangle}{s_{25}s_{14}} + \frac{\langle L_{34}L_{51}V^2\rangle}{s_{51}s_{43}} + \frac{\langle L_{23}L_{14}V^5\rangle}{s_{14}s_{32}} + \frac{\langle L_{25}L_{34}V^1\rangle}{s_{43}s_{25}} + \frac{\langle L_{51}L_{23}V^4\rangle}{s_{32}s_{51}}\\
\mathcal{A}(1,3,4,2,5) &= \frac{\langle L_{25}L_{13}V^4\rangle}{s_{25}s_{13}} - \frac{\langle L_{34}L_{51}V^2\rangle}{s_{51}s_{34}} + \frac{\langle L_{13}L_{42}V^5\rangle}{s_{13}s_{42}} - \frac{\langle L_{25}L_{34}V^1\rangle}{s_{34}s_{25}} + \frac{\langle L_{42}L_{51}V^3\rangle}{s_{42}s_{51}}\\
\mathcal{A}(1,2,4,3,5) &= \frac{\langle L_{35}L_{12}V^4\rangle}{s_{35}s_{12}} + \frac{\langle L_{42}L_{51}V^3\rangle}{s_{51}s_{43}} - \frac{\langle L_{12}L_{34}V^5\rangle}{s_{12}s_{43}} + \frac{\langle L_{35}L_{42}V^1\rangle}{s_{42}s_{35}} - \frac{\langle L_{34}L_{51}V^2\rangle}{s_{43}s_{51}}\\
\mathcal{A}(1,4,2,3,5) &= \frac{\langle L_{35}L_{14}V^2\rangle}{s_{35}s_{14}} - \frac{\langle L_{42}L_{51}V^3\rangle}{s_{51}s_{24}} - \frac{\langle L_{23}L_{14}V^5\rangle}{s_{14}s_{23}} - \frac{\langle L_{35}L_{42}V^1\rangle}{s_{24}s_{35}} - \frac{\langle L_{51}L_{23}V^4\rangle}{s_{23}s_{51}}
\end{aligned} \tag{2.8}$$

are BRST-closed. One can also check that all sub-amplitudes in (2.8) are related to $A(1,2,3,4,5)$ by index relabeling, taking into account the antisymmetry of L_{ij} and its fermionic nature. The signs in (2.8) precisely match the ones presented in equation (4.5) of [5], so one can identify

$$\begin{gathered}
n_1 = \langle L_{45}L_{12}V^3\rangle,\ n_2 = \langle L_{51}L_{23}V^4\rangle,\ n_3 = \langle L_{12}L_{34}V^5\rangle,\ n_4 = \langle L_{23}L_{45}V^1\rangle\\
n_5 = \langle L_{34}L_{51}V^2\rangle,\ n_6 = \langle L_{25}L_{14}V^3\rangle,\ n_7 = \langle L_{23}L_{14}V^5\rangle,\ n_8 = \langle L_{25}L_{34}V^1\rangle\\
n_9 = \langle L_{25}L_{13}V^4\rangle,\ n_{10} = \langle L_{13}L_{42}V^5\rangle,\ n_{11} = \langle L_{42}L_{51}V^3\rangle,\ n_{12} = \langle L_{35}L_{12}V^4\rangle\\
n_{13} = \langle L_{35}L_{42}V^1\rangle,\quad n_{14} = \langle L_{35}L_{14}V^2\rangle,\quad n_{15} = \langle L_{45}L_{13}V^2\rangle.
\end{gathered} \tag{2.9}$$

As will be mentioned in the appendix, the above "solution" for the n_i's of [5] do not satisfy the strict Bern-Carrasco-Johansson (BCJ) kinematic identities, but they do satisfy the generalized BCJ's of [12,13]. As explained in [12,13], a general parametrization of the sub-amplitudes in terms of poles does not necessarily satisfy the BCJ Jacobi-like identities of [5]. They must however satisfy "generalized BCJ identities", for which the original BCJ relations are just one out of many possible solutions.

The amplitudes in (2.8) will now be obtained from the field theory limit of a BRST-equivalent expression of the pure spinor superstring amplitude computed in [11].

3. First principles derivation of the 5-pt ansatz (2.8)

The massless 5-point open superstring amplitude is given by [11]

$$\mathcal{A}_5(1,2,3,4,5) = L_{2131}V^4V^5K_1 - L_{2134}V^5K_2 - L_{2434}V^1V^5K_1' + L_{2431}V^5K_3$$

$$-L_{2331}V^4V^5K_5 - L_{2334}V^1V^5K_4' + L_{2314}V^1V^4V^5K_6, \tag{3.1}$$

where K_j and K_j' denote integrals which satisfy [20]

$$s_{34}K_2 = s_{13}K_1 + s_{23}K_4, \quad s_{24}K_3 = s_{12}K_1 - s_{23}K_5, \quad K_1 = K_4 - K_5$$

$$s_{12}K_2 = s_{24}K_1' + s_{23}K_4', \quad s_{13}K_3 = s_{34}K_1' - s_{23}K_5', \quad K_1' = K_4' - K_5'$$

$$(1+s_{23})K_6 = s_{34}K_4' - s_{13}K_5 = s_{12}K_4 - s_{24}K_5'. \tag{3.2}$$

and the various L_{ijkl} have the following pure spinor superspace expressions

$$L_{2131} = +\langle L_{12}((k^1+k^2)\cdot A^3)\rangle + \langle(\lambda\gamma^m W^3)\big[A^1_m(k^1\cdot A^2) + A^{1\,n}\mathcal{F}^2_{mn} - (W^1\gamma_m W^2)\big]\rangle$$

$$+s_{12}\langle\big[(A^1W^3)V^2 - (A^2W^3)V^1\big]V^4V^5\rangle + (s_{13}+s_{23})\langle(A^1W^2)V^3V^4V^5\rangle,$$

$$L_{2134}V^5 = \tilde{L}_{12}\tilde{L}_{43}V^5 + s_{12}\langle(A^4W^3)V^1V^2V^5\rangle + s_{34}\langle(A^1W^2)V^3V^4V^5\rangle$$

$$L_{2314}V^1V^4V^5 = (1+s_{23})\langle\big[(A^2W^3) + (A^3W^2) - (A^2\cdot A^3)\big]V^1V^4V^5\rangle. \tag{3.3}$$

Furthermore, the following BCJ identities [11]

$$L_{2331}V^4V^5 = L_{3121}V^4V^5 - L_{2131}V^4V^5, \quad L_{2334}V^1V^5 = L_{3424}V^1V^5 - L_{2434}V^1V^5 \tag{3.4}$$

can be used to obtain $L_{2331}V^4V^5$ and $L_{2334}V^1V^5$ from (3.3) (the other L's are obtained by simple relabeling of the above ones). Using (3.2) one can show that all terms of the form $s_{ij}\langle(A^kW^l)V^mV^nV^p\rangle$ appearing in (3.3) and (3.4) cancel out from the amplitude (3.1). As an illustration, the terms containing (A^1W^2) are

$$\big[(s_{13}+s_{23})K_1 - s_{34}K_2 + s_{23}K_5\big](A^1W^2)V^3V^4V^5 =$$

$$= (s_{13}K_1 - s_{34}K_2)(A^1W^2)V^3V^4V^5 + s_{23}(K_1+K_5)(A^1W^2)V^3V^4V^5$$

$$= -s_{23}K_4(A^1W^2)V^3V^4V^5 + s_{23}K_4(A^1W^2)V^3V^4V^5 = 0.$$

All other cases can be similarly proved. From now on when we refer to L_{ijkl} it means the kinematic factors of (3.3) without those terms. Using the integral relation for K_6 and the expression for L_{2314},

$$L_{2314}V^4V^5K_6 = -(1+s_{23})K_6(A^2\cdot A^3)V^4V^5 = (s_{13}K_5 - s_{34}K_4')(A^2\cdot A^3)V^4V^5$$

and therefore the amplitude (3.1) becomes

$$\mathcal{A}_5(1,2,3,4,5) = L_{2131}V^4V^5K_1 - L_{2134}V^5K_2 - L_{2434}V^4V^5K_1' + L_{2431}V^5K_3$$

$$-(L_{2331} - s_{13}(A^2\cdot A^3)V^1)V^4V^5K_5 - (L_{2334} - s_{34}(A^2\cdot A^3)V^4)V^1V^5K_4'. \qquad (3.5)$$

If one defines[6]

$$L_{ijkj} = K_{ijkj} + S_{ijkj} \qquad (3.6)$$

where

$$S_{ijkj} = \frac{1}{2}s_{ji}((A^i\cdot A^k)V^j - (A^j\cdot A^k)V^i) - \frac{1}{2}(s_{jk}+s_{ik})(A^j\cdot A^i)V^k, \qquad (3.7)$$

it is then a straightforward exercise to use the relations (3.2) and the definition (3.6) together with the kinematic factors of (3.3) to show that (3.5) becomes

$$\mathcal{A}_5 = (K_{2131}K_1 - K_{2331}K_5)V^4V^5 - (K_{2434}K_1' + K_{2334}K_4')V^1V^5 + (L_{42}L_{13}K_3 + L_{12}L_{34}K_2)V^5 \qquad (3.8)$$

where we used that (and similarly for other labels)

$$L_{2331} = L_{3121} - L_{2131} = K_{3121} - K_{2131} - S_{2131} + S_{3121}$$

$$L_{2134}V^5 = \tilde{L}_{12}\tilde{L}_{43}V^5 = -L_{12}L_{34}V^5 - \frac{s_{12}}{2}(A^3\cdot A^4)V^1V^2V^5 - \frac{s_{34}}{2}(A^1\cdot A^2)V^3V^4V^5.$$

To find a BRST-equivalent expression of (3.8) one uses the fact that $Q(L_{45}/s_{45}) = -V^4V^5$ to rewrite $\langle K_{ijkl}V^4V^5\rangle$ as $-\langle K_{ijkl}Q(L_{45}/s_{45})\rangle$, integrates the BRST-charge by parts and uses the following relation

$$QK_{ijkj} = s_{ji}(L_{ik}V^j - L_{jk}V^i) - (s_{jk}+s_{ik})L_{ji}V^k, \qquad (3.9)$$

[6] I thank Dimitrios Tsimpis for suggesting the relevance of using this definition in the context of an ansatz for the 6-pt amplitude. It turns out to clean up the 5-pt formulæ too.

which is easily obtained from the first expression in (3.3) and the definition (3.6). Doing that one gets

$$\langle K_{2131}V^4V^5\rangle = \langle L_{45}L_{12}V^3\rangle - \frac{s_{12}}{s_{45}}\langle L_{45}L_{\{12}V_{3\}}\rangle = n_1 - \frac{s_{12}}{s_{45}}(n_1 - n_4 - n_{15}) \tag{3.10}$$

$$\langle K_{2434}V^1V^5\rangle = \langle L_{42}L_{51}V^3\rangle + \frac{s_{24}}{s_{51}}\langle L_{51}L_{\{23}V_{4\}}\rangle = n_{11} + \frac{s_{24}}{s_{51}}(n_2 - n_{11} - n_5) \tag{3.11}$$

$$\langle K_{2331}V^4V^5\rangle = -\langle L_{23}L_{45}V^1\rangle - \frac{s_{23}}{s_{45}}\langle L_{45}L_{\{12}V_{3\}}\rangle = -n_4 - \frac{s_{23}}{s_{45}}(n_1 - n_4 - n_{15}) \tag{3.12}$$

$$\langle K_{2334}V^1V^5\rangle = -\langle L_{51}L_{23}V^4\rangle + \frac{s_{23}}{s_{51}}\langle L_{51}L_{\{23}V_{4\}}\rangle = -n_2 + \frac{s_{23}}{s_{51}}(n_2 - n_{11} - n_5) \tag{3.13}$$

Plugging the above relations in (3.8) and using the relations (3.2),

$$\mathcal{A}_5(1,2,3,4,5) = n_1K_1 + n_3K_2 - n_{11}K_1' + n_4K_5 + n_2K_4' - n_{10}K_3$$

$$-\frac{s_{12}}{s_{51}}K_2(n_2 - n_{11} - n_5) - \frac{s_{24}}{s_{45}}K_3(n_1 - n_4 - n_{15}). \tag{3.14}$$

Once the integrals K_j are written in terms of the basis (T, K_3) [20], as in the appendix of [11], the amplitude (3.14) becomes

$$\mathcal{A}_5(1,2,3,4,5) = T\,A_{\rm YM}(\theta) + K_3\,A_{F^4}(\theta), \tag{3.15}$$

where $A_{YM}(\theta)$ and $A_{F^4}(\theta)$ are the superfields,

$$A_{\rm YM}(\theta) = \frac{\langle L_{45}L_{12}V^3\rangle}{s_{45}s_{12}} + \frac{\langle L_{51}L_{23}V^4\rangle}{s_{51}s_{23}} + \frac{\langle L_{12}L_{34}V^5\rangle}{s_{12}s_{34}} + \frac{\langle L_{23}L_{45}V^1\rangle}{s_{23}s_{45}} + \frac{\langle L_{34}L_{51}V^2\rangle}{s_{34}s_{51}} \tag{3.16}$$

and

$$A_{F^4}(\theta) = -\langle L_{45}L_{12}V^3\rangle\left(\frac{s_{23}}{s_{45}} + \frac{s_{34}}{s_{12}}\right) - \langle L_{51}L_{23}V^4\rangle\left(\frac{s_{34}}{s_{15}} + \frac{s_{45}}{s_{23}}\right) \tag{3.17}$$

$$-\langle L_{12}L_{34}V^5\rangle\left(\frac{s_{45}}{s_{12}} + \frac{s_{51}}{s_{34}}\right) - \langle L_{23}L_{45}V^1\rangle\left(\frac{s_{51}}{s_{23}} + \frac{s_{12}}{s_{45}}\right) - \langle L_{34}L_{51}V^2\rangle\left(\frac{s_{12}}{s_{34}} + \frac{s_{23}}{s_{51}}\right)$$

$$+\langle L_{12}L_{34}V^5 + L_{51}L_{23}V^4 - L_{13}L_{42}V^5 + L_{23}L_{45}V^1\rangle + \frac{s_{13}}{s_{51}}\langle L_{51}L_{\{23}V_{4\}}\rangle - \frac{s_{24}}{s_{45}}\langle L_{45}L_{\{12}V_{3\}}\rangle$$

In the field theory limit $T \to 1$ and $K_3 \to 0$ [20], so the first principles derivation of (2.8) is completed.

3.1. Higher-point amplitudes

It is worth checking whether the simple mappings between the cubic Feynman diagrams and pure spinor building blocks persist at higher-points. The discussion in section 2 suggests a way to write down n-point field theory amplitudes. For each one of the $2^{n-2}(2n-5)!!/(n-1)!$ color-ordered diagrams specifying the kinematic poles [5], a ghost-number-three numerator whose BRST transformation is proportional to those poles should be written down. One then tries to find a combination with the correct dimension of a n-point amplitude such that the sum of all diagrams is BRST-closed.

To help finding candidates for superfield building blocks, the first principles tree-level superstring amplitude prescription [2,21] can be used as guide. For example, the superfield $\tilde{L}_{ij}$ appears in the OPE of $V^i(z)U^j(w)$ in the 4-pt string amplitude [18], and its BRST transformation $Q\tilde{L}_{ij} = -s_{ij}V^iV^j$ has precisely the Mandelstam variable to cancel poles in the 5-pt amplitude. Similarly, the superfield L_{jiki} comes from the numerator of the $1/z_{ij}z_{ik}$ pole in the OPE $V^i(z_i)U^j(z_j)U^k(z_k)$ appearing in the 5-pt computation [11], and its BRST transformation has the required Mandelstam variables to cancel poles in the 6-pt amplitude,

$$QL_{jiki} = s_{ij}(\tilde{L}_{jk}V^i - \tilde{L}_{ik}V^j + \tilde{L}_{ij}V^k) - (s_{jk} + s_{ki} + s_{ij})\tilde{L}_{ij}V^k, \tag{3.18}$$

or, defining $T_{ijk} \equiv K_{jiki}$,

$$QT_{ijk} = s_{ij}L_{\{ij}V_{k\}} - (s_{jk} + s_{ki} + s_{ij})L_{ij}V_k. \tag{3.19}$$

Following the above procedure for the 14 color-ordered diagrams of the 6-point amplitude[7], a BRST-closed expression with the correct pole structure looks like[8]

$$\mathcal{A}_6(1,2,3,4,5,6) = \frac{\langle L_{12}L_{34}L_{56}\rangle}{3s_1s_3s_5} + \frac{\langle L_{23}L_{45}L_{61}\rangle}{3s_2s_4s_6} \tag{3.20}$$

$$+\frac{1}{2}\frac{\langle T_{123}}{s_1t_1}\Big(\frac{V^4L_{56}}{s_5} + \frac{L_{45}V^6\rangle}{s_4}\Big) - \frac{1}{2}\frac{\langle T_{126}}{s_1t_3}\Big(\frac{V^3L_{45}}{s_4} + \frac{L_{34}V^5\rangle}{s_3}\Big) + \text{cyclic}(1\ldots6)$$

where $s_1 = s_{12}, s_2 = s_{23}, \ldots, s_6 = s_{61}$, $t_1 = (s_{12} + s_{23} + s_{13})$, $t_2 = (s_{23} + s_{34} + s_{24})$ and $t_3 = (s_{34} + s_{45} + s_{35})$ are the 6-point Mandelstam variables of [6]. The full component

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[7] Work is currently in progress to obtain the 6-pt field theory limit of the open superstring amplitude [10].

[8] I thank Oliver Schlotterer and Dimitrios Tsimpis for many valuable discussions.

expansion for the 6-gluon amplitude obtained from (3.20) contains 6706 terms [8] and it was checked to be gauge invariant. The first few terms of this expansion are given in Appendix B.

Similarly, an ansatz for the 42 color-ordered 7-point diagrams which is BRST-closed and has the correct pole structure is given by

$$\mathcal{A}_7(1,2,3,4,5,6,7) = +\frac{\langle T_{231} L_{45} L_{67}\rangle}{s_2 t_1 s_4 s_6} + \frac{\langle T_{123} T_{564} V_7\rangle}{s_1 t_1 s_5 t_4} + \frac{\langle T_{127} T_{345} V_6\rangle}{s_1 t_7 s_3 t_3} \tag{3.21}$$

$$-\frac{\langle T_{123} T_{456} V_7\rangle}{s_1 t_1 s_4 t_4} - \frac{\langle T_{127} T_{453} V_6\rangle}{s_1 t_7 s_4 t_3} - \frac{\langle T_{123} L_{45} L_{67}\rangle}{s_1 t_1 s_4 s_6} + \text{cyclic}(1\ldots 7)$$

where $s_1, \ldots, s_7$ and $t_1, \ldots, t_7$ are the 7-point Mandelstam variables of [7]. The ten-dimensional 7-gluon expansion of (3.21) contains more than 130 thousand terms [8] and a few are written in appendix B. As the results of [7] are written in the four-dimensional helicity formalism, a direct comparison with the results quoted there is not possible.

The simplicity of the above Ansätze is remarkable and claims for a first principles formalism. The compact results presented here provide strong evidence that the language of pure spinor superspace is well-suited for writing down ten-dimensional scattering amplitudes. Furthermore, having these compact supersymmetric expressions is interesting because there is no need to treat amplitudes differently, depending on whether the helicity configuration is MHV or NMHV.

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Appendix A. The Bern-Carrasco-Johansson kinematic identities

The 4-pt BCJ kinematic relation $n_u = n_s - n_t$ is mapped to the superspace expression $\langle L_{13} V^2 V^4\rangle = \langle L_{12} V^3 V^4\rangle - \langle L_{41} V^2 V^3\rangle$. Using $\langle L_{41} V^2 V^3\rangle = -\langle L_{23} V^1 V^4\rangle$ it can be rewritten as

$$\langle L_{\{12} V_{3\}} V^4\rangle = 0, \tag{A.1}$$

where $\{ijk\}$ means to sum over the cyclic permutation of the labels. Furthermore, one can check that (A.1) is true by expanding it in components. Note that

$$QL_{\{ij}V_{k\}} = -(s_{ij} + s_{jk} + s_{ik})V^i V^j V^k, \tag{A.2}$$

which vanishes for the 4-pt amplitude because $s_{ik} = -s_{ij} - s_{jk}$.

The 5-pt *extended BCJ relations* of [12][13] are given by

$$\frac{n_4 - n_1 + n_{15}}{s_{45}} - \frac{n_{10} - n_{11} + n_{13}}{s_{24}} - \frac{n_3 - n_1 + n_{12}}{s_{12}} - \frac{n_5 - n_2 + n_{11}}{s_{51}} = 0 \tag{A.3}$$

$$\frac{n_7 - n_6 + n_{14}}{s_{14}} - \frac{n_{10} - n_{11} + n_{13}}{s_{24}} - \frac{n_8 - n_6 + n_9}{s_{25}} - \frac{n_5 - n_2 + n_{11}}{s_{51}} = 0 \tag{A.4}$$

$$\frac{n_{10} - n_9 + n_{15}}{s_{13}} + \frac{n_5 - n_2 + n_{11}}{s_{51}} - \frac{n_4 - n_2 + n_7}{s_{23}} + \frac{n_8 - n_6 + n_9}{s_{25}} = 0 \tag{A.5}$$

$$\frac{n_4 - n_1 + n_{15}}{s_{45}} - \frac{n_{10} - n_9 + n_{15}}{s_{13}} - \frac{n_5 - n_2 + n_{11}}{s_{51}} - \frac{n_3 - n_5 + n_8}{s_{34}} = 0. \tag{A.6}$$

Using the mappings of (2.9) they become

$$-\frac{L_{45}}{s_{45}}L_{\{12}V_{3\}} + \frac{L_{42}}{s_{24}}L_{\{13}V_{5\}} - \frac{L_{12}}{s_{12}}L_{\{34}V_{5\}} + \frac{L_{51}}{s_{51}}L_{\{23}V_{4\}} = 0, \tag{A.7}$$

$$-\frac{L_{14}}{s_{14}}L_{\{23}V_{5\}} + \frac{L_{42}}{s_{24}}L_{\{13}V_{5\}} - \frac{L_{25}}{s_{25}}L_{\{13}V_{4\}} + \frac{L_{51}}{s_{51}}L_{\{23}V_{4\}} = 0, \tag{A.8}$$

$$+\frac{L_{13}}{s_{13}}L_{\{25}V_{4\}} - \frac{L_{51}}{s_{51}}L_{\{23}V_{4\}} - \frac{L_{23}}{s_{23}}L_{\{14}V_{5\}} + \frac{L_{25}}{s_{25}}L_{\{13}V_{4\}} = 0, \tag{A.9}$$

$$-\frac{L_{45}}{s_{45}}L_{\{12}V_{3\}} - \frac{L_{13}}{s_{13}}L_{\{25}V_{4\}} + \frac{L_{51}}{s_{51}}L_{\{23}V_{4\}} + \frac{L_{34}}{s_{34}}L_{\{12}V_{5\}} = 0, \tag{A.10}$$

which one can check to hold true when expanding in components. Using the momentum conservation relations

$$s_{13} = s_{45} - s_{12} - s_{23}, \quad s_{14} = s_{23} - s_{51} - s_{45}, \quad s_{24} = s_{51} - s_{23} - s_{34}$$

$$s_{25} = s_{34} - s_{12} - s_{51}, \quad s_{35} = s_{12} - s_{45} - s_{34}, \tag{A.11}$$

one finds that the LHS of (A.7) – (A.10) are BRST-closed. Roughly speaking, the extended BCJ identities are BRST-closed expressions which do not have the correct pole structure to be amplitudes.

Appendix B. The 5-, 6- and 7-gluon amplitudes

The 5-gluon amplitude is easily obtained by using [8], and one can check that the first few terms are

$$2880\,\mathcal{A}_5(1,2,3,4,5) = \tag{B.1}$$

$$-(k^1\cdot e^2)(k^1\cdot e^3)(k^1\cdot e^4)(e^1\cdot e^5)s_1^{-1}s_4^{-1} + (k^1\cdot e^2)(k^1\cdot e^3)(k^1\cdot e^5)(e^1\cdot e^4)s_1^{-1}s_4^{-1}$$

$$-(k^1\cdot e^2)(k^1\cdot e^3)(k^2\cdot e^4)(e^1\cdot e^5)s_1^{-1}s_4^{-1} + (k^1\cdot e^2)(k^1\cdot e^3)(k^2\cdot e^5)(e^1\cdot e^4)s_1^{-1}s_4^{-1}$$

$$-(k^1\cdot e^2)(k^1\cdot e^3)(k^3\cdot e^4)(e^1\cdot e^5)s_1^{-1}s_3^{-1} + \ldots$$

The 6-gluon component expansion from the ansatz (3.20) generates 6706 terms of which the first few are [8]

$$2880\,\mathcal{A}_6(1,2,3,4,5,6) = \tag{B.2}$$

$$\big[(k^1\cdot e^2)(k^1\cdot e^3)(k^1\cdot e^4)(k^1\cdot e^6)(e^1\cdot e^5) - (k^1\cdot e^2)(k^1\cdot e^3)(k^1\cdot e^4)(k^1\cdot e^5)(e^1\cdot e^6)$$

$$-(k^1\cdot e^2)(k^1\cdot e^3)(k^1\cdot e^4)(k^2\cdot e^5)(e^1\cdot e^6) + (k^1\cdot e^2)(k^1\cdot e^3)(k^1\cdot e^4)(k^2\cdot e^6)(e^1\cdot e^5)$$

$$-(k^1\cdot e^2)(k^1\cdot e^3)(k^1\cdot e^4)(k^3\cdot e^5)(e^1\cdot e^6) + (k^1\cdot e^2)(k^1\cdot e^3)(k^1\cdot e^4)(k^3\cdot e^6)(e^1\cdot e^5)\big]\,s_1^{-1}s_5^{-1}t_1^{-1}$$

$$-(k^1\cdot e^2)(k^1\cdot e^3)(k^1\cdot e^4)(k^4\cdot e^5)(e^1\cdot e^6)s_1^{-1}s_4^{-1}t_1^{-1} + \ldots$$

Similarly, the 7-gluon component expansion of (3.21) has 134460 terms[9] and the first ones are

$$2880\,\mathcal{A}_7(1,2,3,4,5,6,7) = \tag{B.3}$$

$$\big[+(k^1\cdot e^2)(k^1\cdot e^3)(k^1\cdot e^4)(k^1\cdot e^5)(k^1\cdot e^6)(e^1\cdot e^7) - (k^1\cdot e^2)(k^1\cdot e^3)(k^1\cdot e^4)(k^1\cdot e^5)(k^1\cdot e^7)(e^1\cdot e^6)$$

$$+(k^1\cdot e^2)(k^1\cdot e^3)(k^1\cdot e^4)(k^1\cdot e^5)(k^2\cdot e^6)(e^1\cdot e^7) - (k^1\cdot e^2)(k^1\cdot e^3)(k^1\cdot e^4)(k^1\cdot e^5)(k^2\cdot e^7)(e^1\cdot e^6)$$

$$+(k^1\cdot e^2)(k^1\cdot e^3)(k^1\cdot e^4)(k^1\cdot e^5)(k^3\cdot e^6)(e^1\cdot e^7)\big]\,s_1^{-1}s_6^{-1}t_1^{-1}t_5^{-1} + \ldots$$

It is curious to note that the coefficient of $\pm 1/2880$ is the same for all the terms in the 5-, 6- and 7-gluon amplitudes alike. This is the same coefficient which was observed in [21] to be the conversion factor required to match the RNS amplitudes at tree-level.

[9] Some of those terms contain ϵ_{10} tensors and are expected to vanish once rules for the vanishing of things like $\epsilon_{10}^{[m_1\ldots m_{10}}\delta_n^{m_{11}]}$ are implemented in [8].

Appendix C. Shortcut to compute QL

There is a shortcut to compute QL's for n-points using only the L's appearing at $(n-1)$-points. The definitions of $\tilde{L}_{ij}$ and L_{jiki} are [11],

$$V^i(z_i)U^j(z_j) \to \frac{\tilde{L}_{ij}}{z_{ij}}, \qquad \tilde{L}_{ij}(z_i)U^k(z_k) \to \frac{L_{jiki}}{z_{ik}}, \tag{C.1}$$

so that $Q\tilde{L}_{ij} = \lim_{z_j \to z_i} z_{ij} Q(V^i(z_i)U^j(z_j))$ and $QL_{jiki} = \lim_{z_k \to z_i} z_{ik} Q(\tilde{L}_{ij}(z_i)U^k(z_k))$ leads to

$$Q\tilde{L}_{ij} = \lim_{z_j \to z_i} z_{ij} \partial V^j(z_j) V^i(z_i) = -s_{ij} V^i V^j,$$

$$\begin{aligned} QL_{jiki} &= -\lim_{z_k \to z_i} z_{ik}(s_{ij} V^i(z_i)V^j(z_i)U^k(z_k) + \tilde{L}_{ij}(z_i)\partial V^k(z_k)) \\ &= -s_{ij}(\tilde{L}_{ik}(z_i)V^j(z_i) + V^i(z_i)\tilde{L}_{jk}(z_i)) + (s_{ik} + s_{jk})V^k(z_i)\tilde{L}_{ij}(z_i), \end{aligned} \tag{C.2}$$

which agree with (2.3) and (3.18), respectively. In the above we used $QU^i(z) = \partial V^i(z) = \Pi^m(z) k^i_m V^i(z) + \partial\theta^\alpha D_\alpha V^i(z) + \partial\lambda^\alpha A^i_\alpha$, which together with the OPE's of the conformal weight-one variables [22,16] implies that

$$\lim_{z_i \to z_j} Q(U^i(z_i)V^j(z_j)) = \lim_{z_i \to z_j} \partial V^i(z_i) V^j(z_j) \to -s_{ij} \frac{V^i(z_i)V^j(z_i)}{z_{ij}}. \tag{C.3}$$

References

[1] S. J. Parke and T. R. Taylor, "An Amplitude for n Gluon Scattering," Phys. Rev. Lett. **56**, 2459 (1986).

[2] N. Berkovits, "Super-Poincare covariant quantization of the superstring," JHEP **0004**, 018 (2000) [arXiv:hep-th/0001035].

[3] N. Berkovits, "Explaining pure spinor superspace," arXiv:hep-th/0612021.

[4] E. Witten, "Twistor - Like Transform In Ten-Dimensions," Nucl. Phys. B **266**, 245 (1986).

[5] Z. Bern, J. J. M. Carrasco and H. Johansson, "New Relations for Gauge-Theory Amplitudes," Phys. Rev. D **78**, 085011 (2008) [arXiv:0805.3993 [hep-ph]].

[6] S. Stieberger and T. R. Taylor, "Multi-gluon scattering in open superstring theory," Phys. Rev. D **74**, 126007 (2006) [arXiv:hep-th/0609175].

[7] S. Stieberger and T. R. Taylor, "Supersymmetry Relations and MHV Amplitudes in Superstring Theory," Nucl. Phys. B **793**, 83 (2008) [arXiv:0708.0574 [hep-th]].

[8] C. R. Mafra, "PSS: A FORM Program to Evaluate Pure Spinor Superspace Expressions," arXiv:1007.4999 [hep-th].

[9] D. Oprisa and S. Stieberger, "Six gluon open superstring disk amplitude, multiple hypergeometric series and Euler-Zagier sums," arXiv:hep-th/0509042.

[10] C. Mafra, O. Schlotterer, S. Stieberger, D. Tsimpis, work in progress

[11] C. R. Mafra, "Simplifying the Tree-level Superstring Massless Five-point Amplitude," JHEP **1001**, 007 (2010) [arXiv:0909.5206 [hep-th]].

[12] H. Tye and Y. Zhang, "Dual Identities inside the Gluon and the Graviton Scattering Amplitudes," arXiv:1003.1732 [hep-th].

[13] N. E. J. Bjerrum-Bohr, P. H. Damgaard, T. Sondergaard and P. Vanhove, "Monodromy and Jacobi-like Relations for Color-Ordered Amplitudes," arXiv:1003.2403 [hep-th].

[14] J. M. Drummond and J. M. Henn, "All tree-level amplitudes in N=4 SYM," JHEP **0904**, 018 (2009) [arXiv:0808.2475 [hep-th]].

[15] P. S. Howe, "Pure Spinors Lines In Superspace And Ten-Dimensional Supersymmetric Theories," Phys. Lett. B **258**, 141 (1991) [Addendum-ibid. B **259**, 511 (1991)]. ;
P. S. Howe, "Pure Spinors, Function Superspaces And Supergravity Theories In Ten-Dimensions And Eleven-Dimensions," Phys. Lett. B **273**, 90 (1991).

[16] N. Berkovits, "ICTP lectures on covariant quantization of the superstring," arXiv:hep-th/0209059.

[17] C. R. Mafra, "Superstring Scattering Amplitudes with the Pure Spinor Formalism," arXiv:0902.1552 [hep-th].

[18] C. R. Mafra, "Pure Spinor Superspace Identities for Massless Four-point Kinematic Factors," JHEP **0804**, 093 (2008) [arXiv:0801.0580 [hep-th]].

[19] N. Berkovits, “Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring,” JHEP **0409**, 047 (2004) [arXiv:hep-th/0406055].

[20] R. Medina, F. T. Brandt and F. R. Machado, “The open superstring 5-point amplitude revisited,” JHEP **0207**, 071 (2002) [arXiv:hep-th/0208121]. ;
L. A. Barreiro and R. Medina, “5-field terms in the open superstring effective action,” JHEP **0503**, 055 (2005) [arXiv:hep-th/0503182].

[21] N. Berkovits and B. C. Vallilo, “Consistency of super-Poincare covariant superstring tree amplitudes,” JHEP **0007**, 015 (2000) [arXiv:hep-th/0004171].

[22] W. Siegel, “Classical Superstring Mechanics,” Nucl. Phys. B **263**, 93 (1986).