Towards Planning with Force Constraints:
On the Mobility of Bodies in Contact

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1. Introduction

We describe a novel configuration-space based approach for analyzing the interactions and mobility of objects in quasi-static contact. This analysis is motivated by a class of articulated robot motion-planning problems which are not handled by current planning systems. Examples are: a "snake-like" robot that crawls inside a tunnel by bracing against the tunnel walls; a limbed robot (analogous to a "monkey") that climbs a trussed structure by pushing and pulling; or a dextrous robotic hand that moves its fingers along an object while holding it stationary. In these examples, one must plan the robot motion to satisfy high-level goals while maintaining *quasistatic stability*. That is, the forces of interaction between the robot and its environment (or grasped object) must sum to zero for stability. We are primarily concerned with planning the "hand-hold" states (analogous to the hand-holds used by rock climbers between dynamically moving states) where the robot mechanism is at a static equilibrium. The results presented in this paper are some of the first steps necessary to develop planning paradigms for this class of problems. While we have the general class of quasi-static planning problems in mind, the rest of this paper uses the language of grasping for discussion.

1.1. Outline of the Method

Consider a rigid body $\mathcal{B}$ and fingers $\mathcal{A}_1, \cdots, \mathcal{A}_d$ (Figure 1) in $\mathbf{R}^n$, where $n = 2, 3$. $\mathcal{B}(q)$ denotes the location of $\mathcal{B}$ in physical space with configuration q. The fingers are attached to a common base via ideal linkages that can place the fingers in any position and deliver any contact force. The fingers contact $\mathcal{B}$ with *frictionless point contact*. For the problems considered in this paper one can focus on the c-space of $\mathcal{B}$, rather than the composite configuration space of the $d+1$ rigid bodies.

Let $\mathcal{C}$ be the configuration space (c-space) of $\mathcal{B}$ and let $\mathcal{F} \subset \mathcal{C}$ be its free c-space. The fingers are represented in $\mathcal{C}$ as forbidden regions (also called c-space obstacles), denoted by $\mathcal{CA}_i$ (the set of all $q \in \mathcal{C}$ where $\mathcal{B}(q)$ intersects $\mathcal{A}_i$). The boundary of each $\mathcal{CA}_i$ is denoted by $\mathcal{S}_i$. We treat the fingers as stationary bodies and analyze instead the mobility of $\mathcal{B}$. If d fingers contact $\mathcal{B}(q_0)$, q_0 lies on the boundary of $\mathcal{F}$, at the intersection of $\mathcal{S}_i$ for $i = 1, ..., d$. Many properties of interest can be determined by the local c-space geometry around q_0.

We first embed real-world contact forces in $\mathcal{C}$. Using this embedding, we show that an equilibrium grasp has a simple c-space characterization. Next we consider the possible *first* and *second order instantaneous free motions* of $\mathcal{B}$, which are related to the first and second order object and finger geometrical properties. The 1'st order free motions are exactly the ones obtained by the *reciprocal/contrary screws principal* of Screw Theory.

The 2'nd order free motions and the ensuing 2'nd order mobility analysis are the principle contributions of this paper. The 2'nd order free motions are determined by the *c-space curvature-form* of $\mathcal{F}$ at q_0, which is a 2'nd order approximation to the boundary of $\mathcal{F}$, expressed in terms of the normals and curvatures of the object and the fingers at the contact points. The 2'nd order mobility analysis of an equilibrium grasp can lead to a different mobility than is predicted by screw theory. We introduce two integer-valued functions, the *1'st and 2'nd order mobility indices*, that measure the respective mobility of $\mathcal{B}$ at a given equilibrium grasp. It can be used to distinguish between maximal and minimal 2-fingered equilibrium grasps (Fig. 3), which have the same 1'st order index and can not be distinguished by Screw Theory. Other important applications are considered in Section 4.

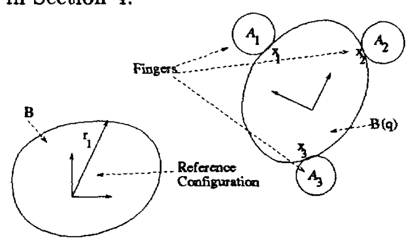

Figure 1: Object and Finger Geometry

1.2. Preliminaries

The c-space of $\mathcal{B}$ is $\mathcal{C} = \mathbf{R}^k$, where $k = n + m$ and $m = \frac{1}{2}n(n-1)$ is the dimension of $SO(n)$. Points in $\mathcal{C}$ are denoted by $q = (d, \theta)$, where $d \in \mathbf{R}^n$ is translation and $\theta \in \mathbf{R}^m$ represents orientation. The mapping of points r in $\mathcal{B}$ to points x in $\mathcal{B}(q)$ is given by the forward kinematics map:

$$x = X(r, q) = R(\theta)r + d \quad r \in B,\, x \in B(q). \qquad (1.1)$$

θ is a convenient covering of $SO(n)$ by $\mathbf{R}^m$ via the exponential map,

$$\exp : \mathbf{R}^m \to SO(n) \quad \text{such that} \quad R(\theta) = e^{\Omega(\theta)},$$

where $\widehat{\theta} = \theta/||\theta||$ is the axis of rotation, $||\theta||$ is the angle of rotation, and $\Omega(\theta)$ is an $m \times m$ skew-symmetric matrix. For $SO(3)$, $\Omega(\theta)$ is the vector-product operation: $\Omega(\theta)v = \theta \times v$ for all $v \in \mathbf{R}^3$. We emphasize this fact with the notation $[\theta\times]v \triangleq \Omega(\theta)v$ (see [McCarthy] for details). We regard $SO(2)$ as a subgroup of $SO(3)$, with $\widehat{\theta}$ normal to the plane. We call the $q = (d, \theta)$ parametrization the *hybrid coordinates* of $SE(3)$.

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We shall denote tangent vectors by $\dot{q} = (v, \omega)$, where $v = \dot{d}$ is translational velocity and $\omega = \dot{\theta}$ is the angular velocity vector. The generalized forces expressed in the hybrid coordinates $q = (d, \theta)$ are termed *c-wrenches*, and are denoted by $w = (f\ \tau)$. The c-wrench due to d real-world forces $F(x_1), ..., F(x_d)$ acting on $\mathcal{B}(q)$ along its normals at the contact points $x_1, ..., x_d$ is denoted by $w(q; F_1, \cdots, F_d)$. Scalar products, such as $w \cdot \dot{q}$, are derived from the standard Euclidean metric.

1.3. Related Literature

There are numerous papers related to the subjects touched upon in this work. Many papers have considered the relation between the force/torque applied by fingers on an object and the associated kinematic constraints. Work on this problem was pioneered by Reuleaux in 1876 [Reuleaux]. More recently, [Ohwovoriole] considers this relation in terms of Screw Theory, while [MK] takes a geometric approach. The central finding of our paper is that *reasoning about the mobility of objects in contact in terms of the instantaneous force/torque leads only to 1'st order mobility.* The actual mobility, however, *is not an infinitesimal notion but a local one.* For example, using what we call 1'st order mobility theory, Mishra et. al [Mishra et. al] found that $k+1$ is the minimum number of frictionless point contact fingers needed to immobilize an object ($k+1 = 4$ for 2D grasps and $k+1 = 7$ for 3D grasps). We show by example in Section 4 that immobility may be achieved with fewer fingers when 2'nd order mobility is taken into account. Others have proposed algorithms for finding grasp configurations. See [Trinkle] for a review of these ideas. Cai&Roth [Cai] and Montana [Montana] have derived formulas governing the motion of the contact point along roll/slide motion which take into account the geometry of the contacting bodies. We draw upon their results in the derivation of a c-space curvature-form.

2. Contact and Grasp Forces

2.1. Embedding Contact Forces in C-space

We derive here a formula for the c-wrench, $w = (f, \tau)$, that arises from normal forces exerted by fingers on $\mathcal{B}$. We shall need two basic facts. The first is the usual Lagrange equation of motion for $\mathcal{B}$,

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}}L - \frac{\partial}{\partial q}L = w(t), \tag{2.1}$$

where $L = K - U$. The kinetic energy, K, is given by $K(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q}$, where $M(q)$ is the $k \times k$ inertia matrix of $\mathcal{B}$. We assume that the potential energy, U, is zero i.e., gravity effects are excluded. Since $L = K$, the change of K along motions of (2.1) is:

$$\frac{d}{dt}K(q, \dot{q}) = w(t) \cdot \dot{q}. \tag{2.2}$$

The second fact, usually termed the *virtual work principal,* relates the effect of real-world forces $F_1(x_1), \cdots, F_d(x_d)$ applied at points $x_1, \cdots, x_d$ to changes in K:

$$\frac{d}{dt}K(q, \dot{q}) = \sum_{i=1}^{d} F_i(x_i) \cdot \dot{x}_i. \tag{2.3}$$

Let $\widehat{\mathrm{n}}_i(q)$ denote the unit normal to $\mathcal{S}_i$ at q, pointing into $\mathcal{F}$. We assume that $\mathcal{S}_i$ is smooth, so that $\widehat{\mathrm{n}}_i(q)$ is well defined. The following theorem relates the net c-wrench on $\mathcal{B}(q)$ to the c-space obstacle normals:

Theorem 2.1 [RB]: *The* **c-wrench** *w due to a single-finger normal force $F_1(x_1)$ pushing on $\mathcal{B}(q)$ is* **normal** *to the finger c-obstacle boundary $\mathcal{S}_1$ at q, and is:*

$$w(q, F_1) = [DX_{r_1}(q)]^T F_1(x_1) = \lambda_1 \widehat{\mathrm{n}}_1(q) \quad \text{for } \lambda_1 \geq 0,$$

where $DX_{r_1}(q) = \frac{\partial}{\partial q}X(r_1, q)$. More generally, let $d \geq 0$ fingers push on $\mathcal{B}(q)$ with normal forces $F_1(x_1), \cdots, F_d(x_d)$. Then the **net c-wrench** *w is* **normal** *to the set $\bigcap_{i=1}^{d} \mathcal{S}_i$ and is given by*

$$w(q; F_1, ..., F_d) = \sum_{i=1}^{d} w(q, F_i) = \sum_{i=1}^{d} \lambda_i \widehat{\mathrm{n}}_i(q) \quad \lambda_i \geq 0.$$

Remark: The condition of being normal to $\bigcap_{i=1}^{d} \mathcal{S}_i$ is equivalent to being orthogonal to the intersection of the tangent spaces $T_q\mathcal{S}_i$, for $i = 1, ..., d$.

It follows from the theorem that the collection of all wrenches attainable by varying the magnitude of the finger forces $\|F(x_1)\|, ..., \|F(x_d)\|$, while the object is at configuration q, is a cone spanned by the outward-pointing normals $\widehat{\mathrm{n}}_1(q), ..., \widehat{\mathrm{n}}_d(q)$. This is captured in the following definition.

Definition 2.2: *Let d fingers contact $\mathcal{B}$ at configuration q. The* **generalized normal** *to $\mathcal{F}$ at q is the convex combination of the outward-pointing unit normals to the fingers' c-obstacles at q,*

$$\mathcal{N}_q(\mathcal{F}) = \sum_{i=1}^{d} \lambda_i \widehat{\mathrm{n}}_i(q) \quad \lambda_i \geq 0 \text{ and } \textstyle\sum_{i=1}^{d} \lambda_i = 1.$$

The **c-wrench cone** *at q is the cone spanned by the generalized normal at q,*

$$\mathcal{W}_q(\mathcal{F}) = \bigcup_{\lambda \geq 0} \lambda \mathcal{N}_q(\mathcal{F}) = \sum_{i=1}^{d} \lambda_i \widehat{\mathrm{n}}_i(q) \quad \lambda_i \geq 0.$$

The notation $\mathcal{N}_q(\mathcal{F})$ and $\mathcal{W}_q(\mathcal{F})$ is shorthand for $\mathcal{N}_q(\mathcal{CA}_1, ..., \mathcal{CA}_d)$ and $\mathcal{W}_q(\mathcal{CA}_1, ..., \mathcal{CA}_d)$, since the boundary of $\mathcal{F}$ at q is the boundary of $\bigcup_{i=1}^{d} \mathcal{CA}_i$. $\mathcal{N}_q(\mathcal{F})$ is a purely geometrical structure that depends only on q and the local geometry of $\mathcal{CA}_1, ..., \mathcal{CA}_d$.

2.2. Equilibrium Grasps

By definition, for $\mathcal{B}(q_0)$ to be at a d-fingered equilibrium grasp, the net c-wrench on $\mathcal{B}$ due to finger forces $F_1(x_1), ..., F_d(x_d)$ must be zero. Theorem 2.1 yields the following simple condition for an equilibrium grasp:

Corollary 2.3: *Let d fingers push on $\mathcal{B}(q_0)$ with normal forces $F_1(x_1), ..., F_d(x_d)$. q_0 can be made an* **equilibrium grasp** *by suitable choice of the finger force magnitudes iff zero lies in the generalized normal at q_0,*

$$0 \in \mathcal{N}(q; \mathcal{CA}_1, ... \mathcal{CA}_d). \tag{2.4}$$

Remark: This simple geometrical condition can be used to locate, in $\mathcal{C}$, equilibrium grasp finger arrangements. A similar result has been derived in [Mishra91].

3. Mobility of Bodies in Contact

By definition, the *free* motions of $\mathcal{B}$ are its instantaneous motions that cannot be prevented by any combination of the force magnitudes of frictionless fingers contacting $\mathcal{B}$. These motions indicate how mobile the object is, with less mobility implying a safer grip.

3.1. 1'st Order Mobility

Consider a single-finger contact between $\mathcal{B}(q_0)$ and $\mathcal{A}_1$. The motions of $\mathcal{B}(q_0)$ must respect the rigidity of the object and finger. If $\mathcal{A}_1$ is stationary the c-space motions of $\mathcal{B}$ must lie in $\mathcal{F}$. Consider the following signed distance of a configuration point q from $\mathcal{S}_1$:

$$d_1(q) \triangleq \begin{cases} \mathtt{dst}(q, \mathcal{S}_1) & \text{if } q \text{ is outside of } \mathcal{CA}_1 \\ 0 & \text{if } q \text{ is on } \mathcal{S}_1 \\ -\mathtt{dst}(q, \mathcal{S}_1) & \text{if } q \in \mathit{int}(\mathcal{CA}_1) \end{cases} \tag{3.1}$$

where $\mathtt{dst}(q, \mathcal{S}_1)$ is the Euclidean distance of q from $\mathcal{S}_1$. The 1'st order free motions are related to the following 1'st order approximation to $d_1(q)$ around $q_0 \in \mathcal{S}_1$:

$$\begin{aligned} d_1(q) &= d_1(q_0) + \nabla d_1(q_0) \cdot (q - q_0) \\ &= \widehat{\mathrm{n}}_1(q_0) \cdot (q - q_0), \end{aligned}$$

since $d_1(q_0) = 0$ and, as can be shown, $\nabla d_1(q_0) = \widehat{\mathrm{n}}_1(q_0)$ [Clarke]. The set of 1'st order free motions is defined as the collection of all infinitesimal motions that respect the 1'st order approximation of the rigidity constraint:

Definition 3.1 *The* **1'st order free motions** *of $\mathcal{B}$, in contact with a frictionless finger $\mathcal{A}_1$ at configuration q_0, is the half space of $T_{q_0}\mathbf{R}^k$ satisfying:*

$$M^1_{q_0}(\mathcal{CA}_1) = \left\{ \dot{q} \in T_{q_0}\mathbf{R}^k : \widehat{\mathrm{n}}_1(q_0) \cdot \dot{q} \geq 0 \right\}.$$

More generally, the **1'st order free motions** *of $\mathcal{B}(q_0)$ in contact with d fingers is the cone in $T_{q_0}\mathbf{R}^k$ generated by intersection of the individual half spaces:*

$$M^1_{q_0}(\mathcal{F}) = \bigcap_{i=1}^{d} M^1_{q_0}(\mathcal{CA}_i),$$

where $M^1_{q_0}(\mathcal{F})$ is a shorthand for $M^1_{q_0}(\mathcal{CA}_1, ..., \mathcal{CA}_d)$.

Remark: 1'st order free motions are those along which $\dot{d}_1 = 0$ (roll/slide motions between $\mathcal{B}$ and the fingers) or $\dot{d}_1 > 0$ (the object breaks contact) up to 1'st order approximation. Further, one can interpret this definition to say that the finger c-obstacle boundary is approximated by a hyperplane tangent to $\mathcal{CA}_1$ at q_0.

Let $\mathcal{B}(q_0)$ be at an equilibrium grasp. Recall that the c-wrench cone $\mathcal{W}_{q_0}(\mathcal{F})$ is the set of all c-wrenches which result from varying the finger force magnitudes. The following lemma relates $\mathcal{W}_{q_0}(\mathcal{F})$ to $M^1_{q_0}(\mathcal{F})$. We shall need the cone ***antipodal*** to $\mathcal{W}_{q_0}(\mathcal{F})$, given by

$$-\mathcal{W}_q(\mathcal{F}) = \sum_{i=1}^{d} \lambda_i(-\widehat{\mathrm{n}}_i(q)) \qquad \lambda_i \geq 0 \text{ for } i = 1, .., d.$$

Lemma 3.2 [RB]: *Let $\mathcal{B}(q_0)$ be held at an* **equilibrium grasp** *by d frictionless fingers. Then $M^1_{q_0}(\mathcal{F})$ is the cone* **polar** *to $-\mathcal{W}_{q_0}(\mathcal{F})$, given by,*

$$M^1_{q_0}(\mathcal{F}) = \left\{ \dot{q} \in T_{q_0}\mathbf{R}^k : w \cdot \dot{q} \leq 0 \text{ for all } w \in -\mathcal{W}_{q_0}(\mathcal{F}) \right\}$$

In particular, if the equilibrium is maintained while all fingers contacting $\mathcal{B}(q_0)$ apply non-zero force, $M^1_{q_0}(\mathcal{F})$ is a **subspace** *of $T_{q_0}\mathbf{R}^k$ orthogonal to $\mathcal{W}_{q_0}(\mathcal{F})$,*

$$M^1_{q_0}(\mathcal{F}) = \mathcal{W}_{q_0}(\mathcal{F})^{\perp} \quad (M^1_{q_0}(\mathcal{F}) \text{ is also } \textstyle\bigcap_{i=1}^{d} T_{q_0}\mathcal{S}_i).$$

Special cases excluded, the dimension of $M^1_{q_0}(\mathcal{F})$ is $k - d + 1$.

Definition 3.3 *The* **1'st order mobility index** *of $\mathcal{B}$ at q_0 is the dimension of the subspace $M^1_{q_0}(\mathcal{F})$:*

$$m^1_{q_0}(\mathcal{F}) \triangleq \dim\left(M^1_{q_0}(\mathcal{F})\right) = k - d + 1,$$

according to Lemma 3.2, where $m^1_{q_0}(\mathcal{F})$ is shorthand notation for $m^1_{q_0}(\mathcal{CA}_1, ..., \mathcal{CA}_d)$.

$m^1_{q_0}(\mathcal{F})$ attains values in the range $0, ..., k-1$ and, since $k = 3$ or 6 is always fixed, depends solely on the number of fingers d. It decreases from its maximal value of $k - 1$ for $d = 2$ fingers to zero as the number of fingers increases to $d = k + 1$. In particular, the object is completely immobilized to 1'st order when $m^1_{q_0}(\mathcal{F}) = 0$. It can be shown that $m^1_{q_0}(\mathcal{F})$ is *coordinate invariant* [RB].

Our 1'st order free motions coincide with the definition of reciprocal/contrary screw pairs in [Ohwovoriole]. This is made precise in the Appendix.

3.2. 2'nd Order Mobility

Our concept of 2'nd mobility is related to the following 2'nd order approximation to $d_1(q)$ around $q_0 \in \mathcal{S}_1$:

$$\begin{aligned} d_1(q) &= d_1(q_0) \;+\; \nabla d_1(q_0) \cdot (q - q_0) \\ &\quad + \tfrac{1}{2}(q - q_0)^T [D^2 d_1(q_0)](q - q_0) \\ &= \widehat{\mathrm{n}}_1(q_0) \cdot (q - q_0) + \tfrac{1}{2}(q - q_0)^T [D\widehat{\mathrm{n}}_1(q_0)](q - q_0), \end{aligned} \tag{3.2}$$

since $d_1(q_0) = 0$ and $\nabla d_1(q_0) = \widehat{\mathrm{n}}_1(q_0)$. 2'nd order free motions are defined analogously to the 1'st order free motions as follows:

Definition 3.4 *The* **2'nd order free motions** *of $\mathcal{B}$, in contact with a frictionless finger $\mathcal{A}_1$ at configuration q_0, is the subset denoted $M^2_{q_0}(\mathcal{CA}_1)$ of $T_{q_0}\mathbf{R}^k$ defined by*

$$\left\{ \dot{q} \in T_{q_0}\mathbf{R}^k : \widehat{\mathrm{n}}_1(q_0) \cdot \dot{q} + \tfrac{1}{2}\dot{q}^T [D\widehat{\mathrm{n}}_1(q_0)]\dot{q} \geq 0 \right\}.$$

And, for d fingers,

$$M^2_{q_0}(\mathcal{F}) = \bigcap_{i=1}^{d} M^2_{q_0}(\mathcal{CA}_i).$$

While 2'nd order free motions are useful for assembly planning (see Section 4), our goal is to derive a closed-form formula for $\dot{q}^T[D\widehat{n}_1(q_0)]\dot{q}$ (the c-space curvature form) and discuss its interpretation. This derivation leads to a coordinate invariant 2'nd order mobility index that captures the 2'nd order free motions of $\mathcal{B}$ at an equilibrium grasp. As we have shown, the 1'st order mobility of $\mathcal{B}$ at an equilibrium grasp reduces to a subspace whose dimension, $m^1_{q_0}(\mathcal{F})$, is determined solely by the number of fingers d and their arrangement. All generic d-fingered equilibrium grasps thus look alike up to 1'st order. Yet, the 2-finger examples in Section 4 show that different equilibrium grasps have different mobility. Thus 1'st order mobility can be a poor approximation to true mobility. A suitably defined 2'nd order mobility index would induce a finer partition of the 1'st order mobility subspace, which in turn would distinguish between equilibrium grasps which are 1'st order identical. Let us sketch the analytical derivation of the curvature-form of $\mathcal{S}_1$ at q_0.

The C-space Curvature Form: Let $R_0 \in SO(3)$ be the orientation of $\mathcal{B}$ at its equilibrium grasp. It is convenient to use the following coordinates for $SO(3)$,

$$\theta \in \mathbf{R}^3 \mapsto R(\theta) = e^{\Omega(\theta)} R_0, \tag{3.3}$$

This is a parametrization of $SO(3)$ centered at R_0, such that the equilibrium grasp configuration is $q_0 = (d_0, 0)$. It can be shown that for fixed vector $r_1 \in \mathbf{R}^3$,

$$\left.\frac{d}{dt}\right|_{t=0} R(\theta(t))\, r_1 = \Omega(\omega) R_0 r_1 = -[R_0 r_1 \times]\omega. \tag{3.4}$$

Now consider the computation of $\dot{q}^T D\widehat{n}_1(q_0)\dot{q}$. Using Theorem 2.1, the outward pointing normal to $\mathcal{S}_1$ is:

$$n_1(q_0) = [DX_{r_1}(q_0)]^T \widehat{N}(x_1), \tag{3.5}$$

where $\widehat{n}_1(q_0) = n_1(q_0)/\|n_1\|$ is equal to $\nabla d_1(q_0)$. Let q_0 be a point on $\mathcal{S}_1$. It can be shown that ∇d_1 is locally constant along the direction normal to $\mathcal{S}_1$. Hence $D^2 d_1(q_0) = D\widehat{n}_1(q_0)$ is fully determined by consideration of paths $\alpha(t)$ in $\mathcal{S}_1$, such that $\alpha(0) = q_0$ and $\dot{\alpha}(0) = (v, \omega) \in T_{q_0}\mathcal{S}_1$. Thus

$$\begin{aligned} &\tfrac{d}{dt}\big|_{t=0} \widehat{n}_1(\alpha(t)) = \tfrac{d}{dt}\big|_{t=0} \tfrac{1}{\|n_1\|} n_1(\alpha(t)) \\ &= \tfrac{1}{\|n_1\|}[I - \widehat{n}_1\widehat{n}_1^T]\, \tfrac{d}{dt}\big|_{t=0} \left([DX_{r_1}(\alpha(t))]^T \widehat{N}(x_1(t))\right), \end{aligned}$$

using the chain rule. In [RB] we use Eq. (3.4) for $\dot{R}_0 = \frac{d}{dt}\big|_{t=0} R$ to obtain an intermediate equation that contains terms such as $\dot{R}_0$, $\frac{d}{dy}\Big|_{y=x_1} \widehat{N}(y)$, and $\dot{x}_1|_{t=0}$.

[Montana] and [Cai] have derived a formula for $\dot{x}_1$ along a general roll-slide motion. This is exactly our case, since $\dot{x}_1$ results from a c-space motion $\alpha(t)$ that lies in $\mathcal{S}_1$. Their formula depends on the curvature of the two bodies, for which some notation is now introduced. Let $\mathcal{B}(q_0)$ and $\mathcal{A}_1$ be described by $\mathcal{B}(q_0) = \{y \in \mathbf{R}^n : \beta(y) \le 0\}$ and $\mathcal{A}_1 = \{y \in \mathbf{R}^n : \alpha_1(y) \le 0\}$, where $\beta(y)$ and $\alpha_1(y)$ are smooth real-valued functions. With an abuse of notation, let $\mathcal{B}(q_0)$ and $\mathcal{A}_1$ also denote their respective boundary. By definition, the curvature of $\mathcal{B}(q_0)$ and $\mathcal{A}_1$ at their contact point x_1 along various tangent directions is determined by the following linear maps (the Weingarten map):

$$\begin{aligned} L_{\mathcal{B}}(x_1) &\triangleq \left.\tfrac{d}{dy}\right|_{y=x_1} \widehat{\nabla\beta}(y) : T_{x_1}\mathcal{B}(q_0) \to T_{x_1}\mathcal{B}(q_0) \\ L_{\mathcal{A}_1}(x_1) &\triangleq \left.\tfrac{d}{dy}\right|_{y=x_1} \widehat{\nabla\alpha_1}(y) : T_{x_1}\mathcal{A}_1 \to T_{x_1}\mathcal{A}_1. \end{aligned}$$

For notational simplicity we shall write $L_{\mathcal{B}_1}$ and $L_{\mathcal{A}_1}$.
Remark: In the planar case $T_{x_1}\mathcal{B}(q_0)$ and $T_{x_1}\mathcal{A}_1$ are 1-dimensional. The action of $L_{\mathcal{B}_1}$ and $L_{\mathcal{A}_1}$ is simply a multiplication by $\kappa_{\mathcal{B}_1}$ and $\kappa_{\mathcal{A}_1}$ respectively, which are the scalar curvatures of the curves bounding the respective shapes.

$\dot{x}_1$ at $t = 0$ is then [Montana, Cai]

$$\begin{aligned} &\dot{x}_1|_{t=0} = \\ &\left[L_{\mathcal{A}_1} + L_{\mathcal{B}_1}\right]^{-1} \left\{L_{\mathcal{B}_1}\left(\dot{R}_0 r_1 + v\right) - \dot{R}_0 R_0^T \widehat{\nabla\beta}(x_1)\right\}. \end{aligned}$$

Substitution for $\dot{x}_1(0)$ and for $\frac{d}{dy}\Big|_{y=x_1} \widehat{N}(y) = L_{\mathcal{A}_1}(x_1)$ in the intermediate equation yields the desired result. In anticipation of the graphical interpretation detailed below, we write the resulting formula for a general object frames located at distance r_1 along the line of the contact normal. By convention, r_1 is positive on $\mathcal{B}$'s side of the contact point and negative on $\mathcal{A}_1$'s side.

Theorem 3.5: *Let $\mathcal{B}$ be in contact with a stationary body $\mathcal{A}_1$. Let q be the c-space parametrization due to object frame located at distance r_1 along the contact normal, and let q_0 be $\mathcal{B}$'s contact configuration. Then the* **curvature-form** *of $\mathcal{S}_1$ at q_0 is given by*

$$\kappa_{q_0}(v, \omega) = (v^T, (\widehat{N}_1 \times \omega)^T) \begin{pmatrix} C_{11} & C_{12} \\ C_{12}^T & C_{22} \end{pmatrix} \begin{pmatrix} v \\ \widehat{N}_1 \times \omega \end{pmatrix} \tag{3.6}$$

where $\dot{q} = (v, \omega) \in T_{q_0}\mathcal{S}_1$ and

$$\begin{aligned} C_{11} &= L_{\mathcal{B}_1}\left[L_{\mathcal{A}_1} + L_{\mathcal{B}_1}\right]^{-1} L_{\mathcal{A}_1} \\ C_{12} &= L_{\mathcal{A}_1}\left[L_{\mathcal{A}_1} + L_{\mathcal{B}_1}\right]^{-1} (r_1 L_{\mathcal{B}_1} - I) \\ C_{22} &= (r_1 L_{\mathcal{B}_1} - I)\left[L_{\mathcal{A}_1} + L_{\mathcal{B}_1}\right]^{-1} (r_1 L_{\mathcal{A}_1} + I) \end{aligned}$$

The following corollary is the basis for a graphical interpretation provided below.

Corollary 3.6: *The curvature of $\mathcal{S}_1$ at q_0 along* **pure rotations** *of $\mathcal{B}$ is given in the planar case by*

$$\kappa_{q_0}(0, \omega) = \frac{\left(r_1\kappa_{\mathcal{B}_1} - 1\right)\left(r_1\kappa_{\mathcal{A}_1} + 1\right)}{\kappa_{\mathcal{A}_1} + \kappa_{\mathcal{B}_1}} \quad \text{for } \|\omega\| = 1. \tag{3.7}$$

Interpretation: We can interpret these results in a practical way by considering the 1'st and 2'nd order mobility of a planar object, $\mathcal{B}$, in contact with a planar finger, $\mathcal{A}_1$, at configuration q_0 (Fig. 2(a)). The radii of curvature of $\mathcal{B}$ and $\mathcal{A}_1$ are $\mu_{\mathcal{B}_1} \triangleq 1/\kappa_{\mathcal{B}_1}$ and $\mu_{\mathcal{A}_1} \triangleq 1/\kappa_{\mathcal{A}_1}$ (Fig. 2(c)). Recall that $\dot{q} \in T_q\mathcal{S}_1$ is a free motion which, to 1'st order, maintains contact between $\mathcal{B}$ and $\mathcal{A}_1$. Using the parametrization of screw theory discussed in the appendix, these free motions can be represented by infinitesimal rotations of $\mathcal{B}$ about *any* axis perpendicular to the plane and passing through the line of the contact normal. Consider one $\dot{q} \in T_q\mathcal{S}_1$ which is an infinitesimal rotation about an axis at distance r_1 from the contact point (Fig. 2(a)).

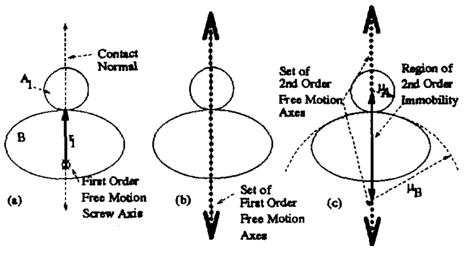

Figure 2: Interpretation of 2'nd Order Mobility

Since $\dot{q} \perp \hat{n}_1(q_0)$, the second order expansion of $d_1(q_0)$ for this $\dot{q}$ is $d_1(q_0) = \frac{1}{2}\dot{q}^T[D\hat{n}_1(q_0)]\dot{q}$. Thus, the sign of $d_1(q_0)$ is solely determined by the curvature-form $\dot{q}^T[D\hat{n}_1(q_0)]\dot{q}$. Let $\mathcal{B}$'s reference frame origin be located where the rotation axis intersects the plane, on the body side of the contact where $r_1 > 0$. In this frame, the instantenous motion is a pure rotation, and thus the curvature form reduces to Eq. (3.7). Its sign, which determines the sign of $d_1(q_0)$ up to 2'nd order, is solely a function of the relative magnitudes of r_1 and $\mu_{\mathcal{B}_1}$. If $r_1 < \mu_{\mathcal{B}_1}$, then $d_1(q_0) < 0$. This implies that $\mathcal{B}$ will penetrate $\mathcal{A}_1$ if we attempt this 1'st-order-free motion. If $r_1 = \mu_{\mathcal{B}_1}$, the two bodies stay in contact (to 2'nd order). If $r_1 > \mu_{\mathcal{B}_1}$, the bodies separate. If the twist axis is located on the finger side of the contact, $r_1 < 0$ and a similar analysis holds, with the curvature of $\mathcal{A}_1$ at the contact point replacing the curvature of $\mathcal{B}$.

Thus, 2'nd order mobility gives us a finer partition of 1'st order mobility predictions. This partition also has a simple geometrical interpretation in the planar case (Fig. 2(c)). All the axes of the 1'st order free twists which lie within $\mu_{\mathcal{B}_1}$ of the contact on $\mathcal{B}$'s side, are 2'nd order immobile–rotation about these axes implies interpenetration of the two bodies. The same is true for those that lie within $\mu_{\mathcal{A}_1}$ on the finger side of the contact. The physical motion associated with twist axes lying further away on the line of the contact normal is breaking of the contact between $\mathcal{B}$ and $\mathcal{A}_1$. The twists lying on the boundary between these regions maintain contact, up to 2'nd order.

3.3. 2'nd Order Mobility Index

It turns out that the 2'nd order free motions of $\mathcal{B}$ can be captured by a coordinated invariant "index". Although it is presented here in the context of 2-fingered equilibrium grasps, reseach currently under progress indicates that it generalizes to all other equilibrium grasps. We shall need the following definition.

Definition The c-space relative curvature *of a two-fingered equilibrium grasp at* q_0 *is*

$$\kappa_1(v,\omega) + \kappa_2(v,\omega) = \dot{q}^T \left[D^2 d_1(q_0) + D^2 d_2(q_0)\right] \dot{q},$$

where $d_i(q)$ *is the signed distance of* q *from* $\mathcal{S}_i$, *for* $i = 1, 2$.

When evaluated on $T_{q_0}\mathcal{S}$, it is shown in [RB] that the signs of the eigenvalues of the c-space relative curvature are ***invariant*** under coordinate transformation. Consequently we define:

Definition 3.7 *The* **2'nd order mobility index** *of* $\mathcal{B}$ *at* q_0 *is the number of non-negative eigenvalues of the relative c-space curvature matrix,*

$$m^2_{q_0}(\mathcal{F}) \triangleq dim\left(\left\{\dot{q} : \dot{q}^T \left[D^2 d_1(q_0) + D^2 d_2(q_0)\right] \dot{q} \geq 0\right\}\right)$$

where $m^2_{q_0}(\mathcal{F})$ *is shorthand notation for* $m^2_{q_0}(\mathcal{CA}_1, \mathcal{CA}_2)$.

$m^2_{q_0}(\mathcal{F})$ attains values in the range $0, 1, 2$ for planar grasps. Let γ_1 and γ_2 be the two eigenvalues of the relative-curvature matrix. If $\gamma_1 > 0$ and $\gamma_2 > 0$, all 1'st order free motions are also 2'nd order free motions. This is the case of the maximal grasp in Figure 3(a). If $\gamma_1 < 0$ and $\gamma_2 < 0$, all 1'st order free motions are 2'nd order immobile (implying $\mathcal{B}$ is immobilized with two frictionless contacts). This can occur for cases where $\mathcal{B}$ is nonconvex, as shown in Figure 3(d). If $\gamma_1 >$ and $\gamma_2 < 0$, the 2'nd order mobility criterion partitions the 1'st order free motions into a 2-dimensional set of motions which cause the bodies to penetrate, a 2-dimensional set which causes the bodies to separate, and a 1-dimensional set which cause roll/slide motions (to 2'nd order). This is the case of the minimal grasp in Figure 3(b). If $\gamma_1 = 0$ and $\gamma_2 < 0$, such as would happen if $\mathcal{B}$ had flat faces at the point of contact (Fig. 3(c)), the 2-dimensional subset of free-motions reduces to a 1-dimensional set. In effect, the geometry of contact reduces the mobility by one degree of freedom.

4. Applications and Discussion

We consider applications of the previous derivations to quasistatic motion analysis and planning. First let us mention a related application domain.

Assembly Planning: The problem of removing a single rigid part (or subassembly) from a given assembly of parts has been addressed in the literature only in the context of 1'st order approximation to the c-space boundary. Furthermore, only translational motions are typically allowed since, it can be shown, 1'st order approximation is too crude for c-space obstacles along their rotational degrees of freedom. We now have a closed-form formula for the 2'nd order approximation in the form of Eq. (3.2). It allows future assembly planners higher quality of approximation, as well as a simple tool for including rotational motions.

Equilibrium Grasp Planning: It can be shown that planar equilibrium grasps are force-closure grasps when the finger/object contact supports arbitrarily small amounts of friction. Force-closure grasps can be robust with respect to external disturbances, and thus planning which employs quasistatic equilibrium grasp configurations is a useful paradigm. Corollary 2.3 gives a precise geometric characterization of equilibrium grasps in c-space, which in turn can be used to construct c-space algorithms for planning equilibrium grasps.

Differentiating Equilibrium Grasps: However, not all equilibrium grasps are alike. Figure 3 depicts the "maximal" (Fig. 3(a)) and "minimal" (Fig. 3(b)) 2-fingered equilibriums grasps of an ellipse (where we assume frictionless contact). Both grasps have the same 1'st order mobility index of $m^1_{q_0}(\mathcal{F}) = 2$. Let q_0 be the equilibrium grasp configuration of $\mathcal{B}$. From Section 3.1, we know that the set of 1'st order free motions is: $M^1_{q_0}(\mathcal{CA}_1, \mathcal{CA}_2) = \bigcap_{i=1}^2 T_{q_0}(\mathcal{S}_i) = T_{q_0}\mathcal{S}_1 = T_{q_0}\mathcal{S}_2$. Let $T_{q_0}\mathcal{S}$ denote $T_{q_0}\mathcal{S}_1$ or $T_{q_0}\mathcal{S}_2$, which are equivalent in this case. Physically, $T_{q_0}\mathcal{S}$ is the 2-dimensional set of instantenous motions free motions which arise when $\mathcal{B}$ is perturbed by external forces.

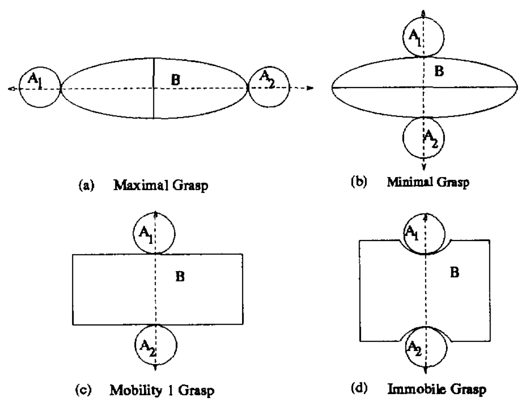

Figure 3: Planar Equilibrium Grasps with Different 2'nd Order Mobility Index

However, intuition suggests that the minimal grasp should be less mobile, and therefore a safer grasp. Figures 4 and 5 show portions of the c-space obstacle boundaries for the maximal and minimal grasps in a region near q_0. For both figures, the reference frame is located at the intersection of the ellipse major and minor axes. Notice in Fig. 5 that the fingers' c-obstacles only touch each other at the equilibrium grasp configuration, while in Fig. 4, the c-obstacles interpenetrate near the equilibrium grasp. It is clear from these c-space pictures that the local mobility of $\mathcal{B}$ at the minimal equilibrium grasp is indeed less than that of the maximal grasp. The 2'nd order index differentiates between these equilibrium grasps. It can be shown that $m^2_{q_0}(\mathcal{F})$ is 2 for the maximal grasp and 1 for the minimal grasp.

A careful planner would use the 2'nd order index to choose equilibrium postures whose degree of mobility is small. Also note that second-order mobility is a function of the object and finger geometry around their contact points. It can possibly be directly measured by suitable tactile sensors.

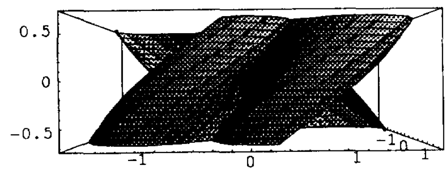

Figure 4: C-space of Minimal 2-Fingered Grasp

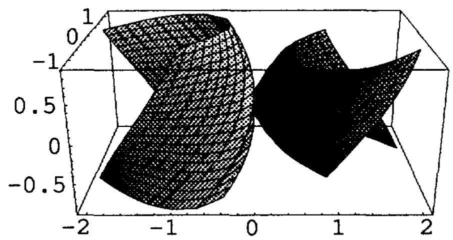

Figure 5: C-space of Maximal 2-Fingered Grasp

The number of frictionless finger contacts required for immobility: The above discussion illustrates that the geometry of contact can effectively lower the mobility of $\mathcal{B}$ at an equilibrium grasp. Figure 6 shows a c-space picture of a 3-fingered equilibrium grasp of an ellipse. The equilibrium configuration is completely surrounded by the finger c-space obstacles, and thus the object is immobile. This example shows that when 2'nd order mobility is taken into account, a planar object can be immobilized with less than the 4 frictionless fingers previously assumed to be necessary [Reuleaux, Mishra et. al.]. Research now under progress supports the following conjecture: if one is free to choose "sufficiently flat" fingers as well as their point of contact, then *almost all bodies can be completely immobilized up to 2'nd order by $n+1$ frictionless fingers*. i.e., 3 fingers (instead of 4) for planar grasps and 4 fingers (instead of 7) for solid grasps.

Active shaping of soft fingers The 2'nd order mobility index, it can be shown, decreases as the fingers' surface become flatter. This provides a justification for active reshaping of fingers curvature. The 2'nd order index also provides a concise tool to determine how much should a given finger surface be flattened to achieve a desired degree of immobility.

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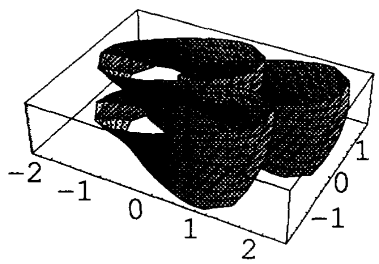

Figure 6: C-space of immobile 3-fingered planar grasp

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6. Appendix: Equivalence of 1'st Order Free Motions and Screw Theory

The 1'st order free motions in $T_q\mathcal{S}_1 \subset M_q^1(\mathcal{CA}_1)$ are equivalent to the ones obtained by the *reciprocal screw principal* of Screw Theory. Screw Theory (see [BR,McCarthy] for details) relies on the fact that a general rigid transformation possesses a unique invariant line, $l \subset \mathbf{R}^3$, with the property that points in l are mapped into l. A motion in which l is an invariant line is called a *screw motion*, and l is called the *screw axis*. Screw motions are further restricted to one parameter groups of motions which have a fixed *pitch*: the ratio of translation along l to rotation about l. Screw motions are straight lines in the *exponential coordinates* parametrization of c-space. This is a parametrization of $SE(3)$ (which is represented by 4×4 homogeneous matrices) by the matrix exponential map $\exp : \mathbf{R}^k \to SE(3)$ such that for $p = (e, \theta)$:

$$p \mapsto \exp\left(\begin{bmatrix} \Omega(\theta) & e \\ 0^T & 0 \end{bmatrix}\right) = \begin{bmatrix} e^{\Omega(\theta)} & d(e,\theta) \\ 0^T & 1 \end{bmatrix} \quad (6.1)$$

where 0^T is a 1×3 row vector of zeroes. The exact form of $d(e,\theta)$ is not important for our purposes. Following convention, the domain $\mathbf{R}^k$ is denoted $se(3)$. For $p = (e,\theta)$, $\exp(p)$ is shorthand for (6.1).

Let $p = (e,\theta)$ be any fixed configuration in exponential coordinates, and consider a straight line $\alpha(t) = p + t\begin{pmatrix} u \\ \omega \end{pmatrix}$, $t \in \mathbf{R}$. (u,ω) determines a unique screw motion, via $\exp(\alpha(t))$, that passes through $\exp(p)$ at $t = 0$. An *instantaneous screw* (or *twist*) at p is any tangent vector $(u,\omega) \in T_p se(3)$. In the traditional kinematics literature, (u,ω) is denoted by $\$$, and is conveniently parametrized in terms of its resulting screw motion as follows [BR]:

$$\$_1 = (\omega, u) = (\widehat{s}_1, \rho_1 \times \widehat{s}_1 + h_1\widehat{s}_1), \quad (6.2)$$

where ω is unit-magnitude, $\widehat{s}_1$ is the direction of the screw axis, $\rho_1 \in \mathbf{R}^3$ is a vector from the origin of a reference frame to any point on the screw axis, and $h_1 = u \cdot \widehat{s}_1$ is the pitch.

Similarly, a covector $\tilde{w} = (\tilde{f}, \tilde{\tau})$ in the dual space $T_p^* se(3)$ is called a *wrench*. $\tilde{w}$ is the generalized force resulting from real-world forces acting on $\mathcal{B}(p)$ when c-space is parametrized by exponential coordinates. The wrench resulting from a unit-magnitude force $F(x_1)$ acting on $\mathcal{B}$ at x_1 is parametrized by

$$\$_2 = \left(\tilde{f}, \tilde{\tau}\right) = (\widehat{s}_2, \rho_2 \times \widehat{s}_2 + h_2\widehat{s}_2) \quad (6.3)$$

where $\widehat{s}_2$ is the direction of $F(x_1)$, $\rho_2 \in \mathbf{R}^3$ is a vector to any point on the line of force, and $h_2 = \tilde{\tau} \cdot \widehat{s}_2$ is the "pitch". For our single-point contact model, $h_2 = 0$, and the wrench is: $\$_2 = (\widehat{s}_2, \rho_2 \times \widehat{s}_2)$.

Consider the object $\mathcal{B}(p)$, where p are exponential coordinates. Let $\$_2 = (\tilde{f}, \tilde{\tau})$ be the wrench resulting from application of force $F(x_1)$. The *reciprocal screws principal* says that instantaneous motion screws $\$_1 = (u,\omega)$ that satisfy $\$_1 \cdot \$_2 \triangleq (\tilde{f},\tilde{\tau}) \cdot (u,\omega) = \tilde{w} \cdot \dot{p} = 0$ are 1'st order free motions with respect to $\$_2$. But $\frac{d}{dt}K(p,\dot{p}) = \tilde{w} \cdot \dot{p}$ according to Eq. (2.3). Since $\tilde{w}$ is directed along the normal, we conclude that the reciprocal screws principal captures the 1'st order free motions in $T_q\mathcal{S}_1$, when exponential coordinates are used to describe $\mathcal{S}_1$.

The screw coordinates are useful for representing the 1'st order free motions of a planar object $\mathcal{B}$. The rotation axis $\widehat{s}_1$ is normal to the plane. Writing the reciprocal screws principal in terms of (6.2) and (6.3) gives: $0 = \$_1 \cdot \$_2 = h_1(\widehat{s}_1 \cdot \widehat{s}_2) + (\rho_1 - \rho_2)\cdot(\widehat{s}_1 \times \widehat{s}_2)$. Since any $\widehat{s}_2 \perp \widehat{s}_1$, the reciprocality condition becomes: $\rho_1 = \rho_2$. Thus, the line containing $\widehat{s}_1$ must intersect the line containing $\widehat{s}_2$. Hence the 1'st order free motions in $T_q\mathcal{S}_1 \subset M_q^1(\mathcal{F})$ are the instantaneous rotations of $\mathcal{B}$ about an axis (perpendicular to the plane) that passes through the line of the force $F(x_1)$ (Fig. 2(b)). When the rotation axis is located "at infinity", we get pure translation perpendicular to the line of $F(x_1)$.