Towards Projectively Equivariant Quantization

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In this paper, we define a natural generalization of the notion of projectively equivariant quantization on a flat space to arbitrary manifolds equipped with arbitrary projective structures. We show how to get such a quantization for the differential operators of order 2 and explain how the method could be adapted to construct a quantization for all the differential operators. In particular, we state a conjecture about the relevant cohomologies that would insure the existence of such quantization in all cases.

§1. Introduction

Basically, a quantization is a way to assign a differential operator to each observable. Observables are smooth functions on the phase space that are polynomial in the momenta. The phase space is the cotangent bundle T^*M of some smooth manifold M. The algebra of observables is thus naturally identified with the space $S(M)$ of smooth contravariant symmetric tensor fields on M. The differential operators associated to observables act on smooth functions on M or on smooth densities, the case of $\frac{1}{2}$-densities being of particular interest. (See Ref. 1) for more informations about that point.) We denote $\mathcal{F}_\lambda$ the space of λ-densities of M and $\mathcal{D}_\lambda(M)$ the algebra of differential operators of M acting between λ-densities.

There are many quantization procedures but none of them is canonical. Technically, this means that no such procedure commutes with the action of the group of diffeomorphisms of the manifold or, in a weaker sense, no quantization procedure is equivariant with respect to the Lie algebra $\mathrm{Vect}(M)$ of vector fields of M acting by Lie derivatives on $\mathcal{S}(M)$ and $\mathcal{D}_\lambda(M)$. In other words, these spaces are isomorphic as vector spaces but not as $\mathrm{Vect}(M)$-modules.

However, on $\mathbb{R}^m$, there are isomorphic as modules over some Lie subalgebras of $\mathrm{Vect}(\mathbb{R}^m)$. Of course, they are isomorphic under the action of the affine Lie subalgebra consisting of the polynomial vector fields of degree at most one. But they are also isomorphic under the action of *maximal* subalgebras of the algebra of polynomial vector fields. For instance, it has been shown [2] that there exists a *unique* sl_{m+1}-equivariant quantization $Q_\lambda : \mathcal{S}(\mathbb{R}^m) \to \mathcal{D}_\lambda(\mathbb{R}^m)$, where sl_{m+1} is the projective embedding of $sl(m+1,\mathbb{R})$, i.e. the Lie subalgebra spanned by the vector fields $\partial_i, x^i\partial_i, x^i \sum_j x^j\partial_j, \quad i,j \in \{1,\cdots,m\}$. A similar result has been obtained [1] for the conformal Lie subalgebra $so(p+1,q+1),\ p+q=m$. See Ref. 4) for a description of the graded maximal subalgebras of the algebra of polynomial vector fields and Ref. 5) for the extentions of the above results to some of these algebras.

Moreover, on an arbitrary manifold M, for the observables of degree 2 and the

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operators of order 2, a bijection $\mathcal{S}^2(M) \to \mathcal{D}^2_\lambda(M)$ has been constructed using a covariant derivation ∇ in such a way that

- the bijection depends only on the projective class of ∇ in the projective case, and on the conformal class of the underlying metric in the conformal case (∇ being then the Levi-Civita covariant derivation),
- the bijection is the restriction of the corresponding sl_{m+1}- or $so(p+1, q+1)$-equivariant quantization when $M = \mathbb{R}^m$ and when ∇ is the canonical flat connection.

(See Ref. 6) in the projective case, and Ref. 8) in the conformal case. A similar result has been obtained for degree 3 observables in Ref. 7).)

These formulae have been found in a purely heuristical and computational way. It is the purpose of the present paper to describe some arguments and methods that could be useful to get generalizations of them to higher degrees, at least in the projective case.

We fix once for all a smooth manifold M of dimension m and a weight λ. The various connections used in the paper are always supposed to be torsion free.

§2. Quantizations

Recall that the space of differential operators $\mathcal{D}_\lambda(M)$ is filtered by the order of differentiation:

$$\mathcal{D}_\lambda(M) = \oplus_{p\geq 0}\mathcal{D}^p_\lambda(M),$$

where $\mathcal{D}^p_\lambda(M)$ denotes the space of differential operators of order at most p. As known, the associated graded space $\mathrm{gr}\mathcal{D}^p_\lambda(M)$ is isomorphic to $\mathcal{S}(M)$: if $A \in \mathcal{D}^p_\lambda(M)$ takes the form

$$A = \sum_{0\leq k\leq p}\ \sum_{i_1,\cdots,i_k} A^{i_1\cdots i_k}\partial_{i_1}\cdots\partial_{i_k}$$

in some local chart, then the element of $\mathcal{S}^p(M)$ defined by

$$\sigma(A)(\xi) = \sum_{i_1,\cdots,i_p} A^{i_1\cdots i_p}\xi_{i_1}\cdots\xi_{i_p}$$

is independent of the chart. It is the *principal symbol* of A and the isomorphism $\mathrm{gr}\mathcal{D}_\lambda(M) \cong \mathcal{S}(M)$ maps A mod $\mathcal{D}^{p-1}_\lambda(M)$ onto $\sigma(A)$.

Remark 1 We usually view $\mathcal{S}(M)$ as a *graded* space, the space $\mathcal{S}^i(M)$ of tensors of degree i being its ith term. It can also be considered as a filtered space, the ith filter being then just $\oplus_{0\leq j\leq i}\mathcal{S}^j(M)$. We shall not use different notations to distinguish the graded space from the filtered space.

Definition 1 A *quantization* of M on λ-densities is a linear map $Q : \mathcal{S}(M) \to \mathcal{D}_\lambda(M)$ such that

- Q is filtered: $Q(\oplus_{0\leq i\leq p}\ \mathcal{S}^i(M)) \subset \mathcal{D}^p_\lambda(M)$,
- Q induces the identity on $\mathrm{gr}\mathcal{D}_\lambda(M)$.

We denote $\mathcal{Q}(M)_\lambda$ the space of all quantizations of M on λ-densities.

It is not hard to construct quantizations. Here is a quite well known way that we call the *Lichnérowicz's construction* as A. Lichnérowicz used it systematically in his development of the theory of star-products. Let ∇ be a covariant derivative of M, associated to some (torsion free) connection. For a given density $\varphi \in \mathcal{F}_\lambda$, the symmetric p-covariant tensor field $\nabla^p\varphi \in \mathcal{S}^p(M) \otimes \mathcal{F}_\lambda$ of M valued in $\mathcal{F}_\lambda$ is defined inductively: $\nabla^1\varphi$ is just the covariant differential $\nabla\varphi$ of φ and $\nabla^{p+1}\varphi$ is the symmetrization of $\nabla(\nabla^p\varphi)$. For each $P \in \mathcal{S}^p(M)$, $\tau_\nabla(P) : \varphi \mapsto \langle P, \nabla^p\varphi\rangle$ is a differential operator with principal symbol P. As easily seen, τ_∇ extends linearly as a quantization on the whole $\mathcal{S}(M)$.

§3. Projective structures on M

Here is a quick introduction to projective structures on arbitrary manifolds. For more details, we refer the reader to Ref. 9).

Definition 2 As geodesics generalize the notion of straight lines in affine spaces, it is quite natural to say that two covariant derivations are *projectively equivalent* if they have the same geodesics (up to parametrizations) and that a *projective structure on M* is an equivalence class of covariant derivations of M.

Denote $\mathcal{C}(M)$ the set of (torsion free) covariant derivations of M. It is an affine space modelled on the vector space $\mathcal{S}^1_2(M)$ of tensor fields of M of type $\binom{1}{2}$ which are symmetric in their covariant indices. For any $S \in \mathcal{S}^1_2(M)$ and any $\nabla \in \mathcal{C}(M)$, we denote $S \cdot \nabla$ the result of the action of S on ∇.

Any 1-form $\omega \in \Omega_1(M)$ of M can be identified with the element $\omega\mathbf{1}$ of $\mathcal{S}^1_2(M)$ defined by

$$\omega\mathbf{1}(X, Y) = \frac{1}{m+1}(\omega(X)Y + \omega(Y)X).$$

($\mathbf{1}$ represents the identity tensor of type $\binom{1}{1}$. The coefficient $\frac{1}{m+1}$ is just a useful normalization factor.) For simplicity, we set $\omega\mathbf{1} \cdot \nabla = \omega \cdot \nabla$. By a theorem of Weyl, two covariant derivations ∇ and ∇' are projectively equivalent if and only if $\nabla' = \omega \cdot \nabla$ for some $\omega \in \Omega_1(M)$. The set of projective structures on M is thus the affine space $\mathcal{C}(M)/\Omega_1(M)$.

The group of diffeomorphisms that preserve a given projective structure on M is a Lie group of dimension at most $(m+1)^2 - 1$. On $\mathbb{R}^m$, the Lie algebra of the Lie group preserving the projective structure of the canonical flat connection is the projective embedding sl_{m+1}.

§4. Projectively equivariant quantizations

Definition 3 A *projectively equivariant quantization* is a map $\mathcal{Q} : \mathcal{C}(M) \to \mathcal{Q}(M)_\lambda$ such that

- a) (*projective invariance*) $\mathcal{Q}_\nabla = \mathcal{Q}_{\nabla'}$, if ∇ and ∇' are projectively equivalent,
- b) (*naturality*) $L_X\mathcal{Q} = 0$ for every $X \in \mathrm{Vect}(M)$.

Combining the above conditions a) and b), we see that if $M = \mathbb{R}^m$ and if ∇ is the canonical flat covariant derivation, then $\mathcal{Q}_\nabla$ is the unique sl_{m+1}-equivariant quantization of λ-densities on $\mathbb{R}^m$. Indeed, since sl_{m+1} is the Lie algebra of the group of diffeomorphisms preserving the projective structure associated to the canonical flat connection, we have then $L_X(\mathcal{Q}_\nabla) = (L_X\mathcal{Q})_\nabla = 0$ for each $X \in sl_{m+1}$.

As we shall see, it is easy to construct mappings from $\mathcal{C}(M)$ into $\mathcal{Q}(M)_\lambda$ that satisfy a) **or** b). The question is to construct such a mapping satisfying **both** conditions.

The quantization τ_∇ given by the Lichnérowicz's construction defines obviously a mapping $\tau : \mathcal{C}(M) \to \mathcal{Q}(M)_\lambda$ satisfying the naturality condition b). It is quite obvious that this map does not verify the projective invariance condition a) (moreover, this will be explicitly shown soon). It can be easily modified in order to satisfy this condition. Doing this without caution, we will however loose the naturality. So, although a map satisfying condition a) can be given immediately, we will explain a method to get it that allow further to restore the naturality, at least for operators of order 2. We believe that it works in general but the question is still open. It is the subject of the thesis under development of one of our students.

§5. Splitting $\mathcal{S}^1_2(M)$

The space $\mathcal{S}^1_2(M) = \Gamma(\vee^2 T^*M \otimes TM)$ has a nice decomposition. It is inherited from a decomposition of each fiber of the bundle $\vee^2 T^*M \otimes TM$ into irreducible components under the action of $\mathrm{gl}(m, \mathbb{R})$. We describe it in terms of the trace mapping.

Definition 4 In any local chart, the *trace* $\mathrm{tr}\ S \in \Omega_1(M) =$ of $S \in \mathcal{S}^1_2(M)$ is defined by contraction: in any local chart of M, it is given by

$$\mathrm{tr}\ S = S^i_{ij} dx^j,$$

if

$$S = S^k_{ij} dx = EE \otimes dx^j \otimes \partial_k.$$

(In the above definition and below, summation over repeated indices is understood.) One has

$$\mathrm{tr}\ (\omega \mathbf{1}) = \omega,\ \forall \omega \in \Omega_1(M).$$

We set $\bar{S} = S - (\mathrm{tr}\ S)\mathbf{1}$ so that $\mathrm{tr}\ \bar{S} = 0$ and

$$S = \bar{S} + (\mathrm{tr}\ S)\mathbf{1},\ \forall S \in \mathcal{S}^1_2(M).$$

The space $\mathcal{S}^1_2(M)$ is then the direct sum of the kernel and the image of tr .

§6. Projective invariance

In order to construct a $\mathcal{Q} : \mathcal{C}(M) \to \mathcal{Q}(M)_\lambda$ verifying the projective invariance condition a) we will compose $\tau : \mathcal{C}(M) \to \mathcal{Q}(M)_\lambda$ with a map defined on $\mathcal{C}(M)$ and valued in the space $\mathcal{B}(\mathcal{S}(M))$ of bijections from $\mathcal{S}(M)$ into itself. As we already said above, we could give this map directly. For later purpose, we will construct it in cohomological fashion.

Definition 5 A map c from $\Omega_1(M)$ into the set of functions from $\mathcal{C}(M)$ into $\mathcal{B}(\mathcal{S}(M))$ is a 1-*cocycle* if

$$c(\omega_1+\omega_2,\nabla)=c(\omega_2,\nabla)\circ c(\omega_1,\omega_2\cdot\nabla),$$

for all $\omega_1,\omega_2\in\Omega_1(M)$ and all $\nabla\in\mathcal{C}(M)$. It is a *coboundary* if it is of the form

$$c(\omega,\nabla)=b_\nabla^{-1}\circ b_{\omega\cdot\nabla}$$

for some $b:\mathcal{C}(M)\to\mathcal{B}(\mathcal{S}(M))$.

Lemma 6 *Each* 1*-cocycle is a coboundary.*

Proof Choose an element $\nabla^0\in\mathcal{C}(M)$. It defines an affine chart $\mathcal{C}(M)\to\mathcal{S}^1_2(M)$: $\nabla\mapsto S_\nabla$, where S_∇ is such that $\nabla=S_\nabla\cdot\nabla^0$. Let c be a cocycle and denote $\hat{c}$ its expression in that chart. Replacing in the cocycle condition ω_1 by ω, ω_2 by tr S_∇ and ∇ by $\overline{S_\nabla}\cdot\nabla^0$, one sees that c is the coboundary of the map $b:\mathcal{C}(M)\to\mathcal{B}(\mathcal{S}(M))$ defined in the chart by

$$\hat{b}(S)=\hat{c}(\mathrm{tr}\ S,\bar{S}).\ \blacksquare \tag{1}$$

The map τ gives rise to a *natural* 1-cocycle

$$c_\tau(\omega,\nabla)=\tau_\nabla^{-1}\circ\tau_{\omega\cdot\nabla}.$$

There is thus some $b:\mathcal{C}(M)\to\mathcal{B}(\mathcal{S}(M))$ such that

$$\tau_{\omega\cdot\nabla}\circ b_{\omega\cdot\nabla}^{-1}=\tau_\nabla\circ b_\nabla^{-1},r$$

meaning that the map $\nabla\mapsto\tau_\nabla\circ b_\nabla^{-1}$ is projectively invariant. From the proof of the above Lemma, it can be easily checked that

$$\tau_\nabla\circ b_\nabla^{-1}=\tau_{\bar{S}\cdot\nabla^0}.$$

Under this form, it is obviously invariant under change of ∇ within the same projective class. Moreover, it is also clear that it is no longer natural. So, we lost naturality in trying to gain projective invariance.

§7. Restoring naturality

From now on, we will assume that $\lambda=0$.

As explained above, if c_τ is the coboudary of b, one can modify τ to get a projectively invariant map from $\mathcal{C}(M)$ into $\mathcal{Q}(M)_\lambda$. To make it both natural and projectively invariant, it suffices to find a map $a:\mathcal{C}(M)\to\mathcal{B}(\mathcal{S}(M))$ *projectively invariant* such that $L_X(a^{-1}\circ b)=0$ for every vector fields X. Indeed, with such a map, we would have

$$c_\tau(\omega,\nabla)=(a_\nabla^{-1}\circ b_\nabla)^{-1}\circ(a_{\omega\cdot\nabla}^{-1}\circ b_{\omega\cdot\nabla}),$$

thus replacing the original b by a natural one. We will show that this is possible at least for the restrictions of the above maps to the polynomials of degree at most 2 and we will explain how this could be generalized.

Any linear map from $\mathcal{S}(M)$ into itself can be represented by an (infinite) matrix. The element of the ith row and the jth column of that matrix is the component of the map that acts from $\mathcal{S}^i(M)$ into $\mathcal{S}^j(M)$. If the map is filtered, then the matrix is upper triangular. In addition, if the map induces the identity on the associated graded space, then the diagonal elements of the matrix are the identity maps of the $\mathcal{S}^i(M)$. (It is convenient to denote them $\mathbf{1}$.) The restriction of the map to the ith filter is of course represented by the $(i+1)\times(i+1)$-submatrix occupying the upper left corner.

Lemma 7 For each $\omega \in \Omega_1(M)$ and each $\nabla \in \mathcal{C}(M)$, the restriction of $c_\mathcal{T}(\omega,\nabla)$ to the polynomials of degree at most 2 is represented by the matrix

$$\begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & -\omega\bar\partial \\ 0 & 0 & \mathbf{1} \end{pmatrix}.$$

(For any 1-form α and any polynomial $P \in \mathcal{S}(M)$, $\alpha\bar\partial P$ denotes the derivative $\sum \alpha_i \frac{\partial P}{\partial \xi_i}$ of P.)

Proof Straightforward computation. ■

It is remarkable that this does not depend on ∇. From the Lemma, we also get immediately the expression of the restriction of the map (1) of which $c_\mathcal{T}$ is a coboundary

$$\begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & -(\mathrm{tr}\ S)\bar\partial \\ 0 & 0 & \mathbf{1} \end{pmatrix}$$

so that, using the notations of the proof of Lemma 6, the restriction of $L_X b$ is represented by the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (\mathrm{tr}\ L_X\nabla^0)\bar\partial \\ 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

We seek for a under the form

$$\begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & \alpha \\ 0 & 0 & \mathbf{1} \end{pmatrix},$$

where, for every $\nabla \in \mathcal{C}(M)$, $\alpha(\nabla)$ is a linear map from $\mathcal{S}^2(M)$ to $\mathcal{S}^1(M)$. The condition $L_X(a^{-1}\circ b) = 0$ is then equivalent to the condition $(L_X\alpha)(\nabla) = (\mathrm{tr}\ L_X\nabla^0)\bar\partial$.

In the following Lemma, we use $\bar\partial^2$ to denote the natural action of any $S \in \mathcal{S}^1_2(M)$ on $\mathcal{S}(M)$, namely

$$S\bar\partial^2 P = S^i_{jk}\xi_i\bar\partial_j\bar\partial_k P.$$

Lemma 8 Let $\nabla \in \mathcal{C}(M)$ be given and denote $D_\nabla = \bar\partial_i\nabla_i : \mathcal{S}(M) \to \mathcal{S}(M)$ the corresponding *divergence* operator. For each $P \in \mathcal{S}(M)$ one has

$$(L_X D_\nabla)P = (\mathrm{tr}\ L_X\nabla)\bar\partial + (L_X\nabla)\bar\partial^2 P. \tag{3}$$

Proof Straightforward computation. ■

Using (3) and

$$(L_X\nabla)\bar{\partial}^2 P = (\overline{L_X\nabla})\bar{\partial}^2 P + \frac{2}{m+1}(\text{tr } L_X\nabla)\bar{\partial}P,$$

we get

$$(\text{tr } L_X\nabla)\bar{\partial} = \frac{m+1}{m+3}L_X D_\nabla - \frac{m+1}{m+3}\overline{L_X\nabla}\bar{\partial}^2.$$

Therefore, $(\text{tr } L_X\nabla^0)\bar{\partial}$ is the Lie derivative in the direction of X of the map α given by

$$\hat{\alpha}(S) = \frac{m+1}{m+3}D_{\nabla^0} + \frac{m+1}{m+3}\bar{S}\bar{\partial}^2.$$

As this map is projectively invariant, we conclude in that way that it is possible to construct a projectively equivariant quantization for the polynomials of degree at most 2.

§8. Going further

The Lie derivative $L_X b$ represented by the matrix (2) is projectively invariant (it even does not depend on ∇). We have shown above that on the polynomials of degree 2, it is the Lie derivative of a projectively invariant map α. In other words, we had a 1-cocycle $X \mapsto L_X b$ of $\mathrm{Vect}(M)$ acting on the space of projectively invariant functions from $\mathcal{C}(M)$ into the mapping from $\mathcal{S}(M)$ into itself and we have seen that it is a coboundary. We did that explicitly. But it could be done directly, using Ref. 3) where the first space of the cohomology of $\mathrm{Vect}(M)$ acting on the space of linear differential operators from $\mathcal{S}^p(M)$ into $\mathcal{S}^q(M)$ itself is computed. It is shown in particular that *it is of dimension* 1 *for* $p = 2$ *and* $q = 1$ so that the non trivial cocycle $(\text{tr } L_X\nabla)\bar{\partial}$ is *a priori* cohomologous to a multiple of the non trivial cocycle $\overline{L_X\nabla}\bar{\partial}^2$.

It is obvious that $L_X b$ is projectively invariant but this is also a corollary of the fact that $c_\mathcal{T}$ is natural. In order to extend the result to higher degree, and ultimately, to get a projectively equivariant quantization, one could try to progressively modify b in such a way that the lines parallel to the main diagonal (these composed by the elements at positions $(i, i+k)$, $i = 0, 1, \cdots$ for $k = 1, 2, \cdots$) in the matrix representing it have natural entries, by induction on k. Assuming that the elements $b_{i,i+k}$, $i = 0, 1, \cdots$ are natural for $k < K$, the naturality of $c_\mathcal{T}$ implies that the $L_X b_{i,i+K}$, $i = 0, 1, \cdots$ are projectively invariant. In order to make them natural, it would suffice to show that these cocycles are coboundaries of projectively invariant maps, as we did above in a very particular case.

The relevant cohomology is that of $\mathrm{Vect}(M)$ acting by Lie derivation on the space of projectively invariant *polynomial* maps from $\mathcal{C}(M)$ valued in the space of linear differential operators from $\mathcal{S}^p(M)$ into $\mathcal{S}^q(M)$. Indeed, as easily seen, $c_\mathcal{T}$ depends in a polynomial fashion on the covariant derivation $\nabla \in \mathcal{C}(M)$. Moreover, the degree of the elements b_{ij} as polynomials in ∇ is easily seen to be controlled by i, j, a feature that could be useful.

We denote

$$\mathrm{Pol}^s(\ \mathcal{C}(M),\ \mathcal{D}(\mathcal{S}^p(M),\mathcal{S}^q(M))\ ),$$

the space of polynomials of degree s on $\mathcal{C}(M)$ valued in the space $\mathcal{D}(\mathcal{S}^p(M),\mathcal{S}^q(M))$ of differential operators between $\mathcal{S}^p(M)$ and $\mathcal{S}^q(M)$, and by

$$\mathrm{Pol}^s(\ \mathcal{C}(M),\ \mathcal{D}(\mathcal{S}^p(M),\mathcal{S}^q(M))\ )^{pi},$$

the subspace of its projectively invariant elements.

Conjecture 9 The inclusion of

$$\mathrm{Pol}^s(\ \mathcal{C}(M),\ \mathcal{D}(\mathcal{S}^p(M),\mathcal{S}^q(M))\ )^{pi} \hookrightarrow \mathrm{Pol}^s(\ \mathcal{C}(M),\ \mathcal{D}(\mathcal{S}^p(M),\mathcal{S}^q(M))\ )$$

induces an isomorphism in cohomology.

If this is true, then we can conclude and produce a projectively equivariant quantization. Indeed, the derivatives $X \mapsto L_X b_{i,i+K}, =\ i = 0, 1, \ldots$ are coboundaries as cochains valued in $\mathrm{Pol}^s(\ \mathcal{C}(M),\ \mathcal{D}(\mathcal{S}^p(M),\mathcal{S}^q(M))\ )$ and we just need them to be coboundaries of elements in $\mathrm{Pol}^s(\ \mathcal{C}(M),\ \mathcal{D}(\mathcal{S}^p(M),\mathcal{S}^q(M))\ )^{pi}$.

Unfortunately, very few is known about the spaces

$$H(\mathrm{Vect}(M), \mathrm{Pol}^s(\ \mathcal{C}(M),\ \mathcal{D}(\mathcal{S}^p(M),\mathcal{S}^q(M))\ ))$$

and

$$H(\mathrm{Vect}(M), \mathrm{Pol}^s(\ \mathcal{C}(M),\ \mathcal{D}(\mathcal{S}^p(M),\mathcal{S}^q(M))\ )^{pi}).$$

In fact, as already said, only the first space of the first is known, for $s = 0$. However, there is some hope to show the conjecture without computing them explicitly, first in $\mathbb{R}^m$ by filtering by the subalgebra slm_{m+1} then globalizing by some classical gluing argument.

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