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Towards the automation of the Local Analytic Sector subtraction

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We present the state of the art of the Local Analytic Sector subtraction. The scheme is now complete at NLO in the massless case for the treatment of initial- and final-state radiations. Its flexibility has been improved by the introduction of *damping factors*, which can be tuned to reduce numerical instabilities, though preserving the simplicity of the algorithm. The same degree of universality has been reached at NNLO for final-state radiation, where we derived fully analytic and compact results for all integrated counterterms. This allows us to explicitly check the cancellation of the virtual infrared singularities in generic processes with massless final-state partons.

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1. Introduction

Automation of QCD computations at NNLO is an ambitious *and* necessary goal, to achieve the required degree of accuracy for theoretical predictions. To this end, one needs to overcome two main bottlenecks: the automation of two-loop corrections and the treatment of infrared (IR) singularities in full generality. While at NLO there are successful general algorithms, such as the *Frixione-Kunzst-Signer (FKS)* [1], *Catani-Seymour (CS)* [2] and *Nagy-Soper* [3, 4] subtraction methods, the complexity of the problem increases at NNLO, where several different methods [5–17] have been developed, however, so far, without reaching the desired degree of generality and automation. We present here recent progress towards the automation of the Local Analytic Sector subtracion [18–20], which attempts to construct a universal local subtraction procedure that is completely analytic and amenable to efficient numerical implementation.

2. Local Analytic Sector subtraction at NLO

Given a partonic process with incoming momenta p_1, p_2 and n particles in the final state, the contribution to the NLO differential cross sections for an IR-safe observable X, can schematically be written as

$$\frac{d\sigma_{\rm NLO} - d\sigma_{\rm LO}}{dX} = \int d\Phi_n \, V \, \delta_n^X + \int d\Phi_{n+1} \, R \, \delta_{n+1}^X + \int d\Phi_n^{x\hat{x}} \, C(x, \hat{x}) \, \delta_n^X \,, \tag{1}$$

with $\delta_i^X \equiv \delta(X - X_i)$, X_i standing for the observable computed with i-body kinematics and $\int d\Phi_n^{x\hat{x}} \equiv \int_0^1 \frac{dx}{x} \int_0^1 \frac{d\hat{x}}{\hat{x}} \int d\Phi_n(xp_1, \hat{x}p_2)$. The first term is the contribution of the virtual corrections, containing explicit ϵ-poles (where $d = 4 - 2\epsilon$ the space-time dimensions), the second term is the phase-space integral in $d\Phi_{n+1}$ of the real squared matrix element, which is affected by phase-space divergences and the last term is the collinear subtraction counterterm stemming from PDF renormalization for the incoming partons. The IR safety of X allows to adopt a subtraction procedure consisting of subtracting a counterterm K from R and of adding it back to V, analytically integrated according to

$$\int d\Phi_{n+1} \, K \, \delta_n^X = \int d\Phi_n \, I \, \delta_n^X + \int d\Phi_n^{x\hat{x}} \, J(x, \hat{x}) \, \delta_n^X \,, \tag{2}$$

and rewrite the contribution of the NLO partonic cross section as

$$\frac{d\sigma_{\rm NLO} - d\sigma_{\rm LO}}{dX} = \int d\Phi_n \, V_{\rm sub} \, \delta_n^X + \int d\Phi_{n+1} \, R_{\rm sub}^X + \int d\Phi_n^{x\hat{x}} \, C_{\rm sub}(x, \hat{x}) \, \delta_n^X \,, \tag{3}$$

where $V_{\rm sub} = V + I$, $R_{\rm sub}^X = R \, \delta_{n+1}^X - K \, \delta_n^X$ and $C_{\rm sub}(x, \hat{x}) = C(x, \hat{x}) + J(x, \hat{x})$ are separately finite in 4 dimensions and free of phase-space divergences. To define the counterterm K, we start with the partition of the $(n+1)$-body phase space through the *sector functions* [1]

$$\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sigma} \,, \qquad \sigma_{ij} = \frac{1}{e_i \, w_{ij}} \,, \qquad \sigma = \sum_{k \in \mathcal{F}} \sum_{l \neq k} \sigma_{kl} \,, \tag{4}$$

where $\mathcal{F}$ $(\mathcal{I})$ is the set of final-(initial-)state partons and $e_i = s_{qi}/s$, $w_{ij} = s \, s_{ij}/s_{qi}/s_{qj}$, where $q = (\sqrt{s}, \vec{0})$ is the partonic center-of-mass momentum. By construction the sector functions sum up to one according to $\sum_{i \in \mathcal{F}} \sum_{j \neq i} \mathcal{W}_{ij} = 1$. Moreover they vanish in all singular limits, except the limit $\mathbf{S}_i$ where particle i becomes soft and the limit $\mathbf{C}_{ij}$ where particles i and j get collinear, so that

$$R - \sum_{i \in \mathcal{F}} \sum_{j \neq i} \Big[\mathbf{S}_i + \mathbf{C}_{ij} (1 - \mathbf{S}_i) \Big] R \, \mathcal{W}_{ij} = \sum_{i \in \mathcal{F}} \sum_{j \neq i} (1 - \mathbf{S}_i)(1 - \mathbf{C}_{ij}) R \, \mathcal{W}_{ij} \; \to \; \text{finite} \,. \tag{5}$$

The combination $\sum_{i\in\mathcal{F}}\sum_{j\neq i}[\mathbf{S}_i+\mathbf{C}_{ij}(1-\mathbf{S}_i)]R\,\mathcal{W}_{ij}$ is thus a good candidate for the required counterterm K, however it factorizes Born squared matrix elements which carry the proper n-body kinematics only in the singular limits. We can overcome the problem by mapping the $(n+1)$-body kinematics into an n-body kinematics and a radiative one, in such a way that the combination $R-K$ remains finite and the analytical integration of K in the radiative phase space is feasible. To this end we conveniently use Catani-Seymour mappings $\{\bar{k}\}^{(abc)}$ [2], which naturally induce a simple factorization of the $d\Phi_{n+1}$ phase space. There are three relevant cases:

Final a,b,c: $\quad s_{ab}=y\bar{s},\quad s_{ac}=z(1-y)\bar{s},\quad s_{bc}=(1-z)(1-y)\bar{s},\qquad \bar{s}\equiv 2\bar{k}_b^{(abc)}\cdot\bar{k}_c^{(abc)},$

$$\int d\Phi_{n+1}(p_1,p_2)=\frac{\varsigma_{n+1}}{\varsigma_n}\,N\int d\Phi_n^{(abc)}(p_1,p_2)\,\bar{s}^{1-\epsilon}\int_0^\pi d\phi\,\sin^{-2\epsilon}\phi\int_0^1 dy\int_0^1 dz\Big[y(1-y)^2\,z(1-z)\Big]^{-\epsilon}(1-y),$$

Final a,b, initial c: $\quad s_{ab}=(1-x)\bar{s},\quad s_{ac}=z\,\bar{s},\quad s_{bc}=(1-z)\bar{s},\qquad \bar{s}\equiv 2\bar{k}_b^{(abc)}\cdot k_c\,,$

$$\int d\Phi_{n+1}(p_1,k_c)=\frac{\varsigma_{n+1}}{\varsigma_n}\,N\int_0^1 dx\int d\Phi_n^{(abc)}(p_1,xk_c)\,\bar{s}^{1-\epsilon}\int_0^\pi d\phi\,\sin^{-2\epsilon}\int_0^1 dz\,[(1-x)\,z(1-z)]^{-\epsilon}\,,$$

Final a, initial b,c: $\quad s_{ab}=(1-x)\bar{s},\quad s_{ac}=(1-x)(1-v)\bar{s},\quad s_{bc}=\bar{s},\qquad \bar{s}\equiv 2k_b\cdot k_c\,,$

$$\int d\Phi_{n+1}(k_b,k_c)=\frac{\varsigma_{n+1}}{\varsigma_n}\,N\int_0^1 dx\int d\Phi_n^{(abc)}(xk_b,k_c)\,\bar{s}^{1-\epsilon}\int_0^\pi d\phi\,\sin^{-2\epsilon}\int_0^1 dv\,[(1-x)^2\,v(1-v)]^{-\epsilon}(1-x)\,,$$

where $N=(4\pi)^{\epsilon-2}/\sqrt{\pi}/\Gamma(1/2-\epsilon)$ and ς_m is the combinatorial factors for a phase space with m final-state particles. To explicitly write the counterterm K, we introduce the *barred* limits $\overline{\mathbf{S}}_i$, $\overline{\mathbf{C}}_{ij}$, to be understood as operators which simultaneously extract the relevant singular limits and convey a specific momentum mapping. Then K and $R_{\rm sub}^X$ are given by

$$K=\sum_{i\in\mathcal{F}}\sum_{j\neq i}\Big[\overline{\mathbf{S}}_i+\overline{\mathbf{C}}_{ij}(1-\overline{\mathbf{S}}_i)\Big]R\,\mathcal{W}_{ij}\,,\qquad R_{\rm sub}^X=R\,\delta_{n+1}^X-K\,\delta_n^X\,. \tag{6}$$

The definition of the barred limits is not unique and we perform our choice with the goal of a trivial analytical integration in the radiative phase spaces $d\Phi_{\rm rad}^{(abc)}$. Of course also an efficient numerical integration of $R_{\rm sub}^X$ is desired, therefore we use again the freedom in defining the barred limits to modulate their behaviour away from the singular region and introduce three parameters $\alpha,\beta,\gamma>0$ in analogy with the α parameter in CS or δ and ξ constants in FKS. This can be done without spoiling the simplicity of the analytical integration, by introducing appropriate *damping factors*. Marking with a bar the quantities depending on barred momenta, we define the barred limits as

$$\begin{aligned}
\overline{\mathbf{S}}_i\,R\,\mathcal{W}_{ij}\;&\equiv\;-2\,\mathcal{N}_1\sum_{c\neq i}\sum_{d\neq i,d<c}\Big\{x^\alpha\theta_{\substack{c\in\mathcal{I}\\ d\in\mathcal{I}}}+(1-z)^\alpha\Big[x^\alpha\theta_{\substack{c\in\mathcal{F}\\ d\in\mathcal{I}}}+(1-y)^\alpha\theta_{\substack{c\in\mathcal{F}\\ d\in\mathcal{F}}}\Big]\Big\}\,\mathcal{E}_{cd}^{(i)}\,\bar{B}_{cd}^{(icd)}\,\mathbf{S}_i\mathcal{W}_{ij}\,,\\
\overline{\mathbf{C}}_{ij}\,R\,\mathcal{W}_{ij}\;&\equiv\;\frac{\mathcal{N}_1}{s_{ij}}\,\Big\{\,\theta_{\substack{j\in\mathcal{I}\\ r\in\mathcal{I}}}\frac{(1-v)^\gamma}{x}\Big(P_{[ij]i}^{(I)}\bar{B}^{(ijr)}+Q_{[ij]i}^{(I)\mu\nu}\bar{B}_{\mu\nu}^{(ijr)}\Big)+\theta_{\substack{j\in\mathcal{I}\\ r\in\mathcal{F}}}\frac{(1-z)^\gamma}{x}\Big(P_{[ij]i}^{(I)}\bar{B}^{(irj)}+Q_{[ij]i}^{(I)\mu\nu}\bar{B}_{\mu\nu}^{(irj)}\Big)\\
&\qquad+\Big[\theta_{\substack{j\in\mathcal{F}\\ r\in\mathcal{I}}}x^\beta+\theta_{\substack{j\in\mathcal{F}\\ r\in\mathcal{F}}}(1-y)^\beta\Big]\Big(P_{ij}\bar{B}^{(ijr)}+Q_{ij}^{\mu\nu}\bar{B}_{\mu\nu}^{(ijr)}\Big)\Big\}\,\mathbf{C}_{ij}\,\mathcal{W}_{ij}\,,\\
\overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}R\,\mathcal{W}_{ij}\;&\equiv\;\mathcal{N}_1\,2\,C_{f_j}\,\mathcal{E}_{jr}^{(i)}\Big\{\,x^\alpha\Big[\theta_{\substack{j\in\mathcal{I}\\ r\in\mathcal{I}}}(1-v)^{\gamma+1}\bar{B}^{(ijr)}+\theta_{\substack{j\in\mathcal{I}\\ r\in\mathcal{F}}}(1-z)^{\gamma-1}\bar{B}^{(irj)}\Big]\\
&\qquad+(1-z)^\alpha\Big[\theta_{\substack{j\in\mathcal{F}\\ r\in\mathcal{I}}}x^\beta+\theta_{\substack{j\in\mathcal{F}\\ r\in\mathcal{F}}}(1-y)^\beta\Big]\bar{B}^{(ijr)}\Big\}\,,
\end{aligned} \tag{7}$$

where $\mathcal{N}_1=8\pi\alpha_{\rm S}(\mu^2e^{\gamma_E}/4/\pi)^\epsilon$ and $\theta_{i\in\mathcal{S}}$ $(\mathcal{S}=\mathcal{I},\mathcal{F})$ is 0 or 1 for particle i belonging to $\mathcal{S}$ or not. The meaning of the Born squared matrix elements B_{cd} and $B_{\mu\nu}$, as well as the expressions for the soft kernel $\mathcal{E}_{cd}^{(i)}=\mathcal{I}_{cd}^{(i)}$ and the final-state Altarelli-Parisi kernels P_{ij}, $Q_{ij}^{\mu\nu}$ can be found in [20]. The initial-state Altarelli-Parisi kernels $P_{[ij]i}^{(I)}$, $Q_{[ij]i}^{(I)\mu\nu}$ read

$$\begin{aligned}
P_{ab}^{(I)}\;&=T_R\left[1-\frac{2x(1-x)}{1-\epsilon}\right](f_a^q f_b^{\bar{q}}+f_a^{\bar{q}}f_b^q)+C_F\left[2\,\frac{x}{1-x}+(1-\epsilon)(1-x)\right](f_a^q+f_a^{\bar{q}})f_b^g\\
&\quad+C_F\left[2\,\frac{1-x}{x}+(1-\epsilon)\,x\right]f_a^g(f_b^q+f_b^{\bar{q}})+2\,C_A\left[\frac{x}{1-x}+\frac{1-x}{x}+x(1-x)\right]f_a^g f_b^g\,,
\end{aligned}$$

$$Q_{ab}^{(I)\mu\nu} = 2\,\frac{1-x}{x}\left[C_F\,(f_a^q f_b^{\bar q} + f_a^{\bar q} f_b^q) - C_A\, f_a^g f_b^g\right]\left[-g^{\mu\nu} + (d-2)\frac{\tilde k_I^\mu \tilde k_I^\nu}{\tilde k_I^2}\right], \qquad \tilde k_I^\mu = \frac{1}{s_{jr}}\left(s_{jr} k_i^\mu - s_{ir} k_j^\mu - s_{ij} k_r^\mu\right),$$

where f_i^* $(* = q, \bar q, g)$ is 1 or 0 for particle i being of type $*$ or not. After having subtracted the counterterm K, we add back its integrated form according to Eq. (2) and get for I and $J(x,\hat x)$:

No initial-state partons: $\quad I = I^F, \qquad J(x,\hat x) = 0,$

$$I^F = \frac{\alpha_S}{2\pi}\left\{\frac{1}{\epsilon^2}\Sigma_C\, B + \frac{1}{\epsilon}\left[\Sigma_\gamma\, B + \sum_{c,d\neq c} L_{cd}\, B_{cd}\right] + \left[\Sigma_\phi - \sum_j \gamma_j^{\rm hc} L_{jr}\right] B + \sum_{c,d\neq c} L_{cd}\left[2 - \frac{L_{cd}}{2}\right] B_{cd} \right.$$
$$\left. + \left[2A_\alpha^{(2)} \sum_j C_{f_j} L_{jr} + \Sigma_C\, A_\alpha^{(2)}\left(A_\alpha^{(2)} - 2A_\beta^{(2)}\right) - \Sigma_C\, A_\alpha^{(3)} + \Sigma_\gamma^{\rm hc} A_\beta^{(2)}\right] B + 2A_\alpha^{(2)} \sum_{c,d\neq c} L_{cd} B_{cd}\right\}; \tag{8}$$

One initial-state parton: $\quad I = I^F + I^I, \qquad J(x,\hat x) = J^I(x)\,\delta(1-\hat x) + J^I(\hat x)\,\delta(1-x),$

$$I^I = \frac{\alpha_S}{2\pi}\, 2C_{f_a}\left[-2 + 2\zeta_2 - A_\alpha^{(2)}\left(A_\gamma^{(1)} - A_\beta^{(2)} - 1\right) + A_\alpha^{(3)}\right] B, \tag{9}$$

$$J^I(x) = \frac{\alpha_S}{2\pi}\left\{-\left[\frac{1}{\epsilon} - L_{ar}\right] P_a(x) + P_a^{(1)}(x) + \left(\frac{x^{1+\beta}}{1-x}\right)_+ \left[\gamma_a^{\rm hc} - \Sigma_\gamma^{\rm hc} + 2(\Sigma_C - C_{f_a})\, A_\alpha^{(2)}\right] - 2\, C_{f_a}\left(\frac{x}{1-x}\right)_+ A_\gamma^{(1)}\right.$$
$$\left. + 2\, C_{f_a}\left[\left(\frac{x\ln(1-x)}{1-x}\right)_+ + \left(\frac{x^{1+\alpha}}{1-x}\right)_+ \left(A_\gamma^{(1)} - A_\alpha^{(2)} - 1 - L_{ar}\right)\right]\right\} B - \frac{\alpha_S}{2\pi}\left(\frac{x^{1+\alpha}}{1-x}\right)_+ \sum_{j\in\mathcal F} 2\, L_{aj} B_{aj};$$

Two initial-state partons: $\quad I = I^F + I^{II}, \qquad J(x,\hat x) = J^{II}(x)\,\delta(1-\hat x) + J^{II}(\hat x)\,\delta(1-x),$

$$I^{II} = \frac{\alpha_S}{2\pi}\left\{(C_{f_a} + C_{f_b})\left[4(\zeta_2 - 1) - A_\alpha^{(2)}\left(2A_\gamma^{(1)} - 2A_\beta^{(2)} + A_\alpha^{(2)}\right) + 3A_\alpha^{(3)}\right] B + 4\left[(\zeta_2 - 1) + A_\alpha^{(3)}\right] B_{ab}\right\}, \tag{10}$$

$$J^{II}(x) = \frac{\alpha_S}{2\pi}\left\{-\left[\frac{1}{\epsilon} - L_{ar}\right] P_a(x) + P_a^{(2)}(x) + \left(\frac{x^{1+\beta}}{1-x}\right)_+ \left[\gamma_a^{\rm hc} + \gamma_b^{\rm hc} - \Sigma_\gamma^{\rm hc} + 2\left(\Sigma_C - C_{f_a} - C_{f_b}\right) A_\alpha^{(2)}\right]\right.$$
$$\left. + 2C_{f_a}\left[2\left(\frac{x\ln(1-x)}{1-x}\right)_+ - \left(\frac{x^{1+\alpha}\ln(1-x)}{1-x}\right)_+ - \left(\frac{x}{1-x}\right)_+ A_\gamma^{(1)} + \left(\frac{x^{1+\alpha}}{1-x}\right)_+ \left(A_\gamma^{(1)} - A_\alpha^{(2)} - 1 - L_{ab}\right)\right]\right\} B$$
$$- \frac{\alpha_S}{2\pi}\left\{2\left[\left(\frac{x^{1+\alpha}\ln(1-x)}{1-x}\right)_+ + \left(\frac{x^{1+\alpha}}{1-x}\right)_+ \left(A_\alpha^{(2)} + 1 + L_{ab}\right)\right] B_{ab} + \left(\frac{x^{1+\alpha}}{1-x}\right)_+ \sum_{j\in\mathcal F} 2\, L_{aj} B_{aj}\right\}.$$

The single pole of $J(x,\hat x)$ is correctly given by the regularized Altarelli-Parisi probabilities (see Ref. [2]) $P_a = (f_a^q + f_a^{\bar q})(P^{qq} + P^{qg}) + f_a^g (P^{gq} + P^{gg})$, cancelling the corresponding one in $C(x,\hat x)$. The other shorthand notations are given by

$$P_a^{(m)}(x) = f_a^g\left\{\left[T_R\left(x^2 + (1-x)^2\right) + 2C_A\left(\frac{1-x}{x} + x(1-x)\right)\right]\left(m\ln(1-x) - A_\gamma^{(1)}\right) + T_R\, 2x(1-x)\right\}$$
$$+ (f_a^q + f_a^{\bar q})\, C_F\left\{\left[\frac{1 + (1-x)^2}{x} + 1 - x\right]\left(m\,\ln(1-x) - A_\gamma^{(1)}\right) + x\right\}, \qquad (m = 1, 2)$$

$$\Sigma_C = \sum_a C_{f_a}, \quad \gamma_a = \frac{3}{2}C_F\,(f_a^q + f_a^{\bar q}) + \frac{1}{2}\beta_0 f_a^g, \quad \Sigma_\gamma = \sum_a \gamma_a, \quad \gamma_a^{\rm hc} = \gamma_a - 2C_{f_a}, \quad \Sigma_\gamma^{\rm hc} = \sum_a \gamma_a^{\rm hc},$$
$$\phi_a^{\rm hc} = \frac{13}{3}C_F\,(f_a^q + f_a^{\bar q}) + \frac{4}{3}\beta_0 f_a^g - \frac{16}{3}C_{f_a}, \quad \Sigma_\phi^{\rm hc} = \sum_a \phi_a^{\rm hc}, \quad \phi_a = \phi_a^{\rm hc} + \left(6 - \frac{7}{2}\zeta_2\right) C_{f_a}, \quad \Sigma_\phi = \sum_a \phi_a,$$
$$L_{ij} = \ln\frac{s_{ij}}{\mu^2}, \quad A_\xi^{(1)} = \gamma_E + \Psi^{(0)}(\xi + 1), \quad A_\xi^{(2)} = \gamma_E - 1 + \Psi^{(0)}(\xi + 2), \quad A_\xi^{(3)} = \Psi^{(1)}(\xi + 2) + 1 - \zeta_2.$$

The subtraction procedure at NLO has been implemented in MadNkLO, a Python framework to automatically generate all ingredients for NLO and NNLO computations. We could check the efficiency of the cancellation of phase-space singularities up to $pp \to 3\,$j and validate the method with MadGraph5 [21] on physical cross-sections, such as $e^+e^- \to 2\,$j and $pp \to Z\,$j.

3. Local Analytic Sector subtraction at NNLO for final-state radiation

Moving to NNLO, we consider a process with n partons in the final state only. In this case, the NNLO contribution to the differential cross section for an IR-safe observable X reads

$$\frac{d\sigma_{\text{NNLO}} - d\sigma_{\text{NLO}}}{dX} = \int d\Phi_n \, VV \, \delta_n^X + \int d\Phi_{n+1} \, RV \, \delta_{n+1}^X + \int d\Phi_{n+2} \, RR \, \delta_{n+2}^X , \tag{11}$$

where RR, RV, and VV are the double-real, real-virtual and double-virtual squared matrix elements respectively. The sum of these three contributions is finite due to the IR safety of X, but separately they feature phase-space singularities in the extra real radiations (RR and RV) and explicit virtual singularities (RV and VV). A subtraction procedure at NNLO consists in modifying Eq. (11) by adding and subtracting appropriate counterterms to get

$$\frac{d\sigma_{\text{NNLO}} - d\sigma_{\text{NLO}}}{dX} = \int d\Phi_n \, VV_{\text{sub}} \, \delta_n^X + \int d\Phi_{n+1} \, RV_{\text{sub}}^X + \int d\Phi_{n+2} \, RR_{\text{sub}}^X , \tag{12}$$

where the three contributions are separately free of phase-space singularities and of ϵ poles. Our first step in setting up the subtracted double real squared matrix element is to make a partition of the double-real phase space introducing the *sector functions*

$$\mathcal{W}_{ijkl} = \frac{\sigma_{ijkl}}{\sigma} , \qquad \sigma = \sum_{a,\, b\neq a} \sum_{c\neq a} \sum_{d\neq a,c} \sigma_{abcd} , \qquad \sigma_{abcd} = \frac{1}{(e_a w_{ab})^\alpha} \frac{1}{(e_c + \delta_{bc} e_a) w_{cd}} , \quad \alpha > 1 , \tag{13}$$

which minimize the number of singular phase-space limits in each sector[1]:

$$\begin{aligned}
(1 - \mathbf{S}_i)\big(1 - \mathbf{C}_{ij}\big)\big(1 - \mathbf{S}_{ij}\big)\big(1 - \mathbf{C}_{ijk}\big)\big(1 - \mathbf{SC}_{ijk}\big) \; RR\, \mathcal{W}_{ijjk} &\quad \to \quad \text{finite} , \\
(1 - \mathbf{S}_i)\big(1 - \mathbf{C}_{ij}\big)(1 - \mathbf{S}_{ik})\big(1 - \mathbf{C}_{ijk}\big)\big(1 - \mathbf{SC}_{ijk}\big)\big(1 - \mathbf{SC}_{kij}\big) \; RR\, \mathcal{W}_{ijkj} &\quad \to \quad \text{finite} , \\
(1 - \mathbf{S}_i)\big(1 - \mathbf{C}_{ij}\big)(1 - \mathbf{S}_{ik})\big(1 - \mathbf{C}_{ijkl}\big)(1 - \mathbf{SC}_{ikl})\big(1 - \mathbf{SC}_{kij}\big) \; RR\, \mathcal{W}_{ijkl} &\quad \to \quad \text{finite} .
\end{aligned} \tag{14}$$

Considering that $RR = \sum_{i,\, j\neq i} \sum_{k\neq i} \sum_{l\neq i,k} RR\, \mathcal{W}_{ijkl}$, we can construct the counterterms, in analogy with the NLO case, by introducing the mappings of the momenta. To this end we choose nested Catani-Seymour final-state mappings $\{k\} \to \{\bar{k}\}^{(abc)} \to \{\bar{k}\}^{(abc,def)}$ (see Ref. [20]), which involve a minimal set of $(n+2)$-momenta and are built in terms of Mandelstam invariants. This allows to define a subtracted double-real squared matrix element RR_{sub}^X as sum over sectors of quantities that are finite in the whole $(n+2)$-body phase space. It is also possible to perform the sum of the sector functions and get the following compact expression for RR_{sub}^X:

$$RR_{\text{sub}}^X = RR\, \delta_{n+2}^X - K^{(\mathbf{1})}\, \delta_{n+1}^X - \Big(K^{(\mathbf{2})} + K^{(\mathbf{12})}\Big)\, \delta_n^X , \tag{15}$$

The counterterm $K^{(\mathbf{1})}$ reads ($r \neq i, j$)

$$K^{(\mathbf{1})} = -\mathcal{N}_1 \sum_i \sum_{c\neq i} \sum_{d\neq i,c} \mathcal{E}_{cd}^{(i)}\, \bar{R}_{cd}^{(icd)} + \mathcal{N}_1 \sum_{i,\, j>i} \frac{P_{ij(r)}^{\text{hc},\mu\nu}}{s_{ij}}\, \bar{R}_{\mu\nu}^{(ijr)} , \tag{16}$$

while for $K^{(\mathbf{2})}$ and $K^{(\mathbf{12})}$ we have ($r \neq i, j, k;\;\; r' \neq i, j, k, l$)

$$\begin{aligned}
K^{(\mathbf{2})} = {} & \sum_{i,j>i} \frac{\mathcal{N}_1^2}{2} \sum_{\substack{c\neq i,j \\ d\neq i,j,c}} \bigg\{ \mathcal{E}_{cd}^{(i)} \bigg[\sum_{\substack{e\neq i,j,c,d \\ f\neq i,j,c,d,e}} \mathcal{E}_{ef}^{(j)}\, \bar{B}_{cdef}^{(icd,jef)} + \sum_{e\neq i,j,c,d} 4\, \mathcal{E}_{ed}^{(j)}\, \bar{B}_{cded}^{(icd,jed)} + 2\, \mathcal{E}_{cd}^{(j)}\, \bar{B}_{cdcd}^{(icd,jcd)} \bigg] + \mathcal{E}_{cd}^{(ij)}\, \bar{B}_{cd}^{(ijcd)} \bigg\} , \\
& - \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ k>j}} \mathcal{N}_1^2 \frac{P_{jk(r)}^{\text{hc},\mu\nu}}{s_{jk}} \bigg\{ \sum_{c\neq i,j,k,r} \bigg[\sum_{d\neq i,j,k,r,c} \frac{\mathcal{E}_{cd}^{(i)}}{2} \bar{B}_{\mu\nu,cd}^{(jkr,icd)} + \mathcal{E}_{cr}^{(i)}\, \bar{B}_{\mu\nu,cr}^{(jkr,icr)} \bigg] + C_{f_{[jk]}}\, \rho_{jk}^{(C)}\, \mathcal{E}_{jr}^{(i)}\, \bar{B}_{\mu\nu}^{(krj,ijr)} \\
& \qquad\qquad + \sum_{c\neq i,j,k} \mathcal{E}_{jc}^{(i)} \Big[\rho_{jk}^{(C)}\, \bar{B}_{\mu\nu,[jk]c}^{(krj,icj)} + \bar{\tilde{B}}_{\mu\nu,[jk]c}^{(krj,icj)} \big(f_j^q f_k^{\bar{q}} - f_j^{\bar{q}} f_k^q\big) \Big] + (j \leftrightarrow k) \bigg\} \\
& + \sum_{i,\, j>i} \sum_{k>j} \mathcal{N}_1^2 \frac{P_{ijk(r)}^{\text{hc},\mu\nu}}{s_{ijk}^2}\, \bar{B}_{\mu\nu}^{(ijkr)} + \sum_{i,\, j>i} \sum_{\substack{k\neq j \\ k>i}} \sum_{\substack{l\neq j \\ l>k}} \mathcal{N}_1^2 \frac{P_{ij(r')}^{\text{hc},\mu\nu}}{s_{ij}} \frac{P_{kl(r')}^{\text{hc},\rho\sigma}}{s_{kl}}\, \bar{B}_{\mu\nu\rho\sigma}^{(ijr',klr')} ,
\end{aligned} \tag{17}$$

[1] We introduce the uniform limits $\mathbf{S}_{ab}$ (where particles a and b become uniformly soft), $\mathbf{C}_{abc}$ (where particles a, b, c become uniformly collinear), $\mathbf{C}_{abcd}$ (where particles a, b and particles c, d become pairwise uniformly collinear) and $\mathbf{SC}_{abc}$ (where particle a becomes soft at the same rate as particles b and c become collinear).

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$$
\begin{aligned}
K^{(\mathbf{12})} = & -\mathcal{N}_1^{\,2}\sum_{i,j\neq i}\,\sum_{\substack{c\neq i,j\\ d\neq i,j,c}}\bigg\{\,\mathcal{E}^{(i)}_{cd}\bigg[\sum_{e\neq i,j,c,d}\bigg(\frac{1}{2}\sum_{f\neq i,j,c,d,e}\bar{\mathcal{E}}^{(j)(icd)}_{ef}\,\bar{B}^{(icd,jef)}_{cdef}+\bar{\mathcal{E}}^{(j)(icd)}_{ed}\,\bar{B}^{(icd,jed)}_{cded}\bigg)\\
&\qquad+\sum_{e\neq i,j,c,d}\bar{\mathcal{E}}^{(j)(idc)}_{ed}\,\bar{B}^{(idc,jed)}_{cded}+\bar{\mathcal{E}}^{(j)(icd)}_{cd}\Big(\bar{B}^{(icd,jcd)}_{cdcd}+C_A\bar{B}^{(icd,jcd)}_{cd}\Big)\bigg]\\
&\qquad-C_A\Big(\mathcal{E}^{(i)}_{jc}\,\bar{\mathcal{E}}^{(j)(icj)}_{cd}\,\bar{B}^{(icj,cjd)}_{cd}+\mathcal{E}^{(i)}_{jd}\,\bar{\mathcal{E}}^{(j)(ijd)}_{cd}\,\bar{B}^{(ijd,cjd)}_{cd}\Big)\bigg\}\\
&+\mathcal{N}_1^{\,2}\sum_{i,j\neq i}\,\sum_{\substack{k\neq i\\ k>j}}\bigg\{\sum_{\substack{c\neq i,j,k\\ d\neq i,j,k,c}}\mathcal{E}^{(i)}_{cd}\,\frac{\bar{P}^{(icd)\mathrm{hc},\mu\nu}_{jk(r)}}{2\,\bar{s}^{(icd)}_{jk}}\,\bar{B}^{(icd,jkr)}_{\mu\nu,cd}+C_{f_{[jk]}}\,\rho^{(C)}_{[jk]}\,\mathcal{E}^{(i)}_{jk}\,\frac{\bar{P}^{(ijk)\mathrm{hc},\mu\nu}_{jk(r)}}{2\,\bar{s}^{(ijk)}_{jk}}\bar{B}^{(ijk,jkr)}_{\mu\nu}\\
&\qquad-C_{f_{[jk]}}\rho^{(C)}_{jk}\mathcal{E}^{(i)}_{jr}\bigg[\frac{\bar{P}^{(ijr)\mathrm{hc},\mu\nu}_{jk(r)}}{2\,\bar{s}^{(ijr)}_{jk}}\Big(\bar{B}^{(ijr,jkr)}_{\mu\nu}-\bar{B}^{(ijr,krj)}_{\mu\nu}\Big)+\frac{\bar{P}^{(irj)\mathrm{hc},\mu\nu}_{jk(r)}}{2\,\bar{s}^{(irj)}_{jk}}\Big(\bar{B}^{(irj,jkr)}_{\mu\nu}-\bar{B}^{(irj,krj)}_{\mu\nu}\Big)\bigg]\\
&\qquad+\sum_{c\neq i,j,k}\mathcal{E}^{(i)}_{jc}\frac{\bar{P}^{(ijc)\mathrm{hc},\mu\nu}_{jk(r)}}{2\,\bar{s}^{(ijc)}_{jk}}\Big[\rho^{(C)}_{jk}\,\bar{B}^{(ijc,krj)}_{\mu\nu,[jk]c}+\tilde{\bar{B}}^{(ijc,krj)}_{\mu\nu,[jk]c}\,(f^q_j f^{\bar q}_k-f^{\bar q}_j f^q_k)\Big]\\
&\qquad+\sum_{c\neq i,j,k}\mathcal{E}^{(i)}_{jc}\frac{\bar{P}^{(icj)\mathrm{hc},\mu\nu}_{jk(r)}}{2\,\bar{s}^{(icj)}_{jk}}\Big[\rho^{(C)}_{jk}\,\bar{B}^{(icj,krj)}_{\mu\nu,[jk]c}+\tilde{\bar{B}}^{(icj,krj)}_{\mu\nu,[jk]c}\,(f^q_j f^{\bar q}_k-f^{\bar q}_j f^q_k)\Big]+(j\leftrightarrow k)\bigg\}\\
&+\mathcal{N}_1^{\,2}\sum_{i,\,j>i}\bigg\{\sum_{c\neq i,j}\,\sum_{d\neq i,j,c}\bigg[\frac{P^{\,\mathrm{hc}}_{ij(r)}}{s_{ij}}\,\bar{\mathcal{E}}^{(j)(ijr)}_{cd}+\frac{Q^{\mu\nu}_{ij(r)}}{s_{ij}}\bigg(\frac{\bar{k}^{(ijr)}_{c,\mu}}{\bar{s}^{(ijr)}_{jc}}-\frac{\bar{k}^{(ijr)}_{d,\mu}}{\bar{s}^{(ijr)}_{jd}}\bigg)\bigg(\frac{\bar{k}^{(ijr)}_{c,\nu}}{\bar{s}^{(ijr)}_{jc}}-\frac{\bar{k}^{(ijr)}_{d,\nu}}{\bar{s}^{(ijr)}_{jd}}\bigg)\bigg]\bar{B}^{(ijr,jcd)}_{cd}\\
&\qquad+\frac{P^{\mathrm{hc},\mu\nu}_{ij(r)}}{s_{ij}}\sum_{k\neq i,j}\bigg[\sum_{\substack{c\neq i,j,k,r\\ d\neq i,j,k,r,c}}\bar{\mathcal{E}}^{(k)(ijr)}_{cd}\bar{B}^{(ijr,kcd)}_{\mu\nu,cd}+\sum_{c\neq i,j,k,r}2\,\bar{\mathcal{E}}^{(k)(ijr)}_{cr}\bar{B}^{(ijr,kcr)}_{\mu\nu,cr}+\sum_{c\neq i,j,k}2\,\bar{\mathcal{E}}^{(k)(ijr)}_{jc}\bar{B}^{(ijr,kcj)}_{\mu\nu,jc}\bigg]\\
&\qquad-\sum_{k>j}\frac{P^{\,\mathrm{hc}}_{ij(r)}}{s_{ij}}\,\frac{\bar{P}^{(ijr)\mathrm{hc},\mu\nu}_{jk(r)}}{\bar{s}^{(ijr)}_{jk}}\,\bar{B}^{(ijr,jkr)}_{\mu\nu}-\sum_{k\neq i,j}\,\sum_{\substack{l\neq i,j,k\\ l>k}}P^{\mathrm{hc},\mu\nu}_{ij(r')}\frac{\bar{P}^{(ijr')\mathrm{hc},\rho\sigma}_{kl(r')}}{\bar{s}^{(ijr')}_{kl}}\,\bar{B}^{(ijr',klr')}_{\mu\nu\rho\sigma}\bigg\},
\end{aligned}
\tag{18}
$$

with $\rho^{(C)}_{ab}=\frac{C_{f_{[ab]}}+C_{f_a}-C_{f_b}}{C_{f_{[ab]}}}$ and $\rho^{(C)}_{[ab]}=\frac{C_{f_{[ab]}}-C_{f_a}-C_{f_b}}{C_{f_{[ab]}}}$. The soft and hard-collinear kernels are

$$
\mathcal{E}^{(ij)}_{cd}=\mathcal{I}^{(ij)}_{cd}-\frac{1}{2}\,\mathcal{I}^{(ij)}_{cc}-\frac{1}{2}\,\mathcal{I}^{(ij)}_{dd}\,,\qquad P^{\mathrm{hc},\mu\nu}_{ij(r)}=P^{\mu\nu}_{ij,\mathcal{F}}\Big(\frac{s_{ir}}{s_{[ij]r}}\Big)+s_{ij}\Big[2\,C_{f_j}\,\mathcal{E}^{(i)}_{jr}+2\,C_{f_i}\,\mathcal{E}^{(j)}_{ir}\Big]g^{\mu\nu}\,,
$$

$$
P^{\mathrm{hc},\mu\nu}_{ijk(r)}=-\Big\{P_{ijk}-s^2_{ijk}\Big[C_{f_k}\Big(4\,C_{f_k}\,\mathcal{E}^{(i)}_{kr}\,\mathcal{E}^{(j)}_{kr}-\mathcal{E}^{(ij)}_{kr}\Big)+(i\leftrightarrow k)+(j\leftrightarrow k)\Big]\Big\}g^{\mu\nu}+Q^{\mu\nu}_{ijk}\,,
$$

where $\mathcal{I}^{(ij)}_{cd}$, P_{ijk} and $Q^{\mu\nu}_{ijk}$ have been computed in [22] and can be found written in our notation in [20] (as can B_{cdef}). We notice the appearance of $\tilde{B}_{cd}=f^g_c\,\mathcal{A}^{(0)*}_n\tilde{\mathbf{T}}_c\cdot\mathbf{T}_d\mathcal{A}^{(0)}_n$ with $(\tilde{T}^A)_{BC}=d_{ABC}$, where $\mathcal{A}^{(0)}_n$ is the Born amplitude. As explained in [20], nested Catani-Seymour mappings lead to a natural factorization of the $(n+2)$-body phase space in an $(n+1)$-body phase space times a single radiation phase space, and in an n-body phase space times a double radiation phase space. These properties allow to analytically integrate $K^{(\mathbf{1})}$ and $K^{(\mathbf{12})}$ in the single radiation phase space and $K^{(\mathbf{2})}$ in the double radiation phase space, following the techniques explained in [20], and thus to compute $I^{(\mathbf{1})}$, $I^{(\mathbf{12})}$ and $I^{(\mathbf{2})}$ defined by

$$
\int d\Phi_{n+1}I^{(\mathbf{1})}\equiv\int d\Phi_{n+2}\,K^{(\mathbf{1})},\qquad\int d\Phi_n I^{(\mathbf{2})}\equiv\int d\Phi_{n+2}\,K^{(\mathbf{2})},\qquad\int d\Phi_{n+1}I^{(\mathbf{12})}\equiv\int d\Phi_{n+2}\,K^{(\mathbf{12})}.\tag{19}
$$

After subtracting $K^{(\mathbf{1})}$, $K^{(\mathbf{2})}$, $K^{(\mathbf{12})}$ to RR and adding them back in their integrated versions $I^{(\mathbf{1})}$, $I^{(\mathbf{2})}$, $I^{(\mathbf{12})}$, the NNLO contribution to the differential cross section reads

$$
\frac{d\sigma_{\mathrm{NNLO}}-d\sigma_{\mathrm{NLO}}}{dX}=\int d\Phi_n\Big[VV+I^{(\mathbf{2})}\Big]\delta^X_n+\int d\Phi_{n+1}\Big[\Big(RV+I^{(\mathbf{1})}\Big)\delta^X_{n+1}+I^{(\mathbf{12})}\delta^X_n\Big]+\int d\Phi_{n+2}\,RR^X_{\mathrm{sub}}\,,\tag{20}
$$

where $RR^X_{\rm sub}$, given in Eq. (15), is free of phase-space singularities and can be safely integrated in four dimensions. The integrand of the $d\Phi_{n+1}$ phase-space integral still features explicit ϵ poles as well as phase-space singularities, but the following properties hold:

$$RV + I^{(\mathbf{1})} \text{ is free of } \epsilon \text{ poles}\,; \qquad I^{(\mathbf{1})}\,\delta^X_{n+1} + I^{(\mathbf{12})}\,\delta^X_n \text{ is phase-space finite}\,. \tag{21}$$

Thanks to these properties of $I^{(\mathbf{1})}$ and $I^{(\mathbf{12})}$, it is evident that, to get a subtracted real-virtual squared matrix element $RV^X_{\rm sub}$, free of ϵ poles and finite in the whole phase space, we can subtract a real-virtual counterterm $K^{(\mathbf{RV})}$ to $\big(RV + I^{(\mathbf{1})}\big)\,\delta^X_{n+1} + I^{(\mathbf{12})}\,\delta^X_n$, which must satisfy

$$K^{(\mathbf{RV})} + I^{(\mathbf{12})} \text{ is free of } \epsilon \text{ poles}\,; \qquad RV\,\delta^X_{n+1} - K^{(\mathbf{RV})}\,\delta^X_n \text{ is phase-space finite}\,. \tag{22}$$

In analogy with the NLO counterterm K, we take $K^{(\mathbf{RV})}$ of the form

$$K^{(\mathbf{RV})} \;\equiv\; \sum_{i,j\neq i} \Big\{ \Big[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} \Big(1 - \overline{\mathbf{S}}_i\Big) \Big] RV\, \mathcal{W}_{ij} + \Delta_{ij} \Big\}\,, \tag{23}$$

where we have introduced a shift Δ_{ij} which is finite under $\mathbf{S}_i$, $\mathbf{C}_{ij}$ and cancels the remaining poles, so that the conditions (22) are satisfied. We can thus define $RV^X_{\rm sub}$ as

$$RV^X_{\rm sub} \;=\; \Big(RV + I^{(\mathbf{1})}\Big)\,\delta^X_{n+1} + \Big(I^{(\mathbf{12})} - K^{(\mathbf{RV})}\Big)\,\delta^X_n\,, \tag{24}$$

where, after summing up the sector functions, we get for $I^{(\mathbf{1})}$ and $I^{(\mathbf{12})}$–$K^{(\mathbf{RV})}$ $(r \neq i,j;\, r' \neq i,j,k)$[2]:

$$I^{(\mathbf{1})} \;=\; \frac{\alpha_s}{2\pi}\Bigg[\frac{\Sigma_C}{\epsilon^2}R + \frac{1}{\epsilon}\Big(\Sigma_\gamma\, R + \sum_{c,d\neq c} L_{cd}R_{cd}\Big) + \Big(\Sigma_\phi - \sum_j \gamma_j^{\rm hc} L_{jr}\Big)R + \sum_{c,d\neq c} L_{cd}\Big(2 - \frac{L_{cd}}{2}\Big)R_{cd}\Bigg], \tag{25}$$

$$\begin{aligned}
I^{(\mathbf{12})} - K^{(\mathbf{RV})} = {}& 4\,\alpha_{\rm S}^2 \sum_{\substack{i,c\neq i\\ d\neq i,c}} \mathcal{E}^{(i)}_{cd} \Bigg\{ \frac{2\pi}{\alpha_{\rm S}}\, \bar V^{(icd)}_{{\rm fin},cd} + \sum_{\substack{e\neq i\\ f\neq i,e}} \Big(L_{ef} - \frac{L_{ef}^2}{4}\Big)\bar B^{(icd)}_{cdef} + 2\sum_{e\neq i,d}\Big(L_{ed} - \frac{L_{ed}^2}{4}\Big)\Big[\bar B^{(icd)}_{cded} - \bar B^{(idc)}_{cded}\Big] \\
&+ \sum_{e\neq i,d} \ln^2\frac{\bar s^{(icd)}_{de}}{s_{de}}\,\bar B^{(icd)}_{cded} - \frac{1}{2}\ln^2\frac{\bar s^{(icd)}_{cd}}{s_{cd}}\,\bar B^{(icd)}_{cdcd} - 2\pi\sum_{e\neq i,c,d} \ln\frac{s_{id}s_{ie}}{\mu^2\, s_{de}}\,\bar B^{(icd)}_{cde} \\
&+ \Bigg[\Big(6 - \frac{7}{2}\zeta_2\Big)\big(\Sigma_C + 2C_{f_d} - 2C_{f_c}\big) + \sum_j \big(\phi_j^{\rm hc} - \gamma_j^{\rm hc} L_{jr}\big) \\
&\quad + C_A\Big(6 - \zeta_2 - \ln\frac{s_{ic}}{s_{cd}}\ln\frac{s_{id}}{s_{cd}} - 2\ln\frac{s_{ic}s_{id}}{\mu^2 s_{cd}}\Big)\Bigg]\bar B^{(icd)}_{cd}\Bigg\} \\
&+ 4\alpha_{\rm S}^2 \sum_{i,j\neq i} \big(\phi_j^{\rm hc} - \gamma_j^{\rm hc} L_{jr}\big)\Bigg[\sum_{c\neq i,j}\mathcal{E}^{(i)}_{cj}\Big(\bar B^{(icj)}_{cj} - \bar B^{(ijc)}_{cj}\Big) + \sum_{c\neq i,j,r}\mathcal{E}^{(i)}_{cr}\Big(\bar B^{(icr)}_{cr} - \bar B^{(irc)}_{cr}\Big)\Bigg] \\
&+ 4\alpha_{\rm S}^2 \sum_{i,j>i} \frac{P^{{\rm hc},\mu\nu}_{ij(r)}}{s_{ij}} \Bigg\{ -\frac{2\pi}{\alpha_{\rm S}}\,\bar V^{(ijr)}_{{\rm fin},\mu\nu} + \sum_{c\neq i,j}\Bigg[\ln^2\frac{\bar s^{(ijr)}_{jc}}{s_{[ij]r}}\,\bar B^{(ijr)}_{\mu\nu,[ij]c} - 2\sum_{d\neq i,j,c}\Big(L_{cd} - \frac{1}{4}L_{cd}^2\Big)\bar B^{(ijr)}_{\mu\nu,cd}\Bigg] \\
&- \sum_{c\neq i,j,r}\Bigg[\ln^2\frac{\bar s^{(ijr)}_{cr}}{s_{cr}}\,\bar B^{(ijr)}_{\mu\nu,cr} + \frac{\rho^{(C)}_{ij}}{2}\,\mathcal{L}_{ijcr}\,\bar B^{(jri)}_{\mu\nu,[ij]c} + \frac{\rho^{(C)}_{ji}}{2}\,\mathcal{L}_{jicr}\,\bar B^{(irj)}_{\mu\nu,[ij]c}\Bigg] \\
&+ \frac{1}{2}\Bigg[\sum_{c\neq i,j}\tilde f^{\,q\bar q}_{ij}\,\tilde{\mathcal{L}}^{ir}_{jc}\,\tilde{\bar B}^{(irj)}_{\mu\nu,[ij]c} + \Big(2\gamma_i^{\rm hc}L_{ir} - C_{f_{[ij]}}\rho^{(C)}_{ij}\big(4L_{ir} - L_{ir}^2\big)\Big)\bar B^{(ijr)}_{\mu\nu} + (i\leftrightarrow j)\Bigg] \\
&- \Bigg[\Big(6 - \frac{7}{2}\zeta_2\Big)\Big(\Sigma_C - C_{f_{[ij]}}\rho^{(C)}_{[ij]}\Big) + \Sigma^{\rm hc}_\phi + C_{f_{[ij]}}\frac{\rho^{(C)}_{[ij]}}{2}\big(4L_{ij} - L_{ij}^2\big)\Bigg]\bar B^{(ijr)}_{\mu\nu}\Bigg\} \\
&- 4\alpha_{\rm S}^2 \sum_{i,j>i}\Bigg\{\Bigg[2\,C_{f_j}C_{f_{[ij]}}\mathcal{E}^{(i)}_{jr}\ln^2\frac{s_{jr}}{s_{[ij]r}}\,\bar B^{(ijr)} + (i\leftrightarrow j)\Bigg] + \frac{\widetilde P^{{\rm hc},\mu\nu}_{{\rm fin},ij}}{s_{ij}}\bar B^{(ijr)}_{\mu\nu} - \sum_{p\neq i,j}\frac{P^{{\rm hc},\mu\nu}_{ij(r')}}{s_{ij}}\gamma_p^{\rm hc}\,L_{pr'}\,\bar B^{(ijr')}_{\mu\nu}\Bigg\},
\end{aligned}$$

[2] We used the shorthand notations $\mathcal{L}_{ijcr} = 2\ln\frac{s_{ic}}{s_{ir}}\Big[2 - L_{ic} + \ln\Big(\bar s^{(jri)}_{ic}/\bar s^{(jri)}_{ir}\Big)\Big]$ and $\tilde{\mathcal{L}}^{ir}_{jc} = 2L_{jc}\Big[2 - L_{jc} + \ln\Big(\bar s^{(irj)}_{jc}/\mu^2\Big)\Big]$.

where B_{cde} is the completely antisymmetric tripole Born squared matrix element and $\widetilde{P}^{\mathrm{hc},\mu\nu}_{\mathrm{fin},ij}$ the finite part of the hard-collinear kernel $\widetilde{P}^{\mathrm{hc},\mu\nu}_{ij} = -g^{\mu\nu}\big[\widetilde{P}_{ij} - s_{ij}\big(2C_{f_j}\widetilde{I}^{(i)}_{jr} + 2C_{f_i}\widetilde{I}^{(j)}_{ir}\big)\big] + \widetilde{Q}^{\mu\nu}_{ij}$, where $\widetilde{P}_{ij}$, $\widetilde{Q}^{\mu\nu}_{ij}$, $\widetilde{I}^{(i)}_{jr}$ have been computed in [23] and can be found in [20]. We perform the integration of $K^{(\mathbf{RV})}$ in the radiation phase space following again the techniques of [20] and compute $I^{(\mathbf{RV})}$ defined as

$$\int d\Phi_n\, I^{(\mathbf{RV})} \;\equiv\; \int d\Phi_{n+1}\, K^{(\mathbf{RV})}\,. \tag{26}$$

Adding back $I^{(\mathbf{RV})}$ to $VV + I^{(\mathbf{2})}$, we can complete the subtraction procedure and get[3]

$$VV^{X}_{\mathrm{sub}} \;=\; \Big(VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})}\Big)\,\delta^{X}_{n}\,, \qquad\qquad I^{(\mathbf{2})} + I^{(\mathbf{RV})} \;=\; I^{(\mathbf{2+RV})}_{\mathrm{poles}} + I^{(\mathbf{2+RV})}_{\mathrm{fin}}\,, \tag{27}$$

$$\begin{aligned} I^{(\mathbf{2+RV})}_{\mathrm{poles}} \;=\; \left(\frac{\alpha_s}{2\pi}\right)^{2}\Bigg\{ & \frac{1}{\epsilon^4}\,\frac{1}{2}\,\Sigma_C^2\, B + \frac{1}{\epsilon^3}\,\Sigma_C\left[\left(\Sigma_\gamma - \frac{3}{8}\,\beta_0\right) B + \sum_{c,d\neq c} L_{cd}\, B_{cd}\right] \\ & + \frac{1}{\epsilon^2}\left[\left(\frac{\widehat{\gamma}^{(2)}_K}{16}\,\Sigma_C + (2\Sigma_\gamma - \beta_0)\frac{\Sigma_\gamma}{4}\right) B + \sum_{c,d\neq c}\frac{L_{cd}}{4}\left((4\Sigma_\gamma - \beta_0)\, B_{cd} + \sum_{e,f\neq e} L_{ef}\, B_{cdef}\right)\right] \\ & + \frac{1}{\epsilon}\left[\frac{\sum_a \gamma^{(2)}_a}{2}\, B + \frac{\widehat{\gamma}^{(2)}_K}{8}\sum_{c,d\neq c} L_{cd}\, B_{cd}\right] + \left(\frac{2\pi}{\alpha_s}\right)\left[\left(\frac{\Sigma_C}{\epsilon^2} + \frac{\Sigma_\gamma}{\epsilon}\right) V + \frac{1}{\epsilon}\sum_{c,d\neq c} L_{cd}\, V_{cd}\right]\Bigg\}, \end{aligned} \tag{28}$$

$$\begin{aligned} I^{(\mathbf{2+RV})}_{\mathrm{fin}} \;=\; \left(\frac{\alpha_s}{2\pi}\right)^{2}\Bigg\{ & \left[I^{(0)} + \sum_j I^{(1)}_j L_{jr} + \sum_j I^{(2)}_j L^2_{jr} + \frac{1}{2}\sum_{j,l\neq j}\gamma^{\mathrm{hc}}_j\,\gamma^{\mathrm{hc}}_l\, L_{jr'} L_{lr'}\right] B + \sum_j\left[I^{(0)}_{jr} + I^{(1)}_{jr} L_{jr}\right] B_{jr} \\ & - 2(1-\zeta_2)\sum_{j,c\neq j,r}\gamma^{\mathrm{hc}}_j (2 - L_{cr}) B_{cr} + \pi\sum_{\substack{c,d\neq c\\ e\neq c,d}}\left[\ln\frac{s_{ce}}{s_{de}}\, L^2_{cd} + \frac{1}{3}\ln^3\frac{s_{ce}}{s_{de}} + 2\,\mathrm{Li}_3\left(-\frac{s_{ce}}{s_{de}}\right)\right] B_{cde} \\ & + \sum_{c,d\neq c} L_{cd}\Bigg[\left(\frac{20}{9} - 2\zeta_2 - \frac{7}{2}\zeta_3\right) C_A + \frac{31}{9}\,\beta_0 + 2\,\Sigma_\phi + 8\,(1-\zeta_2)\, C_{f_d} \\ & \qquad - \left(\frac{C_A}{3} - \frac{\zeta_2}{2} C_A + \frac{\beta_0}{12}(11 + L_{cd}) + \frac{\Sigma_\phi}{2}\right) L_{cd} - \frac{1}{2}(4 - L_{cd})\sum_j \gamma^{\mathrm{hc}}_j L_{jr}\Bigg] B_{cd} \\ & + \sum_{c,d\neq c}\left[-2 + \zeta_2 + 2\,\zeta_3 - \frac{5}{4}\,\zeta_4 + 2(1-\zeta_3)\, L_{cd}\right] B_{cdcd} \\ & + \sum_{c,d\neq c} L_{cd}\left[(1-\zeta_2)\sum_{e\neq d} L_{ed}\, B_{cded} + \sum_{e,f\neq e} L_{ef}\left(1 - \frac{1}{2}\, L_{cd} + \frac{1}{16}\, L_{cd} L_{ef}\right) B_{cdef}\right]\Bigg\} \\ + \left(\frac{\alpha_s}{2\pi}\right)\Bigg\{ & \left[\Sigma_\phi - \sum_j \gamma^{\mathrm{hc}}_j\, L_{jr}\right] V^{\mathrm{fin}} + \sum_{c,d\neq c} L_{cd}\left(2 - \frac{1}{2} L_{cd}\right) V^{\mathrm{fin}}_{cd}\Bigg\}, \end{aligned} \tag{29}$$

where the anomalous dimensions $\widehat{\gamma}^{(2)}_K$ and $\gamma^{(2)}_a$ read

$$\widehat{\gamma}^{(2)}_K = \left(\frac{8}{3} - 4\zeta_2\right) C_A + \frac{10}{3}\,\beta_0\,, \qquad \begin{aligned} \gamma^{(2)}_a \;=\; & (f^q_a + f^{\bar q}_a)\, C_F\left[\left(\frac{3}{8} - 3\zeta_2 + 6\zeta_3\right) C_F + \left(\frac{41}{36} - \frac{13}{2}\zeta_3\right) C_A + \left(\frac{65}{72} + \frac{3}{4}\zeta_2\right)\beta_0\right] \\ & + f^g_a\left\{C_A\left[-\frac{11}{4}\, C_F - \left(\frac{1}{9} + \frac{1}{2}\zeta_3\right) C_A\right] + \beta_0\left[\frac{3}{4}\, C_F + \left(\frac{16}{9} - \frac{1}{4}\zeta_2\right) C_A\right]\right\}. \end{aligned}$$

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[3] The coefficients $I^{(0)}$, $I^{(1)}_j$, $I^{(2)}_j$, $I^{(0)}_{jr}$, $I^{(1)}_{jr}$ are given by (with N_g and N_q the number of gluons and quarks respectively):

$$\begin{aligned} I^{(0)} \;=\; & N_q^2 C_F^2\left[\frac{101}{8} - \frac{141}{8}\zeta_2 + \frac{245}{16}\zeta_4\right] + N_q C_F\left[C_F\left(\frac{53}{32} - \frac{57}{8}\zeta_2 + \frac{1}{2}\zeta_3 + \frac{21}{4}\zeta_4\right) + C_A\left(\frac{677}{432} + \frac{5}{3}\zeta_2 - \frac{25}{2}\zeta_3 + \frac{47}{8}\zeta_4\right) + \beta_0\left(\frac{5669}{864} - \frac{85}{24}\zeta_2 - \frac{11}{12}\zeta_3\right)\right] \\ & + N_g N_q C_F\left[C_A\left(\frac{13}{3} - \frac{125}{6}\zeta_2 + \frac{245}{8}\zeta_4\right) + \beta_0\left(\frac{77}{12} - \frac{53}{12}\zeta_2\right)\right] + N_g^2\left[C_A^2\left(\frac{20}{9} - \frac{13}{3}\zeta_2 + \frac{245}{16}\zeta_4\right) + \beta_0^2\left(\frac{73}{72} - \frac{1}{8}\zeta_2\right) + C_A\beta_0\left(-\frac{1}{9} - \frac{11}{3}\zeta_2\right)\right] \\ & + N_g\left[C_F C_A\left(-\frac{737}{48} + 11\zeta_3\right) + C_F\beta_0\left(\frac{67}{16} - 3\zeta_3\right) + \beta_0^2\left(\frac{73}{72} - \frac{3}{8}\zeta_2\right) + C_A^2\left(-\frac{4289}{216} + \frac{15}{2}\zeta_2 - 14\zeta_3 + \frac{89}{8}\zeta_4\right) + C_A\beta_0\left(\frac{647}{54} - \frac{53}{8}\zeta_2 - \frac{11}{12}\zeta_3\right)\right], \\ I^{(1)}_j \;=\; & (f^q_j + f^{\bar q}_j) C_F\left[N_q C_F\left(\frac{5}{2} - \frac{7}{4}\zeta_2\right) + N_g C_A\left(\frac{1}{3} - \frac{7}{4}\zeta_2\right) + \frac{2}{3} N_g\beta_0 - C_F\left(\frac{3}{8} + 4\zeta_2 - 2\zeta_3\right) + C_A\left(\frac{25}{12} - 3\zeta_2 + 3\zeta_3\right) + \beta_0\left(\frac{1}{24} + \zeta_2\right)\right] \\ & + f^g_j\Bigg[N_q C_F C_A (10 - 7\zeta_2) - N_q C_F\beta_0\left(\frac{5}{2} - \frac{7}{4}\zeta_2\right) + N_g C_A^2\left(\frac{4}{3} - 7\zeta_2\right) + N_g C_A\beta_0\left(\frac{7}{3} + \frac{7}{4}\zeta_2\right) \\ & \qquad - \frac{2}{3}(N_g + 1)\beta_0^2 + \frac{11}{4} C_F C_A - \frac{3}{4} C_F\beta_0 + C_A^2\left(\frac{28}{3} - \frac{23}{2}\zeta_2 + 5\zeta_3\right) - C_A\beta_0\left(\frac{2}{3} - \frac{5}{2}\zeta_2\right)\Bigg], \\ I^{(2)}_j \;=\; & \frac{1}{8}(15\, C_A - 7\beta_0 - 15) C_{f_j} - \frac{1}{4}(5\, C_A - 2\beta_0)\gamma_j + 2\,\zeta_2\, C^2_{f_j}\,, \end{aligned}$$

$$I^{(0)}_{jr} = (-1 + 3\zeta_2 - 2\zeta_3) C_A - \frac{1}{2}(13 + 10\zeta_2 + 2\zeta_3) C_{f_j} + (5 + 2\zeta_3)\gamma_j\,, \qquad I^{(1)}_{jr} = (1 - \zeta_2) C_A + \frac{1}{2}(4 + 7\zeta_2) C_{f_j} - (2 + \zeta_2)\gamma_j\,.$$

The pole part $I^{(\mathbf{2+RV})}_{\text{poles}}$ exactly cancels not only the poles of VV, but also the finite contributions coming from linear and quadratic terms in the ϵ expansion of V and V_{cd}, whose knowledge is not necessary in a universal subtraction scheme, as pointed out in [24].

4. Summary

We have presented the latest developments of the Local Analytic Sector subtraction. The method takes advantage of the partition of the phase space through sector functions and allows to easily identify counterterms by using the known singular limits of squared matrix elements, accompanied by proper mappings of momenta. We have shown that the method can be applied to initial- and final-state radiation at NLO in the massless case, producing a local subtraction procedure for generic processes. The main achievement of the method is its application at NNLO in the massless case, where we could derive a universal subtraction formula for final-state radiation, completely analytic and suitable for direct implementation in any numerical code.

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