Towards a Theory of Stochastic Hybrid Systems

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Abstract. In this paper, we present a scheme of stochastic hybrid system which introduces randomness to the deterministic framework of the traditional hybrid systems by allowing the flow inside each invariant set of the discrete state variables to be governed by stochastic differential equation (SDE) rather than the deterministic ones. The notion of embedded Markov chains is proposed for such systems and some illustrative example from high way model is presented. As an important application, these ideas are then applied to the state space discretization of one dimensional SDE to obtain the natural discretized stochastic hybrid system together with its embedded MC. The invariant distribution and exit probability from interval of the MC are studied and it is shown that they converge to their counterparts for the solution process of the original SDE as the discretization step goes to zero. As a result, the discretized stochastic hybrid system provides a useful tool for studying various sample path properties of the SDE.

1 Introduction

In the conventional formulation of hybrid system (See, for example, [6]), there is no place for randomness. Although the deterministic framework captures many characteristics of the real systems in practice, in other cases, the missing flavor of randomness will indeed be a fatal flaw because of the inherent uncertainty in the environment of most real world applications. The idea of introducing stochastic hybrid system is not new. Different researchers have tried to propose different models from their own perspectives. For the most recent and relevant literature, the readers are referred to $[1, 5, 10, 2, 11, 9]$. The most important difference lies in where to introduce the randomness.

One obvious choice is to replace the deterministic jumps between discrete states by random jumps governed by some prescribed probabilistic law. Hence the evolution of the discrete states constitutes a time homogeneous Markov chain. The question remained then is when does such jump occur? In [1], the jumps occur every ϵ time, and the effect when $\epsilon \to 0$ is studied. In [10], however, the transitions follow a continuous time Markov process. In both papers, the discrete random transitions are assumed to be independent of the continuous dynamics, therefore the models can actually be better viewed as an extension of Markov process with some continuous states attached whose evolutions follow state-dependent deterministic differential equations.

Another choice is to replace the deterministic dynamics inside the invariant set of each discrete state by a stochastic differential equation (SDE). Therefore, even if we keep the deterministic discrete transition part, starting from a fixed initial state, different guards can be activated depending on the realization of the solution stochastic process, thus different discrete transitions occur randomly. More general models can be proposed by blending the above two choices.

This paper is organized as following: in Section 2, we will try to give a general definition of stochastic hybrid system based on the second choice mentioned above. An example will be shown in Section 3 together with its analysis. In Section 4, the idea will be applied to a more general problem, in which we will approximate the solution of the SDE in $\mathbb{R}^1$ by the stochastic hybrid automata obtained from state space discretization. And finally we will discuss the special case of gradient system in the last section. The proofs of the theorems are not included due to the limit of space and will appear in subsequent paper.

2 General Definition

Definition 1 (Stochastic Hybrid System). *A stochastic hybrid system (or automata) is a collection* $H = (Q, X, Inv, f, g, G, R)$ *where*

- Q *is a discrete variable taking countably many values in* $\mathbf{Q} = \{q_1, q_2, \cdots\}$*;*
- X *is a continuous variable taking values in* $\mathbf{X} = \mathbb{R}^N$ *for some* $N \in \mathbb{N}$*;*
- $Inv : \mathbf{Q} \to 2^{\mathbf{X}}$ *assigns to each* $q \in \mathbf{Q}$ *an invariant open subset of* $\mathbf{X}$*;*
- $f, g : \mathbf{Q} \times \mathbf{X} \to T\mathbf{X}$ *are vector fields;*
- $G : E = \mathbf{Q} \times \mathbf{Q} \to 2^{\mathbf{X}}$ *assigns to each* $e \in E$ *a guard* $G(e)$ *such that*

  - *For each $e = (q, q') \in E$, G(e) is a measurable subset of $\partial Inv(q)$ (possibly empty);*
  - *For each $q \in \mathbf{Q}$, the family $\{G(e) : e = (q, q')$ for some $q' \in \mathbf{Q}\}$ is a disjoint partition of $\partial Inv(q)$.*
- *$R : E \times \mathbf{X} \to \mathcal{P}(\mathbf{X})$ assigns to each $e = (q, q') \in E$ and $x \in G(e)$ a reset probability kernel on $\mathbf{X}$ concentrated on $Inv(q')$. Here $\mathcal{P}(\mathbf{X})$ denote the family of all probability measures on $\mathbf{X}$. Furthermore, for any measurable set $A \subset Inv(q')$, $R(e, x)(A)$ is a measurable function in x.*

Remark 1. The measurability assumption on R in the preceding definition is made to ensure that the events we encounter later are measurable w.r.t. the underlying σ-field, hence their probabilities make sense.

Definition 2 (Stochastic Execution). *A stochastic process $(X(t), Q(t)) \in \mathbf{X} \times \mathbf{Q}$ is called a stochastic execution iff there exists a sequence of stopping times $\tau_0 = 0 \leq \tau_1 \leq \tau_2 \leq \cdots$ such that for each $n \in \mathbb{N}$,*

- *In each interval $[\tau_n, \tau_{n+1})$, $Q(t) \equiv Q(\tau_n)$ is constant, $X(t)$ is a (continuous) solution to the SDE:*

$$dX(t) = f(Q(\tau_n), X(t))\, dt + g(Q(\tau_n), X(t))\, dB_t\,,$$

  where B_t is the standard Brownian motion in $\mathbb{R}$;
- *$\tau_{n+1} = \inf\{t \geq \tau_n : X(t) \notin Inv(Q(\tau_n))\}$;*
- *$X(\tau_{n+1}^-) \in G(Q(\tau_n), Q(\tau_{n+1}))$ where $X(\tau_{n+1}^-)$ denotes $\lim_{t\uparrow\tau_{n+1}} X(t)$;*
- *The probability distribution of $X(\tau_{n+1})$ given $X(\tau_{n+1}^-)$ is governed by the law $R(e_n, X(\tau_{n+1}^-))$, where $e_n = (Q(\tau_n), Q(\tau_{n+1})) \in E$.*

Definition 3 (Embedded Markov Process). *In the notation of the previous definition, define $Q_n \triangleq Q(\tau_n)$, $X_n \triangleq X(\tau_n)$. Then $\{(Q_n, X_n), n \geq 0\}$ is called the embedded Markov process for the stochastic execution $(X(t), Q(t))$.*

Under these definitions, for example, a typical stochastic execution starts from (Q_0, X_0) and the continuous state $X(t)$ evolves according to the SDE

$$dX(t) = f(Q_0, X(t))\, dt + g(Q_0, X(t))\, dB_t, \quad X(0) = X_0$$

until time τ_1 when $X(t)$ first hits $\partial Inv(Q_0)$. Then depending on the hitting position $X(\tau_1^-)$, (say, $X(\tau_1^-) \in G(e)$ where $e = (Q_0, Q_1)$ for some $Q_1 \in \mathbf{Q}$), the discrete state jumps to $Q(\tau_1) = Q_1$ and the continuous state is reset randomly to $X(\tau_1) = X_1$ according to the conditional probability distribution $R(e, X(\tau_1^-))(\cdot)$ and the same process is repeated with (Q_1, X_1) replacing (Q_0, X_0) and so on.

Lemma 1. *$\{(Q_n, X_n)\}$ defined above is indeed a Markov process with transition probability:*

$$P(Q_{n+1} = q', X_{n+1} \in dx' | Q_n = q, X_n = x) = \int_{y\in G(e)} R(e, y)(dx')\, P(Y_x(\eta) = dy)\,, \tag{1}$$

where $e = (q, q')$, $Y_x(t)$ *is the solution to the SDE*

$$dY(t) = f(q, Y(t))\,dt + g(q, Y(t))\,dB_t, \quad Y(0) = x\,,$$

and $\eta = \inf\{t \geq 0 : Y_x(t) \notin Inv(q)\}$ *is the first escape time of* $Y_x(t)$ *from* $Inv(q)$.

Lemma 2. *If the reset kernel* $R((q, q'), x) = R(q')$ *does not depend on* q *nor* x, *then* $\{Q_n\}$ *itself is a Markov chain (MC) with transition probability* ($n \geq 1$)*:*

$$P(Q_{n+1} = q' | Q_n = q) = \int_{x \in Inv(q)} P(Y_x(\eta) \in G(e)) R(q)(dx)\,, \tag{2}$$

where e, Y_x, η *is defined in the previous lemma. For* $n = 0$*, the transition probability depends on the initial distribution of* $X(0)$.

Remark 2. The condition in Lemma 2 is fairly restrictive and excludes many general stochastic hybrid systems. The point of imposing this condition is to make calculation tractable. Furthermore, as we will see in the later sections, this special class of systems is still rich enough to admit many important applications.

The reason we introduce the embedded MC is that in most cases, it is hard if not impossible to get an explicit expression of the stochastic execution for a stochastic hybrid system. If all we are interested in is the reachability analysis of the discrete states transitions, then $\{Q_n\}$ will capture all the necessary information. This is the case if a subset of the discrete states is defined to be the "bad" states and a controller is designed to minimize the probability of reaching these states within a given time horizon. Or alternatively, some states are defined to be safe and we want to maximize the probability that the execution will remain in these states for as long as possible. At first sight these observation does not seem to be applicable in general, since in most cases, the definition of bad states and safe states involve both the discrete and continuous states. However, by breaking up the corresponding invariant sets and adding more discrete states and trivial reset kernels, we can always reduce the original system to a new one satisfying the above conditions, at least in the case when the support of any reset kernel is contained exclusively in safe or bad set.

3 A Simple Example

To clarify the above concepts, consider the following simple example. Two cars, labeled 1 and 2 with car 2 in the lead, are moving from left to right on a highway (see Figure 1). Due to various random factors such as road condition, wind, and the presence of human operators, the motions of both cars are stochastic. If we absorb all the randomness into the motion of car 1 and ignore the possible occurrence of emergency braking, then the motion of car 2 can be modeled as having a constant speed v_2. Let Δx be the distance between the two cars. Let $d_0 > d_1 > d_2 > d_3 > 0$ be four thresholds. We propose the following hybrid control scheme for car 1 (see the diagram in Figure 2): It consists of 3 discrete states $\{1, 2, 3\}$ corresponding to chasing, keeping and braking respectively.

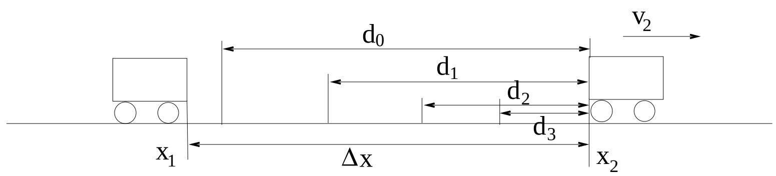

Fig. 1. A two-car platoon on the highway

1. **Chasing**: In this stage, $\Delta x \geq d_2$, and car 1 will try to catch car 2 at speed $v_1 > v_2$. So the perturbed motion of car 1 is governed by $\dot{x}_1 = v_1 + dB_t$, where B_t is a standard 1-D BM;
2. **Keeping**: In this stage $d_3 \leq \Delta x \leq d_1$, and car 1 will try to move at v_2 under the perturbation dB_t;
3. **Braking**: If $\Delta x \leq d_3$, then car 1 will brakes according to some prescribed procedure until $\Delta x = d_0$. For simplicity, we ignore the presence of noise during braking.

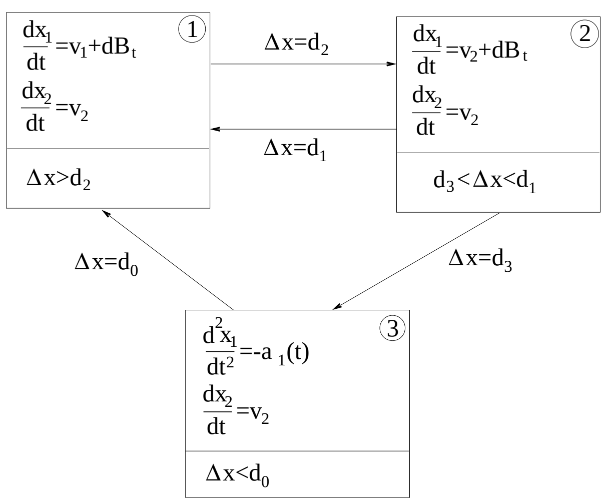

Fig. 2. Diagram for the stochastic hybrid system

The invariant sets and guards for each discrete state are also shown in Figure 2. The reset kernels are trivial, or more precisely, $R(e, x)$ is concentrated at x for any $e = (q, q') \in E$ and any $x \in G(e)$. It is easily seen that H satisfies the condition of Lemma 2. Hence the successive visits to the discrete states $\{Q_n\}$ is a MC. Actually its probability transition matrix is

$$P = \begin{pmatrix} 0 & 1 & 0 \\ p & 0 & 1-p \\ 1 & 0 & 0 \end{pmatrix} ,$$

where $p = (d_2 - d_3)/(d_1 - d_3)$. The first and third row of P is obvious and the second row follows from ([3]):

Lemma 3. *Let B_t be a standard BM starting from 0. For $a < 0 < b$, define $T_a = \inf\{t \geq 0 : B_t = a\}$, $T_b = \inf\{t \geq 0 : B_t = b\}$. Then $P(T_a < T_b) = \frac{b}{b-a}$ and $E(T_a \wedge T_b) = -ab$. Here $T_a \wedge T_b$ denotes* $\min(T_a, T_b)$.

Calculation shows that the stationary distribution for P is $(\frac{1}{3-p}, \frac{1}{3-p}, \frac{1-p}{3-p})$. Therefore the fraction of time the system spends in each discrete state is proportional to:

$$(\frac{ET_1}{3-p}, \frac{ET_2}{3-p}, \frac{(1-p)ET_3}{3-p}),$$

where $ET_1 = (d_0 - d_2)/(v_1 - v_2)$, $ET_2 = (d_1 - d_2)(d_2 - d_3)$, $ET_3 = t_3$ are the expected sojourn time in each discrete state respectively.

In practice, we want to maximize the time the stochastic hybrid system spends in the keeping state and minimize the time it spends in the braking state. This can be done by adjusting the thresholds d_0, d_1, d_2, d_3 properly. Sometimes this choice is restricted by other physical constraints. However, we can always use more thresholds and thus more complex stochastic hybrid controller to achieve the goal within the various physical constraints. This technique will be illustrated in the next section.

4 State Discretization of 1-D Stochastic Differential Equation

4.1 Motivation and Definition

Consider the following stochastic differential equation in $\mathbb{R}$:

$$\frac{dX(t)}{dt} = f(X(t)) + dB_t, \quad X(0) = 0, \tag{3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is smooth and dB_t is white noise with spectral density 1. Define a series of stopping times τ_n inductively as: $\tau_0 = 0$, $\tau_n = \inf\{t \geq \tau_{n-1} : |X(t) - X(\tau_{n-1})| = \delta\}$, $n = 1, 2, 3, \cdots$. Let $S_n = X(\tau_n)$. Then $\{S_n\}$ is a MC taking values in $\delta \cdot \mathbb{Z}$. S_n captures many sample path properties of the solution process $X(t)$, for example, whether $X(t)$ is recurrent. or less obviously, whether $X(t)$ crosses an interval of length less than δ infinitely many times,

Define $\tau_t = \sup_n\{\tau_n : \tau_n \leq t\}$ and let $Y_t = X(\tau_t)$. Then Y_t is piecewise constant with value S_n in time interval $[\tau_n, \tau_{n+1})$. Define $Z(t)$ to be the solution process to the stochastic differential equation:

$$\frac{dZ(t)}{dt} = f(Y_t) + dB_t\,. \tag{4}$$

Comparing equation (3) and (4) and noticing that during time interval $[\tau_n, \tau_{n+1})$, $|X(t) - Y_t| \leq \delta$ by the definition of τ_n's and f is continuous, we can expect that

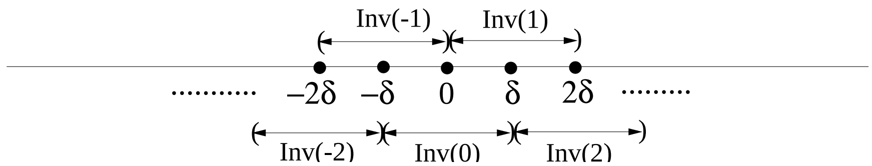

Fig. 3. Discretization of state space

as $\delta \to 0$, $Z(t)$ approaches $X(t)$ in distribution, hence $Z(t)$ is a good approximation to $X(t)$ which is often impossible to calculate explicitly. However, it is still difficult to solve equation (4) since Y_t depend on the original solution process $X(t)$ through τ_n's and S_n's. So to solve equation (4), theoretically we still have to solve equation (3) first.

One way to get out of this loop is to use the fact that $X(t)$ can be approximated by $Z(t)$, hence τ_n's and S_n's can also be approximated by the corresponding random variables defined from $Z(t)$. This will lead to the discretized stochastic hybrid system (DSHS) defined below.

Definition 4 (Discretized Stochastic Hybrid System). *The discretized stochastic hybrid system for equation (3) is* $H = (Q, X, Inv, f, g, G, R)$ *where* $\mathbf{Q} = \mathbb{Z}$, $\mathbf{X} = \mathbb{R}$, *and*
— $Inv(k) = ((k-1)\delta, (k+1)\delta)$ *for any* $k \in \mathbf{Q}$*;*
— $f(k,\cdot) = f(k\delta)$, $g(k,\cdot) = 1$ *are constant functions;*
— $G(k, k-1) = \{(k-1)\delta\}$, $G(k, k+1) = \{(k+1)\delta\}$ *are singletons and* $G(k,l) = \emptyset$ *for all other* l*;*
— *Reset kernels are trivial.*

Since H satisfies the condition of Lemma 2, $\{Q_n\}$ defined as in Section 2 is a MC. By discussion at the beginning of this section, it is expected that $\{Q_n\}$ approximates the MC $\{S_n\}$ defined from the solution $X(t)$ to equation (3). (In the following development, we will use H^δ to stress the dependency of H on the discretization step δ only if necessary).

Obviously the probability transition matrix Q for $\{Q_n\}$ satisfies: $Q_{i,j} = p_i$, if $j = i+1$; $Q_{i,j} = q_i = 1 - p_i$ if $j = i-1$ and $Q_{i,j} = 0$ otherwise. Such a chain is called a *death and birth* chain and we will calculate p_k's and q_k's as follows: The solution to the stochastic differential equation $dY(t) = f(k\delta)\,dt + dB_t$ with initial condition $Y(0) = k\delta$ is $Y(t) = k\delta + f(k\delta)t + B_t$, *i.e.* the BM starting from $k\delta$ and with drift $\mu = f(k\delta)$. If we use B_t^μ to denote the BM starting from 0 and with drift μ, then

$$p_k = P(B_t^\mu \text{ reaches } \delta \text{ before it reaches } -\delta)\,. \tag{5}$$

So the problem becomes calculating the exit distribution of B_t^μ from $(-\delta, \delta)$. We will derive the probability in a more general setting. Assume $\mu \neq 0$ since the case when $\mu = 0$ has already been considered in Lemma 3. Let $a < 0 < b$. Denote $T_a = \inf\{t \geq 0 : B_t^\mu = a\}$, $T_b = \inf\{t \geq 0 : B_t^\mu = b\}$.

Lemma 4. *B_t^μ first exits (a, b) from b with probability*

$$P(T_b < T_a) = \frac{e^{-2\mu a} - 1}{e^{-2\mu a} - e^{-2\mu b}}\,. \tag{6}$$

Therefore by taking $a = -\delta$, $b = \delta$ and $\mu = f(k\delta)$, we have $p_k = \phi[\delta f(k\delta)]$ where ϕ is the monotonically increasing function defined by

$$\phi(x) = \frac{e^{2x} - 1}{e^{2x} - e^{-2x}}, \quad x \neq 0\,. \tag{7}$$

For a plot of function ϕ, see Figure 4.

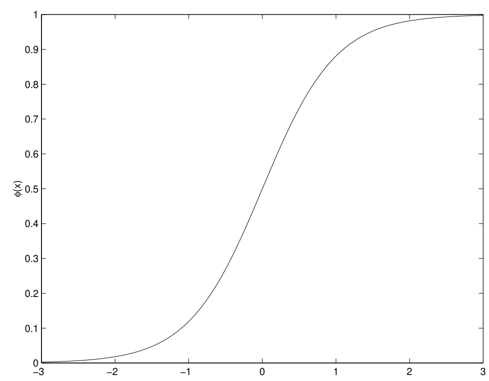

Fig. 4. Plot of ϕ

4.2 Recurrence vs. Stability

Having obtained the probability transition matrix Q, the natural question we will ask ourselves is: what is the relation between the deterministic part of differential equation (3), *i.e.*

$$\frac{dx}{dt} = f(x), \quad x(0) = 0\,, \tag{8}$$

and the embedded MC $\{Q_n\}$? For (8) we have notions such as equilibrium and various kinds of stability. What are their counterparts for $\{Q_n\}$? Intuitively if equation (8) has a globally stable equilibrium then the sample paths of $\{Q_n\}$ should also be centered around the equilibrium most of the time even if the starting point is far away, thus stability in some probabilistic sense can be expected. It turns out that the notions of recurrence and transience are good candidates for this. Assume MC $\{Q_n\}$ is irreducible, *i.e.* starting from any state, there is a positive probability of jumping to any other state in finite steps.

Definition 5 (Recurrent and Transient MC). *A MC $\{Q_n\}$ on a countable state space S is called recurrent if and only if starting from any state $x \in S$, it will return to x in finite time with probability 1, or more precisely, if and only if*

$$P(T_x < \infty | Q_0 = x) = 1 \quad \forall x \in S\,,$$

where $T_x \triangleq \inf\{n \geq 1 : Q_n = x\}$. Otherwise $\{Q_n\}$ is called transient.

Definition 6 (Positive Recurrent MC). *A recurrent MC $\{Q_n\}$ on a countable state space S is called positive recurrent if and only if $E[T_x | Q_0 = x] < \infty$ for all $x \in S$.*

An important characteristic of a positive recurrence chain is that its invariant distribution exists and is unique ([3]). In general, positive recurrence implies recurrence, but not the other way around, since symmetric random walk on integer grid $\mathbb{Z}$ is an example of recurrent but not positive recurrent chain.

Now consider the MC $\{Q_n\}$ obtained in subsection 4.1. Obviously it is irreducible. Let $\{Q_n^+\}$ and $\{Q_n^-\}$ be the MC's obtained by observing $\{Q_n\}$ on the subset $\mathbb{N}^+ = \{0, 1, 2, \cdots\}$ and $\mathbb{N}^- = \{0, -1, -2, \cdots\}$ respectively. Both $\{Q_n^+\}$ and $\{Q_n^-\}$ are irreducible. The following lemma justifies our interest in them.

Lemma 5. *$\{Q_n\}$ is (positive) recurrent iff both $\{Q_n^+\}$ and $\{Q_n^-\}$ are (positive) recurrent respectively. Furthermore, if π^+ is the stationary distribution of $\{Q_n^+\}$ on $\mathbb{N}^+$, π^- is the stationary distribution of $\{Q_n^-\}$ on $\mathbb{N}^-$, then $\pi \triangleq \alpha\pi^+ + (1 - \alpha)\pi^-$ is the stationary distribution of $\{Q_n\}$ on $\mathbb{Z}$, where*

$$\alpha = \frac{\pi^-(0)p_0}{\pi^-(0)p_0 + \pi^+(0)q_0}\,.$$

Notice that the transition matrix Q^+ has the property $Q^+(i,j) = 0$ when $|i-j| > 1$, hence it is a death and birth chain. The following lemma is a standard result from probability theory (see [3]):

Lemma 6. *$\{Q_n^+\}$ is recurrent if and only if $\sum_{m=0}^{\infty} \prod_{j=1}^{m} q_j/p_j = \infty$, $\{Q_n^+\}$ is positive recurrent if and only if $\sum_{m=0}^{\infty} \prod_{j=1}^{m} p_{j-1}/q_j < \infty$ (here $p_0 = 1$). In the latter case, the stationary distribution π^+ of $\{Q^+\}$ is:*

$$\pi^+(i) = \prod_{j=1}^{i} \frac{p_{j-1}}{q_j} \Bigg/ \sum_{m=0}^{\infty} \prod_{j=1}^{m} \frac{p_{j-1}}{q_j}, \quad i = 0, 1, 2, \cdots$$

Note the products are interpreted as 1 whenever $m = 0$.

Similar argument for $\{Q_n^-\}$ can be established by symmetry. Assembling Lemma 5, Lemma 6, equation (7) together, we get

Theorem 1 (Recurrence of DSHS). *The embedded MC $\{Q_n\}$ of the discretized stochastic hybrid system of (3) is recurrent if and only if*

$$\sum_{m=0}^{\infty} \prod_{j=1}^{m} \frac{1 - \exp[-2\delta f(j\delta)]}{\exp[2\delta f(j\delta)] - 1} = \infty \quad \text{and} \quad \sum_{m=0}^{\infty} \prod_{j=-m}^{-1} \frac{\exp[2\delta f(j\delta)] - 1}{1 - \exp[-2\delta f(j\delta)]} = \infty\,. \tag{9}$$

$\{Q_n\}$ is positive recurrent if and only if

$$\sum_{m=0}^{\infty}\prod_{j=1}^{m}\frac{\phi[\delta f((j-1)\delta)]}{1-\phi[\delta f(j\delta)]}<\infty \quad \text{and} \quad \sum_{m=0}^{\infty}\prod_{j=-m}^{-1}\frac{1-\phi[\delta f((j+1)\delta)]}{\phi[\delta f(j\delta)]}<\infty\,. \tag{10}$$

In the latter case, the stationary distribution π of $\{Q_n\}$ is given by Lemma 5.

4.3 Boundary between Recurrence and Transience

From Theorem 1, it is evident that whether $\{Q_n\}$ is (positive) recurrent depends only on the "tail" of function f, *i.e.* the asymptotic behavior of $f(x)$ when $x \to \pm\infty$. In general, we have

Lemma 7 (Comparison Lemma). *Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are two smooth vector fields such that*

$$f(x) > g(x), \quad f(-x) < g(-x) \quad \text{for } x \text{ sufficiently large,}$$

Then if $\{Q_n(f)\}$ is (positive) recurrent, so is $\{Q_n(g)\}$. Conversely, if $\{Q_n(g)\}$ is transient, $\{Q_n(f)\}$ is also transient.

Inspired by [3], let us look at f of the form

$$f(x) = \begin{cases} Cx^{-r} & x \geq M \\ -C(-x)^{-r} & x \leq -M \\ \text{do not care} & |x| < M \end{cases} \tag{11}$$

for some constant C and $r > 0$. Note we have deliberately made f to be an odd function outside $(-M, M)$ such that the corresponding MC $\{Q_n^+\}$ and $\{Q_n^-\}$ are mirror image of each other. So by Lemma 5 we need only to consider one of them, say, $\{Q_n^+\}$. If $C \leq 0$, then by the Comparison Lemma and the previous paragraph, $\{Q_n\}$ is recurrent, so we assume $C > 0$ here.

Proposition 1. *Assuming $C > 0$. The DSHS $\{Q_n\}$ corresponding to f in (11) is recurrent if $r > 1$ or if $r = 1$ and $C < 0.5$. $\{Q_n\}$ is transient if $r < 1$ or if $r = 1$ and $C > 0.5$.*

Note the above conclusion is independent of the discretization step δ. Next we will discuss the boundary of positive recurrence. Suppose f is of the form:

$$f(x) = \begin{cases} -Cx^{-r} & x \geq M \\ C(-x)^{-r} & x \leq -M \\ \text{do not care} & |x| < M \end{cases} \tag{12}$$

where C, r are positive constants. A similar argument generates:

Proposition 2. *Assuming $C > 0$. The DSHS $\{Q_n\}$ corresponding to f in (12) is positive recurrent if $r < 1$ or if $r = 1$ and $C > 0.5$. $\{Q_n\}$ is not positive recurrent if $r > 1$ or if $r = 1$ and $C < 0.5$.*

5 DSHS of Gradient System

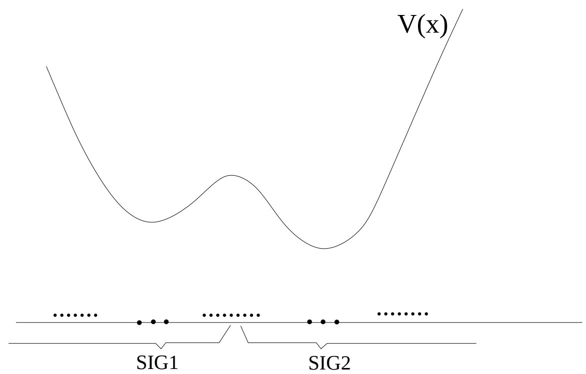

Fig. 5. DSHS for a gradient system

If equation (8) is a gradient system ([8]) of the form:

$$\frac{dx}{dt} = f(x) = -\nabla V(x) \tag{13}$$

for some $V \in C^2(\mathbb{R})$, then each local minimum of $V(x)$ is an equilibrium of (13) and in the embedded MC $\{Q_n\}$ of the corresponding DSHS, states in the vicinity of each equilibrium constitute an *strongly interacting group* (SIG) in the sense that in any typical execution of $\{Q_n\}$, once the state jumps into an SIG, it will stay inside it for a relatively long period before jumping to another SIG. (See Figure 5). In many applications it is often the case that we want to choose some suitable control so as to make the system evolve inside some desired valleys for as long as possible while avoiding some undesired trap.

Under this setting, the conclusion of Proposition 1 and Proposition 2 in the last subsection translates into: $\{Q_n\}$ is recurrent (transient) if $V(x)$ approaches $-\infty$ slower (faster) than $-\frac{1}{2}\ln(|x|)$ as $|x| \to \infty$ respectively; $\{Q_n\}$ is (not) positive recurrent if $V(x)$ approaches ∞ faster (slower) than $\frac{1}{2}\ln(|x|)$ as $|x| \to \infty$ respectively. Therefore instead of the clear cut boundary between stability and non-stability in the deterministic system, the DSHS have a blurred boundary between positive recurrence and transience, with $V(x)$ growing asymptotically between $-\frac{1}{2}\ln(|x|)$ and $\frac{1}{2}\ln(|x|)$ corresponding to recurrent but not positive recurrent $\{Q_n\}$. In this subsection, we will always assume that $V(x)$ is chosen such that for δ small enough, the corresponding $\{Q_n\}$ is positive recurrent and hence has a stationary distribution π. We will elaborate on the asymptotic behavior of π as $\delta \to 0$ and reveal its relation with $V(x)$.

From Lemma 5 and Lemma 6, π can be written as: $\pi(i) = \alpha\pi^+(i) + (1 - \alpha)\pi^-(i)$ for all $i \in \mathbb{Z}$ with

$$\alpha = \frac{\pi^-(0)\phi[\delta f(0)]}{\pi^-(0)\phi[\delta f(0)] + \pi^+(0)(1 - \phi[\delta f(0)])}$$

and

$$\begin{aligned} \pi^+(i) &= \prod_{j=1}^{i} \frac{\phi[\delta f((j-1)\delta)]}{1 - \phi[\delta f(j\delta)]} \Big/ \sum_{m=0}^{\infty} \prod_{j=1}^{m} \frac{\phi[\delta f((j-1)\delta)]}{1 - \phi[\delta f(j\delta)]}, \\ \pi^-(i) &= \prod_{j=-i}^{-1} \frac{1 - \phi[\delta f((j+1)\delta)]}{\phi[\delta f(j\delta)]} \Big/ \sum_{m=0}^{\infty} \prod_{j=-m}^{-1} \frac{1 - \phi[\delta f((j+1)\delta)]}{\phi[\delta f(j\delta)]}, \quad \forall i \in \mathbb{Z}\,. \end{aligned} \tag{14}$$

This messy-looking expression takes an especially simple form as $\delta \to 0$. To reveal this, for each $\delta > 0$ denote π^δ the stationary distribution of $\{Q_n\}$ for the discretized stochastic hybrid system H^δ with discretization step δ. Define function $u^\delta : \mathbb{R} \to \mathbb{R}$ as: $u^\delta(x) = \pi^\delta(k)/\delta$, if $x \in [k\delta, (k+1)\delta)$ for some $k \in \mathbb{Z}$. Then it can be easily checked that u^δ satisfies: $\int_{-\infty}^{\infty} u^\delta(x)\, dx = 1$, and u^δ has roughly the same shape as π^δ. Therefore the discrete distribution π^δ is converted to a continuous density function u^δ. Moreover,

Lemma 8. *Suppose $V(x)$ is chosen such that π^δ exists for $\delta > 0$ small enough. Then*

$$\lim_{\delta \to 0} \frac{u^\delta(y)}{u^\delta(x)} = e^{-2[V(y) - V(x)]} \quad \forall x, y \in \mathbb{R}\,.$$

We need the following notion to ensure that u^δ converges to a probability density.

Definition 7 (Tightness). *A family $\{u_\alpha, \alpha \in \Lambda\}$ of probability densities indexed by Λ is tight if and only if for each $\epsilon > 0$, there exists an M such that $\int_{-M}^{M} u_\alpha(x)\, dx > 1 - \epsilon$ for all $\alpha \in \Lambda$.*

Theorem 2. *Suppose $V(x)$ is chosen such that π^δ exists for $\delta > 0$ small enough and the resulting $\{u^\delta, \delta > 0\}$ is tight, then $\int_{-\infty}^{\infty} e^{-2V(x)}\, dx < \infty$ and*

$$u^\delta(x) \to u^0(x) \triangleq \frac{e^{-2V(x)}}{\int_{-\infty}^{\infty} e^{-2V(y)}\, dy} \quad \text{as } \delta \to 0\,,$$

where the convergence is pointwise.

Shown in Figure 6 are the plots of u^δ for different δ when $V(x) = (x^4 + 20(x-5) + c(x-5)^2)/100$ and $c = 275$. Here we choose $\delta = 40/N$, *i.e.* $[-20, 20]$ is discretized into N subintervals. Notice that the convergence speed is fast: even if the discretization is coarse, the resulting u^δ is still close to the final limit. In

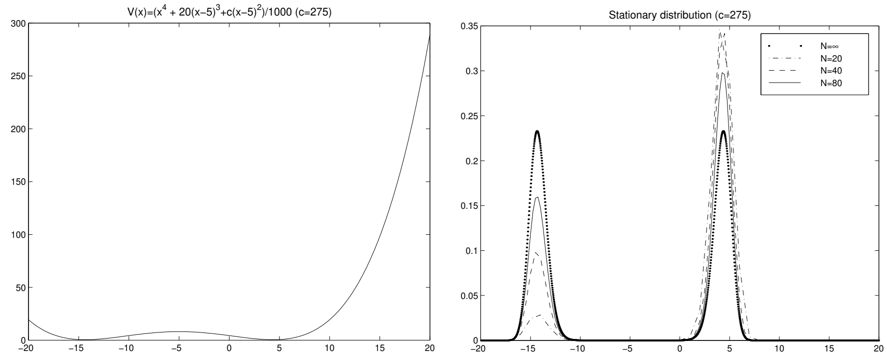

Fig. 6. Left: $V(x)$; Right: u^δ for different $\delta = \frac{40}{N}$

Figure 6, the two local minimums are at roughly the same level. By changing the value of c slightly, we can make one valley slightly deeper than the other. However, due to the exponential inverse relation of u^0 to V, this small change will be considerably amplified in u^0.

It is expected that the limiting distribution u^0 in Theorem 2 will also be the stationary distribution of the original stochastic differential equation: $dX_t = -\nabla V(X_t)dt + dB_t$ in the sense that if $X(0)$ is distributed as u^0 independently of $\{B_t\}$, then for any $t > 0$, the solution process X_t has the same distribution. We illustrated this in the following example.

Example 1. (**Ornstein-Uhlenbeck process**) Solution X_t to the SDE $dX_t = \mu X_t + \sigma dB_t$ is called the Ornstein-Uhlenbeck precess ([7]). Consider the case when $\sigma = 1$, $\mu = -a$ for some $a > 0$. Then by Ito formula, $X_t = X_0 e^{-at} + \int_0^t e^{-a(t-s)} dB_s$. If X_0 is Gaussian $N(0, \sigma)$ independently of $\{B_t\}$, then for each $t > 0$, X_t is also Gaussian with mean 0 and variance $\frac{1}{2a} + (\sigma^2 - \frac{1}{2a})e^{-2at}$. Let $\sigma^2 = \frac{1}{2a}$, then we can see that X_t has stationary distribution $N(0, \frac{1}{\sqrt{2a}})$ with density function predicted by Theorem 2.

Next we will discuss the limit behavior of first exit distribution of MC $\{Q_n\}$ from an interval. Consider MC $\{Q_n^+\}$ obtained in subsection 4.2.

Lemma 9. *Suppose $i_1 < i_0 < i_2$ are nonnegative integers. Then the probability that Q_n^+ starting from i_0 hits i_2 first than it hits i_1 is:*

$$\sum_{m=i_1+1}^{i_0-1} \prod_{j=i_1+1}^{m} \frac{q_j}{p_j} \Big/ \sum_{m=i_1+1}^{i_2-1} \prod_{j=i_1+1}^{m} \frac{q_j}{p_j}. \tag{15}$$

Suppose $a, b, c \in \mathbb{R}$ and $a < b < c$. For each $\delta > 0$, define $i_a^\delta = [a/\delta]$, $i_b^\delta = [b/\delta]$, $i_c^\delta = [c/\delta]$. Then for the corresponding embedded MC $\{Q_n\}$, the probability $P_{i_b^\delta}(T_{i_c^\delta} < T_{i_a^\delta})$ can be calculated by Lemma 9. The next theorem characterize the limiting behavior of such probability when $\delta \to 0$.

Theorem 3. *Using the same notation as in the above paragraph. Then as $\delta \to 0$,*

$$P_{i_b^\delta}(T_{i_c^\delta} < T_{i_a^\delta}) \to \frac{\int_a^b e^{-2V(x)}\,dx}{\int_a^c e^{-2V(x)}\,dx}\,.$$

It can be shown that the above asymptotic expression coincides with the corresponding probability of the original diffusion process (see [4]). Furthermore, under some proper assumptions, the expected escape time from an interval of the embedded MC can be studied as well and can be shown to converge to the corresponding value of the original diffusion process. Therefore the DSHS presents a powerful tool for studying the sample path properties of the SDE, at least when the discretization step is small enough.

The advantage of having closed form formulae for various properties of the stochastic hybrid systems is that it can greatly facilitate the design and evaluation of such systems. These topics will be pursued in future work.

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