TRACE FORMULAE ASSOCIATED WITH THE POLAR DECOMPOSITION OF OPERATORS

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Abstract

Let $T = X + iY$ be the Cartesian decomposition of an invertible operator T on a Hilbert space with trace class self-commutator $[T^*, T]$. Carey–Pincus introduced the principal function g and proved a trace formula associated with the Cartesian decomposition $T = X + iY$. Applying the ordered C^∞-functional calculus for (X, Y) to their trace formula, we define the principal function g^P and prove a trace formula associated with the polar decomposition $T = U|T|$. Using this formula, we show that $g(x, y) = g^P(e^{i\theta}, r)$ almost everywhere $x + iy = re^{i\theta}$ on $\mathbf{C}$.

1. Introduction

Let $B(\mathcal{H})$ be the set of all bounded linear operators on a complex separable Hilbert space $\mathcal{H}$, and let $\mathcal{C}_1$ be the set of trace-class operators of $B(\mathcal{H})$. In [4], Carey-Pincus defined the principal function g and proved a trace formula associated with the Cartesian decomposition $T = X + iY$ with $[T^*, T] \in \mathcal{C}_1$ (see also [12]). It is known that the principal functions are useful for the operator theory; for example, relating the size of the principal function to the existence of cyclic vectors, Berger [3] proved that, for a hyponormal operator T, the operator T^n has a non-trivial invariant subspace for sufficiently high n (see other examples, [6; 9; 13; 14; 15; 16]. We also have two different trace formulae and the principal functions g and g^P associated with the decomposition $T = X + iY$ and the polar decomposition $T = U|T|$, respectively [4; 15; 16]. The relation between g and g^P is that if there exists a trace formula for the polar decomposition, then there exists g by a transformation of variables, and g essentially coincides with g^P. An operator T is called p-hyponormal if $(T^*T)^p \geq (TT^*)^p$ [1]. If $p = 1$ and $\frac{1}{2}$, then T is called hyponormal and semi-hyponormal, respectively. The principal function g has been studied well.

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For example, if T is hyponormal, then $g \geq 0$ (see, for example [13; 16]). If T is semi-hyponormal, then $g^P \geq 0$. Applying this property for g^P, we have that $g \geq 0$.

The existences of the trace formulae and g and g^P [4] have been shown separately (see also [15; 16]). In this paper, by the ordered C^∞-functional calculus, we give a trace formula of $|T|$ and U for an invertible operator $T = U|T|$ such that $[T^*, T] \in \mathcal{C}_1$. Using this result, we show a trace formula of a non-invertible semi-hyponormal operator $T = U|T|$ with unitary U such that $[|T|, U] \in \mathcal{C}_1$. Finally, we show a relation between two principal functions g and g^P for such an operator T. We remark that for an operator $T = U|T|$, it is easy to see that if $[|T|, U] \in \mathcal{C}_1$, then $[T^*, T] \in \mathcal{C}_1$.

Let $\mathcal{S}(\mathbf{R}^2)$ be the Schwartz space of rapidly decreasing functions at infinity. For $T = X + iY$, let $\mathcal{E}$ and $\mathcal{F}$ be the spectral measures of self-adjoint operators X and Y, respectively. We define τ on $\mathcal{S}(\mathbf{R}^2)$ by

$$\tau(\phi) = \int\int \phi(x, y) d\mathcal{E}(x) d\mathcal{F}(y) \quad (\phi \in \mathcal{S}(\mathbf{R}^2)). \qquad (*)$$

By a standard argument, we have

$$\int\int e^{itX} e^{isY} \hat{\phi}(t, s) dt ds = \int\int \phi(x, y) d\mathcal{E}(x) d\mathcal{F}(y),$$

where

$$\hat{\phi}(t, s) = \frac{1}{2\pi} \int\int e^{-i(tx+sy)} \phi(x, y) dx dy$$

is the Fourier transform of the function ϕ (see, for example, [13, p. 237]).

Put $\nu(E) = \int\int_E \hat{\phi}(t, s) dt ds$ for a measurable set $E \subset \mathbf{R}^2$. Since $\hat{\phi}(t, s) \in \mathcal{S}(\mathbf{R}^2)$, we have

$$\int\int (1 + |t|)(1 + |s|)|\hat{\phi}(t, s)| dt ds < \infty.$$

Following Carey–Pincus [4], put $G(x, y) = \int\int e^{itx+isy} d\nu(t, s)$ and define

$$G(X, Y) = \int\int G(x, y) d\mathcal{E}(x) d\mathcal{F}(y).$$

Then

$$\tau(\phi) = \int\int e^{itX} e^{isY} \nu(t, s) dt ds = G(X, Y).$$

Note here that we have $\tau(\psi) = \tau(\phi)$ for any smooth function $\psi(x, y)$ that coincides with $\phi(x, y)$ on $\mathrm{supp}(\tau)$.

The map $\tau : \mathcal{S}(\mathbf{R}^2) \to B(\mathcal{H})$ has the following properties [13, chapter X, §2];

(1) τ is linear, continuous and $\mathrm{supp}\,(\tau) \subseteq \sigma(X) \times \sigma(Y)$,
(2) $\tau(1) = I, \tau(p + q) = p(X) + q(Y)$ for polynomials p and q of one variable of x and y, respectively.
(3) $\tau(\phi)\tau(\psi) - \tau(\phi\psi) \in \mathcal{C}_1$ for $\phi, \psi \in \mathcal{S}(\mathbf{R}^2)$,
(4) $\tau(\phi)^* - \tau(\bar{\phi}) \in \mathcal{C}_1$.

By (3) we have an important property $[\tau(\phi), \tau(\psi)] \in \mathcal{C}_1$ for $\phi, \psi \in \mathcal{S}(\mathbf{R^2})$.

Let $\mathcal{A}$ be the linear space of all Laurent polynomials $\mathcal{P}(r, z)$ with polynomial coefficients such that $\mathcal{P}(r, z) = \sum_{k=-N}^{N} p_k(r)z^k$, where N is a non-negative integer and each $p_k(r)$ is a polynomial. For the polar decomposition $T = U|T|$ of T, let $\mathcal{P}(|T|, U) = \sum_{k=-N}^{N} p_k(|T|)U^k$. For differentiable functions P, Q of two variables (x, y), let $J(P, Q)(x, y) = \dfrac{\partial P}{\partial x}\dfrac{\partial Q}{\partial y} - \dfrac{\partial P}{\partial y}\dfrac{\partial Q}{\partial x}$. For a trace-class operator $T \in \mathcal{C}_1$, we denote the trace of T by $\mathrm{Tr}(T)$.

In this paper, we prove the following trace formula of an invertible operator $T = X + iY = U|T|$ with $[|T|, U] \in \mathcal{C}_1$ by the above Cartesian functional calculus of τ with X and Y. For $\mathcal{P}, \mathcal{Q} \in \mathcal{A}$,

$$\mathrm{Tr}([\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)]) = \frac{1}{2\pi}\int\int J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta})e^{i\theta}g^P(e^{i\theta}, r)drd\theta.$$

The function g^P in the above formula is called the principal function associated with the polar decomposition of T. As a corollary of this result, we show that the same formula holds for a non-invertible semi-hyponormal operator $T = U|T|$ with unitary U and $[|T|, U] \in \mathcal{C}_1$. For an operator T, let $\sigma(T)$ be the spectrum of T. The following theorem [4, theorem 5.1] is a basis of this paper (see [12] also):

Theorem 1 (Carey–Pincus). *Let $T = X + iY$ be an operator with $[T^*, T] \in \mathcal{C}_1$. Let $\mathcal{E}, \mathcal{F}$ be the spectral measures of X and Y, respectively, and τ be given by $(*)$. Then there exists a summable function g such that, for $\phi, \psi \in \mathcal{S}(\mathbf{R}^2)$,*

$$Tr([\tau(\phi), \tau(\psi)]) = \frac{1}{2\pi i}\int\int J(\phi, \psi)(x, y)g(x, y)dxdy.$$

Moreover, if T is hyponormal, then $g \geq 0$ and $g(x, y) = 0$ for $x + iy \notin \sigma(T)$.

The function g in Theorem 1 is called *the principal function associated with the Cartesian decomposition of T.*

2. Function calculus and trace

Let $||A||_1 = \mathrm{Tr}(|A|)$ for $A \in \mathcal{C}_1$, that is, $||A||_1$ is the trace norm of A. Let $A \in \mathcal{C}_1$ and B be an operator. Then it holds that

$$|\mathrm{Tr}(A)| \leq ||A||_1, \mathrm{Tr}(AB) = \mathrm{Tr}(BA), ||AB||_1 \leq ||A||_1||B|| \text{ and } ||BA||_1 \leq ||B||||A||_1.$$

We use an elementary property that if operators A, B and C satisfy $[A, C], [B, C] \in \mathcal{C}_1$ and $A - B \in \mathcal{C}_1$, then $[AB, C], [BA, C] \in \mathcal{C}_1$ and

$$\mathrm{Tr}([AB, C]) = \mathrm{Tr}([BA, C]).$$

Our standard reference on trace is [11].

We begin with two lemmas that are key tools in this paper.

Lemma 2. *Let A be a positive invertible operator and operators D, E, F satisfying $[A, D], [E, D], [F, D] \in \mathcal{C}_1$. Then for any real number α, we have*

$$[EA^\alpha F, D] \in \mathcal{C}_1.$$

Proof. We use the following expansion known as the binomial series: For $|z| < 1$, it holds

$$(1+z)^\alpha = \sum_{m=0}^{\infty} \binom{\alpha}{m} z^m,$$

where $\binom{\alpha}{m} = \dfrac{\alpha(\alpha-1)\cdots(\alpha-m+1)}{m!}$. Considering $||\beta A|| < 1$ with some positive number β, we may assume that $||A|| < 1$. Since A is an invertible positive operator and $\|A\| < 1$, we have $||A - I|| < 1$ and

$$A^\alpha = (I+(A-I))^\alpha = \lim_{n\to\infty} \sum_{m=0}^{n} \binom{\alpha}{m} (A-I)^m. \tag{1}$$

Let $A_n = \left[\sum_{m=0}^{n} \binom{\alpha}{m} (A-I)^m, D\right]$ for $n = 1, 2, 3, \cdots$. Then $\lim_{n\to\infty} A_n = [A^\alpha, D]$ with respect to the operator norm. By [12, p. 158 (3.3)], for a positive integer m, it holds that

$$||[(A-I)^m, D]||_1 \le m||A-I||^{m-1}||[A, D]||_1,$$

so that

$$||A_n||_1 \le \left(\sum_{m=1}^{n} \left|\binom{\alpha}{m}\right| m||A-I||^{m-1}\right) ||[A, D]||_1.$$

Since $\|A - I\| < 1$, (1) converges absolutely. Hence $\{A_n\}$ is a Cauchy sequence with respect to the norm $|| \cdot ||_1$. Let B denote the limit of the sequence $\{A_n\}$ in $\mathcal{C}_1$. For any unit vector $\xi \in \mathcal{H}$, we define an operator C on $\mathcal{H}$ by $C\eta = (\eta, \xi)\xi$ for $\eta \in \mathcal{H}$. Let $\{e_j\}$ be a complete orthonormal basis of $\mathcal{H}$ such that $e_1 = \xi$. Since $\mathrm{Tr}(SC) = \sum_{j=1}^{\infty} (SCe_j, e_j) = (S\xi, \xi)$, then

$$(B\xi, \xi) = \mathrm{Tr}(BC) = \lim_{n\to\infty} \mathrm{Tr}(A_nC) = \lim_{n\to\infty} (A_n\xi, \xi) = ([A^\alpha, D]\xi, \xi).$$

Since ξ is an arbitrary vector, it follows that

$$[A^\alpha, D] = B \in \mathcal{C}_1.$$

We have

$$[EA_nF, D] = [E, D]A_nF + E[A_n, D]F + EA_n[F, D].$$

Since $\lim_{n\to\infty} A_n = A^\alpha$ with respect to the operator norm,

$$\lim_{n\to\infty}[E, D]A_nF = [E, D]A^\alpha F, \quad \lim_{n\to\infty} E[A_n, D]F = E[A^\alpha, D]F,$$

$$\text{and } \lim_{n\to\infty} EA_n[F, D] = EA^\alpha[F, D],$$

so that

$$\lim_{n\to\infty}[EA_nF, D] = [EA^\alpha F, D]$$

with respect to $\mathcal{C}_1$. ■

The proof of Lemma 2 is based on an idea of [8, theorem 2].

Let $T = X + iY$ be the Cartesian decomposition of T. For the spectral measures $\mathcal{E}$ and $\mathcal{F}$ of self-adjoint operators X and Y, respectively, we recall

$$\tau(\phi) = \int\int \phi(x, y)d\mathcal{E}(x)d\mathcal{F}(y) \quad (\phi \in \mathcal{S}(\mathbf{R}^2)).$$

Lemma 3. [4, p. 158] *Let $T = X + iY$ be an invertible operator such that $[T^*, T] \in \mathcal{C}_1$. Let $\psi \in \mathcal{S}(\mathbf{R}^2)$, $D = \tau(\psi)$ and operators E, F satisfy $[E, D], [F, D] \in \mathcal{C}_1$. Then, for $\phi(x, y) = (x^2 + y^2)^\alpha$ with a real number α,*

$$Tr([E\tau(\phi)F, D]) = Tr([E|T|^{2\alpha}F, D]).$$

PROOF. We may assume that $||T|| < d < \frac{1}{2}$. Then $||X^2+Y^2|| = |||T|^2 - \frac{1}{2}[T^*, T]|| < 1$. Hence, $X^2 + Y^2 < I$. Since T is invertible, we choose a positive number c such that $0 < c \leq X^2 + Y^2$. Hence, we may assume that f of $\tau(f)$ is a function on $\{(x, y) \,|\, c \leq x^2 + y^2 < 1\}$. Also we choose $\varphi \in C_0^\infty(\mathbf{R}^2)$ and d_1 such that $d < d_1 < 1$, $\varphi(x, y) = 1$ on $\{(x, y) \,|\, c \leq x^2 + y^2 \leq d\}$ and $\text{supp}(\varphi) \subset \{(x, y) \,|\, x^2 + y^2 < d_1\}$. Then

$$\tau(\phi\varphi) = \sum_{m=0}^{\infty} \binom{\alpha}{m} \int\int ((x^2+y^2)-1)^m d\mathcal{E}(x)d\mathcal{F}(y) = \sum_{m=0}^{\infty} \binom{\alpha}{m} \tau\big(((x^2+y^2)-1)^m\big)$$

with respect to the operator norm. Since

$$\tau((x^2 + y^2) - 1) = X^2 + Y^2 - I \ \text{ and } \ |T|^2 = X^2 + Y^2 + \tfrac{1}{2}[T^*, T],$$

we get

$$\tau((x^2 + y^2) - 1) - (|T|^2 - I) \in \mathcal{C}_1.$$

Since by property (3) of τ and the above it holds that

$$\tau\big(((x^2 + y^2) - 1)^m\big) - \tau\big((x^2 + y^2) - 1\big)^m \in \mathcal{C}_1,$$

we have for $m > 0$

$$\tau\big(((x^2+y^2)-1)^m\big) - (|T|^2 - I)^m$$
$$= \tau\big(((x^2+y^2)-1)^m\big) - \tau\big((x^2+y^2)-1\big)^m + \tau((x^2+y^2)-1)^m - (|T|^2-I)^m \in \mathcal{C}_1.$$

Hence, it holds that

$$\mathrm{Tr}([\tau\big(((x^2+y^2)-1)^m\big), D]) = \mathrm{Tr}([(|T|^2 - I)^m, D])$$

and

$$\Big[\sum_{m=0}^{n} \begin{pmatrix} \alpha \\ m \end{pmatrix} \tau\big(((x^2+y^2)-1)^m\big), D\Big] \in \mathcal{C}_1.$$

Therefore, we see

$$\mathrm{Tr}\Big(\Big[\sum_{m=0}^{n} \begin{pmatrix} \alpha \\ m \end{pmatrix} \tau\big(((x^2+y^2)-1)^m\big), D\Big]\Big) = \mathrm{Tr}\Big(\Big[\sum_{m=0}^{n} \begin{pmatrix} \alpha \\ m \end{pmatrix} ((X^2+Y^2)-I)^m, D\Big]\Big).$$

Let

$$\varphi_\infty(r) = r^\alpha = \sum_{m=0}^{\infty} \begin{pmatrix} \alpha \\ m \end{pmatrix} (r-1)^m \qquad (0 < |r| < 1),$$
$$\varphi_n(r) = \sum_{m=0}^{n} \begin{pmatrix} \alpha \\ m \end{pmatrix} (r-1)^m,$$
$$\phi_n(x,y) = \varphi_n(x^2+y^2) = \sum_{m=0}^{n} \begin{pmatrix} \alpha \\ m \end{pmatrix} ((x^2+y^2)-1)^m.$$

Put $\tilde{\phi_n} = \phi_n\varphi$ and $\tilde{\phi} = \phi\varphi$. Then for some $f_k \in C^\infty$ with $\mathrm{supp}(f_k) \subset \{(x,y)\,|\,x^2+y^2 < 1\}$ $(k = 0, \cdots, m)$, we have

$$\frac{\partial^m}{\partial x^j \partial y^{m-j}}(\tilde{\phi_n} - \tilde{\phi})(x,y)$$
$$= (\varphi_n^{(m)}(r^2) - \varphi_\infty^{(m)}(r^2))f_m(x,y) + (\varphi_n^{(m-1)}(r^2) - \varphi_\infty^{(m-1)}(r^2))f_{m-1}(x,y)$$
$$+ \cdots + (\varphi_n(r^2) - \varphi_\infty(r^2))f_0(x,y),$$

where $r^2 = x^2 + y^2$. We remark that each f_k depends on $\dfrac{\partial^m}{\partial x^j \partial y^{m-j}}$ and is independent of $\tilde{\phi}_n$. Hence we obtain $\tilde{\phi}_n \to \tilde{\phi}$ in $\mathcal{S}(\mathbf{R}^2)$. By [13, chapter X, corollary 2.3], it holds that

$$[\tau(\tilde{\phi}_n), D] \to [\tau(\tilde{\phi}), D] \quad \text{in } \mathcal{C}_1.$$

Since

$$[E\tau(\tilde{\phi}_n)F, D] = [E, D]\tau(\tilde{\phi}_n)F + E[\tau(\tilde{\phi}_n), D]F + E\tau(\tilde{\phi}_n)[F, D]$$

and $\lim_{n\to\infty} \tau(\tilde{\phi}_n) = \tau(\tilde{\phi})$ with respect to the operator norm, in $\mathcal{C}_1$ it holds that

$$\begin{aligned} \lim_{n\to\infty} [E, D]\tau(\tilde{\phi}_n)F &= [E, D]\tau(\tilde{\phi})F, \\ \lim_{n\to\infty} E[\tau(\tilde{\phi}_n), D]F &= E[\tau(\tilde{\phi}), D]F, \\ \lim_{n\to\infty} E\tau(\tilde{\phi}_n)[F, D] &= E\tau(\tilde{\phi})[F, D]. \end{aligned}$$

Hence in $\mathcal{C}_1$ we obtain

$$\lim_{n\to\infty} [E\tau(\tilde{\phi}_n)F, D] = [E\tau(\tilde{\phi})F, D].$$

Since $T \ \to \ \mathrm{Tr}(T)$ is continuous in $\mathcal{C}_1$, we have

$$\begin{aligned} \mathrm{Tr}([E\tau(\tilde{\phi})F, D]) &= \lim_{n\to\infty} \mathrm{Tr}(E[\tau(\tilde{\phi}_n)F, D]) \\ &= \lim_{n\to\infty} \mathrm{Tr}(E \sum_{m=0}^{n} \begin{pmatrix} \alpha \\ m \end{pmatrix} [(|T|^2 - I)^m F, D]) \\ &= \mathrm{Tr}([E|T|^{2\alpha} F, D]). \end{aligned}$$

■

3. Main theorem

First we show the following:

Theorem 4. *Let $T = U|T|$ be an invertible operator with $[T^*, T] \in \mathcal{C}_1$ and let g be the principal function associated with the Cartesian decomposition of $T = X + iY$. Then there exists a summable function g^P such that, for $\mathcal{P}, \mathcal{Q} \in \mathcal{A}$,*

$$Tr([\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)]) = \frac{1}{2\pi} \int\int J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta}) e^{i\theta} g^P(e^{i\theta}, r) dr d\theta,$$

and $g^P(e^{i\theta}, r) = g(x, y)$ almost everywhere $x + iy = re^{i\theta}$ on $\mathbf{C}$.

PROOF. Since T is invertible, there exists a number $c > 0$ such that $c \le X^2 + Y^2$. Then $\frac{c}{2} \le X^2$ or $\frac{c}{2} \le Y^2$, so that, if $\zeta \in \mathcal{S}(\mathbf{R}^2)$ satisfies $\zeta(x, y) = 0$ for $\frac{c}{2} > |x|^2$ or $\frac{c}{2} > |y|^2$, then $\tau(\zeta) = 0$. With $g(x, y)$ in Theorem 1, we know that $g(x, y) = 0$ for $x + iy$ with $x^2 + y^2 < \frac{c}{2}$. Let $w(x, y)$ and $h(x, y)$ be in $\mathcal{S}(\mathbf{R}^2)$ such that $w(x, y) = (x + iy)(x^2 + y^2)^{-\frac{1}{2}}$ and $h(x, y) = (x^2 + y^2)^{\frac{1}{2}}$ on the support of g. For $\psi, \phi_l, \phi_r \in \mathcal{S}(\mathbf{R}^2)$, let $D = \tau(\psi), E = \tau(\phi_l)$ and $F = \tau(\phi_r)$. By property (3) of τ and Lemma 3, for a positive integer k we obtain

$$\begin{aligned} \mathrm{Tr}([EU^k F, D]) &= \mathrm{Tr}([E(T|T|^{-1})^k F, D]) = \mathrm{Tr}([E(T\tau(h^{-1}))^k F, D]) \\ &= \mathrm{Tr}\big([E\big(\tau(x + iy)\tau(h^{-1})\big)^k F, D]\big) \\ &= \mathrm{Tr}\big([E\tau\big(\big((x + iy)(x^2 + y^2)^{-\frac{1}{2}}\big)^k\big)F, D]\big) = \mathrm{Tr}\big([E\tau(w^k)F, D]\big) \end{aligned}$$

and

$$\begin{aligned}\mathrm{Tr}\big([EU^{-k}F,D]\big) &= \mathrm{Tr}\big([E(|T|T^{-1})^{k}F,D]\big) = \mathrm{Tr}\Big(\big[E\big(\tau(h)\tau\big(1/(x+iy)\big)\big)^{k}F,D\big]\Big)\\ &= \mathrm{Tr}\Big(\big[E\tau\big(\big((x-iy)(x^2+y^2)^{-\frac{1}{2}}\big)^{k}\big)F,D\big]\Big) = \mathrm{Tr}\big([E\tau(w^{-k})F,D]\big).\end{aligned}$$

Then for integers m,s and non-negative integers n,t, we have

$$\begin{aligned}\mathrm{Tr}([U^m|T|^n,U^s|T|^t]) &= \mathrm{Tr}([\tau(w^m)\tau(h^n),\tau(w^s)\tau(h^t)])\\ &= \mathrm{Tr}([\tau(w^m h^n),\tau(w^s h^t)]).\end{aligned}$$

By Theorem 1, there exists a summable function g such that

$$\mathrm{Tr}([\tau(w^m h^n),\tau(w^s h^t)]) = \frac{1}{2\pi i}\int\int J(w^m h^n, w^s h^t)(x,y)g(x,y)dxdy.$$

By the transformation $x=r\cos\theta$ and $y=r\sin\theta$,

$$\begin{aligned}&\frac{1}{2\pi i}\int\int J(w^m h^n, w^s h^t)(x,y)g(x,y)dxdy\\ =&\frac{1}{2\pi i}\int\int J(w^m h^n, w^s h^t)(r\cos\theta,r\sin\theta)g(r\cos\theta,r\sin\theta)rdrd\theta.\end{aligned}$$

Hence we have, for Laurent polynomials $\mathcal{P}$ and $\mathcal{Q}$,

$$\mathrm{Tr}([\mathcal{P}(|T|,U),\mathcal{Q}(|T|,U)])$$

$$=\frac{1}{2\pi i}\int\int J(\mathcal{P}(h,w),\mathcal{Q}(h,w))(r\cos\theta,r\sin\theta)rg(r\cos\theta,r\sin\theta)drd\theta.$$

For $x+iy\in\sigma(T)$, let $x=r\cos\theta$ and $y=r\sin\theta$. Since $w(x,y)=(x+iy)(x^2+y^2)^{-\frac{1}{2}}$ and $h(x,y)=(x^2+y^2)^{\frac{1}{2}}$, then $w(r\cos\theta,r\sin\theta)=e^{i\theta}$, $h(r\cos\theta,r\sin\theta)=r$,

$$\frac{\partial(h,w)}{\partial(r,\theta)} = \frac{\partial(r,e^{i\theta})}{\partial(r,\theta)} = ie^{i\theta} \quad\text{and}\quad \frac{\partial(x,y)}{\partial(r,\theta)} = \frac{\partial(r\cos\theta,r\sin\theta)}{\partial(r,\theta)} = r.$$

Also, it holds that

$$\frac{\partial\big(\mathcal{P}(r,e^{i\theta}),\mathcal{Q}(r,e^{i\theta})\big)}{\partial(r,\theta)} = \frac{\partial\big(\mathcal{P}(h,w),\mathcal{Q}(h,w)\big)}{\partial(h,w)}(r,e^{i\theta})\cdot\frac{\partial(h,w)}{\partial(r,\theta)}$$

$$= ie^{i\theta}\cdot\frac{\partial\big(\mathcal{P}(h,w),\mathcal{Q}(h,w)\big)}{\partial(h,w)}(r,e^{i\theta}) = ie^{i\theta}\cdot J\big(\mathcal{P},\mathcal{Q}\big)(r,e^{i\theta})$$

and

$$\frac{\partial\big(\mathcal{P}\big(h(r\cos\theta, r\sin\theta), w(r\cos\theta, r\sin\theta)\big), \mathcal{Q}\big(h(r\cos\theta, r\sin\theta), w(r\cos\theta, r\sin\theta)\big)\big)}{\partial(r,\theta)}$$

$$= \frac{\partial\big(\mathcal{P}(h(x,y), w(x,y)), \mathcal{Q}(h(x,y), w(x,y))\big)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= r \cdot \frac{\partial\big(\mathcal{P}(h(x,y), w(x,y)), \mathcal{Q}(h(x,y), w(x,y))\big)}{\partial(x,y)}$$

$$= r \cdot J\big(\mathcal{P}(h,w), \mathcal{Q}(h,w)\big)(r\cos\theta, r\sin\theta).$$

Hence we have

$$\mathrm{Tr}([\mathcal{P}(|T|,U), \mathcal{Q}(|T|,U)]) = \frac{1}{2\pi i}\int\int i\frac{\partial(\mathcal{P}(h,w), \mathcal{Q}(h,w))}{\partial(h,w)}(r, e^{i\theta})e^{i\theta}g(r\cos\theta, r\sin\theta)drd\theta.$$

Put $g^P(e^{i\theta}, r) = g(r\cos\theta, r\sin\theta)$. Then

$$\mathrm{Tr}([\mathcal{P}(|T|,U), \mathcal{Q}(|T|,U)]) = \frac{1}{2\pi}\int\int J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta})e^{i\theta}g^P(e^{i\theta}, r)drd\theta.$$

■

The function g^P in Theorem 4 is called *the principal function associated with the polar decomposition $T = U|T|$ of T*. An invertible operator T is said to be log-hyponormal if $\log T^*T \geq \log TT^*$ [10]. Lemma 2 and Theorem 4 give another proof of a trace formula of log-hyponormal operators in [5].

For the proof of the next result, we need the following two lemmas. For an operator T, let $\sigma_{ap}(T)$ and $\sigma_p(T)$ be the approximate point spectrum and the point spectrum of T, respectively.

Lemma 5. *Let $T = U|T|$ be an invertible semi-hyponormal operator with $[|T|, U] \in \mathcal{C}_1$. Then the principal function g^P associated with the polar decomposition $T = U|T|$ of T satisfies $g^P(e^{i\theta}, r) = 0$ for $re^{i\theta} \notin \sigma(T)$.*

PROOF. Put $S = U|T|^{\frac{1}{2}}$. Then S is hyponormal and $[S^*, S] = [|T|, U]U^* \in \mathcal{C}_1$. Let g_S^P be the principal function associated with the polar decomposition of $S = U|T|^{\frac{1}{2}}$. Then by Theorems 1 and 4 it holds that $g_S^P(e^{i\theta}, r) = 0$ for $re^{i\theta} \notin \sigma(S)$. By [16, lemma VI 3.6] and $S = U|T|^{\frac{1}{2}}$, we also have

$$\sigma(T) = \{r^2e^{i\theta} : re^{i\theta} \in \sigma(S)\} \quad \text{and} \quad g^P(e^{i\theta}, r) = g_S^P(e^{i\theta}, r^2).$$

Hence, g^P has the desired property. ■

Lemma 6. *Let $T = U|T|$ be an operator with unitary U and put $S = U(|T| + I)$. If $z \in \partial\sigma(S)$, then $|z| \geq 1$. Therefore, if $z \in \sigma(S)$, then $|z| \geq 1$.*

PROOF. Since U and $|T| + I$ are invertible, so is S. Since $z \in \partial\sigma(S)$ and $\partial\sigma(S) \subseteq \sigma_{ap}(S)$, we have $z \in \sigma_{ap}(S)$. Hence, let $\pi : B(\mathcal{H}) \to B(\mathcal{K})$ denote the Berberian representation [2]. Since $\sigma_{ap}(S) = \sigma_p(\pi(S))$, there exists $\mathbf{x} \in \mathcal{K}$ such that

$$z\boldsymbol{x} = \pi(S)\boldsymbol{x} = \pi(U)\pi(|T| + I)\boldsymbol{x}.$$

Since $\pi(U)$ is unitary, there exists $\mathbf{y} \in \mathcal{K}$ such that $\pi(U)^*\mathbf{y} = \boldsymbol{x}$. Hence

$$||\boldsymbol{y}||^2 = (\boldsymbol{y}, \boldsymbol{y}) \leq (\pi(U)\pi(|T| + I)\pi(U)^*\boldsymbol{y}, \boldsymbol{y}) = (z\boldsymbol{x}, \boldsymbol{y}) \leq |z|\, ||\boldsymbol{x}||\, ||\boldsymbol{y}|| = |z|\, ||\boldsymbol{y}||^2,$$

so that $1 \leq |z|$. Let $z_0 \in \sigma(S)$ such that $|z_0| = \inf\{|\mu| : \mu \in \sigma(S)\}$. Since S is invertible, we have

$$z_0 \in \partial\sigma(S).$$

By the above argument, we obtain $1 \leq |z_0|$. ■

Now we give another proof of [7, theorem 9].

Theorem 7. *Let $T = U|T|$ be a semi-hyponormal operator with unitary U and $[|T|, U] \in \mathcal{C}_1$. Then there exists a summable function g^P such that, for $\mathcal{P}, \mathcal{Q} \in \mathcal{A}$,*

$$Tr([\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)]) = \frac{1}{2\pi} \int\int J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta})e^{i\theta}g^P(e^{i\theta}, r)drd\theta.$$

PROOF. Since by the assumption $[|T|, U] \in \mathcal{C}_1$ it holds that $[T^*, T] \in \mathcal{C}_1$, by Theorem 4 we may only prove the theorem when T is not invertible. Put $|\tilde{T}| = |T|+I$ and $\tilde{T} = U|\tilde{T}|$. Then $\tilde{T}$ is semi-hyponormal. For Laurent polynomials $\mathcal{P}$ and $\mathcal{Q}$, put $\tilde{\mathcal{P}}(r, z) = \mathcal{P}(r - 1, z)$ and $\tilde{\mathcal{Q}}(r, z) = \mathcal{Q}(r - 1, z)$. Then

$$\begin{aligned}\mathrm{Tr}([\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)]) &= \mathrm{Tr}([\mathcal{P}(|\tilde{T}| - I, U), \mathcal{Q}(|\tilde{T}| - I, U)]) \\ &= \mathrm{Tr}([\tilde{\mathcal{P}}(|\tilde{T}|, U), \tilde{\mathcal{Q}}(|\tilde{T}|, U)]).\end{aligned}$$

Since $\tilde{T}$ is invertible and $[|\tilde{T}|, U] = [|T|, U] \in \mathcal{C}_1$, by Theorem 4 there exists a summable function $\tilde{g}^P$ such that

$$\mathrm{Tr}([\tilde{\mathcal{P}}(|\tilde{T}|, U), \tilde{\mathcal{Q}}(|\tilde{T}|, U)]) = \frac{1}{2\pi} \int\int J(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}})(r, e^{i\theta})e^{i\theta}\tilde{g}^P(e^{i\theta}, r)drd\theta.$$

By Lemma 5, it holds that $\tilde{g}^P(e^{i\theta}, r) = 0$ for $re^{i\theta} \notin \sigma(\tilde{T})$. We have

$$\frac{1}{2\pi} \int\int J(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}})(r, e^{i\theta})e^{i\theta}\tilde{g}^P(e^{i\theta}, r)drd\theta$$

$$= \frac{1}{2\pi} \int\int_{\sigma(\tilde{T})} J(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}})(r, e^{i\theta})e^{i\theta}\tilde{g}^P(e^{i\theta}, r)drd\theta$$

$$= \frac{1}{2\pi}\int\int_{\sigma(\tilde{T})} J(\mathcal{P},\mathcal{Q})(r-1,e^{i\theta})e^{i\theta}\tilde{g}^P(e^{i\theta},r)drd\theta$$

$$= \frac{1}{2\pi}\int\int_{A} J(\mathcal{P},\mathcal{Q})(\rho,e^{i\theta})e^{i\theta}\tilde{g}^P(e^{i\theta},\rho+1)d\rho d\theta \quad \text{(by the transformation } \rho = r-1\text{)},$$

where $A = \{(r-1)e^{i\theta} : re^{i\theta} \in \sigma(\tilde{T})\}$. We remark that, by Lemma 6, $r-1 \geq 0$ for $re^{i\theta} \in \sigma(\tilde{T})$. We define g^P by $g^P(e^{i\theta},r) = \tilde{g}^P(e^{i\theta},r+1)$. Then g^P is the desired function. ■

Finally, we show a relation between g and g^P.

Theorem 8. *Let $T = X+iY = U|T|$ be a semi-hyponormal operator with unitary U and $[|T|,U] \in \mathcal{C}_1$. If g and g^P are the principal function associated with the Cartesian decomposition of T and the summable function in Theorem 7, respectively, then*

$$g(x,y) = g^P(e^{i\theta},r)$$

almost everywhere $x+iy = re^{i\theta}$ on $\mathbf{C}$.

PROOF. Since $[|T|,U] \in \mathcal{C}_1$, by Lemma 2 we have $[|T|^2,U] \in \mathcal{C}_1$. Hence

$$2i[X,Y] = T^*T - TT^* = |T|^2 - U|T|^2U^* = [|T|^2,U]U^* \in \mathcal{C}_1.$$

Let $\mathcal{Q}_0(x,y) = y$. For the polynomial $\mathcal{Q}_0(x,y) = y$ and an arbitrary polynomial $\mathcal{P}(x,y)$, by Theorem 1 and [4, theorem 5.2] we have

$$\begin{aligned}\mathrm{Tr}([\mathcal{P}(X,Y),\mathcal{Q}_0(X,Y)]) &= \frac{1}{2\pi i}\int\int_{\sigma(T)} J(\mathcal{P},\mathcal{Q}_0)g(x,y)dxdy \\ &= \frac{1}{2\pi i}\int\int_{\sigma(T)} \mathcal{P}_x(x,y)g(x,y)dxdy \\ &= \frac{1}{2\pi i}\int\int_{\mathrm{M}} \mathcal{P}_x(r\cos\theta,\, r\sin\theta)g(r\cos\theta,\, r\sin\theta)rdrd\theta, \quad (2)\end{aligned}$$

where $\mathrm{M} = \{(r,\theta) : re^{i\theta} \in \sigma(T),\ 0 \leq \theta < 2\pi\}$. Let

$$\tilde{\mathcal{P}}(r,z) = \mathcal{P}\left(\frac{zr+rz^{-1}}{2}, \frac{zr-rz^{-1}}{2i}\right) \quad \text{and} \quad \tilde{\mathcal{Q}}_0(r,z) = \frac{zr-rz^{-1}}{2i}.$$

Then

$$\begin{aligned}J(\tilde{\mathcal{P}},\tilde{\mathcal{Q}}_0) = &\left(\mathcal{P}_x\cdot\frac{z+z^{-1}}{2} + \mathcal{P}_y\cdot\frac{z-z^{-1}}{2i}\right)\left(\frac{r}{2i}(1+\frac{1}{z^2})\right) \\ &-\frac{r}{2}\left\{\mathcal{P}_x\cdot(1-\frac{1}{z^2}) + \frac{1}{i}\mathcal{P}_y\cdot(1+\frac{1}{z^2})\right\}\frac{z-z^{-1}}{2i}.\end{aligned}$$

Hence

$$J\big(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}_0\big)(r, e^{i\theta}) \cdot e^{i\theta} = (\mathcal{P}_x \cdot \cos\theta + \mathcal{P}_y \cdot \sin\theta)(-ir\cos\theta) - r(i\mathcal{P}_x \cdot \sin\theta - i\mathcal{P}_y \cdot \cos\theta)\sin\theta$$
$$= -ir\mathcal{P}_x.$$

Theorem 7 implies

$$\mathrm{Tr}\Big(\Big[\mathcal{P}\big(\frac{U|T|+|T|U^{-1}}{2}, \frac{U|T|-|T|U^{-1}}{2i}\big), \frac{U|T|-|T|U^{-1}}{2i}\Big]\Big)$$
$$= \frac{1}{2\pi}\int\int_{\mathrm{M}} J\big(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}_0\big)(r, e^{i\theta})e^{i\theta}g^P(e^{i\theta}, r)drd\theta$$
$$= \frac{1}{2\pi}\int\int_{\mathrm{M}} -ir\mathcal{P}_x(r\cos\theta,\, r\sin\theta)g^P(e^{i\theta}, r)drd\theta$$
$$= \frac{1}{2\pi i}\int\int_{\mathrm{M}} \mathcal{P}_x(r\cos\theta,\, r\sin\theta)g^P(e^{i\theta}, r)rdrd\theta. \tag{3}$$

Since

$$\mathrm{Tr}\big([\mathcal{P}(X,Y), \mathcal{Q}_0(X,Y)]\big) = \mathrm{Tr}\Big(\Big[\mathcal{P}\big(\frac{U|T|+|T|U^{-1}}{2}, \frac{U|T|-|T|U^{-1}}{2i}\big), \frac{U|T|-|T|U^{-1}}{2i}\Big]\Big),$$

we have (2) = (3) and

$$\int\int_{\mathrm{M}} \mathcal{P}_x(r\cos\theta,\, r\sin\theta)g(r\cos\theta,\, r\sin\theta)rdrd\theta$$
$$= \int\int_{\mathrm{M}} \mathcal{P}_x(r\cos\theta,\, r\sin\theta)g^P(e^{i\theta}, r)rdrd\theta.$$

Since $\mathcal{P}$ is an arbitrary polynomial, we obtain the desired relation between g and g^P. ■

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