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Trace formulae for p-hyponormal operators

by

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Dedicated to Professor W. Żelazko on his 70th birthday with respect

Abstract. The purpose of this paper is to introduce mosaics and principal functions of p-hyponormal operators and give a trace formula. Also we introduce p-nearly normal operators and give trace formulae for them.

1. Introduction. In Carey–Pincus [2] and Pincus–Xia [6], the trace formulae for pairs of operators associated with the polar decomposition are studied. In this paper, in a situation similar to [6] we introduce mosaics and principal functions of p-hyponormal operators for $0 < p \le 1/2$ and give trace formulae for p-hyponormal and p-nearly normal operators.

Let $\mathcal{H}$ be a complex separable Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is said to be *p-hyponormal* if $(T^*T)^p - (TT^*)^p \ge 0$ (see [1]). If $p = 1$, T is called *hyponormal*, and if $p = 1/2$, T is called *semi-hyponormal*. The set of all semi-hyponormal operators in $B(\mathcal{H})$ is denoted by SH. The set of all p-hyponormal operators in $B(\mathcal{H})$ is denoted by p-H. Let SHU and p-HU denote the sets of all operators in SH and in p-H with equal defect and nullity (cf. [7, p. 4]), respectively. Hence we may assume that the operator U in the polar decomposition $T = U|T|$ is unitary if $T \in \mathrm{SHU} \cup p\text{-HU}$. Throughout this paper, p satisfies $0 < p \le 1/2$.

Let $\mathbb{T} = \{e^{i\theta} \mid 0 \le \theta < 2\pi\}$, Σ be the set of all Borel sets in $\mathbb{T}$, m be a measure on the measurable space $(\mathbb{T}, \Sigma)$ such that $dm(\theta) = (2\pi)^{-1} d\theta$ and $\mathcal{D}$ be a separable Hilbert space. The Hilbert space of all vector-valued, strongly measurable and square-integrable functions with values in $\mathcal{D}$ and

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with inner product

$$(f,g)=\int_{\mathbb{T}}(f(e^{i\theta}),g(e^{i\theta}))_{\mathcal{D}}\,dm(\theta)$$

is denoted by $L^2(\mathcal{D})$; the Hardy space is denoted by $H^2(\mathcal{D})$, and the projection from $L^2(\mathcal{D})$ to $H^2(\mathcal{D})$ by $\mathcal{P}$. If $f\in L^2(\mathcal{D})$, then

$$(\mathcal{P}(f))(e^{i\theta})=\lim_{r\to 1-0}\frac{1}{2\pi i}\int_{|z|=1} f(z)(z-re^{i\theta})^{-1}\,dz.$$

Let ν be a singular measure on $(\mathbb{T},\Sigma)$, and $F\in\Sigma$ be a set such that $\nu(\mathbb{T}\backslash F)=0$ and $m(F)=0$. Put $\mu=m+\nu$. Let $R(\cdot)$ be a standard operator-valued strongly measurable function defined on $\Omega=(\mathbb{T},\Sigma,\mu)$ whose values are projections in $\mathcal{D}$, $L^2(\Omega,\mathcal{D})$ be the Hilbert space of all $\mathcal{D}$-valued strongly measurable and square-integrable functions on Ω with inner product $(f,g)=\int_{\mathbb{T}}(f(e^{i\theta}),g(e^{i\theta}))_{\mathcal{D}}\,d\mu$, and

$$\widetilde{H}=\{f\mid f\in L^2(\Omega,\mathcal{D}),\ R(e^{i\theta})f(e^{i\theta})=f(e^{i\theta}),\ e^{i\theta}\in\mathbb{T}\}.$$

Then $\widetilde{H}$ is a subspace of $L^2(\Omega,\mathcal{D})$. The space $L^2(\mathcal{D})$ is identified with a subspace of $L^2(\Omega,\mathcal{D})$. Hence $\mathcal{P}$ extends to $L^2(\Omega,\mathcal{D})$ so that

$$\mathcal{P}f=0\qquad\text{for } f\in L^2(\Omega,\mathcal{D})\ominus L^2(\mathcal{D}).$$

We define an operator $\mathcal{P}_0$ from $L^2(\Omega,\mathcal{D})$ to $\mathcal{D}$ as follows:

$$\mathcal{P}_0(f)=\int f(e^{i\theta})\,dm(\theta).$$

Then $\mathcal{P}_0$ is the projection from $L^2(\Omega,\mathcal{D})$ onto $\mathcal{D}$ (cf. [7, p. 50]). Let $\alpha(\cdot)$ and $\beta(\cdot)$ be operator-valued, uniformly bounded, and strongly measurable functions on Ω such that $\alpha(e^{i\theta})$ and $\beta(e^{i\theta})$ are linear operators in $\mathcal{D}$ satisfying

$$R(e^{i\theta})\alpha(e^{i\theta})=\alpha(e^{i\theta})R(e^{i\theta})=\alpha(e^{i\theta}),$$
$$R(e^{i\theta})\beta(e^{i\theta})=\beta(e^{i\theta})R(e^{i\theta})=\beta(e^{i\theta})$$

and $\beta(e^{i\theta})\ge 0$.

Furthermore, suppose that $\alpha(e^{i\theta})=0$ if $e^{i\theta}\in F$. We write $(\alpha f)(e^{i\theta})=\alpha(e^{i\theta})f(e^{i\theta})$. An operator $\widetilde{U}$ in $\widetilde{\mathcal{H}}$ is defined by

$$(\widetilde{U}f)(e^{i\theta})=e^{i\theta}f(e^{i\theta}).$$

Since $\beta(e^{i\theta})\ge 0$ and $\mathcal{P}$ is a projection on $L^2(\mathcal{D})$, we have

$$(\alpha(e^{i\theta})^*(\mathcal{P}(\alpha f))(e^{i\theta})+\beta(e^{i\theta})f(e^{i\theta}),f(e^{i\theta}))_{\mathcal{D}}\ge 0.$$

Therefore, we can define the operator $(\alpha^*\mathcal{P}\alpha+\beta)^{1/(2p)}$. See the details in [7]. Moreover, the following results hold.

Theorem A (Chō, Huruya and Itoh [3, Th. 1]). *With the above notations, let $\widetilde{T}$ be an operator in $\widetilde{\mathcal{H}}$ defined by*

$$(\widetilde{T}f)(e^{i\theta}) = e^{i\theta}(Af)(e^{i\theta}),$$

where $(A^{2p}f)(e^{i\theta}) = \alpha(e^{i\theta})^*(\mathcal{P}(\alpha f))(e^{i\theta}) + \beta(e^{i\theta})f(e^{i\theta})$. *Then $\widetilde{T}$ is p-hyponormal and the corresponding polar differential operator* $|\widetilde{T}| - \widetilde{U}|\widetilde{T}|\widetilde{U}^*$ *is*

$$((|\widetilde{T}| - \widetilde{U}|\widetilde{T}|\widetilde{U}^*)f)(e^{i\theta}) = \alpha(e^{i\theta})^*\mathcal{P}_0(\alpha f).$$

Theorem B (Chō, Huruya and Itoh [3, Th. 3]). *Let $T = U|T|$ be a p-hyponormal operator in $\mathcal{H}$ such that U is unitary. Then there exist a function space $\widetilde{\mathcal{H}}$, and operators $\widetilde{T}$ and $\widetilde{U}$ in $\widetilde{\mathcal{H}}$ which have the forms in Theorem* A *such that*

$$WTW^{-1} = \widetilde{T} \quad \textit{and} \quad WUW^{-1} = \widetilde{U},$$

where W is a unitary operator from $\mathcal{H}$ to $\widetilde{\mathcal{H}}$. Moreover $\alpha(\cdot) \geq 0$.

$\widetilde{T}$ is said to be the *singular integral model* of T.

2. Mosaics of operators $T \in p$-HU. For the singular integral model of a semi-hyponormal operator $T = U|T|$, the following holds:

Theorem C (Xia [7, Th. V.2.5]). *With the above notations, let $T = U|T|$ be in* SHU *and $\alpha(\cdot)$, $\beta(\cdot)$ be as in Theorems* A *and* B *for the singular integral model of T. Then the following statements hold.*

(1) *There exists a unique $B(\mathcal{D})$-valued measurable function of two variables,* $\mathrm{B}(e^{i\theta}, r)$ $(e^{i\theta} \in \mathbb{T}$, $r \in [0,\infty))$ *satisfying*

$$0 \leq \mathrm{B}(e^{i\theta}, r) \leq I$$

such that

$$I + \alpha(e^{i\theta})(\beta(e^{i\theta}) - l)^{-1}\alpha(e^{i\theta}) = \exp\int_0^\infty \frac{\mathrm{B}(e^{i\theta}, r)}{r - l}\, dr.$$

(2) *For any bounded Baire function ψ on $\sigma(|T|)$, the function* $\mathrm{B}(e^{i\theta}, r)$ *satisfies*

$$\int \psi(r)\mathrm{B}(e^{i\theta}, r)\, dr = \alpha(e^{i\theta})\int_0^1 \psi(\beta(e^{i\theta}) + k\cdot\alpha(e^{i\theta})^2)\, dk\, \alpha(e^{i\theta}).$$

In particular,

$$\int \frac{\mathrm{B}(e^{i\theta}, r)}{r - l}\, dr = \alpha(e^{i\theta})\int_0^1 (\beta(e^{i\theta}) + k\cdot\alpha(e^{i\theta})^2 - l)^{-1}\, dk\, \alpha(e^{i\theta}).$$

Definition 1. The function $\mathrm{B}(\cdot,\cdot)$ in Theorem C is said to be the *mosaic* of T. We denote the mosaic of T by $\mathrm{B}_T(\cdot,\cdot)$.

For $T \in p$-HU, we define $T_p = U|T|^{2p}$. Since T_p is in SHU, the mosaic $\mathrm{B}_{T_p}(\cdot,\cdot)$ of T_p exists.

Definition 2. For $T = U|T| \in p$-HU $(0 < p < 1/2)$, we define

$$\mathcal{B}_T(e^{i\theta}, r) = \mathrm{B}_{T_p}(e^{i\theta}, r^{2p}).$$

We call the function $\mathcal{B}_T(\cdot,\cdot)$ appearing in Definition 2 the *mosaic* of $T \in p$-HU. The essential support of $\mathcal{B}_T(\cdot,\cdot)$ is called the *determining set* of T. We denote this set by $\mathrm{D}(T)$, i.e.,

$$\mathrm{D}(T) = \mathbb{C} - \bigcup \{\mathrm{G} : \mathrm{G} \text{ is open in } \mathbb{C} \text{ and } \mathcal{B}_T(e^{i\theta}, r) = 0 \text{ for a.e. } re^{i\theta} \in \mathrm{G}\}.$$

Then we have the following

Theorem 1. *Let* $T = U|T|$ *be in* p-HU. *Then*

$$\mathrm{D}(T) \subset \sigma(T).$$

Moreover, *if* T *is completely nonnormal*, *then* $\mathrm{D}(T) = \sigma(T)$.

Proof. Since $T_p = U|T|^{2p}$ is semi-hyponormal, Theorem V.3.2 of [7] yields

$$\mathrm{D}(T_p) \subset \sigma(T_p).$$

By the definition of $\mathrm{D}(T)$ for a p-hyponormal operator T, we have

$$re^{i\theta} \in \mathrm{D}(T) \;\Leftrightarrow\; r^{2p}e^{i\theta} \in \mathrm{D}(T_p).$$

Since Theorem 3 of [4] implies that $r^{2p}e^{i\theta} \in \sigma(T_p)$ if and only if $re^{i\theta} \in \sigma(T)$, we have $\mathrm{D}(T) \subset \sigma(T)$.

If T is completely nonnormal, then Theorem 5 of [5] shows that T_p is completely nonnormal. Also $\mathrm{D}(T_p) = \sigma(T_p)$ by Theorem V.3.2 of [7]. Hence $\mathrm{D}(T) = \sigma(T)$.

Theorem 2. *Let* $T = U|T|$ *be in* p-HU. *Then*

$$\big\||T|^{2p} - |T^*|^{2p}\big\| \le \frac{p}{\pi} \iint_{\mathrm{D}(T)} r^{2p-1}\, dr\, d\theta.$$

Proof. Since $T_p = U|T|^{2p}$ is semi-hyponormal, by Theorem V.3.5 of [7] we have

$$\big\||T|^{2p} - |T^*|^{2p}\big\| \le \frac{1}{2\pi} \iint_{\mathrm{D}(T_p)} d\varrho\, d\theta.$$

By the transformation $\varrho = r^{2p}$, we have

$$\big\||T|^{2p} - |T^*|^{2p}\big\| \le \frac{p}{\pi} \iint_{\mathrm{D}(T)} r^{2p-1}\, dr\, d\theta.$$

Hence we have the following corollary.

COROLLARY 3. *Let T be in p-HU. If $\mathrm{m}_2(\mathrm{D}(T)) = 0$, then T is normal, where $\mathrm{m}_2(\cdot)$ is the planar Lebesgue measure.*

3. Principal functions. In this section, we introduce principal functions of operators T in p-HU. First we prepare some notations. If ψ is analytic in the upper half plane and with range in the closed upper half plane, ψ is called a *Pick function* ([7, p. 129]). ψ is a Pick function if and only if it has the following unique canonical representation:

$$\psi(z) = az + b + \int \left[\frac{1}{x - z} - \frac{x}{x^2 + 1} \right] d\mu(x),$$

where $a \geq 0$, b is a real number, and μ is a nonnegative Borel measure on the real line $\mathbb{R}$ which satisfies

$$\int \frac{1}{1 + x^2}\, d\mu(x) < \infty.$$

For a bounded closed set E of the real line $\mathbb{R}$, let P(E) be the set of all Pick functions with representation measure $\mu(\mathrm{E}^{\mathrm{c}}) = 0$. Moreover, let PM(E) be the set of all Pick functions ψ in P(E) such that

$$\psi'(t) = a + \int_{\mathrm{E}} \frac{1}{(t - x)^2}\, d\mu(x) < \infty$$

([7, pp. 129, 166]). Let $\mathrm{Tr}_{\mathcal{D}}(\cdot)$ be the trace on $\mathcal{D}$. Subscripts will usually be suppressed when clear from the context.

DEFINITION 3. (1) For $T \in$ SHU, we define the *principal function* $g_T(e^{i\theta}, r)$ of T by

$$g_T(e^{i\theta}, r) = \mathrm{Tr}_{\mathcal{D}}(\mathrm{B}_T(e^{i\theta}, r)),$$

where $\mathrm{B}_T(\cdot, \cdot)$ is the mosaic of T.

(2) For an operator $T \in p$-HU, we define the *principal function* $g_T(e^{i\theta}, r)$ by

$$g_T(e^{i\theta}, r) = \mathrm{Tr}_{\mathcal{D}}(\mathcal{B}_T(e^{i\theta}, r)) \quad (= \mathrm{Tr}_{\mathcal{D}}(\mathrm{B}_{T_p}(e^{i\theta}, r^{2p}))),$$

where $\mathcal{B}_T(\cdot, \cdot)$ is the mosaic of $T \in p$-HU $(0 < p \leq 1/2)$.

Hence, for $0 < p \leq 1/2$, we have $g_T(e^{i\theta}, r) = g_{T_p}(e^{i\theta}, r^{2p})$.

THEOREM 4. *Let $T = U|T|$ and $S = V|S|$ be in p-HU. If T and S are unitarily equivalent, then*

$$g_T(e^{i\theta}, r) = g_S(e^{i\theta}, r).$$

Proof. If $p = 1/2$, the assertion holds by Theorem VII.2.4 of [7]. Hence we need only prove that T_p and S_p are unitarily equivalent. We assume that $W^*TW = S$ for a unitary operator W. Since $W^*|T|W = |S|$, we have

$$W^*UW|S| = W^*UWW^*|T|W = W^*TW = S = V|S|.$$

Hence $W^*UWx = Vx$ for $x \in \operatorname{ran}(|S|)$. Therefore,

$$\begin{aligned} W^*T_pW &= W^*U|T|^{2p}W = W^*UWW^*|T|^{2p}W = W^*UW|S|^{2p} \\ &= V|S|^{2p} = S_p. \end{aligned}$$

So the proof is complete.

Hence, the principal function $g_T(\cdot,\cdot)$ of T is independent of the concrete model of T.

Now we would like to give a trace formula for p-hyponormal operators. First we give a trace formula for semi-hyponormal operators. This formula is slightly different from Theorem VII.2.4 of [7]. The proof is based on an idea of the proof of Theorem VII.2.2 of [7] about hyponormal operators.

THEOREM 5. *Let* $T = U|T| \in \mathrm{SHU}$,

$$\varphi(z) = e^{i\lambda}\,\frac{z-\overline{a}}{az-1} \quad \textit{with } |a| < 1 \textit{ and } \lambda \in \mathbb{R},$$

$\psi \in \mathrm{PM}(\sigma(|T|))$ *and* $g_T(\cdot,\cdot)$ *be the principal function of* T. *Then*

$$\operatorname{Tr}(\psi(|T|) - \varphi(U)\psi(|T|)\varphi(U)^*) = \iint |\varphi'(e^{i\theta})|\psi'(r)g_T(e^{i\theta}, r)\,dr\,dm(\theta).$$

Proof. We may assume that $T = U|T|$ is represented by the singular integral model. We define $|T|_+$ and $|T|_-$ by

$$|T|_+ = \operatorname*{s-lim}_n U^{*n}|T|U^n, \qquad |T|_- = \operatorname*{s-lim}_n U^n|T|U^{*n}.$$

For $\alpha(\cdot)$ and $\beta(\cdot)$ of the singular integral model of T, by Theorem III.1.3 of [7] we have

$$|T|_+ = \beta(\cdot) + \alpha(\cdot)^2, \qquad |T|_- = \beta(\cdot).$$

Let $S = U\psi(|T|)$. Put $\psi_1 = \psi + a$ with $a > 0$. Since

$$\begin{aligned} \psi_1(|T|) - \varphi(U)\psi_1(|T|)\varphi(U)^* &= (\psi(|T|) + a) - \varphi(U)(\psi(|T|) + a)\varphi(U)^* \\ &= \psi(|T|) - \varphi(U)\psi(|T|)\varphi(U)^* \end{aligned}$$

and $\psi_1' = \psi'$, we may assume that $\psi \ge 0$. Since ψ is operator monotone on $\sigma(|T|)$, we have $\psi(|T|) \ge \psi(U|T|U^*) = U\psi(|T|)U^* \ge 0$, so that $S \in \mathrm{SHU}$ (cf. [7, Theorem VI.3.2]). Let $\alpha_1(\cdot)$ and $\beta_1(\cdot)$ come from the singular integral model of S. Since U is unitary, we have

$$\begin{aligned} |S|_+ = \psi(|T|)_+ &= \operatorname*{s-lim}_n U^{*n}\psi(|T|)U^n = \psi(\operatorname*{s-lim}_n U^{*n}|T|U^n) \\ &= \psi(|T|_+) = \psi(\beta(\cdot) + \alpha(\cdot)^2) \end{aligned}$$

and also

$$|S|_- = \psi(|T|)_- = \operatorname*{s-lim}_n U^n\psi(|T|)U^{*n} = \psi(|T|_-) = \psi(\beta(\cdot)).$$

Since $\alpha_1 = (\psi(|T|)_+ - \psi(|T|)_-)^{1/2}$ and $\beta_1 = \psi(|T|)_-$, we have

$$\alpha_1(z) = (\psi(\beta(z) + \alpha(z)^2) - \psi(\beta(z)))^{1/2}, \qquad \beta_1(z) = \psi(\beta(z)). \tag{1}$$

Since $\varphi(U)(z)\beta_1(z)\varphi(U)^*(z) = \beta_1(z)$, by (1) and Theorem A we have

$$\begin{aligned}
((\psi(|T|) - \varphi(U)\psi(|T|)\varphi(U)^*)f)(z) &= \alpha_1(z)\mathcal{P}(\alpha_1 f)(z) + \beta_1(z)f(z) \\
&\quad - (\varphi(U)\alpha_1(z)\mathcal{P}(\varphi(U)^*\alpha_1 f)(z) + \varphi(U)(z)\beta_1(z)\varphi(U)^*(z)f(z)) \\
&= \alpha_1(z)\mathcal{P}(\alpha_1 f)(z) - \varphi(U)(z)\alpha_1(z)\mathcal{P}(\varphi(U)^*\alpha_1 f)(z) \\
&= \frac{1}{2\pi i}\,\alpha_1(z) \lim_{r\to 1-0} \int_{|\zeta|=1} \left(\frac{1}{\zeta - rz} - \frac{z-\overline{a}}{az-1}\cdot\frac{1}{\zeta - rz}\cdot\frac{\overline{\zeta}-a}{\overline{a}\overline{\zeta}-1}\right)\alpha_1(\zeta)f(\zeta)\,d\zeta \\
&= \frac{1-|a|^2}{2\pi i}\,\alpha_1(z) \int_{|\zeta|=1} \frac{1}{(az-1)(\overline{a}\overline{\zeta}-1)}\,\overline{\zeta}\alpha_1(\zeta)f(\zeta)\,d\zeta \\
&= (1-|a|^2)\alpha_1(z)\int \frac{1}{(az-1)(\overline{a}e^{-i\theta}-1)}\,\alpha_1(e^{i\theta})f(e^{i\theta})\,dm(\theta),
\end{aligned}$$

where we put $\zeta = e^{i\theta}$. Hence

$$\begin{aligned}
(2)\quad ((\psi(|T|) &- \varphi(U)\psi(|T|)\varphi(U)^*)f, f) \\
&= (1-|a|^2)\iint \frac{1}{(ae^{i\theta_1}-1)(\overline{a}e^{-i\theta}-1)} \\
&\qquad \times (\alpha_1(e^{i\theta})f(e^{i\theta}), \alpha_1(e^{i\theta_1})f(e^{i\theta_1}))_{\mathcal{D}}\,dm(\theta)\,dm(\theta_1) \\
&= (1-|a|^2)\left\|\int \frac{1}{\overline{a}e^{-i\theta}-1}\,\alpha_1(e^{i\theta})f(\zeta)\,dm(\theta)\right\|_{\mathcal{D}}^2.
\end{aligned}$$

Let $\{e_i\}$ and $\{h_j(\cdot)\}$ be orthonormal bases of $\mathcal{H}$ and $L^2(\mathbb{T}, \Sigma, m)$. Put

$$g_{jk}(e^{i\theta}) = \frac{1}{\overline{a}e^{-i\theta}-1}\,(\alpha_1(e^{i\theta})e_j, e_k) \in L^2(\mathbb{T}, m).$$

Then by (2) we have

$$\begin{aligned}
(3)\quad &\mathrm{Tr}(\psi(|T|) - \varphi(U)\psi(|T|)\varphi(U)^*) \\
&= (1-|a|^2)\sum_{i,j}\left\|\int \frac{1}{\overline{a}e^{-i\theta}-1}\,\alpha_1(e^{i\theta})e_i h_j(e^{i\theta})\,dm(\theta)\right\|_{\mathcal{D}}^2 \\
&= (1-|a|^2)\sum_{i,j,k}\left|\left(\int \frac{1}{\overline{a}e^{-i\theta}-1}\,\alpha_1(e^{i\theta})e_i h_j(e^{i\theta})\,dm(\theta), e_k\right)\right|^2 \\
&= (1-|a|^2)\sum_{i,j,k}\left|\int \frac{1}{\overline{a}e^{-i\theta}-1}\,(\alpha_1(e^{i\theta})e_i, e_k)h_j(e^{i\theta})\,dm(\theta)\right|^2 \\
&= (1-|a|^2)\sum_{i,j,k}\left|\int g_{jk}(e^{i\theta})h_i(e^{i\theta})\,dm(\theta)\right|^2 = (1-|a|^2)\sum_{i,j,k}|(\overline{g}_{jk}, h_i)|^2
\end{aligned}$$

$$= (1-|a|^2)\sum_{j,k}\|\overline{g}_{jk}\|^2 = (1-|a|^2)\sum_{j,k}\int\left|\frac{1}{\overline{a}e^{-i\theta}-1}\,(\alpha_1(e^{\theta})e_j, e_k)\right|^2 dm(\theta)$$

$$= (1-|a|^2)\sum_{j}\int\left\|\frac{1}{\overline{a}e^{-i\theta}-1}\,\alpha_1(e^{i\theta})e_j\right\|^2 dm(\theta)$$

$$= (1-|a|^2)\sum_{j}\int\left|\frac{1}{\overline{a}e^{-i\theta}-1}\right|^2 \|\alpha_1(e^{i\theta})e_j\|^2\, dm(\theta)$$

$$= (1-|a|^2)\int\left|\frac{1}{ae^{i\theta}-1}\right|^2 \mathrm{Tr}_{\mathcal{D}}(\alpha_1(e^{i\theta})^2)\, dm(\theta).$$

Putting $\psi(r) = (r-x)^{-2}$ in Theorem C, we have

$$\mathrm{Tr}_{\mathcal{D}}\left(\int\frac{\mathrm{B}(z,r)}{(r-x)^2}\,dr\right) = \mathrm{Tr}_{\mathcal{D}}\Big(\alpha(z)\int_0^1(\beta(z)+k\alpha(z)^2-x)^{-2}\,dk\,\alpha(z)\Big)$$

$$= \mathrm{Tr}_{\mathcal{D}}\Big(\int_0^1(\beta(z)+k\alpha(z)^2-x)^{-1}\alpha(z)^2(\beta(z)+k\alpha(z)^2-x)^{-1}\,dk\Big).$$

Considering $\alpha(z)+\varepsilon$ for a small positive number ε, we may assume that $\alpha(z)$ is invertible. We have

$$(x-(\beta(z)+k\alpha(z)^2))^{-1} = \alpha(z)^{-1}(x\alpha(z)^{-2}-\alpha(z)^{-1}\beta(z)\alpha(z)^{-1}-k)^{-1}\alpha(z)^{-1},$$

so that

$$\begin{aligned}
&\frac{d}{dk}(x-(\beta(z)+k\alpha(z)^2))^{-1}\\
&\qquad= \alpha(z)^{-1}(x\alpha(z)^{-2}-\alpha(z)^{-1}\beta(z)\alpha(z)^{-1}-k)^{-2}\alpha(z)^{-1}\\
&\qquad= \alpha(z)^{-1}(x\alpha(z)^{-2}-\alpha(z)^{-1}\beta(z)\alpha(z)^{-1}-k)^{-1}\\
&\qquad\quad\times(x\alpha(z)^{-2}-\alpha(z)^{-1}\beta(z)\alpha(z)^{-1}-k)^{-1}\alpha(z)^{-1}\\
&\qquad= (x\alpha(z)^{-1}-\alpha(z)^{-1}\beta(z)-k\alpha(z))^{-1}\\
&\qquad\quad\times(x\alpha(z)^{-1}-\beta(z)\alpha(z)^{-1}-k\alpha(z))^{-1}\\
&\qquad= (x-\beta(z)-k\alpha(z)^2)^{-1}\alpha(z)\cdot\alpha(z)(x-\beta(z)-k\alpha(z)^2)^{-1}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
(4)\qquad &\int_0^1(\beta(z)+k\alpha(z)^2-x)^{-1}\alpha(z)^2(\beta(z)+k\alpha(z)^2-x)^{-1}dk\\
&\qquad= (x-(\beta(z)+\alpha(z)^2))^{-1}-(x-\beta(z))^{-1}.
\end{aligned}$$

By Definition 3 and (4) we have

$$\begin{aligned}\int \frac{g_T(z,r)}{(x-r)^2}\,dr &= \mathrm{Tr}_{\mathcal{D}}\left(\int \frac{\mathrm{B}(z,r)}{(r-x)^2}\,dr\right)\\ &= \mathrm{Tr}_{\mathcal{D}}((x-\beta(z)-\alpha(z)^2)^{-1}-(x-\beta(z))^{-1}).\end{aligned}$$

Putting $\psi(r)\equiv 1$ in Theorem C, by Definition 3 we have

$$\int g_T(z,r)\,dr = \mathrm{Tr}_{\mathcal{D}}(\alpha(z)^2). \tag{5}$$

Let $\mathrm{E}=\sigma(|T|)$. Since $\psi\in\mathrm{PM}(\mathrm{E})$, we can put

$$\psi(t) = ct + d + \int_{\mathrm{E}}\left(\frac{1}{x-t}-\frac{x}{1+x^2}\right)d\mu(x)$$

and hence

$$\psi'(t) = c + \int_{\mathrm{E}} \frac{1}{(x-t)^2}\,d\mu(x).$$

Therefore

$$\begin{aligned}&\psi(\beta(z)+\alpha(z)^2)-\psi(\beta(z))\\ &\qquad= c(\beta(z)+\alpha(z)^2-\beta(z))\\ &\qquad\quad+\int_{\mathrm{E}}\{(x-\beta(z)-\alpha(z)^2)^{-1}-(x-\beta(z))^{-1}\}\,d\mu(x)\\ &\qquad= c(\alpha(z)^2)+\int_{\mathrm{E}}\{(x-\beta(z)-\alpha(z)^2)^{-1}-(x-\beta(z))^{-1}\}\,d\mu(x).\end{aligned}$$

Since $c\geq 0$ and $\mathrm{Tr}_{\mathcal{D}}(\int_{\mathrm{E}}\{(x-\beta(z)-\alpha(z)^2)^{-1}-(x-\beta(z))^{-1}\}\,d\mu(x))\geq 0$, we have

$$\begin{aligned}&\mathrm{Tr}_{\mathcal{D}}(\psi(\beta(z)+\alpha(z)^2)-\psi(\beta(z)))\\ &\quad= \mathrm{Tr}_{\mathcal{D}}\Big(c\alpha(z)^2+\int_{\mathrm{E}}\{(x-\beta(z)-\alpha(z)^2)^{-1}-(x-\beta(z))^{-1}\}\,d\mu(x)\Big)\\ &\quad= c\,\mathrm{Tr}_{\mathcal{D}}(\alpha(z)^2)+\mathrm{Tr}_{\mathcal{D}}\Big(\int_{\mathrm{E}}\{(x-\beta(z)-\alpha(z)^2)^{-1}-(x-\beta(z))^{-1}\}\,d\mu(x)\Big)\\ &\quad= c(\mathrm{Tr}_{\mathcal{D}}(\alpha(z)^2))+\int_{\mathrm{E}}\{\mathrm{Tr}_{\mathcal{D}}((x-\beta(z)-\alpha(z)^2)^{-1}-(x-\beta(z))^{-1})\}\,d\mu(x)\\ &\quad= c\,\mathrm{Tr}_{\mathcal{D}}(\alpha(z)^2)+\iint_{\mathrm{E}}\frac{g_T(z,t)}{(x-t)^2}\,dt\,d\mu(x)\qquad\text{(by (4))}\\ &\quad= c\int g_T(z,t)\,dt+\iint_{\mathrm{E}}\frac{1}{(x-t)^2}\,d\mu(x)g_T(z,t)\,dt\\ &\quad= \int\Big(c+\int_{\mathrm{E}}\frac{1}{(x-t)^2}\,d\mu(x)\Big)g_T(z,t)\,dt = \int\psi'(t)g_T(z,t)\,dt.\end{aligned}$$

Hence

(6) $\quad \mathrm{Tr}_{\mathcal{D}}(\alpha_1(z)^2) = \mathrm{Tr}_{\mathcal{D}}(\psi(\beta(z)+\alpha(z)^2) - \psi(\beta(z))) = \int \psi'(r) g_T(z,r)\,dr.$

Since $\varphi(e^{i\theta}) = e^{i\lambda}\frac{e^{i\theta}-\bar{a}}{ae^{i\theta}-1}$, we have $\varphi'(e^{i\theta}) = e^{i\lambda}\frac{|a|^2-1}{(ae^{i\theta}-1)^2}$. Therefore, by (3) and (6),

$$\begin{aligned}\mathrm{Tr}(\varphi(|T|) - \psi(U)\varphi(|T|)\psi(U)^*) &= (1-|a|^2)\int \left|\frac{1}{ae^{i\theta}-1}\right|^2 \mathrm{Tr}_{\mathcal{D}}(\alpha_1(e^{i\theta})^2)\,dm(\theta)\\ &= \int\int |\varphi'(e^{i\theta})|\psi'(r) g_T(e^{i\theta}, r)\,dr\,dm(\theta).\end{aligned}$$

So the proof is complete.

In the case of p-HU operators, we have the following

Theorem 6. *Let $T = U|T| \in p$-HU. For $|a| < 1$ and a real number λ, let $\varphi(z) = e^{i\lambda}\frac{z-\bar{a}}{az-1}$, $\psi \in \mathrm{PM}(\sigma(|T|^{2p}))$ and $g_T(\cdot,\cdot)$ be the principal function of T. Then*

$$\begin{aligned}&\mathrm{Tr}(\psi(|T|^{2p}) - \varphi(U)\psi(|T|^{2p})\varphi(U)^*)\\ &\qquad = 2p\int\int r^{2p-1}|\varphi'(e^{i\theta})|\psi'(r^{2p}) g_T(e^{i\theta}, r)\,dr\,dm(\theta).\end{aligned}$$

Proof. Let $T_p = U|T|^{2p}$. Then $T_p \in$ SHU and $g_T(e^{i\theta}, r) = g_{T_p}(e^{i\theta}, r^{2p})$. Hence, by Theorem 5 and the transformation $\varrho^{2p} = r$ we have the assertion.

4. Trace formulae for commutators associated with polar decompositions. We denote the trace class of operators by $\mathcal{C}_1$. For operators A and B, the commutator $AB - BA$ is denoted by $[A, B]$. In this section, we give a trace formula for $[|T|^m, U^n]$ for a semi-hyponormal operator $T = U|T|$ with unitary U. First we give the following theorem.

Theorem 7. *Let $T = U|T| \in$ SHU and $g_T(\cdot,\cdot)$ be the principal function of T. Assume that $[|T|, U] \in \mathcal{C}_1$. Then, for any integer $n \geq 1$,*

$$\mathrm{Tr}([|T|, U^n]) = \int\int ne^{in\theta} g_T(e^{i\theta}, r)\,dr\,dm(\theta).$$

Proof. For $n \geq 1$, since

$$[|T|, U^n] = [|T|, U]U^{n-1} + U[|T|, U]U^{n-2} + \ldots + U^{n-1}[|T|, U],$$

we have

$$\mathrm{Tr}([|T|, U^n]) = \mathrm{Tr}(nU^{n-1}[|T|, U]).$$

Using the singular integral model of T, we obtain

$$\begin{aligned}((|T|U - U|T|)f)(z) &= \alpha(z)\,\frac{1}{2\pi i}\int_{|\zeta|=1} \alpha(\zeta) f(\zeta)\,d\zeta\\ &= \alpha(z)\int \alpha(e^{i\theta})e^{i\theta} f(e^{i\theta})\,dm(\theta).\end{aligned}$$

Let $\{e_j\}$ and $\{h_k(\cdot)\}$ be the orthonormal bases of $\mathcal{H}$ and $L^2(\mathbb{T},\Sigma,m)$, respectively. By (5), $\alpha(z)$ is Hilbert–Schmidt; put

$$F(e^{i\theta}) = \mathrm{Tr}_{\mathcal{D}}(\alpha(e^{i\omega})e^{i\theta}\alpha(e^{i\theta})).$$

Then

$$\begin{aligned}
&\mathrm{Tr}(nU^{n-1}[|T|,U])\\
&\quad= \sum_{j,k}\int\Big(ne^{i(n-1)\omega}\alpha(e^{i\omega})\int\alpha(e^{i\theta})e_jh_k(e^{i\theta})\,dm(\theta), e_jh_k(e^{i\omega})\Big)\,dm(\omega)\\
&\quad= \sum_k\int ne^{i(n-1)\omega}\int\sum_j(\alpha(e^{i\omega})\alpha(e^{i\theta})e^{i\theta}e_j,e_j)h_k(e^{i\theta})\,dm(\theta)\,\overline{h_k(e^{i\omega})}\,dm(\omega)\\
&\quad= \sum_k\int ne^{i(n-1)\omega}\int\mathrm{Tr}_{\mathcal{D}}(\alpha(e^{i\omega})\alpha(e^{i\theta})e^{i\theta})h_k(e^{i\theta})\,dm(\theta)\,\overline{h_k(e^{i\omega})}\,dm(\omega)\\
&\quad= \int ne^{i(n-1)\omega}\Big(\sum_k(F,\overline{h}_k)_{L^2(\mathbb{T},m)}\overline{h_k(e^{i\omega})}\Big)\,dm(\omega)\\
&\quad= \int ne^{i(n-1)\omega}F(e^{i\omega})\,dm(\omega) = \int ne^{i(n-1)\omega}\,\mathrm{Tr}_{\mathcal{D}}(\alpha(e^{i\omega})e^{i\omega}\alpha(e^{i\omega}))\,dm(\omega)\\
&\quad= \int ne^{in\omega}\int g_T(e^{i\omega},r)\,dr\,dm(\omega) \qquad \text{(by (5))}.
\end{aligned}$$

Therefore,

$$\mathrm{Tr}([|T|,U^n]) = \int\int ne^{in\theta}g_T(e^{i\theta},r)\,dr\,dm(\theta).$$

So the proof is complete.

Next we give a trace formula for $[|T|^k,U^n]$.

THEOREM 8. *Let $T=U|T|\in\mathrm{SHU}$ and $g_T(\cdot,\cdot)$ be the principal function of T. If $[|T|,U]\in\mathcal{C}_1$, then for $k=1,2,\dots$ and $n=\pm1,\pm2,\dots$,*

$$\mathrm{Tr}([|T|^k,U^n]) = \int\int kne^{in\theta}r^{k-1}g_T(e^{i\theta},r)\,dr\,dm(\theta).$$

Proof. If $k,n\ge1$, then $(\mathrm{Tr}([|T|^k,U^n]))^* = -\mathrm{Tr}([|T|^k,U^{-n}])$ and

$$\begin{aligned}
&\Big(\int\int kne^{in\theta}r^{k-1}g_T(e^{i\theta},r)\,dr\,dm(\theta)\Big)^*\\
&\qquad\qquad= -\int\int k(-n)e^{i(-n)\theta}r^{k-1}g_T(e^{i\theta},r)\,dr\,dm(\theta).
\end{aligned}$$

Hence it is sufficient to prove the equalities for $k,n\ge1$. By Theorem 5, we have

$$\mathrm{Tr}(|T|-U|T|U^*) = \int\int g_T(e^{i\theta},r)\,dr\,dm(\theta) < \infty.$$

For $|\lambda|>\|T\|$, let $\psi(r)=1/(\lambda-r)$. By Theorem 5, we have

$$\mathrm{Tr}(\psi(|T|)-U\psi(|T|)U^*) = \int\int\frac{1}{(\lambda-r)^2}\,g_T(e^{i\theta},r)\,dr\,dm(\theta),$$

so that $\psi(|T|) - U\psi(|T|)U^* \in \mathcal{C}_1$. Hence $\psi(|T|)U - U\psi(|T|) \in \mathcal{C}_1$. Let $S = U\psi(|T|)$. Applying Theorem 7 to S, for $n \geq 1$ we obtain

$$\mathrm{Tr}(\psi(|T|)U^n - U^n\psi(|T|)) = \int ne^{in\theta} \int g_S(e^{i\theta}, r)\, dr\, dm(\theta).$$

Since in the proof of Theorem 5 we have $\psi(|T|)_+ = \psi(|T|_+)$ and $\psi(|T|)_- = \psi(|T|_-)$, by (1), (2) and (6) we obtain

$$\begin{aligned}\int g_S(e^{i\theta}, r)\, dr &= \mathrm{Tr}(\psi(|T|)_+ - \psi(|T|)_-) = \mathrm{Tr}(\psi(|T|_+) - \psi(|T|_-)) \\ &= \int \psi'(r) g_T(e^{i\theta}, r)\, dr = \int \frac{1}{(\lambda - r)^2}\, g_T(e^{i\theta}, r)\, dr.\end{aligned}$$

By Theorem 1, if $r > \|T\|$, then $g_T(e^{i\theta}, r) = 0$. Hence

$$\begin{aligned}\mathrm{Tr}(\psi(|T|)U^n - U^n\psi(|T|)) &= \int ne^{in\theta} \int \frac{1}{(\lambda - r)^2}\, g_T(e^{i\theta}, r)\, dr\, dm(\theta) \\ &= \sum_{k=0}^{\infty} \int ne^{in\theta} \int \frac{(k+1)r^k}{\lambda^{k+2}}\, g_T(e^{i\theta}, r)\, dr\, dm(\theta) \\ &= \sum_{k=0}^{\infty} \frac{1}{\lambda^{k+2}} \int\int n(k+1) e^{in\theta} r^k g_T(e^{i\theta}, r)\, dr\, dm(\theta).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\psi(|T|)U^n - U^n\psi(|T|) &= (\lambda - |T|)^{-1}U^n - U^n(\lambda - |T|)^{-1} \\ &= (\lambda - |T|)^{-1}[|T|, U^n](\lambda - |T|)^{-1}.\end{aligned}$$

Since $[|T|, U] \in \mathcal{C}_1$, we have $[|T|, U^n] \in \mathcal{C}_1$. Hence $\mathrm{Tr}((\cdot)[|T|, U^n])$ is a bounded linear functional on the bounded linear operators on the Hilbert space. By the same argument of the first part of the proof of Theorem 7,

$$\mathrm{Tr}((k+1)|T|^k[|T|, U^n]) = \mathrm{Tr}([|T|^{k+1}, U^n]).$$

Then

$$\begin{aligned}&\mathrm{Tr}(\psi(|T|)U^n - U^n\psi(|T|)) = \mathrm{Tr}((\lambda - |T|)^{-2}[|T|, U^n]) \\ &\quad = \sum_{k=0}^{\infty} \frac{1}{\lambda^{k+2}}\, \mathrm{Tr}((k+1)|T|^k[|T|, U^n]) = \sum_{k=0}^{\infty} \frac{1}{\lambda^{k+2}}\, \mathrm{Tr}([|T|^{k+1}, U^n]).\end{aligned}$$

Therefore, by comparing the coefficients of λ^{k+1} we have

$$\mathrm{Tr}([|T|^k, U^n]) = \int\int kne^{in\theta} r^{k-1} g_T(e^{i\theta}, r)\, dr\, dm(\theta).$$

So the proof is complete.

5. Trace formulae for p-nearly normal operators. In this section, we give trace formulae for p-nearly normal operators. Let $\mathcal{A}_1$ be the set of all polynomials of one variable. By $\mathcal{A}_2$ we denote the set of all Laurent polynomials of two variables r and z which have the form

$$p(r,z) = \sum_{j=0}^{N} \sum_{k=-N}^{N} a_{jk} r^j z^k,$$

where N is a positive integer and a_{jk} are constant coefficients. If $h \in \mathcal{A}_1$ and $p \in \mathcal{A}_2$, we define

$$(h \circ p)(r,z) = h(p(r,z)).$$

For a bilinear form $(\cdot,\cdot)$ on $\mathcal{A}_2$, we consider the following property:

$$(*) \qquad (p \circ r, q \circ r) = 0$$

for all $p, q \in \mathcal{A}_1$ and $r \in \mathcal{A}_2$. Condition $(*)$ is called the *collapsing property* ([7, p. 171]). Let X be an operator and Y be an invertible operator. For $p(r,z) = \sum_{j=0}^{N} \sum_{k=-N}^{N} a_{jk} r^j z^k$, we define

$$p(X,Y) = \sum_{j=0}^{N} \sum_{k=-N}^{N} a_{jk} X^j Y^k.$$

We denote the Jacobian for $p, q \in \mathcal{A}_2$ by $J(p,q)$, that is,

$$J(p,q)(r,e^{i\theta}) = \frac{\partial p}{\partial r}(r,e^{i\theta}) \cdot \frac{\partial q}{\partial z}(r,e^{i\theta}) - \frac{\partial p}{\partial z}(r,e^{i\theta}) \cdot \frac{\partial q}{\partial r}(r,e^{i\theta}).$$

Definition 4. For $T = U|T|$ with U unitary, T is called *p-nearly normal* if $[|T|^{2p}, U] \in \mathcal{C}_1$ (cf. [7, p. 170]).

It is easy to see that if $T = U|T|$ is p-nearly normal, then, for $p, q \in \mathcal{A}_2$, $[p(|T|^{2p}, U), q(|T|^{2p}, U)] \in \mathcal{C}_1$ and $\mathrm{Tr}([p(|T|^{2p}, U), q(|T|^{2p}, U)])$ is independent of the order of multiplication of the factors $|T|^{2p}$ and U (see [7, p. 174]). First we give a proof of Theorem VII.3.3 of [7] for a trace formula for a $\frac{1}{2}$-nearly normal operator.

Theorem 9. *Let $T = U|T| \in \mathrm{SHU}$ and $g_T(\cdot,\cdot)$ be the principal function of T. If T is $\frac{1}{2}$-nearly normal, then, for $p, q \in \mathcal{A}_2$,*

$$\mathrm{Tr}([p(|T|, U), q(|T|, U)]) = \iint J(p,q)(r,e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r)\, dr\, dm(\theta).$$

Proof. We define a bilinear form on $\mathcal{A}_2$ by

$$(p,q) = \mathrm{Tr}([p(|T|, U), q(|T|, U)])$$

for $p, q \in \mathcal{A}_2$. Then it is easy to see that $(\cdot,\cdot)$ has the collapsing property. For $q \in \mathcal{A}_2$, we choose $q_1, q_2 \in \mathcal{A}_2$ such that $\partial q_1/\partial r = q = \partial q_2/\partial r$. Then $q_1 - q_2$ is a Laurent polynomial of variable z. Let $h(r,z) = z$. By definition of $(\cdot,\cdot)$ we have $(h, q_1 - q_2) = 0$. Hence we can define a linear functional ℓ on $\mathcal{A}_2$ by

$$\ell(q) = (h, q_1)$$

where $\partial q_1/\partial r = q$. From now on, if $p(r,z) = r^j z^k$, then we simply denote (p,q) by $(r^j z^k, q)$ and so on. Hence $\ell(\partial q/\partial r) = (z,q)$. We define an auxiliary bilinear form $(\cdot,\cdot)_1$ on $\mathcal{A}_2$ by

$$(\cdot,\cdot)_1 = (\cdot,\cdot) + \ell(J(\cdot,\cdot)).$$

Since $J(p \circ s, q \circ s) = 0$ for any $p, q \in \mathcal{A}_1$ and $s \in \mathcal{A}_2$, the bilinear form $(\cdot,\cdot)_1$ has the collapsing property.

We show that $(\cdot,\cdot)_1 \equiv 0$. For each $q \in \mathcal{A}_2$, we have

$$(z,q)_1 = (z,q) + \ell(J(z,q)) = (z,q) + \ell\left(-\frac{\partial q}{\partial r}\right) = (z,q) - (z,q) = 0, \tag{7}$$

$$(z^{-1},q)_1 = 0. \tag{8}$$

In fact, since $J(z^{-1},q) = z^{-2}\partial q/\partial r$, we have

$$\begin{aligned}
\ell\left(z^{-2}\,\frac{\partial q}{\partial r}\right) &= (z, z^{-2}q) = \mathrm{Tr}(UU^{-2}q(|T|,U) - U^{-2}q(|T|,U)U) \\
&= \mathrm{Tr}(U^{-1}q(|T|,U) - U^{-2}q(|T|,U)U) \\
&= \mathrm{Tr}(U^{-1}(q(|T|,U)U^{-1} - U^{-1}q(|T|,U))U) \\
&= \mathrm{Tr}([q(|T|,U),U^{-1}]) = (q,z^{-1}) = -(z^{-1},q).
\end{aligned}$$

Hence

$$\begin{aligned}
(z^{-1},q)_1 &= (z^{-1},q) + \ell(J(z^{-1},q)) \\
&= (z^{-1},q) + \ell\left(z^{-2}\,\frac{\partial q}{\partial r}\right) = (z^{-1},q) - (z^{-1},q) = 0.
\end{aligned}$$

Now, for $\alpha \in \mathbb{C}$ and $n \geq 1$, using (7) we have

$$0 = ((r+\alpha z),(r+\alpha z)^n)_1 = (r,(r+\alpha z)^n)_1 = \sum_{j=1}^{n} {}_n\mathrm{C}_j\, \alpha^j (r, r^{n-j}z^j)_1,$$

so that

$$(r, r^{n-j}z^j)_1 = 0 \qquad (j = 1,\ldots,n).$$

Therefore, we have

$$(r, r^j z^k)_1 = 0 \qquad (j,k = 1,2,\ldots).$$

Since (8) holds, we have $(r, r^j z^{-k})_1 = 0 \quad (j,k = 1,2,\ldots)$. Hence for all $q \in \mathcal{A}_2$ we have

$$(r,q)_1 = 0. \tag{9}$$

Next, we prove that if $s, t \in \mathcal{A}_2$ satisfy $(s,q)_1 = (t,q)_1 = 0$ for all $q \in \mathcal{A}_2$, then

$$(st,q)_1 = 0. \tag{10}$$

In fact, let $q \in \mathcal{A}_2$ and $\alpha, \beta \in \mathbb{C}$. By the collapsing property we have

$$((\alpha s + \beta t + q)^2, (\alpha s + \beta t + q))_1 = 0.$$

Since $(u^2, u)_1 = 0$ for $u \in \mathcal{A}_2$, we have

$$\alpha^2(s^2, q)_1 + \beta^2(t^2, q)_1 + 2\alpha\beta(st, q)_1 + 2\alpha(sq, q)_1 + 2\beta(tq, q)_1 = 0.$$

Since α and β are arbitrary, the coefficient of $\alpha\beta$ must vanish: i.e.,

$$(st, q)_1 = 0.$$

By (7)–(10), we have

$$(rz, q)_1 = (rz^{-1}, q)_1 = 0,$$

so that

$$(r^2 z, q)_1 = (r^2 z^{-1}, q)_1 = (rz^2, q)_1 = (rz^{-2}, q)_1 = 0.$$

Repeating this procedure, we have

$$(\cdot, \cdot)_1 \equiv 0.$$

Therefore, for $p, q \in \mathcal{A}_2$ we have

$$(p, q) = -\ell(J(p, q)). \tag{11}$$

Since $g_T(e^{i\theta}, r) \geq 0$, $\int\int g_T(e^{i\theta}, r)\, dr\, dm(\theta) = \mathrm{Tr}(|T| - U|T|U^{-1}) < \infty$ and $g_T(e^{i\theta}, r) = 0$ for $r > \|T\|$, we can define a linear functional ℓ_0 on $\mathcal{A}_2$ by

$$\ell_0(p) = \int\int p(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r)\, dr\, dm(\theta).$$

Since

$$\begin{aligned} (r^m, z^n) &= \mathrm{Tr}(|T|^m U^n - U^n |T|^m) \quad \text{(by Theorem 8)} \\ &= mn \int\int (e^{i\theta})^{n-1} r^{m-1} e^{i\theta} g_T(e^{i\theta}, r)\, dr\, dm(\theta) \\ &= mn\ell_0(z^{n-1} r^{m-1}), \end{aligned}$$

it follows from (11) that

$$-\ell(r^{m-1} z^{n-1}) = \ell_0(r^{m-1} z^{n-1}) \quad (m \geq 1,\, n \neq 0).$$

For $|\lambda| > \|T\|$, let $\psi(r) = 1/(\lambda - r)$. By Theorem 5,

$$\mathrm{Tr}(\psi(|T|) - U\psi(|T|)U^{-1}) = \int\int \frac{1}{(\lambda - r)^2}\, g_T(e^{i\theta}, r)\, dr\, dm(\theta).$$

Since

$$\psi(|T|) - U\psi(|T|)U^{-1} = ((\lambda - |T|)^{-1}[|T|, U](\lambda - |T|)^{-1})U^{-1}$$

and $[|T|, U] \in \mathcal{C}_1$, we have

$$\begin{aligned}
\mathrm{Tr}(\psi(|T|) - U\psi(|T|)U^{-1}) &= \mathrm{Tr}((\lambda - |T|)^{-1}[|T|, U](\lambda - |T|)^{-1}U^{-1}) \\
&= \mathrm{Tr}([|T|, U]((\lambda - |T|)^{-1}U^{-1}(\lambda - |T|)^{-1})) \\
&= \sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \frac{1}{\lambda^{2+s+t}} \mathrm{Tr}([|T|, U]\,|T|^{s}U^{-1}|T|^{t}) \\
&= \sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \frac{1}{\lambda^{2+s+t}} \mathrm{Tr}((|T|^{t}[|T|, U]|T|^{s})U^{-1}) \\
&= \sum_{m=0}^{\infty} \frac{1}{\lambda^{2+m}} \mathrm{Tr}([|T|^{m+1}, U]U^{-1}) \\
&= \sum_{m=0}^{\infty} \frac{1}{\lambda^{m+2}} \mathrm{Tr}(|T|^{m+1} - U|T|^{m+1}U^{-1}),
\end{aligned}$$

because

$$\begin{aligned}
[|T|^{n}, U] = |T|^{n-1}[|T|, U] + |T|^{n-2}[|T|, U]|T| \\
+ \ldots + |T|[|T|, U]|T|^{n-2} + [|T|, U]|T|^{n-1}.
\end{aligned}$$

Therefore,

$$\mathrm{Tr}(\psi(|T|) - U\psi(|T|)U^{-1}) = \sum_{m=0}^{\infty} \frac{1}{\lambda^{m+2}} \mathrm{Tr}(|T|^{m+1} - U|T|^{m+1}U^{-1}).$$

Since

$$\begin{aligned}
&\int\!\!\int \frac{1}{(\lambda - r)^{2}} g_{T}(e^{i\theta}, r)\, dr\, dm(\theta) \\
&\qquad = \sum_{m=0}^{\infty} \frac{1}{\lambda^{m+2}} \int\!\!\int (m+1) r^{m} g_{T}(e^{i\theta}, r)\, dr\, dm(\theta),
\end{aligned}$$

comparing the coefficients of λ^{m+1}, we have

$$\mathrm{Tr}(|T|^{m} - U|T|^{m}U^{-1}) = \int\!\!\int m r^{m-1} g_{T}(e^{i\theta}, r)\, dr\, dm(\theta) \qquad (m \geq 1).$$

We also have

$$\begin{aligned}
-\ell(r^{m}z^{-1}) &= -\frac{1}{m+1}(z, r^{m+1}z^{-1}) \\
&= -\frac{1}{m+1} \mathrm{Tr}(U|T|^{m+1}U^{-1} - |T|^{m+1}U^{-1}U) \\
&= -\frac{1}{m+1} \mathrm{Tr}(U|T|^{m+1}U^{-1} - |T|^{m+1}) \\
&= \frac{1}{m+1} \mathrm{Tr}(|T|^{m+1} - U|T|^{m+1}U^{-1}) \\
&= \frac{1}{m+1} \int\!\!\int (m+1) r^{m} g_{T}(e^{i\theta}, r)\, dr\, dm(\theta)
\end{aligned}$$

$$= \iint r^m g_T(e^{i\theta}, r)\, dr\, dm(\theta)$$
$$= \iint r^m e^{-i\theta} e^{i\theta} g_T(e^{i\theta}, r)\, dr\, dm(\theta) = \ell_0(r^m z^{-1}),$$

so that $\ell_0 = -\ell$. Consequently, we obtain

$$\begin{aligned}\mathrm{Tr}([p(|T|, U), q(|T|, U)]) &= (p, q) = \ell_0(J(p, q)) \\ &= \iint J(p, q)(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r)\, dr\, dm(\theta).\end{aligned}$$

So the proof is complete.

Finally, we have

THEOREM 10. *Let m be a positive integer. Let $T = U|T| \in \frac{1}{2m}$-HU. If T is $\frac{1}{2m}$-nearly normal, then*

$$\mathrm{Tr}([p(|T|, U), q(|T|, U)]) = \iint J(p, q)(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r)\, dr\, dm(\theta)$$

for $p, q \in \mathcal{A}_2$.

Proof. Put $\widetilde{p}(r, z) = p(r^m, z)$, $\widetilde{q}(r, z) = q(r^m, z) \in \mathcal{A}_2$ and $S = U|T|^{1/m}$. Since S is in SHU and $\frac{1}{2}$-nearly normal, by Theorem 9 we have

$$\mathrm{Tr}([p(|T|^{1/m}, U), q(|T|^{1/m}, U)]) = \iint J(p, q)(r, e^{i\theta}) e^{i\theta} g_S(e^{i\theta}, r)\, dr\, dm(\theta)$$

and

$$\begin{aligned}\mathrm{Tr}([p(|T|, U), q(|T|, U)]) &= \mathrm{Tr}([p((|T|^{1/m})^m, U), q((|T|^{1/m})^m, U)]) \\ &= \iint J(\widetilde{p}, \widetilde{q})(r, e^{i\theta}) e^{i\theta} g_S(e^{i\theta}, r)\, dr\, dm(\theta).\end{aligned}$$

Since $g_T(e^{i\theta}, r) = g_S(e^{i\theta}, r^{1/m})$, from the translation $r = \varrho^{1/m}$ we have

$$\begin{aligned}&\iint J(\widetilde{p}, \widetilde{q})(r, e^{i\theta}) e^{i\theta} g_S(e^{i\theta}, r)\, dr\, dm(\theta) \\ &\qquad = \iint J(p, q)(\varrho, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, \varrho)\, d\varrho\, dm(\theta).\end{aligned}$$

So the proof is complete.

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