ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

DONATELLA DANIELLI
NICOLA GAROFALO
DUY-MINH NHIEU

Trace inequalities for Carnot-Carathéodory spaces and applications

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 27, nº 2 (1998), p. 195-252

<http://www.numdam.org/item?id=ASNSP_1998_4_27_2_195_0>

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXVII (1998), pp. 195-252

Trace Inequalities for Carnot-Carathéodory Spaces and Applications

DONATELLA DANIELLI - NICOLA GAROFALO - DUY-MINH NHIEU

In ricordo di Filippo Chiarenza

Abstract. Given a distribution belonging to a sub-elliptic Sobolev space with respect to a system of locally Lipschitz vector fields, we study the problem of its membership to sharp trace spaces with respect to a given Borel measure. Various applications to geometric trace inequalities and to optimal regularity theorems for solutions of quasilinear sub-elliptic equations are presented.

Mathematics Subject Classification (1991): 49Q15 (primary), 28C, 35H05 (secondary).

1. - Introduction and statement of the results

In recent years there has been an explosion of interest in the study of Carnot-Carathéodory spaces. These are metric spaces whose distance is generated by the sub-unit curves relative to a system of non-commuting vector fields. They arise naturally in the study of boundary value problems in several complex variables, in CR geometry, in the study of quasi-conformal mappings between nilpotent Lie groups and in control theory. For all these aspects we refer the reader to the recent monographs [53], [97], [59], [6], and also to the bibliography of this paper for an extensive account. There exists also a continuously growing literature in the analysis of the relevant partial differential equations arising in this context, but in this direction the advances are not, to present day, as susbstantial as one might desire. This is mainly due on one hand to

The first author was supported by a grant of the Purdue Research Foundation and subsequently by a Fellowship of the Italian C.N.R.
The second author was supported by NSF Grants No. DMS-9404358 and No. DMS-9706892
The third author was supported by a grant of the Purdue Research Foundation, by the second author's NSF Grant No. DMS-9404358, and also by the Italian C.N.R., G.N.A.F.A.
Pervenuto alla Redazione il 16 aprile 1998.

the complexity of the underlying sub-Riemannian geometry, on the other hand to the considerable difficulties posed by the presence of characteristic points. As a result many basic questions remain nowadays open.

In the study of boundary value problems for sub-elliptic equations it is important to know when the a priori integrability requirements on the horizontal derivatives of the solution guarantee that the latter possess a trace in some L^p space on the boundary of the relevant domain. This question is subtle already in the classical setting of ordinary Sobolev spaces since the boundary of the domain is a set of measure zero. In the sub-elliptic context the presence of characteristic points of the system X on the boundary complicates matters considerably. The problem of traces also arises naturally in the study of the regularity of minimal surfaces [57]. For Carnot-Carathéodory spaces the existence of such surfaces has been recently proved in [54]. This paper represents a first step in trying to understand the trace problem in Carnot-Carathéodory spaces. We focus on functions which possess integrable generalized derivatives with respect to a system $X = \{X_1, \dots, X_m\}$ of vector fields in $\mathbb{R}^n$, with $X_j = \sum_{k=1}^n b_{jk} \frac{\partial}{\partial x_k}$ having real-valued locally Lipschitz coefficients b_{jk}. The interest for working with minimally smooth vector fields stems from the following considerations: It includes on one hand the important case of C^∞ systems of Hörmander type, on the other hand it also incorporates the general subelliptic operators studied in [84], [35] since by the results in [85] the factorization matrix of a smooth positive semi-definite matrix has in general at most Lipschitz continuous entries. A further motivation comes from the fact that there are interesting classes of operators (such as, e.g., the Baouendi-Grushin ones) which arise from systems of non-smooth vector fields.

For a given open set $\Omega \subset \mathbb{R}^n$, we consider the space of weak Sobolev functions

$$\mathcal{L}^{1,p}(\Omega) = \{ f \in L^p(\Omega) \mid X_j f \in L^p(\Omega), j = 1, \dots, m \}, \quad 1 \le p \le \infty,$$

which, endowed with the norm

$$\|u\|_{\mathcal{L}^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|Xu\|_{L^p(\Omega)},$$

is a Banach space. Hereafter, the notation $X_j f$ stands for the distributional derivative of f along the vector field X_j defined by

$$\langle X_j f, \phi \rangle = \int_\Omega f X_j^* \phi \, dx \quad \text{for any } \phi \in C_0^\infty(\Omega),$$

where $X_j^* = -\sum_{k=1}^n \frac{\partial}{\partial x_k}(b_{jk} \cdot)$ is the formal adjoint of X_j. We also need to introduce the Sobolev spaces of functions having strong derivatives in $L^p(\Omega)$ along the vector fields $X_1, \dots, X_m$

$$S^{1,p}(\Omega) = \overline{C^\infty(\Omega) \cap \mathcal{L}^{1,p}(\Omega)}^{\| \|_{\mathcal{L}^{1,p}(\Omega)}},$$

and those with zero generalized trace on $\partial\Omega$

$$\overset{\circ}{S}{}^{1,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\,\|_{\mathcal{L}^{1,p}(\Omega)}}.$$

A piecewise C^1 curve $\gamma : [0, T] \to \mathbb{R}^n$ is called *sub-unit* [34] if whenever $\gamma'(t)$ exists one has

$$\langle \gamma'(t), \xi \rangle^2 \le \sum_{j=1}^{m} \langle X_j(\gamma(t)), \xi \rangle^2 \quad \text{for every } \xi \in \mathbb{R}^n.$$

We note explicitly that the above inequality forces $\gamma'(t)$ to belong to the span of $\{X_1(\gamma(t)), \dots, X_m(\gamma(t))\}$. The sub-unit length of γ is by definition $l_s(\gamma) = T$. Given $x, y \in \mathbb{R}^n$, we denote by $\mathcal{S}(x, y)$ the collection of all sub-unit curves joining x to y. Throughout this paper we will make the following basic assumption: $\mathcal{S}(x, y) \neq \emptyset$ *for any* $x, y \in \mathbb{R}^n$. Such hypothesis implies that if one lets

$$d(x, y) = \inf\{l_s(\gamma) \mid \gamma \in \mathcal{S}(x, y)\},$$

then $d(x, y) < \infty$ for any $x, y \in \mathbb{R}^n$. It is then easy to recognize that d defines a distance on $\mathbb{R}^n$, usually known as the *control*, or *Carnot-Carathéodory distance*, associated to the system X.

The main properties of d are discussed in, e.g., [83] or [97]. It is easy to recognize that when $X = \{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$, then $d(x, y) = d_e(x, y) := |x - y|$. We will denote $B(x, R) = \{y \in \mathbb{R}^n \mid d(x, y) < R\}$ and $B_e(x, R) = \{y \in \mathbb{R}^n \mid d_e(x, y) < R\}$, respectively the metric and Euclidean ball centered at x with radius R. When $B = B(x, R)$, we simply write αB for $B(x, \alpha R)$.

Our primary concern is the following

PROBLEM. For which nonnegative Borel measures μ and $1 \le p \le q < \infty$ does the a priori inequality

$$\left(\int_B |u - u_{B,\mu}|^q \, d\mu\right)^{\frac{1}{q}} \le C \left(\int_{2B} |Xu|^p \, dx\right)^{\frac{1}{p}}, \quad u \in C^\infty(2B), \tag{1.1}$$

hold?

In (1.1) we have let $u_{B,\mu} = \frac{1}{\mu(B)} \int_B u \, d\mu$, whereas $Xu = (X_1 u, \dots, X_m u)$ denotes the horizontal gradient of u. Clearly, $|Xu| = (\sum_{j=1}^m (X_j u)^2)^{\frac{1}{2}}$. Besides its obvious relevance as a generalized Sobolev type inequality (with respect to two different measures), the interest of an estimate such as (1.1) lies in the fact it can be thought of as a trace inequality. Indeed, this is easily understood in the classical situation when $X = \{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$, in which case one has $d = d_e$. If $\mathcal{M}$ is a compact C^1 manifold with volume element $d\sigma$, and we let

$$\mu(E) \overset{\text{def}}{=} \sigma(E \cap \mathcal{M}),$$

then (1.1) reads

$$(1.2)\qquad \left(\int_{B\cap\mathcal{M}} |u - u_{B,\sigma}|^q \, d\sigma\right)^{\frac{1}{q}} \leq C \left(\int_B |Du|^p \, dx\right)^{\frac{1}{p}}, \quad u \in C^\infty(B),$$

where $u_{B,\sigma} = \frac{1}{\sigma(B\cap\mathcal{M})} \int_{B\cap\mathcal{M}} u \, d\sigma$. In this case it is a classical result that the trace theorem (1.2) holds for any $1 \leq p < n$ with $q = p\frac{n-1}{n-p}$, see, e.g., [71, Theorem 6.4.1, p. 316].

For a Carnot-Carathéodory space the validity of an inequality such as (1.1) is a more delicate question. As it will be clear from the proofs, the full thrust of the elaborate theory developed in [54] will be needed. The case $p = 1$ has a special geometric relevance. In this context we note that no improvement in the exponent of integrability is possible in the example considered in (1.2), which now reads

$$\int_{B\cap\mathcal{M}} |u - u_{B,\sigma}| \, d\sigma \leq C \int_B |Du| \, dx, \quad u \in C^\infty(B).$$

In the (geometric) case $p = 1$ the main motivation for studying the above problem comes from the study of isoperimetric and Sobolev inequalities in Carnot-Carathéodory spaces, BV functions, minimal surfaces and their regularity, see [54], [95]. In the nongeometric case $p > 1$, one is naturally led to trace inequalities such as (1.1) when studying the Dirichlet and Neumann problems in stratified, nilpotent Lie groups, or more generally in CR geometry. In either cases, the presence of characteristic points makes surface measure unsuitable since the latter does not scale correctly with respect to the underlying anisotropic group dilations and finding the *ad hoc* replacement for it becomes a crucial question. The inequality (1.1) is also related to the Fefferman-Phong trace inequality established by one of us in [28]

$$(1.3)\qquad \int_B |u - u_B|^p \, V dx \leq C \int_B |Xu|^p \, dx, \quad u \in C^\infty(B), \quad 1 < p,$$

where V is a nonnegative function in the generalized Morrey space $M^{s,ps}$, and to its various applications to linear and nonlinear subelliptic pde's. In the Euclidean setting there exists a large literature on this subject, going back to some pioneering work of Maz'ya [80], and continuing up to recently, see [33], [20], [21], [70], [23], [81]. We also refer the reader to the enlightening paper of F. Chiarenza [22] for various open problems connected with trace inequalities. It is clear that when $d\mu = V dx$, then (1.1) is related to (1.3) although as we will see neither inequality implies the other.

To state our main results we need to introduce the relevant assumptions. Throughout the paper we will use extensively the openness of the Carnot-Carathéodory balls in the (Euclidean) topology of $\mathbb{R}^n$. When the vector fields are C^∞ and satisfy the finite rank condition: $\text{rankLie}[X_1, \ldots, X_m](x) = n$ for every $x \in \mathbb{R}^n$, then this property can be deduced from the following estimate obtained in [83]: *For every connected $\Omega \subset\subset \mathbb{R}^n$ there exist $C, \epsilon > 0$ such that*

$$C d_e(x, y) \le d_\Omega(x, y) \le C^{-1} d_e(x, y)^\epsilon, \qquad x, y \in \Omega.$$

Here, d_Ω is the Carnot-Carathéodory distance defined by sub-unit curves whose trace lies in Ω. We note that Chow's accessibility theorem guarantees that for any $x, y \in \Omega$ there exists such a curve connecting x to y, see, e.g., [6]. Since, obviously, $d(x, y) \le d_\Omega(x, y)$ for any $x, y \in \Omega$, we obtain

$$d(x, y) \le C^{-1} d_e(x, y)^\epsilon, \qquad x, y \in \Omega.$$

This implies the (Euclidean) openness of the metric balls. This fact, however, is not guaranteed in our general framework (see [6, p. 18]), therefore we will introduce it as a hypothesis:

(H.1) $\qquad i : (\mathbb{R}^n, d_e) \to (\mathbb{R}^n, d)$ *is continuous*.

(H.1) is a qualitative assumption (a stronger (localized) quantitative version of (H.1) appears in [34]). Interestingly, it implies that the metric topology is in fact equivalent to the Euclidean one, see Remark 1.3 below. An important consequence of (H.1) is Proposition 2.3 below. We also need to make the following two basic quantitative assumptions. Given a function u and a Lebesgue measurable set E, let $u_E = |E|^{-1} \int_E u\, dx$.

(H.2) *For every bounded set $U \subset \mathbb{R}^n$ there exist constants $C_1, R_0 > 0$ such that for $x_0 \in U$ and $0 < R < R_0$ one has*

$$\big|B(x_0, 2R)\big| \le C_1 \big|B(x_0, R)\big|.$$

(H.2) expresses the familiar "doubling condition", which is requested to in any space of homogeneous type [26]. Along with (H.2), we also assume the following weak-L^1 Poincaré type inequality

(H.3) *Given U as in (H.2), there exist constants $C_2, R_0 > 0$ and $\alpha \ge 1$ such that for any $x_0 \in U$, $0 < R < R_0$ and $u \in C^1(B(x_0, \alpha R))$, one has*

$$\sup_{\lambda > 0} [\lambda\, |\{x \in B \mid |u(x) - u_B| > \lambda\}|] \le C_2 R \int_{\alpha B} |Xu|\, dx.$$

Henceforth, if $U \subset \mathbb{R}^n$ is a fixed bounded set the relative quantities C_1, C_2, R_0 and α in the above hypothesis will always denote what we call the *characteristic local parameters of U*. Of special relevance will be the (positive) number

$$Q = \log_2 C_1,$$

which we call the *homogeneous dimension* of U. It is easy to see that (H.1) implies

$$(1.4) \quad |B(x_0, tR)| \geq C_1^{-1} t^Q |B(x_0, R)|, \quad x_0 \in U, \quad 0 < t \leq 1, \quad 0 < R \leq R_0 .$$

REMARK 1.1. It is important to observe that in many situations of interest the number R_0 can be taken to be $+\infty$ in (H.1), (H.2). This is the case, e.g., of all stratified nilpotent Lie groups, where we take as dx a bi-invariant Haar measure on the group, see [37], [38], or, more in general, of groups of polynomial type [97]. We recall that in this setting a simple proof of a Poincaré inequality stronger than (H.2) can be found in [96]. When X is a system of C^∞ vector fields satisfying the finite rank condition [65]: rankLie$[X_1, \dots, X_m] \equiv n$, then (H.1), (H.2) respectively follow from the deep works [83] and [67]. Our results also cover the case of a Riemannian manifold M^n with nonnegative Ricci tensor. In this context, with dx denoting the Riemannian volume form, (H.2) follows from Bishop's comparison theorem [24], whereas (H.3) was proved by Buser [13] for $X = \nabla_M$, the Riemannian gradient. Finally, we give some interesting examples of non-smooth vector fields to which our results apply. Consider, for instance, the simplest model of Baouendi-Grushin vector fields [5], [60]

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = |x|^\beta \frac{\partial}{\partial y} \quad \text{in } \mathbb{R}^2 .$$

The system $X = \{X_1, X_2\}$ is not of Hörmander type if $\beta \neq 2k$, nonetheless (H.1), (H.2) hold thanks to the works [42], [43], [44]. (H.1), (H.2) also hold in the more general settings of the papers [47], [39], [41].

The general assumptions (H.1), (H.2) present some subtle topological issues that need to be dealt with. We refer the reader to the books [6], [53] for this aspect, confining ourselves to recall the following elementary, yet useful result from [54].

PROPOSITION 1.2. *Let* $(\mathbb{R}^n, d)$ *be a Carnot-Carathéodory space as above. The inclusion* $i : (\mathbb{R}^n, d) \to (\mathbb{R}^n, d_e)$ *is continuous.*

REMARK 1.3. As a consequence of Proposition 1.2 we note that, if we assume (H.1), then the Euclidean topology coincides with that generated by the Carnot-Carathéodory metric.

At this point we can state our main results.

THEOREM 1.4 (Geometric trace inequality). *Suppose* (H.1)-(H.3) *hold. Fix a bounded, open set* $U \subset \mathbb{R}^n$, *and let* μ *be a nonnegative Borel measure on* $\mathbb{R}^n$. *Suppose that for* $x \in U$ *and* $0 < r < R_0$ *one has*

$$(1.5) \qquad \mu(B(x, r)) \leq M \frac{|B(x, r)|}{r}$$

for some $M > 0$. There exist positive constants $C = C(C_1, C_2)$, $d_0 = d_0(U)$ such that for any $x_0 \in U$, $0 < R \le d_0$ one has for $B = B(x_0, R)$, $B^* = B(x_0, 2R)$:

1. *If $u \in \mathcal{L}^{1,1}(B^*)$, then there exists a uniquely determined $\tilde{u} \in L^1(B^*, d\mu)$ such that*

$$\int_B |\tilde{u} - \tilde{u}_{B,\mu}|\, d\mu \le CM \int_{B^*} |Xu|\, dx \,. \tag{1.6}$$

2. *Furthermore, when $u \in C^1(B^*) \cap \mathcal{L}^{1,1}(B^*)$, then $\tilde{u} = u$ in (1.6).*
3. *Finally, if $u \in \overset{\circ}{S}{}^{1,1}(B)$ then we have*

$$\int_B |\tilde{u}|\, d\mu \le CM \int_B |Xu|\, dx \,. \tag{1.7}$$

Condition (1.5) is also necessary for (1.6), hence for (1.7).

The proof of Theorem 1.4 rests on various delicate facts. A crucial ingredient is the following covering theorem which extends to the setting of this paper a basic result of Federer [32, Section 4.5.4]. Given a measurable set $E \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$ we let

$$\overline{D}(E, x) = \limsup_{r \to 0} \frac{|E \cap B(x, r)|}{|B(x, r)|} \,,$$

and call the function $\overline{D}(x, r)$ the *upper density* of E at x. We also need to recall the notion of X-perimeter introduced in [15], see also [54]. We mention that a slightly different notion of subelliptic perimeter was set forth in [40], simultaneously to (and independently from) the work [15]. Such two notions are in fact equivalent, see [48]. Consider an open set $\Omega \subset \mathbb{R}^n$ and define

$$\mathcal{F}(\Omega) = \left\{ \phi = (\phi_1, \dots, \phi_m) \in C_0^1(\Omega; \mathbb{R}^m) \mid \|\phi\|_\infty = \sum_{x \in \Omega} \left(\sum_{j=1}^m |\phi_j(x)|^2 \right)^{\frac{1}{2}} \le 1 \right\} .$$

For a given $u \in L^1_{\mathrm{loc}}(\Omega)$ the *X-variation* of u with respect to Ω is defined as

$$\mathrm{Var}_X(u; \Omega) = \sup_{\phi \in \mathcal{F}(\Omega)} \int_\Omega u(x) \sum_{j=1}^m X_j^* \phi_j(x)\, dx \,.$$

If $E \subset \mathbb{R}^n$ is measurable, then the *X-perimeter* of E relative to Ω is defined by

$$P_X(E; \Omega) = \mathrm{Var}_X(\chi_E; \Omega) \,,$$

where χ_E denotes the characteristic function of E. We say that a measurable set $E \subset \mathbb{R}^n$ is a *X-Caccioppoli set* if $P_X(E; \Omega) < \infty$ for any $\Omega \subset\subset \mathbb{R}^n$.

THEOREM 1.5. *Suppose* (H.1)-(H.3) *hold. Fix a bounded set* $U \subset \mathbb{R}^n$. *There exists a positive number* $d_0 = d_0(U) > 0$ *such that for* $x_0 \in U$, $0 < R \le d_0$, *and for any measurable set* $E \subset B = B(x_0, R)$ *one has: If*

$$\overline{D}(E, x) > \tau$$

for every $x \in E$ *and for some* $\frac{1}{4} < \tau < \frac{1}{2}$, *then there exist a constant* $C = C(C_1, C_2, \tau) > 0$ *and a sequence of balls* $\{B(x_i, kr_i)\}_{i\in\mathbb{N}}$ *such that*

$$\text{(i)} \quad E \subset \bigcup_{i=1}^{\infty} B(x_i, kr_i);$$

$$\text{(ii)} \quad B(x_i, r_i) \cap B(x_j, r_j) = \emptyset \quad \text{if } i \neq j;$$

$$\text{(iii)} \quad \sum_{i=1}^{\infty} \frac{|B(x_i, r_i)|}{r_i} \le C P_X(E; \mathbb{R}^n).$$

The key to the proof of Theorem 1.5 is the following basic relative isoperimetric inequality established in [54, Theorem 1.28]:

THEOREM 1.6. *We assume* (H.1)-(H.3). *Let* $E \subset \mathbb{R}^n$ *be a* X*-Caccioppoli set. For any bounded set* $U \subset \mathbb{R}^n$, *with local homogeneous dimension* Q, *there exists* $C_3 = C_3(C_1, C_2) > 0$ *and* $R_0 > 0$ *such that for* $B = B(x_0, R)$, *with* $x_0 \in U$ *and* $0 < R < R_0$, *one has*

$$\min\big(|E \cap B|, |E^c \cap B|\big)^{\frac{Q-1}{Q}} \le C_3 R |B|^{-\frac{1}{Q}} P_X(E; B). \tag{1.8}$$

It is worthwile observing that when $X = \{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$, then (H.2) holds with $U = \mathbb{R}^n$, $R_0 = \infty$, and $C_1 = 2^n$. In this case Theorem 1.6 gives back the classical relative isoperimetric inequality due to De Giorgi, Federer, Maz'ya et al. It should be mentioned that related isoperimetric and Sobolev inequalities inequalities have been independently obtained in [8], [45], [78]. The results in these papers, however, do not seem to fully cover the context of Theorem 1.6. We also mention the note [62], in which the authors announce interesting generalizations of sharp Sobolev imbeddings to the setting of Sobolev spaces on metric spaces [61], and the preprint "Sobolev met Poincaré", by the same authors.

We stress the crucial presence of the scaling factor $R|B|^{-\frac{1}{Q}}$ in the right hand side of (1.8). Such factor allows to connect in an important way the notion of X-perimeter to a special Hausdorff measure generated by the Carnot-Carathéodory metric $d(x, y)$. To make this point precise we introduce a definition. Given an open set $\Omega \subset \mathbb{R}^n$ we say that Ω admits an interior corkscrew at $x_0 \in \partial\Omega$ if for some $K > 0$ and $R_0 > 0$, and any $0 < r < R_0$, one can find $A_r(x_0) \in \Omega$ such that

$$\frac{r}{K} < d\big(A_r(x_0), x_0\big) \le r, \quad \operatorname{dist}\big(A_r(x_0), \partial\Omega\big) > \frac{r}{K}.$$

If in the above the same K and R_0 work for every $x_0 \in \partial\Omega$, then we say that Ω has the *uniform interior corkscrew condition.* Finally, Ω is said to satisfy the *uniform corkscrew condition* if both Ω and Ω^c fulfill the uniform interior corkscrew condition. Next, for $s > 0$ we denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure in $\mathbb{R}^n$ constructed using the Carnot-Carathéodory metric $d(x, y)$, see, e.g., [32] or [79]. We recall that when $X = \{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$, and therefore $d(x, y) = d_e(x, y)$, then the following important result holds, see, e.g., [32] or [31]: Suppose that $E \subset \mathbb{R}^n$ is a bounded Lipschitz domain. There exist constants $\alpha, \beta > 0$, depending on $n \in \mathbb{N}$ and on the Lipschitz character of E, such that

$$\alpha P(E) \leq \mathcal{H}^{n-1}(\partial E) \leq \beta P(E)\,. \tag{1.9}$$

Here, $P(E)$ denotes the perimeter of E according to De Giorgi. We note, in passing, that Lipschitz domains possess the uniform corkscrew condition with respect to the Euclidean metric. More in general, every NTA domain as in [68] has this property, see also [18] for a study of NTA domains in the context of Carnot-Carathéodory spaces and for a discussion of various basic examples. In our setting we have the following

THEOREM 1.7. *We suppose that* (H.1)-(H.3) *hold. Let $U \subset \mathbb{R}^n$ be a bounded, open set having homogeneous dimension Q. If $E \subset \bar{E} \subset U$ satisfies the uniform corkscrew condition with relative parameters K and R_0, then there exists a constant $C = C(U, X, C_1, C_2, K, n) > 0$ such that*

$$\mathcal{H}^{Q-1}(\partial E) \leq C P_X(E; \mathbb{R}^n)\,.$$

Influenced by (1.9) one might think that a two-sided inequality in Theorem 1.7 be possible, but this intuition fails miserably. A counter-example is given after the proof of Theorem 1.7 in Section 3. However, we conjecture that the opposite inequality should hold for the important class of stratified, nilpotent Lie groups. Strong evidence in support of this conjecture comes from the following

THEOREM 1.8. *Let $\mathbb{H}^n$ be the Heisenberg group with homogeneous dimension $Q = 2n + 2$, see* [38], [92]. *Suppose that $E \subset \mathbb{H}^n$ is a bounded open set such that ∂E is a C^2 compact manifold. There exist $\alpha, \beta > 0$ such that*

$$\alpha P_X(E; \mathbb{H}^n) \leq \mathcal{H}^{Q-1}(\partial E) \leq \beta P_X(E; \mathbb{H}^n)\,.$$

We warn the unfamiliar reader that Theorems 1.7 and 1.8 rest on a number of nontrivial results.

After this interlude on the connection between Hausdorff measure and the X-perimeter we resume the discussion of the trace inequalities. Our objective now is the (non-geometric) case $p > 1$ of (1.1). We mention that in the Euclidean setting results of this type were first proved in a beautiful pioneering paper of D. Adams [1], see also [2], [3] and the recent monograph [4]. In opposition to the geometric case $p = 1$, the assumption on the Borel measure μ is necessarily stronger. However, there is a trade-off in that the exponent q in the left-hand side of (1.1) is strictly bigger than p.

THEOREM 1.9 (Trace inequality: The non-geometric case). *Suppose that* (H.1)-(H.3) *hold, and consider a bounded, open set* $U \subset \mathbb{R}^n$, *with local homogeneous dimension* Q. *For* $1 \leq p < Q$ *let* μ *be a nonnegative Borel measure on* $\mathbb{R}^n$ *such that for some* $M > 0$ *and* $0 < \epsilon \leq p$ *one has*

$$\mu\big(B(x,r)\big) \leq M \frac{|B(x,r)|}{r^{p-\epsilon}}, \quad x \in U,\ 0 < r \leq R_0. \tag{1.10}$$

There exist positive constants $C = C(C_1, C_2, p, \epsilon)$ *and* $\beta = \beta(C_1, C_2) \geq 1$ *such that for any* $x_0 \in U$, $0 < R \leq R_0$, $B = B(x_0, R)$, $B^* = B(x_0, \beta R)$ *the following holds:*

1. *If* $u \in \mathcal{L}^{1,p}(B^*)$, *then*

$$\left(\int_B |u - u_{B,\mu}|^q \, d\mu\right)^{\frac{1}{q}} \leq C M^{\frac{1}{q}} \left(\frac{R}{|B|^{\frac{1}{Q}}}\right)^{\frac{\epsilon Q}{p(Q-p+\epsilon)}} \left(\int_{B^*} |Xu|^p \, dx\right)^{\frac{1}{p}}, \tag{1.11}$$

where $q = p\frac{Q-p+\epsilon}{Q-p} > p$.

2. *If* $u \in \overset{\circ}{S}{}^{1,p}(B^*)$, *one has*

$$\left(\int_B |u|^q \, d\mu\right)^{\frac{1}{q}} \leq C M^{\frac{1}{q}} \left(\frac{R}{|B|^{\frac{1}{Q}}}\right)^{\frac{\epsilon Q}{p(Q-p+\epsilon)}} \left(\int_{B^*} |Xu|^p \, dx\right)^{\frac{1}{p}}. \tag{1.12}$$

Finally, (1.10) *is also necessary for* (1.11), (1.12) *to hold.*

REMARK 1.10. It is important to observe that, in general, when $p > 1$, (1.10) cannot be improved (as in the case $p = 1$, see Theorem 1.4) to the following

$$\mu(B(x,r)) \leq M \frac{|B(x,r)|}{r^p} \quad x \in U, 0 < r \leq R.$$

This obstacle is quite serious indeed and it is ultimately related to a subelliptic version, due to Vodop'yanov [98], of an important characterization of Hedberg and Wolff [64] of those measures which belong to the dual of the Sobolev space $\mathcal{L}^{1,p}$.

An important consequence of Theorem 1.9 is the following

THEOREM 1.11 (Poincaré type inequality). *Under the assumptions in Theorem 1.9, for any* $u \in \mathcal{L}^{1,p}(B^*)$ *we have*

$$\int_B |u - u_{B,\mu}|^p \, d\mu \leq C R^\epsilon \int_{B^*} |Xu|^p \, dx$$

for some $C = C(C_1, C_2, M, p, \epsilon) > 0$.

We remark that when $d\mu = dx$, then (1.10) holds with $M = 1$ and $\epsilon = p$. In such case we respectively recover from Theorems 1.9 and 1.11 Sobolev embedding and the Poincaré inequality in [54], except that in the latter paper such results did not rely on fractional integration, but were directly deduced from the geometric case $p = 1$. This unified geometric approach does not seem to work for trace inequalities and one has to treat the cases $p = 1$, and $p > 1$ separately.

From the point of view of the applications it is of interest to extend the above results to a class of domains which is as large as possible. We have in mind the class of X-(PS) domains in [54]. To motivate the subsequent discussion we point out that in situations connected with rectifiability problems, see [29], [30], the Borel measures μ in Theorems 1.4 and 1.9 display a two-sided control such as

$$M_1 \frac{|B(x,r)|}{r^d} \le \mu\big(B(x,r)\big) \le M_2 \frac{|B(x,r)|}{r^d}, \quad x \in U,\ 0 < r < R_0,$$

for some $d > 0$. When this happens, then in view of (H.2) we see that the measure μ is doubling, i.e.: *There exists $C_4 = C_4(U) > 0$ such that for any $x \in U$ and $0 < r < R_0$*

$$\mu\big(B(x,2r)\big) \le C_4 \mu\big(B(x,r)\big). \tag{H.4}$$

These considerations bring us to study trace inequalities with respect to a measure μ which in addition to an estimate such as

$$\mu\big(B(x,r)\big) \le M \frac{|B(x,r)|}{r^d},$$

also satisfies (H.4). To this purpose we introduce the following

DEFINITION 1.12. An open set Ω is called a X-(PS) domain if there exist a covering $\{B\}_{B\in\mathcal{F}}$ of Ω by metric balls, and numbers $N > 0$, $\gamma \ge 1$, $\nu \ge 1$ such that

(i) $\sum_{B\in\mathcal{F}} \chi_{\gamma B} \le N\chi_\Omega$.
(ii) There exists a (central) ball $B_0 \in \mathcal{F}$ such that for any $B \in \mathcal{F}$ one can find a chain $B_0, B_1, \dots, B_{s(B)} = B$, with $B_i \cap B_{i+1} \supset \tilde{B}_i$ for some ball $\tilde{B}_i$ for which $N\tilde{B}_i \supset B_i \cup B_{i+1}$.
(iii) For any $i = 0, \dots, s(B)$, one has $B \subset \nu B_i$.

We emphasize that the notion of X-(PS) domain is of a purely metrical nature. (PS) domains are also known in the literature as *Boman domains*, see [74], [41]. In the Euclidean setting they were introduced by J. Boman in his (unfortunately) unpublished manuscript [10]. We would like to thank J. Boman for kindly discussing with the second named author his results and providing us with a copy of [10]. The importance of such domains rests in the following chain of inclusions valid when $d(x,y) = d_e(x,y)$ and $X = \{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$

$$\mathrm{Lip} \subset NTA \subset (\epsilon,\delta) \subset \mathrm{John} \subset PS. \tag{1.13}$$

Such inclusions (except the former) continue to hold for $(\mathbb{R}^n, d)$ when d is the Carnot-Carathéodory metric generated by a system $X = \{X_1, \dots, X_m\}$, but now matters become much more intricated. We refer the reader to the papers [54], [55], [18] for a detailed study and for examples. Here, we confine ourselves to mention that for any complete length-space which is also a space of homogeneous type according to [26], the latter inclusion in (1.13) is, in fact, a set theoretical equality, i.e.,

$$\text{John} = \text{PS}. \tag{1.14}$$

Such theorem was proved in [54] and also, independently and simultaneously, in [11]. The notation *John* stands for the class of domains introduced by F. John in his seminal paper [69].

THEOREM 1.13. *Under the assumptions of Theorem* 1.4 *assume, in addition, that* μ *satisfy* (H.4). *Then there exist positive constants* $C = C(C_1, C_2, C_4, N, \gamma, \nu)$, $d_0 = d_0(U, X, n)$ *such that for any* X-(PS) *domain* $\Omega \subset \overline{\Omega} \subset U$ *with parameters* N, γ, ν *and* $\operatorname{diam}(\Omega) \le d_0$, *one has:*

1. *For any* $u \in \mathcal{L}^{1,1}(\Omega)$ *there exists a uniquely determined* $\tilde{u} \in L^1(\Omega, d\mu)$, *with* $\tilde{u}_{\Omega,\mu} = \frac{1}{\mu(\Omega)} \int_\Omega \tilde{u}\, d\mu = 0$, *for which*
$$\int_\Omega |\tilde{u}|\, d\mu \le C \int_\Omega |Xu|\, dx. \tag{1.15}$$
2. *When* $u \in C^1(\Omega) \cap \mathcal{L}^{1,1}(\Omega)$, *then we can take* $\tilde{u} = u - u_{\Omega,\mu}$ *in* (1.15).

In [40], see also [54], it was proved that in any complete length-space, which is also of homogeneous type according to [26], the metric balls are X-(PS) domains. This follows by combining the ideas in [67] with the important fact that in any locally compact, complete length-space the geodesic segment property holds. Putting this result together with Theorem 1.13 we see that the latter now holds with Ω replaced by a Carnot-Carathéodory ball. This improves on Theorem 1.4 by allowing the same ball on both sides of the trace inequality. Rather than stating again Theorem 1.13 with $\Omega = B$, we confine ourselves to display the ensuing a priori trace inequality on smooth functions

$$\int_B |u - u_{B,\mu}|\, d\mu \le C \int_B |Xu|\, dx, \quad u \in C^1(B) \cap \mathcal{L}^{1,1}(B). \tag{1.16}$$

In the non-geometric case $p > 1$ we obtain an analogous global version of Theorem 1.9 when μ is doubling.

THEOREM 1.14. *Suppose that the assumptions in Theorem* 1.9 *are fulfilled and that, in addition,* μ *satisfies* (H.4). *There exists a constant* $C = C(C_1, C_2, C_4, M, p, \epsilon, N, \gamma, \nu) > 0$ *such that for any X-(PS) domain* $\Omega \subset \overline{\Omega} \subset U$ *(with parameters* N, γ, ν*), for which* $\operatorname{diam}(\Omega) < \frac{R_0}{2}$, *and any* $u \in \mathcal{L}^{1,p}(\Omega)$, *one has*

$$\left(\int_\Omega |u - u_{\Omega,\mu}|^q\, d\mu\right)^{\frac{1}{q}} \le C \left(\frac{\operatorname{diam}(\Omega)}{|\Omega|^{\frac{1}{Q}}}\right)^{\frac{\epsilon Q}{p(Q-p+\epsilon)}} \left(\int_\Omega |Xu|^p\, dx\right)^{\frac{1}{p}} \tag{1.17}$$

where $q = p\frac{Q-p+\epsilon}{Q-p}$.

REMARK 1.15. It is important to observe that the restriction on diam(Ω) in Theorems 1.13, 1.14 does not subsist in those situations in which R_0 in (H.2) can be taken infinite. This is the case, for instance, of all stratified, nilpotent Lie groups.

In closing, we briefly describe the plan of the paper. In Section 2 we establish various basic properties of Carnot-Carathéodory spaces which are needed in the proofs of the main theorems. Section 3 is devoted to proving Theorems 1.5, 1.4, 1.7 and 1.8. In Section 4 we prove Theorems 1.9 and 1.14. In the study of Carnot-Carathéodory spaces the question of geometrically significant examples is of fundamental importance. Without them, the theory would be devoid of meaning. Providing such examples turns out to be a task of considerable difficulty, due to the presence of characteristic points. A major effort in this direction was made in the papers [18], [19], to which we refer the reader. In Section 5 we prove the existence of measures $d\mu$ which satisfy the assumptions of the trace theorems in this paper, see Theorem 5.1. As a by-product of Theorems 1.4, 1.9 and 5.1, we obtain the existence of traces for Sobolev functions in the Heisenberg group on compact manifolds of codimension one. Our results in this direction are motivated by the study of boundary value problems for sub-Laplacians [19]. We also mention that for the Heisenberg group similar results, along with a sharp end-point one, are contained in the interesting thesis of M. Mekias [82]. Finally, in Section 6 we give some applications of trace inequalities to the regularity theory of quasilinear partial differential equations which arise in geometry. In two famous papers in the sixties Serrin developed the local regularity of weak solutions and studied the precise asymptotic behavior of singular solutions to a class of general quasilinear equations whose prototype is the p-Laplacian div $(|Du|^{p-2}Du) = 0$, see [89], [90]. In [14] and [17] Capogna and two of us obtained results which generalized Serrin's cited ones to equations generated by a system of smooth vector fields satisfying the finite rank condition. Here, the assumptions on the lower order terms are optimal in the scale of Lebesgue spaces. In the applications, however, one often incurs the need to work with Morrey-Campanato, rather than L^p spaces, see [56]. In Section 6 we apply the trace Theorem 1.9 to obtain the local boundedness, the Harnack inequality and the Hölder continuity of weak solutions to equations with structure as in [14], [17], but with lower order terms which belong to function spaces which are optimal in the scale of Morrey type spaces.

2. – Some preliminaries and basic properties

In this section we develop some basic material which will be needed in the rest of the paper.

DEFINITION 2.1. A metric space $(\mathbb{R}^n, d)$ is called a *length-space* ([58]) if for any $x,\ y \in S$ one has

$$d(x, y) = \inf l(\gamma_{xy}),$$

where $\gamma_{xy} : [a, b] \to \mathbb{R}^n$ is a continuous, rectifiable curve joining x to y, and $l(\gamma_{xy})$ denotes its metric length, i.e., $l(\gamma_{xy}) = \sup \sum_{i=1}^{l} d(\gamma(t_i), \gamma(t_{i+1}))$, the supremum being taken on all finite partitions $a = t_1 < t_2 < \cdots < t_l < t_{l+1} = b$ of the interval $[a, b]$.

Proposition 2.2. *Every Carnot-Carathéodory space is a length-space.*

Proof. Let $x,\ y \in \mathbb{R}^n$ be given and denote with $\mathcal{R}(x, y)$ the collection of all continuous, rectifiable curves joining x to y. Clearly, by the triangle inequality, we have $d(x, y) \leq \inf_{\gamma \in \mathcal{R}(x,y)} l(\gamma)$. To prove the converse inequality, we consider $\gamma \in \mathcal{S}(x, y)$. We claim that

$$l(\gamma) \leq l_s(\gamma).$$

Suppose we have proved the claim. The latter implies

$$\inf_{\gamma \in \mathcal{R}(x,y)} l(\gamma) \leq \inf_{\gamma \in \mathcal{S}(x,y)} l(\gamma) \leq \inf_{\gamma \in \mathcal{S}(x,y)} l_s(\gamma) = d(x, y),$$

which, together with the above inequality, proves the proposition. Thus, let $\gamma \in \mathcal{S}(x, y)$, $\gamma : [0, T] \to \mathbb{R}^n$. Let $\mathcal{P} = \{0 = t_1 < t_2 < \cdots < t_{l+1} = T\}$ be a partition of $[0, T]$ and for each $i = 1, \cdots, l+1$, set $x_i = \gamma_i(0)$. If we define $\gamma_i(t) = \gamma(t_i + t)$ for $0 \leq t \leq t_{i+1} - t_i$, then one easily checks that $\gamma_i \in \mathcal{S}(x_i, x_{i+1})$, and therefore

$$d(x_i, x_{i+1}) \leq l_s(\gamma_i) = t_{i+1} - t_i.$$

This gives

$$\sum_{i=1}^{l} d\big(\gamma(t_i), \gamma(t_{i+1})\big) \leq \sum_{i=1}^{l} (t_{i+1} - t_i) = T = l_s(\gamma). \tag{2.1}$$

Taking the supremum on all partitions $\mathcal{P}$ of $[0, T]$ in (2.1) one obtains the claim. □

Proposition 2.3. *We assume* (H.1). *Then,* $(\mathbb{R}^n, d)$ *is locally compact. Furthermore, for any bounded set* $U \subset \mathbb{R}^n$ *there exists* $R_0 = R_0(U) > 0$ *such that the closed balls* $\bar{B}(x_0, R)$*, with* $x_0 \in U$ *and* $0 < R < R_0$*, are compact.*

We refer the reader to [55] and [53] for the proof of Proposition 2.3.

Remark 2.4. The number $R_0 = R_0(U)$ in (H.2), (H.3) will always be chosen to accommodate Proposition 2.3. By this we mean that for those balls involved in assumptions (H.2), (H.3) we can (and will) assume in view of Proposition 2.3 that they have compact closure.

Using Propositions 2.2, 2.3, a version of the Hopf-Rinow theorem due to Cohn-Vossen [25], see also [12], together with the important fact that in any locally compact, complete length-space the geodesic segment property holds, we obtain the following

PROPOSITION 2.5 (see [41], [54]). *Assume* (H.1), (H.2) *and let* $U \in \mathbb{R}^n$ *be a bounded set,* R_0 *as in Remark* 2.4. *Then the balls* $B(x, R)$ *are PS-domains for* $x \in U$ *and* $0 < R < R_0$.

PROPOSITION 2.6. *Suppose* (H.1), (H.2). *If* Ω *satisfies the uniform interior corkscrew condition, then* $|\partial\Omega| = 0$.

PROOF. For every $q \in \partial\Omega$ and $0 < r \leq r_0$, the uniform interior corkscrew condition implies the existence of a ball $B(A_r(q), \frac{r}{4})$ such that $B(A_r(q), \frac{r}{2M}) \subseteq B(q, r) \cap \Omega$. We also have then $B(q, r) \subset B(A_r(q), 2r)$. Thus

$$\frac{|B(q,r) \cap \Omega|}{|B(q,r)|} \geq \frac{|B(A_r(q), \frac{r}{2M})|}{|B(q,r)|} \ (\text{by } (1.4)) \ \geq C = C(M, C_1) > 0\,.$$

Since $q \in \partial\Omega$ is arbitrary, we infer $\partial\Omega \subseteq \mathcal{S}$ where

$$\mathcal{S} = \left\{ q \in \mathbb{R}^n \mid \frac{|B(q,r) \cap \Omega|}{|B(q,r)|} = \frac{1}{|B(q,r)|} \int_{B(q,r)} \chi_\Omega \not\to \chi_\Omega \quad \text{as } r \downarrow 0 \right\}.$$

By the Lebesgue differentiation theorem for spaces of homogeneous type (see [92]) we conclude that

$$|\partial\Omega| \leq |\mathcal{S}| = 0\,. \qquad \square$$

It is proved in [67], [18] that the Carnot-Carathéodory balls satisfy the uniform interior corkscrew condition. In view of this and of Proposition 2.6 we obtain the following

COROLLARY 2.7. *Under the hypothesis* (H.1), (H.2), *for any* $x \in \mathbb{R}^n$ *and* $R > 0$ *one has* $|\partial B(x, R)| = 0$.

PROPOSITION 2.8. *Suppose* (H.1), (H.2) *hold. Then for each fixed* $x \in U$ *and Lebesgue measurable set* $E \subset \mathbb{R}^n$ *the function*

$$r \mapsto \frac{|E \cap B(x,r)|}{|B(x,r)|}$$

is continuous.

Although Proposition 2.8 appear natural, its proof rests, via Corollary 2.7, on some deep consequences of (H.2) such as, for instance, Lebesgue differentiation theorem in a space of homogeneous type.

PROOF. By the absolute continuity of the Lebesgue integral, it suffices to prove that

$$f(r) = |B(x,r)|$$

is continuous. We will show that

$$\lim_{r \to r_0^+} f(r) = \lim_{r \to r_0^-} f(r) = f(r_0)\,.$$

In any metric space one has

$$\overline{B(x,r_0)} = \bigcap_{r_0<r} B(x,r) \tag{2.3}$$

and

$$B(x,r_0) = \bigcup_{0\le r\le r_0} B(x,r)\,. \tag{2.4}$$

By properties of Lebesgue measure, we infer

$$\lim_{r\to r_0^+} |B(x,r)| = \quad (\text{by } (2.3)) \quad \left|\overline{B(x,r_0)}\right| \quad (\text{by Corollary 2.7}) \quad = \left|B(x,r_0)\right|,$$

and

$$\lim_{r\to r_0^-} \left|B(x,r)\right| = \quad (\text{by } (2.4)) \quad \left|B(x,r_0)\right|. \qquad \square$$

We will also need the following

Proposition 2.9. *We assume* (H.1) *and* (H.2). *Fix* $0 < R \le R_0$. *The function*

$$x \mapsto \left|B(x,R)\right|$$

is continuous in $(\mathbb{R}^n, d)$. *Therefore, it is also continuous in* $(\mathbb{R}^n, d_e)$.

Proof. Let $x_0 \in \mathbb{R}^n$ and $\epsilon > 0$ be given. We estimate

$$\begin{aligned}
\big||B(x_0,R)| - |B(x,R)|\big| &\\
&\le \big||B(x_0,R+d(x,x_0))| - |B(x_0,R)|\big| \\
&\quad + \big||B(x_0,R+d(x,x_0))| - |B(x,R)|\big| \\
&= I + II\,.
\end{aligned}$$

Since

$$B\big(x_0, R-d(x,x_0)\big) \subset B(x,R) \subset B\big(x_0, R+d(x,x_0)\big)\,,$$

we can expand II further as

$$\begin{aligned}
II &= \big||B(x_0,R+d(x,x_0))| - |B(x,R)|\big| \\
&\le |B(x_0,R+d(x,x_0))| - |B(x_0,R-d(x,x_0))| \\
&= |B(x_0,R+d(x,x_0))| - |B(x_0,R)| + |B(x_0,R)| \\
&\quad - |B(x_0,R-d(x,x_0))|\,.
\end{aligned}$$

Now we choose $\delta > 0$ from Proposition 2.8 so that if $d(x,x_0) < \delta$ then

$$\left|B(x_0,R+d(x,x_0))\right| - |B(x_0,R)|,\, |B(x_0,R)| - \left|B(x_0,R-d(x,x_0))\right| \le \frac{\epsilon}{3}$$

to reach the conclusion. $\square$

COROLLATY 2.10. *Under the same hypothesis of Proposition* 2.9, *let* $U \subset \mathbb{R}^n$ *be a connected and bounded set with* $|U| > 0$, *and let* R_0 *be the relative doubling parameter in* (H.2). *For any* $0 < R \leq R_0$ *we have*

$$C_R = \inf_{x \in \overline{U}} |B(x, R)| > 0 .$$

PROOF. We argue by contradiction. Suppose that for some $O < R \leq R_0$, we have $C_R = 0$. There exist x_0, $\{x_k\} \in \bar{U}$ such that $x_k \to x_0$ and $|B(x_k, R)| \to 0$. By the continuity in $(\mathbb{R}^n, d_e)$ of the map $x \mapsto |B(x, R)|$, see Proposition 2.9, we infer $|B(x_k, R)| \to |B(x_0, R)|$, and hence $|B(x_0, R)| = 0$. Next, we let $\Omega = \{x \in \overline{U} \mid |B(x, R)| = 0\}$. Clearly, $\Omega \neq \emptyset$. By Proposition 2.9, Ω is closed. To prove that Ω is open, pick $\bar{x} \in \Omega$. We want to show that $B(\bar{x}, \frac{R}{2}) \subset \Omega$. If $y \in B(\bar{x}, \frac{R}{2})$, then $B(y, \frac{R}{2}) \subset B(\bar{x}, R)$. Therefore, $|B(y, \frac{R}{2})| = 0$. By (H.2) we infer $|B(y, R)| = 0$, hence Ω is open. By connectedness, $\Omega = \overline{U}$ and we reach the contradiction $|\overline{U}| = 0$. □

PROPOSITION 2.11. *Assume* (H.1), (H.2). *For any bounded set* $U \subset \mathbb{R}^n$ *let* $f(r) = \sup_{x \in \bar{U}} |B(x, r)|$. *One has* $f(r) \to 0$ *as* $r \to 0^+$.

PROOF. Arguing by contradiction we assume the existence of $\epsilon > 0$ and of a sequence of positive numbers $0 < r_k < R_0$, with $r_k \to 0$ as $k \to \infty$, such that $f(r_k) > \epsilon$. For each $k \in \mathbb{N}$ there exists $x_k \in \bar{U}$ such that

$$\big|B(x_k, r_k)\big| \geq \epsilon . \tag{2.5}$$

Since $\bar{U}$ is compact, one can find $x_0 \in \bar{U}$ and a subsequence still denoted by $\{x_k\}$ such that $x_k \to x_0$ in $(\mathbb{R}^n, d_e)$ (hence in $(\mathbb{R}^n, d)$ by (H.1)). Thus, for large enough k we have by the triangle inequality $|B(x_k, r_k)| \leq |B(x_0, 2r_k)|$. By the proof of Proposition 2.8, $|B(x_0, 2r_k)|$ (and hence $|B(x_k, r_k)|$) $\to 0$ as $k \to \infty$. This contradicts (2.5) and the proposition is proved. □

3. – Trace inequalities: The geometric case

This section is devoted to proving Theorems 1.4, 1.5, 1.7 and 1.8. We begin by proving Theorem 1.5. The latter plays a crucial role in the proof of Theorem 1.4. In preparation for the proof of Theorem 1.5, for every bounded set $U \subset \mathbb{R}^n$ we fix two constants $C_5 = C_5(U)$ and $d_0 = d_0(U)$ as follows. First, we let $C_5 = \inf_{x \in \bar{U}} |B(x, R_0)| > 0$ (see Corollary 2.10) where R_0 is the constant in Remark 2.4. Next, by Proposition 2.11, we choose d_0 such that $\sup_{x \in \bar{U}} |B(x, r)| \leq \frac{C_5}{4}$ for all $r \leq d_0$. We make the following important

REMARK 3.1. Whenever the parameter R_0 in (H.2) can be allowed to be $+\infty$, then we can take $d_0 = +\infty$. (Since C_5 in this case can be taken to be ∞). This is the case, e.g., of all stratified nilpotent Lie groups [38]. When

(M, g) is an n-dimensional complete Riemannian manifold with nonnegative Ricci tensor, then Bishop's comparison theorem guarantees that (H.2) holds for the geodesic balls with $R_0 = +\infty$, see e.g. [24], and therefore we can take $d_0 = +\infty$. In this example Lebesgue measure in (H.2) must be replaced by the Riemannian volume dv_g.

PROOF OF THEOREM 1.5. For each $x \in E$ there exists $0 < \bar{r}_x \leq d_0$ such that

$$\frac{|B(x, \bar{r}_x) \cap E|}{|B(x, \bar{r}_x)|} > \tau . \tag{3.1}$$

Otherwise, we would have $\overline{D}(E, x) \leq \tau$, against the assumptions. On the other hand, we have for $0 < r \leq R_0$ and $d(E) = \operatorname{diam}(E)$

$$\frac{|B(x, r) \cap E|}{|B(x, r)|} \leq \frac{|E|}{|B(x, r)|} \leq \frac{|B(x, d(E))|}{|B(x, r)|} .$$

Letting $r \to R_0$ in the above and using Proposition 2.8 we obtain

$$\begin{aligned} \lim_{r \to R_0} \frac{|B(x, r) \cap E|}{|B(x, r)|} &\leq \lim_{r \to R_0} \frac{|B(x, d(E))|}{|B(x, r)|} = \frac{|B(x, d(E))|}{|B(x, R_0)|} \\ &\leq \frac{|B(x, d_0)|}{C_5} \leq \frac{1}{C_5} \sup_{x \in \bar{U}} |B(x, d_0)| \leq \frac{1}{4} , \end{aligned}$$

by our choice of C_5 and d_0. By (3.1) and again by Proposition 2.8 we infer the existence of $r_x \in (0, R_0)$ such that

$$\frac{|B(x, r_x) \cap E|}{|B(x, r_x)|} = \tau . \tag{3.2}$$

Since, by hypothesis, $\tau < \frac{1}{2}$, we must have

$$\min\big(|B(x, r_x) \cap E|, |B(x, r_x) \cap E^c|\big) = |B(x, r_x) \cap E| . \tag{3.3}$$

At this point we use the relative isoperimetric inequality in Theorem 1.6 which gives

$$\begin{aligned} &\min\big(|B(x, r_x) \cap E|, |B(x, r_x) \cap E^c|\big)^{\frac{Q-1}{Q}} \\ &\qquad \leq C_3 r_x |B(x, r_x)|^{-\frac{1}{Q}} P_X\big(E; B(x, r_x)\big) . \end{aligned} \tag{3.4}$$

From (3.2)-(3.4) one obtains

$$\big(\tau |B(x, r_x)|\big)^{\frac{Q-1}{Q}} \leq C_3 r_x |B(x, r_x)|^{-\frac{1}{Q}} P_X\big(E; B(x, r_x)\big) ,$$

hence

$$\frac{|B(x,r_x)|}{r_x} \le C_3 \tau^{\frac{1-Q}{Q}} P_X\big(E; B(x,r_x)\big) \le C_3' P_X\big(E; B(x,r_x)\big)\,. \tag{3.5}$$

The estimate (3.5) plays an important role in the conclusion of the proof of Theorem 1.5. We now cover E as follows

$$E \subset \bigcup_{x\in E} B(x,r_x)\,.$$

Theorem 1.2 in [26] allows to extract from $\{B(x,r_x)\}_{x\in E}$ a sequence of pairwise disjoint balls $\{B(x_i,r_i)\}_{i\in\mathbb{N}}$ such that

$$E \subset \bigcup_{i=1}^{\infty} B(x_i, kr_i)$$

for some $k > 0$ depending only on C_1 in (H.2). By (3.5) we conclude

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{|B(x_i,r_i)|}{r_i} &\le C_3' \sum_{i=1}^{\infty} P_X\big(E; B(x_i,r_i)\big) = C_3' P_X\left(E; \bigcup_{i=1}^{\infty} B(x_i,r_i)\right)\\ &\le C_3' P_X(E;\mathbb{R}^n)\,.\end{aligned}$$

(Notice that we have used the fact that the balls $B(x_i,r_i)$ are pairwise disjoint). This completes the proof. □

Next, we recall an interesting generalization of Federer's co-area formula. This result was established in [54] and also, independently, in [48].

THEOREM 3.2. *Let $u \in BV_X(\Omega)$ and for $t \in \mathbb{R}$ denote $E_t = \{x \in \Omega \mid u(x) > t\}$. Then*

(i) $P_X(E_t;\Omega) < \infty$ *for* $a.e.\, t \in \mathbb{R}$;

(ii) $\mathrm{Var}_X(u;\Omega) = \int_{-\infty}^{\infty} P_X(E_t;\Omega)\,dt$;

(iii) *Conversely, if $u \in L^1(\Omega)$ and $\int_{-\infty}^{\infty} P_X(E_t;\Omega)\,dt < \infty$, then $u \in BV_X(\Omega)$.*

We also recall the following basic result about the existence of cut-off functions which fit the sub-Riemannian geometry of the geodesic balls, see [55] and also [49].

THEOREM 3.3. *Assume* (H.1). *Let $B(x_0,R)$ be a bounded metric ball. Then, for every $0 < s < t < R$ there exists a d-Lipschitz continuous function $\phi : \mathbb{R}^n \to [0,\infty)$ such that $\phi \in \mathcal{L}^{1,p}(\mathbb{R}^n)$ for every $1 \le p < \infty$. Furthermore, we have*

(i) $\phi \equiv 1$ *on* $B(x_0,s)$ *and* $\phi \equiv 0$ *outside* $B(x_0,t)$,

(ii) $|X\phi| \le \frac{C}{t-s}$ *for a.e.* $x \in \mathbb{R}^n$,

where $C > 0$ is a constant which depends only on C_1 in (H.1).

With these results in hand we turn to the

PROOF OF THEOREM 1.4. The proof is accomplished in several steps.

STEP 1. We start by considering a nonnegative Lipschitz continuous function u having compact support in B^*. Let $0 \le t$ and set $E_t = \{x \in B^* \,|\, u(x) > t\}$. Then E_t is open in B^*, when the latter is endowed with the metric induced by d_e, and therefore, thanks to (H.1), it is also open in $(\mathbb{R}^n, d_e)$. By Proposition 1.2, E_t is then open in $(\mathbb{R}^n, d)$. This implies that for every $x \in E_t$

$$\overline{D}(E_t, x) = 1 .$$

Applying Theorem 1.5 we find a sequence of balls $\{B(x_i, kr_i)\}_{i\in\mathbb{N}}$ satisfying properties (i)-(iii) in the statement of the theorem with $E = E_t$. This gives

$$\begin{aligned}
\mu(E_t) &\le \mu\left(\bigcup_{i=1}^{\infty} B(x_i, kr_i)\right) \le \sum_{i=1}^{\infty} \mu(B(x_i, kr_i)) \\
&\le \text{ (by (1.5)) } M \sum_{i=1}^{\infty} \frac{|B(x_i, kr_i)|}{kr_i} \\
\text{(by (H.2))} \quad &\le CM \sum_{i=1}^{\infty} \frac{|B(x_i, r_i)|}{r_i} \le \text{ (by (iii)) } CMP_X(E_t, \mathbb{R}^n) .
\end{aligned}$$

Observing that $u \in \mathcal{L}^{1,1}(B^*)$ (hence $u \in BV_X(B^*)$, and $\mathrm{Var}_X(u; B^*) = \int_{B^*} |Xu|\, dx$, see [54]), we can apply (ii) of Theorem 3.2 to obtain

$$\begin{aligned}
\int_{B^*} u\, d\mu = \int_0^{\infty} \mu(E_t)\, dt &\le CM \int_0^{\infty} P_X(E_t, \mathbb{R}^n)\, dt \\
&= CM \int_{B^*} |Xu|\, dx .
\end{aligned}$$

STEP 2. We now remove the assumption $u \ge 0$. Let then u be Lipschitz continuous with compact support in B^*. We write $u = u^+ - u^-$ and note that we can apply step 1 to u^+, u^-. We thus infer

$$\int_{B^*} |u|\, d\mu = \int_{B^*} (u^+ + u^-)\, d\mu \le CM \int_{B^*} |X(u^+) + X(u^-)|\, dx \le CM \int_{B^*} |Xu|\, dx .$$

STEP 3. $u \in \overset{\circ}{S}{}^{1,1}(B^*)$. Let $u_k \in C_0^{\infty}(B^*)$, $u_k \to u$ in $S^{1,1}(B^*)$ as $k \to \infty$. By Step 2 we have for $h, k \to \infty$

$$\int_{B^*} |u_k - u_h|\, d\mu \le CM \int_{B^*} |Xu_k - Xu_h|\, dx \longrightarrow 0 .$$

We infer that $\{u_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^1(B^*, d\mu)$. Hence, there exists $\tilde{u} \in L^1(B^*, d\mu)$ such that $u_k \to \tilde{u}$ in $L^1(B^*, d\mu)$ as $k \to \infty$. Passing to the limit as $k \to \infty$ in

$$\int_{B^*} |u_k| \, d\mu \le CM \int_{B^*} |Xu_k| \, dx$$

we obtain

$$\int_{B^*} |\tilde{u}| \, d\mu \le CM \int_{B^*} |Xu| \, dx \,. \tag{3.6}$$

STEP 4. The map $u \mapsto \tilde{u}$ is well-defined. Let $u \in \overset{\circ}{S}{}^{1,1}(B^*)$. We show that the function $\tilde{u}$ in (3.6) is uniquely determined. To this purpose, let $u_k, v_k \in \mathrm{Lip}_0(B^*)$ such that $u_k \to u$, $v_k \to u$ in $S^{1,1}(B^*)$ as $k \to \infty$. We have for $k \to \infty$

$$\int_{B^*} |u_k - v_k| \, d\mu \le CM \int_{B^*} |Xu_k - Xv_k| \, dx \to 0 \,.$$

If $\tilde{u} = \lim_{k\to\infty} u_k$, $\tilde{v} = \lim_{k\to\infty} v_k$ in $L^1(B^*, d\mu)$, we infer

$$\int_{B^*} |\tilde{u} - \tilde{v}| \, d\mu \le \lim_{k\to\infty} \int_{B^*} |u_k - v_k| \, d\mu \le CM \lim_{k\to\infty} \int_{B^*} |Xu_k - Xv_k| \, dx = 0 \,.$$

This shows that $\tilde{u} = \tilde{v}$ (μ-almost everywhere in B^*).

STEP 5. Poincaré type inequality. This step relies on the previous ones as well as on the delicate existence result for cut-off functions of Theorem 3.3. According to the latter we can find a d-Lipschitz continuous function ϕ with compact support in B^* such that $0 \le \phi \le 1$, $\phi \equiv 1$ on B and $|X\phi| \le \frac{C}{R}$. We now consider $u \in \mathcal{L}^{1,1}(B^*)$. By the Meyers-Serrin type approximation theorem (see [54, Theorem 1.29]) we know that $u \in S^{1,1}(B^*)$. But then $\phi u \in \overset{\circ}{S}{}^{1,1}(B^*)$ and it is easy to recognize that $\widetilde{\phi u} = \phi \tilde{u}$. By (3.6) in Step 3 we infer

$$\begin{aligned}\int_B |\tilde{u}| \, d\mu &\le \int_{B^*} |\phi \tilde{u}| \, d\mu = \int_{B^*} |\widetilde{\phi u}| \, d\mu \le CM \int_{B^*} |X(\phi u)| \, dx \\ &\le CM \left[\int_{B^*} |Xu| \, dx + \int_{B^*} |u||X\phi| \, dx \right] .\end{aligned}$$

Using the control on $|X\phi|$ we conclude

$$\int_B |\tilde{u}| \, d\mu \le CM \left(\int_{B^*} |Xu| \, dx + \frac{1}{R} \int_{B^*} |u| \, dx \right) . \tag{3.7}$$

Before proceeding we recall a result from [54, Theorem 1.15] which was proved under the hypothesis (H.1)-(H.3)

$$\int_{B^*} |u - u_{B^*}| \, dx \;\le\; CR \int_{B^*} |Xu| \, dx \,. \tag{3.8}$$

The Poincaré inequality in (3.8) plays a crucial role in the subsequent arguments. Returning to (3.7), we see that the latter now holds for any $u \in \mathcal{L}^{1,1}(B^*)$. Next, consider $u \in \mathcal{L}^{1,1}(B^*)$ and apply (3.7) to $u - u_B$. Since $\widetilde{u - u_B} = \tilde{u} - u_B$ we obtain

$$\begin{aligned}
\int_B |\tilde{u} - u_B|\, d\mu = \int_B |\widetilde{u - u_B}|\, d\mu &\leq CM \left(\int_{B^*} |Xu|\, dx + \frac{1}{R} \int_{B^*} |u - u_B| dx \right) \\
&\leq CM \left(\int_{B^*} |Xu| dx + \frac{1}{R} \int_{B^*} |u - u_{B^*}| dx + \frac{1}{R} |B^*| |u_B - u_{B^*}| \right) \\
\text{(by (3.8))} &\leq CM \left(\int_{B^*} |Xu| dx + C \int_{B^*} |Xu| dx + \frac{1}{R} \frac{|B^*|}{|B|} \int_{B^*} |u - u_{B^*}| dx \right) \\
\text{(by (3.8))} &\leq CM \left(\int_{B^*} |Xu| dx + C \int_{B^*} |Xu| dx + \frac{|B^*|}{|B|} \int_{B^*} |Xu| dx \right) \\
\text{(by (H.2))} &\leq CM \int_{B^*} |Xu|\, dx\,.
\end{aligned}$$

We have thus proved for any $u \in \mathcal{L}^{1,1}(B^*)$

$$\int_B |\tilde{u} - u_B|\, d\mu \leq CM \int_{B^*} |Xu|\,. \tag{3.9}$$

Finally, one has

$$\begin{aligned}
\int_B |\tilde{u} - \tilde{u}_{B,\mu}|\, d\mu &\leq \int_B |\tilde{u} - u_B|\, d\mu + \mu(B) |u_B - \tilde{u}_{B,\mu}| \\
\text{(by (3.9))} \quad &\leq CM \int_{B^*} |Xu|\, dx + \int_B |\tilde{u} - u_B|\, d\mu \\
\text{(by (3.9))} \quad &\leq CM \int_{B^*} |Xu|\, dx\,.
\end{aligned}$$

This completes the proof. □

We now turn to the

Proof of Theorem 1.7. We make the following considerations. First, observe that the uniform corkscrew condition on E implies that for $x \in \partial E$ and $0 < R \leq R_0$

$$C|B(x,R)| \leq \min\left(|B(x,R) \cap E|, |B(x,R) \cap E^c|\right)$$

for some constant $C = C(M) > 0$. Next, by Corollary 2.10 and (1.4) we have

$$R^Q \leq \frac{C_1}{C_5} R_0^Q |B(x,R)| \tag{3.10}$$

for $0 < R \leq R_0$, $x \in U$. Now, fix $\delta > 0$ and let $\mathcal{F} = \{B(x_i, k\delta) \,|\, x_i \in \partial E\}$ be a covering of ∂E by a sequence of countable balls such that $B(x_i, \delta) \cap B(x_j, \delta) = \emptyset$

if $i \neq j$, see [26]. We can take $\mathcal{F}$ to be a finite set since ∂E is compact. Then we have with $B_i = B(x_i, \delta)$, $B_i^* = B(x_i, k\delta)$

$$\begin{aligned}
\sum_i \operatorname{diam}(B_i^*)^{Q-1} &\leq \text{ (by (3.10)) } C \sum_i \frac{|B_i^*|}{\delta} \leq \text{ (by (H.2))} C \sum_i \frac{|B_i|^{\frac{1}{Q}}}{\delta} |B_i|^{\frac{Q-1}{Q}} \\
&\leq C \sum_i \frac{|B_i|^{\frac{1}{Q}}}{\delta} \min\left(|B_i \cap E|, |B_i \cap E^c|\right)^{\frac{Q-1}{Q}} \\
\text{(by (Theorem 1.6))} \quad &\leq C \sum_i P_X(E; B_i) \\
&\leq C P_X(E; \mathbb{R}^n),
\end{aligned}$$

since the balls $B_i \in \mathcal{F}$ are pairwise disjoint. Passing to the infimum on all coverings of ∂E by balls whose radius does not exceed δ and letting $\delta \to 0$, we conclude that

$$\mathcal{H}^{Q-1}(\partial E) \leq C P_X(E; \mathbb{R}^n)$$

where $C = C(M, R_0, C_1, C_3, C_5) > 0$. This proves the theorem. □

REMARK 3.4. It is easy to see that the reverse inequality in Theorem 1.7 is false by the following example. We consider the Baouendi-Grushin vector fields on $\mathbb{R}^2$ given by $X = \{X_1, X_2\}$ where

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x \frac{\partial}{\partial y}.$$

Let d be the Carnot-Carathéodory metric associated to the system X, $U = B((0,0), 1)$ and E be a square centered at $(\frac{1}{2}, 0)$ and having side-length equal to $\frac{1}{16}$. We recall that the Carnot-Carathéodory metric is equivalent to the following pseudo-metric found in [66],

$$d((x,y),(x',y')) \sim |x - x'| + \min\left(\frac{|y - y'|}{|x|}, |y - y'|^{\frac{1}{2}}\right).$$

Now for $(x, y), (x', y')$ in a small neighborhood of E, which stays away from the y-axis, $d((x, y), (x', y'))$ is essentially the Euclidean distance. Since $E \subset U$ satisfies the uniform corkscrew condition with respect to the Euclidean metric, we conclude that E also satisfies the uniform corkscrew condition with, say, $R_0 = \frac{1}{64}$ with respect to d. It is known that in this case the homogeneous dimension Q of the set U is $Q = 3$, hence

$$\mathcal{H}^{Q-1}(\partial E) = \mathcal{H}^2(\partial E) = 0.$$

Nevertheless, if we consider the smooth portion of ∂E, $E_1 = \{(\frac{1}{2} - \frac{1}{32}, y) \in \mathbb{R}^2 \mid -\frac{1}{16} \leq y \leq \frac{1}{16}\}$, and η denotes the unit normal to E_1, then an easy calculation gives

$$P_X(E;\mathbb{R}^2) \geq P_X(E_1;\mathbb{R}^2) = \int_{E_1} \left(\sum_{i=1}^{2} < X_j, \eta >^2 \right)^{\frac{1}{2}} d\mathcal{H}^1 > 0 .$$

This proves that the inequality

$$P_X(E;\mathbb{R}^n) \leq C\mathcal{H}^{Q-1}(\partial E)$$

cannot hold in general.

We now present the

PROOF OF THEOREM 1.8. We begin by recalling the following result from the statement of Theorem 2.24 in [18]: *Let $\Omega \subset \mathbb{H}^n$ be a domain of class $C^{1,1}$, then Ω satisfies the uniform corkscrew condition.* Thanks to this theorem we can apply Theorem 1.7 to the set E and obtain for it the inequality

$$\mathcal{H}^{Q-1}(\partial E) \leq \alpha P_X(E;\mathbb{H}^n) .$$

To show the opposite inequality, we can, without loss of generality (since the problem is local in nature), assume that E is given by the set $\{(z,t) \mid \rho(z,t) < 0\}$ for some function $\rho \in C^2(\mathbb{H}^n)$. If X_j, $j = 1, \dots, 2n$ indicate the $2n$ left invariant horizontal vector fields on $\mathbb{H}^n$ which generate the Lie algebra, and $\eta = \frac{\nabla\rho}{|\nabla\rho|}$ denotes the outer unit normal to ∂E, then we have (see (3.2) in [15])

$$P_X(E;\mathbb{H}^n) = \int_{\partial E} \left(\sum_{i=1}^{2n} \langle X_j, \eta\rangle^2 \right)^{\frac{1}{2}} d\sigma = \int_{\partial E} \frac{|X\rho|}{|\nabla\rho|} d\sigma , \tag{3.11}$$

with $d\sigma$ denoting the Lebesgue surface measure on ∂E.

To continue we exploit a delicate result due to Mekias [82] which states that if $B((z,t),r)$ is a metric ball corresponding to the Carnot-Carathéodory metric associated to the system $X = \{X_1, \dots, X_{2n}\}$, and if $(z,t) \in \partial E$, then

$$\int_{B((z,t),r)\cap\partial E} |X\rho|\, d\sigma \cong r^{2n+1} = r^{Q-1} . \tag{3.12}$$

Since Mekias's thesis is not available in print, in Section 5 we provide complete details of the proof of that direction in (3.12) which is needed in the subsequent argument. Returning to (3.11) we observe that by the compactness of ∂E we can assume that $|\nabla\rho| > C > 0$. For any ball B with center in ∂E and radius r we therefore have

$$\begin{aligned} P_X(E; B) = \text{ (by (3.11)) } &\leq C \int_{\partial E \cap B} |X\rho| d\sigma \\ \text{(by the direction } \leq \text{ in (3.12)) } &\leq C r^{Q-1} . \end{aligned} \tag{3.13}$$

Fix $\epsilon > 0$ and let now $\{B(p_i, \delta_i) \mid \delta_i < \epsilon,\ i = 1, 2, \dots\}$ be a covering of ∂E by metric balls. Then for each $B(p_i, \delta_i)$ in the covering, and $q_i \in B(p_i, \delta_i) \cap \partial E$, $B(p_i, \delta_i) \subset B(q_i, 2\delta_i)$ holds. Hence, we have

$$\begin{aligned} P_X(E; \mathbb{H}^n) &\leq \sum_i P_X\big(E; B(p_i, \delta_i)\big) \leq \sum_i P_X\big(E; B(q_i, 2\delta_i)\big) \\ \text{(by (3.13))} &\leq C \sum_i {\delta_i}^{Q-1} \end{aligned}$$

where C in the above is independent of the number of balls in the covering, and of δ_i's. Since

$$\mathcal{H}^{Q-1}(\partial E) = \liminf_{\epsilon \to 0} \left\{ \sum_i {\delta_i}^{Q-1} \middle| \partial E \subset \bigcup B(p_i, \delta_i),\ \delta_i < \epsilon \right\}$$

the result follows.

4. – Trace inequality: The nongeometric case $p > 1$

This section is devoted to proving Theorems 1.9 and 1.14. Our first step will be to establish a delicate generalization of the Hardy-Littlewood-Sobolev theorem of fractional integration in a space of homogeneous type proved independently in [27], and [73], see also [14]. In the Euclidean setting such generalization was first established in two beautiful papers by D. Adams [1], [2], and subsequently extended to weighted spaces. Our approach is based on D. Adams' one, but incorporates some modifications, inspired by a paper of Hedberg [63], see also the recent monograph [4]. Such modifications of Adam's arguments are necessary to handle our general setting and, at the same time, slightly simplify the proofs.

Henceforth, we assume that $U \subset \mathbb{R}^n$ is a given bounded set and let C_1, R_0 be the relative parameters in (H.2). Let Q denote the homogeneous dimension of U. For $0 < \alpha \leq Q$, we recall the definition of the fractional integration operator of order α [27]

$$I_\alpha f(x) = \int_B |f(y)| \frac{d(x, y)^\alpha}{|B(x, d(x, y))|} dy,\ x \in B ,$$

where $B = B(x_0, R)$, $x_0 \in U$, $0 < R \leq R_0$. Then we have the following

THEOREM 4.1. *Assume* (H.1), (H.2). *Let* $1 \leq p < \frac{Q}{\alpha}$ *and suppose that* μ *is a nonnegative Borel measure on* $\mathbb{R}^n$ *satisfying*

$$\mu\big(B(x,r)\big) \leq M\frac{|B(x,r)|}{r^{\alpha(p-\epsilon)}},\ x \in U,\ 0 < r \leq R_0\,, \tag{4.1}$$

for some $M > 0$ *and* $0 < \epsilon \leq p$. *Then,* $I_\alpha : L^p(B, dx) \to L^{q,\infty}(B, d\mu)$ *with* $q = p\frac{Q-\alpha(p-\epsilon)}{Q-\alpha p} > p$. *Furthermore, there exists* $C = C(C_1, \alpha, p, \epsilon) > 0$ *such that*

$$\|I_\alpha f\|_{L^{q,\infty}(B,d\mu)} \leq C M^{\frac{1}{q}} \left(\frac{R}{|B|^{\frac{1}{Q}}}\right)^{\frac{\epsilon\alpha Q}{p(Q-\alpha(p-\epsilon))}} \|f\|_{L^p(B,dx)} \tag{4.2}$$

for any $f \in L^p(B, dx)$.

REMARK 4.2. 1) When $1 < p < \frac{Q}{\alpha}$ one can actually prove that $I_\alpha : L^p(B, dx) \to L^q(B, d\mu)$ and that in (4.2) the weak L^q norm can be replaced by the strong one. This requires proving a strong version of Corollary 4.1 below. For the latter we refer to [28]. Since the weak continuity result in Theorem 4.1 is enough to prove the strong trace inequality in Theorem 1.9, we will not prove the stronger version of (4.2).

2) It is interesting to observe that when $d\mu = dx$, Lebesgue measure in $\mathbb{R}^n$, then (4.1) holds with $M = 1$ and $\epsilon = p$. In such case $q = \frac{pQ}{Q-\alpha p}$, i.e., $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$, and Theorem 4.1 is nothing but the weak version of the fractional integration theorem in [27], [14], [73].

Before proving Theorem 4.1 we establish a generalization of a result in [36]. For $0 < \gamma \leq Q$ we introduce the *fractional maximal operator* of order γ of a nonnegative measure μ

$$M_\gamma \mu(x) = \sup_{\rho > 0} \frac{\rho^\gamma}{|B(x,\rho)|} \mu\big(B(x,\rho)\big)\,.$$

When $d\mu = f dx$, for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then we will write $M_\gamma f$ instead of $M_\gamma \mu$. We have the following

LEMMA 4.3. *Let* $f \in L^1(B)$, *where* $B = B(x_0, R)$, $x_0 \in U$, $0 < R \leq R_0$, *and* μ *be a nonnegative Borel measure. Then, for* $0 < \gamma \leq Q$ *one has*

$$\sup_{t>0}\big[t\mu(\{x \in B \mid M_\gamma f(x) > t\})\big] \leq C \int_B |f(x)| M_\gamma \mu(x)\, dx \tag{4.3}$$

for some constant $C = C(C_1, \gamma) > 0$.

PROOF. If the right-hand side of (4.3) is infinite there is nothing to prove, therefore it suffices to assume that it be finite. In what follows we think of f as defined in $\mathbb{R}^n$ by letting $f = 0$ in B^c. Set $E_t = \{x \in B \mid M_\gamma f(x) > t\}$. If $x \in E_t$ there exists $r_x > 0$ such that

$$\frac{r_x^\gamma}{|B(x, r_x)|} \int_{B(x,r_x)} |f(y)|\, dy > t\,. \tag{4.4}$$

Since $E_t \subset \bigcup_{x\in E_t} B(x, r_x)$, by the Vitali type covering Theorem 1.2 on p. 69 of [26] we can find sequences $x_i \in E_t$, $r_i = r(x_i) > 0$, $i \in \mathbb{N}$, such that $B(x_i, r_i)$ satisfies (4.4)

$$E_t \subset \bigcup_{i=1}^{\infty} B(x_i, kr_i)$$

and $B(x_i, r_i)$ are pairwise disjoint. We have

$$\begin{aligned}
\mu(E_t) &\le \sum_{i=1}^{\infty} \mu\big(B(x_i, kr_i)\big)\\
\text{(by (H.2) and (4.4))} \quad &\le \frac{C}{t}\sum_{i=1}^{\infty} \frac{(kr_i)^\gamma \mu(B(x_i, kr_i))}{|B(x_i, kr_i)|} \int_{B(x_i,r_i)} |f(y)|\, dy \qquad (4.5)\\
&= \frac{C}{t}\sum_{i=1}^{\infty} \int_{B(x_i,r_i)} |f(y)| \frac{(kr_i)^\gamma \mu(B(x_i, kr_i))}{|B(x_i, kr_i)|}\, dy\,.
\end{aligned}$$

Now for each $y \in B(x_i, r_i)$ we obtain from (H.2)

$$\begin{aligned}
\frac{(kr_i)^\gamma \mu(B(x_i, kr_i))}{|B(x_i, kr_i)|} &\le \frac{(kr_i)^\gamma \mu(B(x_i, kr_i))}{|B(y, \frac{kr_i}{2})|}\\
&\le C\frac{(2kr_i)^\gamma \mu(B(y, 2kr_i))}{|B(y, 2kr_i)|} \le C M_\gamma \mu(y)\,.
\end{aligned}$$

Substitution of this estimate in the right-hand side of (4.5) yields

$$\mu(E_t) \le \frac{C}{t}\sum_{i=1}^{\infty} \int_{B(x_i,r_i)} |f(y)| M_\gamma \mu(y)\, dy \le \frac{C}{t}\int_B |f(y)| M_\gamma \mu(y)\, dy\,,$$

since the balls $B(x_i, r_i)$ are pairwise disjoint. This completes the proof. □

COROLLARY 4.4. *Let $0 < \gamma \le Q$, $1 \le p \le \frac{Q}{\gamma}$. Suppose that $f \in L^p(B)$, and that μ is a nonnegative Borel measure on $\mathbb{R}^n$. Then, for every $t > 0$ one has*

$$\mu\big(\{x \in B \mid M_\gamma f(x) > t\}\big) \le \frac{C}{t^p}\int_B |f(x)|^p M_{\gamma p}\mu(x)\, dx\,.$$

for some $C = C(C_1, \gamma, p) > 0$.

PROOF. We note the estimate for $g \ge 0$, $g \in L^1_{\text{loc}}(\mathbb{R}^n)$

$$\frac{r^\gamma}{|B(x,r)|}\int_{B(x,r)} g^{\frac{1}{p}}\,dy \le \left(\frac{r^{\gamma p}}{|B(x,r)|}\int_{B(x,r)} g\,dy\right)^{\frac{1}{p}} \le \left(M_{\gamma p} g(x)\right)^{\frac{1}{p}}.$$

Taking the supremum in $r > 0$ in the left-hand side gives

$$M_\gamma(g^{\frac{1}{p}})(x) \le \left(M_{\gamma p} g(x)\right)^{\frac{1}{p}}.$$

The latter inequality implies

$$\begin{aligned} \mu\left(\{x \in B \mid M_\gamma(g^{\frac{1}{p}})(x) > t\}\right) &\le \mu\left(\{x \in B \mid M_{\gamma p} g(x) > t^p\}\right) \\ \text{(by (4.3))} \quad &\le \frac{C}{t^p}\int_B g(x) M_{\gamma p}\mu(x)\,dx. \end{aligned} \tag{4.6}$$

If now $f \in L^p(B)$, then applying (4.6) with $g = |f|^p$ we reach the conclusion. □

PROOF OF THEOREM 4.1. We adopt an idea of Hedberg [63] which has already been exploited in [27], see also [14] for complete details. For any $0 < \delta < R$ write for $x \in B = B(x_0, R)$

$$\begin{aligned} I_\alpha f(x) &= \int_{B(x,\delta)\cap B} |f(y)|\frac{d(x,y)^\alpha}{|B(x,d(x,y))|}dy + \int_{B(x,\delta)^c\cap B} |f(y)|\frac{d(x,y)^\alpha}{|B(x,d(x,y))|}dy \\ &= I^1_\alpha f(x) + I^2_\alpha f(x), \end{aligned}$$

where we assume that f has been extended with zero outside B. As in [14, formula (2.11)] we estimate

$$I^2_\alpha f(x) \le C\|f\|_{L^p(B)} R^{\frac{Q}{p}} |B(x,R)|^{-\frac{1}{p}} \delta^{-\frac{Q}{p}+\alpha}.$$

Since by (H.2) we have

$$|B(x,R)|^{-\frac{1}{p}} \le C|B(x,2R)|^{-\frac{1}{p}} \le C|B|^{-\frac{1}{p}},$$

we obtain

$$I^2_\alpha f(x) \le C\|f\|_{L^p(B)} R^{\frac{Q}{p}} |B|^{-\frac{1}{p}} \delta^{-\frac{Q}{p}+\alpha}. \tag{4.7}$$

Next, we use the fractional maximal operator of order γ, $0 < \gamma < 1$, $M_\gamma f$, to estimate $I^1_\alpha f$ as follows

$$\begin{aligned} I^1_\alpha f(x) &\le \sum_{k=0}^{\infty}\int_{2^{-(k+1)}\delta \le d(x,y) < 2^{-k}\delta} |f(y)|\frac{d(x,y)^\alpha}{|B(x,d(x,y))|}\,dy \\ \text{(by (H.2))} \quad &\le C\sum_{k=0}^{\infty}\frac{(2^{-k}\delta)^\alpha}{|B(x,2^{-k}\delta)|}\int_{B(x,2^{-k}\delta)} |f(y)|\,dy \\ &\le C M_{\gamma\alpha} f(x)\delta^{\alpha(1-\gamma)}, \end{aligned} \tag{4.8}$$

for some $C = C(C_1, \alpha, \gamma) > 0$. From (4.7), (4.8) we infer

$$I_\alpha f(x) \le C M_{\gamma\alpha} f(x) \delta^{\alpha(1-\gamma)} + C \|f\|_{L^p(B)} R^{\frac{Q}{p}} |B|^{-\frac{1}{p}} \delta^{-\frac{Q}{p}+\alpha} \tag{4.9}$$

for $0 < \delta < R$. Minimizing the right-hand side with respect to δ yields

$$I_\alpha f(x) \le C \left(\frac{R^Q}{|B|} \right)^{\frac{Q(1-\gamma)}{Q-\gamma\alpha p}} \|f\|_{L^p(B)}^{\frac{\alpha p(1-\gamma)}{Q-\gamma\alpha p}} \left(M_{\gamma\alpha} f(x) \right)^{\frac{Q-\alpha p}{Q-\gamma\alpha p}} \tag{4.10}$$

for some constant $C = C(C_1, p, \alpha, \gamma) > 0$.

At this point we use Corollary 4.4 to estimate the distribution function of $(M_{\gamma\alpha} f)^{\frac{Q-\alpha p}{Q-\gamma\alpha p}}$ with respect to μ. We choose $\gamma = \frac{p-\epsilon}{p} < 1$, so that the exponent q in the statement of Theorem 4.1 is given by $q = p\frac{Q-\gamma\alpha p}{Q-\alpha p}$. For $t > 0$ one has

$$\begin{aligned} \mu\left(\{x \in B \mid \left(M_{\gamma\alpha} f(x)\right)^{\frac{Q-\alpha p}{Q-\gamma\alpha p}} > t\}\right) &= \mu\left(\{x \in B \mid M_{\gamma\alpha} f(x) > t^{\frac{q}{p}}\}\right) \\ \text{(by Corollary 4.4) } &\le C t^{-q} \int_B |f(x)|^p M_{\gamma\alpha p}\mu(x)\, dx . \end{aligned} \tag{4.11}$$

In terms of the fractional maximal function of the measure μ we see that (4.1) can be reformulated as follows

$$M_{\gamma\alpha p}\mu(x) \le M, \quad x \in U .$$

But then, (4.11) gives

$$\mu\left(\{x \in B \mid \left(M_{\gamma\alpha} f(x)\right)^{\frac{Q-\alpha p}{Q-\gamma\alpha p}} > t\}\right) \le C M t^{-q} \|f\|^p_{L^p(B)} .$$

The latter estimate implies

$$\left\| \left(M_{\gamma\alpha} f\right)^{\frac{p}{q}} \right\|_{L^{q,\infty}(B, d\mu)} \le (CM)^{\frac{1}{q}} \|f\|^{\frac{p}{q}}_{L^p(B)} . \tag{4.12}$$

Finally, (4.10) and (4.12) imply (note that $1 - \gamma = \frac{\epsilon}{p}$)

$$\begin{aligned} \|I_\alpha f\|_{L^{q,\infty}(B,d\mu)} &\le C \left(\frac{R}{|B|^{\frac{1}{Q}}} \right)^{\frac{Q\alpha\epsilon}{p(Q-\gamma\alpha p)}} \|f\|_{L^p(B)}^{\frac{\alpha\epsilon}{Q-\gamma\alpha p}} \| \left(M_{\gamma\alpha} f\right)^{\frac{p}{q}} \|_{L^{q,\infty}(B,d\mu)} \\ &\le C M^{\frac{1}{q}} \left(\frac{R}{|B|^{\frac{1}{Q}}} \right)^{\frac{Q\alpha\epsilon}{p(Q-\gamma\alpha p)}} \|f\|_{L^p(B)}^{\frac{\alpha\epsilon}{Q-\gamma\alpha p}+\frac{p}{q}} . \end{aligned}$$

It is now enough to observe that $\frac{\alpha\epsilon}{Q-\gamma\alpha p} + \frac{p}{q} = \frac{\alpha\epsilon}{Q-\alpha(p-\epsilon)} + \frac{Q-\alpha p}{Q-\alpha(p-\epsilon)} = 1$, to reach the conclusion. $\square$

We are now ready to give the proof of Theorem 1.9. The latter will be obtained from Theorem 4.1 and from the following important result.

THEOREM 4.5. *Suppose* (H.1)-(H.3) *hold. Then, there exists constants $\beta, C > 0$ depending on C_1, C_2 such that for $B = B(x_0, R)$, $x_0 \in U$, $0 < R \le R_0$, and $u \in \mathcal{L}^{1,p}(B(x_0, \beta R))$, one has for a.e. $x \in B$*

$$|u(x) - u_B| \le C \int_{\beta B} |Xu(y)| \frac{d(x,y)}{|B(x, d(x,y))|} \, dy \,. \tag{4.13}$$

Theorem 4.5 was first proved in [45] for Hörmander type operators by an argument based on the Rothschild-Stein lifting theorem. A more elementary proof, based on (H.2), (H.3), and estimates of the fundamental solution of sub-Laplacians, was found in [16]. Subsequently, Franchi, Lu and Wheeden [46] made the interesting observation that one needs only (H.1), (H.2), the Poincaré inequality, and the following size estimate for the metric balls: for any $0 < t < 1$

$$|B(x, tR)| \le C_1^{-1} t^{Q(x)} |B(x, R)| \ \text{ with } \ \inf x \in U Q(x) > 1 \,, \tag{4.14}$$

to obtain (4.13). Franchi and Wheeden [50] have recently made the important discovery that, even for a system of locally Lipschitz vector fields, (4.14) holds with $\inf_{x \in U} Q(x) = 1$, and that moreover such linear growth is enough to deduce Theorem 4.5 from (H.1), (H.2) and the Poincaré inequality.

COROLLARY 4.6. *Under the assumption of Theorem* 4.5, *suppose that $u \in \overset{\circ}{S}{}^{1,p}(B)$. Then for a.e. $x \in B$ we have*

$$|u(x)| \le C I_1(|Xu|)(x) \,,$$

where

$$I_1(|Xu|)(x) = \int_B |Xu(y)| \frac{d(x,y)}{|B(x, d(x,y))|} \, dy \,.$$

Another fact about $\mathcal{L}^{1,p}$ needed in the proof of Theorem 1.9 as well as in many applications to PDEs is stated in the following

LEMMA 4.7. *Let F be a piecewise smooth function on $\mathbb{R}$ with $F' \in L^\infty(\mathbb{R})$. Then if $u \in \mathcal{L}^{1,p}(U)$ we have $F \circ u \in \mathcal{L}^{1,p}(U)$. Furthermore, if L denotes the set of corner points of F, we have*

$$X(F \circ u) = \begin{cases} F'(u) Xu & \text{if } u \notin L \,, \\ 0 & \text{if } u \in L \,. \end{cases}$$

The proof is standard and follows from Lemma 3.5 parts (I) and (II) of [54]. We now give the

PROOF OF THEOREM 1.9. We start with $u \in \overset{\circ}{S}{}^{1,p}(B^*)$. Corollary 4.6 and Theorem 4.1 allow to conclude for any $t > 0$

$$\mu\big(\{x \in B^* \mid |u(x)| > t\}\big) \leq \frac{CM}{t^q}\left(\frac{R}{|B|^{\frac{1}{Q}}}\right)^{\frac{\epsilon Q}{Q-p}}\left(\int_{B^*}|Xu|^p\,dx\right)^{\frac{q}{p}}. \tag{4.15}$$

For each $k \in \mathbb{Z}$ define

$$F_k(t) = \begin{cases} 0 & \text{if } |t| < 2^k \\ |t| - 2^k & \text{if } 2^k \leq |t| < 2^{k+1} \\ 2^k & \text{if } 2^{k+1} \leq |t|\,. \end{cases}$$

Clearly, F is piecewise smooth with corners at $\pm 2^k, \pm 2^{k+1}$ and $|F'| \leq 1$ at the non corner points. Next, take $u \in \overset{\circ}{S}{}^{1,p}(B^*)$ and let

$$u_k(x) = F_k \circ u(x) = \begin{cases} 0 & \text{if } |u(x)| < 2^k \\ |u(x)| - 2^k & \text{if } 2^k \leq |u(x)| < 2^{k+1} \\ 2^k & \text{if } 2^{k+1} \leq |u(x)|\,. \end{cases}$$

By Lemma 4.7 we have $u_k \in \overset{\circ}{S}{}^{1,p}(B^*)$ and $|Xu_k| = \chi_{\{2^k \leq |u| < 2^{k+1}\}}|Xu|$ for $a.e.\, x \in B^*$. This gives

$$\begin{aligned}
\int_{B^*}|u|^q\,d\mu &= q\int_0^\infty t^{q-1}\mu(\{x \in B^* \mid |u(x)| > t\})\,dt \\
&= q\sum_{k\in\mathbb{Z}}\int_{2^{k+1}}^{2^{k+2}} t^{q-1}\mu(\{x \in B^* \mid |u(x)| > t\})\,dt \\
&\leq q2^{2q-1}\sum_{k\in\mathbb{Z}} 2^{kq}\mu(\{x \in B^* \mid u_k(x) > 2^{k-1}\}) \\
\text{(by (4.15))}\quad &\leq q2^{2q-1}CM\left(\frac{R}{|B|^{\frac{1}{Q}}}\right)^{\frac{\epsilon Q}{Q}-p}\sum_{k\in\mathbb{Z}}2^{kq}2^{-(k-1)q}\left(\int_{B^*}|Xu_k|^p\,dx\right)^{\frac{q}{p}} \\
&\leq CM\left(\frac{R}{|B|^{\frac{1}{Q}}}\right)^{\frac{\epsilon Q}{Q-p}}\left(\sum_{k\in\mathbb{Z}}\int_{B^*}|Xu_k|^p\,dx\right)^{\frac{q}{p}} \\
&\leq CM\left(\frac{R}{|B|^{\frac{1}{Q}}}\right)^{\frac{\epsilon Q}{Q-p}}\left(\sum_{k\in\mathbb{Z}}\int_{\{2^k\leq|u|<2^{k+1}\}}|Xu|^p\,dx\right)^{\frac{q}{p}} \\
&= CM\left(\frac{R}{|B|^{\frac{1}{Q}}}\right)^{\frac{\epsilon Q}{Q-p}}\left(\int_{B^*}|Xu|^p\,dx\right)^{\frac{q}{p}}
\end{aligned}$$

We remark that in the above chain of inequalities we have used the fact $q > p$. Summarizing, we have shown that for every $u \in \overset{\circ}{S}^{1,p}(B^*)$

$$\left(\int_{B^*} |u|^q \, d\mu\right)^{\frac{1}{q}} \leq C M^{\frac{1}{q}} \left(\frac{R}{|B|^{\frac{1}{Q}}}\right)^{\frac{\epsilon Q}{p(Q-(p-\epsilon))}} \left(\int_{B^*} |Xu|^p \, dx\right)^{\frac{1}{p}} . \tag{4.16}$$

Arguing as in the proof of Theorem 1.4 we establish for any $u \in \mathcal{L}^{1,p}(B^*)$

$$\left(\int_{B} |u|^q \, d\mu\right)^{\frac{1}{q}} \leq C M^{\frac{1}{q}} \left(\frac{R}{|B|^{\frac{1}{Q}}}\right)^{\frac{\epsilon Q}{p(Q-(p-\epsilon))}} \left[\frac{1}{R^p}\int_{B^*} |u|^p \, dx + \int_{B^*} |Xu|^p \, dx\right]^{\frac{1}{p}} .$$

From the latter and from the Poincaré inequality

$$\int_{B^*} |u - u_{B^*}|^p \, dx \leq C R^p \int_{B^*} |Xu|^p \, dx$$

for functions in $\mathcal{L}^{1,p}(B^*)$ (see Corollary 1.6, part (II), and Theorem 1.15, part (I), in [54]) we reach the desired conclusion. □

We now turn to the proof of Theorem 1.14. But first, we recall a Lemma from [54].

LEMMA 4.8 (see Lemma 3.1 in [54]). *Let $\Omega \subset \mathbb{R}^n$ be a measurable set with* $\operatorname{diam}(\Omega) \leq R_0/2$. *Then there exists a constant $C > 0$, depending only on C_1 in* (H.2), *such that for any ball $B \subset \Omega$, with radius $r(B)$, we have*

$$|B|^{1-k} \leq C \left(\frac{\operatorname{diam}(\Omega)}{r(B)}\right)^k |\Omega|^{1-k} ,$$

where $k = \frac{Q}{Q-1}$.

PROOF OF THEOREM 1.14. Since the computations are similar to the one in [54, Theorem 1.5], we only highlight the main differences and refer the reader to the above cited work. Let $u \in \mathcal{L}^{1,p}(\Omega)$ be given and denote βB by B^*. We estimate

$$\begin{aligned}
\int_{\Omega} |u - u_{B_0,\mu}|^q \, d\mu &\leq \sum_{B \in \mathcal{F}} \int_{B} |u - u_{B_0,\mu}|^q \, d\mu \\
&\leq 2^q \sum_{B \in \mathcal{F}} \left(\int_{B} |u - u_{B,\mu}|^q \, d\mu + \int_{B} |u_{B,\mu} - u_{B_0,\mu}|^q \, d\mu\right) \\
&= 2^q (I + II) .
\end{aligned}$$

It is easy to see by using Lemma 4.8 and Theorem 1.9 that

$$I = \sum_{B\in\mathcal{F}} \int_B |u - u_{B,\mu}|^q \, d\mu \le C \left(\frac{\operatorname{diam}(\Omega)}{|\Omega|^{\frac{1}{Q}}} \right)^{\frac{\epsilon Q q}{p(Q-p+\epsilon)}} \left(\int_\Omega |Xu|^p \, dx \right)^{\frac{q}{p}} .$$

To estimate II, we consider for each $B \in \mathcal{F}$ a chain $B_0, B_1, \dots, B_{s(B)} = B$ as in Definition 1.12. Then

$$\begin{aligned}
\|u_{B,\mu} - u_{B_0,\mu}\|_{L^q(B,d\mu)} &\le \sum_{i=0}^{s(B)-1} \|u_{B_{i+1},\mu} - u_{B_i,\mu}\|_{L^q(B,d\mu)} \\
&= \sum_{i=0}^{s(B)-1} \left(\frac{\mu(B)}{\mu(B_i \cap B_{i+1})} \right)^{\frac{1}{q}} \|u_{B_i,\mu} - u_{B_{i+1},\mu}\|_{L^q(B_i\cap B_{i+1},d\mu)} \\
&\qquad \text{(by (ii) in Definition 1.12)} \\
&\le C \left[\sum_{i=0}^{s(B)-1} \left(\frac{\mu(B)}{\mu(B_i)} \right)^{\frac{1}{q}} \|u - u_{B_i,\mu}\|_{L^q(B_i,d\mu)} \right. \\
&\qquad \left. + \sum_{i=0}^{s(B)-1} \left(\frac{\mu(B)}{\mu(B_{i+1})} \right)^{\frac{1}{q}} \|u - u_{B_{i+1},\mu}\|_{L^q(B_{i+1},d\mu)} \right] \\
&= C\,[III + IV] .
\end{aligned} \tag{4.17}$$

Since the analysis of III and IV is the same and, at the end, they are majorized by the same quantity, we analyze III only.

$$\begin{aligned}
III &= \sum_{i=0}^{s(B)-1} \left(\frac{\mu(B)}{\mu(B_i)} \right)^{\frac{1}{q}} \|u - u_{B_i,\mu}\|_{L^q(B_i,d\mu)} \quad \text{(by Theorem 1.9)} \\
&\le \sum_{i=0}^{s(B)-1} \left(\frac{\mu(B)}{\mu(B_i)} \right)^{\frac{1}{q}} \left(\frac{r(B_i)}{|B_i|} \right)^{\frac{\epsilon Q}{p(Q-p+\epsilon)}} \|Xu\|_{L^p(B_i^*)} \quad \text{(by Lemma 4.8)} \\
&\le \left(\frac{\operatorname{diam}(\Omega)}{|\Omega|} \right)^{\frac{\epsilon Q}{p(Q-p+\epsilon)}} \sum_{i=0}^{s(B)-1} \left(\frac{\mu(B)}{\mu(B_i)} \right)^{\frac{1}{q}} \|Xu\|_{L^p(B_i^*)} .
\end{aligned} \tag{4.18}$$

Now for $B \in \mathcal{F}$ fixed, let $\mathcal{F}_B = \{\tilde{B} \in \mathcal{F} \mid \nu\tilde{B} \supseteq B\}$. Note that every B_j in the chain connecting B_0 to B is in $\mathcal{F}_B$. For any $B \in \mathcal{F}$ set

$$a_B = \frac{1}{\mu(B)^{\frac{1}{q}}} \|Xu\|_{L^p(B^*)} ,$$

then

$$\begin{aligned}
\sum_{i=0}^{s(B)-1} \left(\frac{\mu(B)}{\mu(B_i)}\right)^{\frac{1}{q}} \|Xu\|_{L^p(B_i^*)} &\le \sum_{\tilde{B}\in\mathcal{F}_B} \left(\frac{\mu(B)}{\mu(\tilde{B})}\right)^{\frac{1}{q}} \|Xu\|_{L^p(B_i^*)} \\
&= \mu(B)^{\frac{1}{q}} \sum_{\tilde{B}\in\mathcal{F}_B} a_{\tilde{B}}\,.
\end{aligned} \tag{4.19}$$

Substitute (4.19) in (4.18) and subsequently (4.18) in (4.17), raising to the q-th power. Using the trivial observation that for every $x \in B$,

$$\sum_{\tilde{B}\in\mathcal{F}_B} a_{\tilde{B}} \le \sum_{\tilde{B}\in\mathcal{F}} a_{\tilde{B}} \chi_{\nu\tilde{B}}(x)$$

we have

$$\begin{aligned}
\int_B |u_{B,\mu} - u_{B_0,\mu}|^q \, d\mu &\le C \left(\frac{\operatorname{diam}(\Omega)}{|\Omega|}\right)^{\frac{\epsilon Q q}{p(Q-p+\epsilon)}} \int_B \left(\sum_{\tilde{B}\in\mathcal{F}_B} a_{\tilde{B}}\right)^q d\mu \\
&\le C \left(\frac{\operatorname{diam}(\Omega)}{|\Omega|}\right)^{\frac{\epsilon Q q}{p(Q-p+\epsilon)}} \int_B \left(\sum_{\tilde{B}\in\mathcal{F}} a_{\tilde{B}} \chi_{\nu\tilde{B}}(x)\right)^q d\mu\,.
\end{aligned}$$

Summing over $B \in \mathcal{F}$ one obtains

$$\begin{aligned}
II = \sum_{B\in\mathcal{F}} \int_B |u_{B,\mu} - u_{B_0,\mu}|^q \, d\mu &\le C \left(\frac{\operatorname{diam}(\Omega)}{|\Omega|}\right)^{\frac{\epsilon Q q}{p(Q-p+\epsilon)}} \\
&\quad \times \int_\Omega \sum_{B\in\mathcal{F}} \chi_B(x) \left(\sum_{\tilde{B}\in\mathcal{F}} a_{\tilde{B}} \chi_{\nu\tilde{B}}(x)\right)^q d\mu \\
\text{(by (i) in Definition1.12)} \quad &\le CN \left(\frac{\operatorname{diam}(\Omega)}{|\Omega|}\right)^{\frac{\epsilon Q q}{p(Q-p+\epsilon)}} \\
&\quad \times \int_\Omega \left(\sum_{\tilde{B}\in\mathcal{F}} a_{\tilde{B}} \chi_{\nu\tilde{B}}(x)\right)^q d\mu
\end{aligned}$$

$$
\begin{aligned}
\text{(by Lemma 4 in [94, p.115]} \quad &\leq CD\left(\frac{\operatorname{diam}(\Omega)}{|\Omega|}\right)^{\frac{\epsilon Qq}{p(Q-p+\epsilon)}} \\
&\quad \times \int_\Omega \left(\sum_{\tilde{B}\in\mathcal{F}} a_{\tilde{B}}\chi_{\tilde{B}}(x)\right)^q d\mu \\
&\leq CN^{q-1}\left(\frac{\operatorname{diam}(\Omega)}{|\Omega|}\right)^{\frac{\epsilon Qq}{p(Q-p+\epsilon)}} \\
&\quad \times \int_\Omega \sum_{\tilde{B}\in\mathcal{F}} a_{\tilde{B}}^q \chi_{\tilde{B}}(x)\, d\mu \\
&= C\left(\frac{\operatorname{diam}(\Omega)}{|\Omega|}\right)^{\frac{\epsilon Qq}{p(Q-p+\epsilon)}} \sum_{\tilde{B}\in\mathcal{F}} a_{\tilde{B}}^q \mu(\tilde{B}) \\
&\leq C\left(\frac{\operatorname{diam}(\Omega)}{|\Omega|}\right)^{\frac{\epsilon Qq}{p(Q-p+\epsilon)}} \left(\sum_{\tilde{B}\in\mathcal{F}} \|Xu\|^p_{L^p(\tilde{B}^*)}\right)^{\frac{q}{p}} \\
&= C\left(\frac{\operatorname{diam}(\Omega)}{|\Omega|}\right)^{\frac{\epsilon Qq}{p(Q-p+\epsilon)}} \left(\int_\Omega \sum_{\tilde{B}\in\mathcal{F}} \chi_{\tilde{B}^*}(x)|Xu|dx\right)^q \\
\text{(by (i) in Definition 1.16)} \quad &\leq CN^q\left(\frac{\operatorname{diam}(\Omega)}{|\Omega|}\right)^{\frac{\epsilon Qq}{p(Q-p+\epsilon)}} \left(\int_\Omega |Xu|^p\, dx\right)^{\frac{q}{p}}.
\end{aligned}
$$

In the above, we have used the fact that the sum $\sum_{\tilde{B}\in\mathcal{F}} a_{\tilde{B}}\chi_{\tilde{B}}(x)$ has at most N terms that are not zero for each $x \in \Omega$ and $\frac{q}{p} > 1$. By the estimate of I and II we infer

$$
\left(\int_\Omega |u - u_{B_0,\mu}|^q\, d\mu\right)^{\frac{1}{q}} \leq C\left(\frac{\operatorname{diam}(\Omega)}{|\Omega|}\right)^{\frac{\epsilon Q}{p(Q-p+\epsilon)}} \left(\int_\Omega |Xu|^p\, dx\right)^{\frac{1}{p}}.
$$

Finally, Hölder's inequality gives

$$
\left(\int_\Omega |u - u_{\Omega,\mu}|^q\, dx\right)^{\frac{1}{q}} \leq C\left(\frac{\operatorname{diam}(\Omega)}{|\Omega|}\right)^{\frac{\epsilon Q}{p(Q-p+\epsilon)}} \left(\int_\Omega |Xu|^p\, dx\right)^{\frac{1}{p}}. \qquad \square
$$

5. – A geometric application of trace inequalities

The purpose of this section is to present some interesting and useful applications of Theorems 1.4,1.9 to manifolds of codimension one in the Heisenberg group $\mathbb{H}^n$. We begin with a known result from Euclidean analysis which serves as motivation.

Consider $\mathbb{R}^n$ with the standard basis $X_j = \frac{\partial}{\partial x_j}$, $1 \le j \le n$. In this case one proves $d = d_e = |\cdot|$ and $B_e = B_e(x,r) = \{y \in \mathbb{R}^n \mid |x-y| < r\}$. The space $\mathcal{L}^{1,1}(B_e)$ is then the ordinary Sobolev space $W^{1,1}(B_e)$. Take $\mathcal{M}$ in Theorem 1.4 to be a Lipschitz $(n-1)$-dimensional manifold. Since our result is local in nature, we can assume that $\mathcal{M}$ be bounded. Define an outer measure μ on $\mathcal{M}$ by letting

$$\mu(E) = \mathcal{H}^{n-1}(E \cap \mathcal{M})$$

where $\mathcal{H}^{n-1}$ is the $n-1$ dimensional Hausdorff measure in $\mathbb{R}^n$ with respect to the metric d_e. It is well-known that for any $x \in \mathcal{M}$, $\mu(B(x,r)) \cong r^{n-1} \cong \frac{|B(x,r)|}{r}$. Hence, Theorem 1.4 applies in this case, yielding the conclusion that every $u \in W^{1,1}(B_e)$ has a trace in $L^1(\mathcal{M} \cap B_e)$. If, instead, we apply Theorem 1.9, we infer that membership of u in $W^{1,p}(B_e)$ implies existence of a trace in $L^q(\mathcal{M} \cap B_e)$, with $q = p\frac{n-1}{n-p}$. In this context, these results are included in a theorem of Gagliardo [51], see also [91].

If we abandon the Euclidean setting and turn the attention to the simplest non-abelian example of a Carnot group, namely the Heisenberg group $\mathbb{H}^n$, the situation becomes suddenly more intricated due to the presence of characteristic points. To clarify this comment we consider a smooth compact manifold $\mathcal{M} \subset \mathbb{H}^n$, and ask the question of when $\mathcal{M}$ supports a Borel measure $d\mu$ which fulfills the growth assumptions in Theorems 1.4 and 1.9. The answer to this question is somewhat delicate, as we will show in what follows. Let us recall that $\mathbb{H}^n$ is the Lie group whose underlying manifold is $\mathbb{R}^{2n} \times \mathbb{R}$ with group law

$$(x,y,t)(\xi,\eta,\tau) = \big(x+\xi,\, y+\eta,\, t+\tau+2(\langle \xi, y\rangle - \langle x, \eta\rangle)\big),$$
$$x, y, \xi, \eta \in \mathbb{R}^n,\ t, \tau \in \mathbb{R},$$

where $\langle\, ,\rangle$ denotes the ordinary inner product in $\mathbb{R}^n$. The corresponding Lie algebra of left-invariant vector fields is generated by the system $X = \{X_1, \dots, X_{2n}\}$, where

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \quad X_{n+j} = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}, \quad j = 1, \dots, n.$$

A basis for such algebra is given by $\{X_1, \dots, X_{2n}, T\}$, where $T = -4\frac{\partial}{\partial t}$. In the sequel we will need the exponential mapping induced by such basis, see the Appendix in [83]. For each $q_0 = (x_0, y_0, t_0)$, $q = (x,y,t) \in \mathbb{H}^n$ a simple calculation gives

$$\mathrm{Exp}_{q_0}(x,y,t) = (x+x_0,\, y+y_0,\, 2(\langle x, y_0\rangle - \langle y, x_0\rangle) - 4t + t_0). \tag{5.1}$$

We assume that the manifold $\mathcal{M}$ has codimension one in $\mathbb{H}^n$, i.e., $\dim(\mathcal{M}) = 2n$, and that it be given by

$$\mathcal{M} = \{q \in \mathbb{H}^n \mid \rho(q) = 0\},$$

where $\rho \in C^\infty(\mathbb{H}^n)$. Consider the angle function

$$w(q) = \left(\sum_{j=1}^{2n} (X_j \rho)^2 (q) \right)^{\frac{1}{2}} = |X\rho(q)| .$$

The characteristic set of the system X on $\mathcal{M}$ is given by $\Sigma = \{q \in \mathcal{M} \mid w(q) = 0\}$.

Returning to the question above, a natural candidate for the measure $d\mu$ on $\mathcal{M}$ would clearly be surface measure $d\sigma$, but the latter does not scale correctly across the characteristic set Σ. Instead, for a set $E \subset \mathbb{H}^n$, we define

$$\mu(E) = \int_{E \cap \mathcal{M}} w(q) d\sigma(q) .$$

The measure $d\mu$ is supported on $\mathcal{M}$. In order to see whether it is an appropriate candidate for Theorems 1.4, 1.9, we need to study how it charges the Carnot-Carathéodory balls. It is known that the Carnot-Carathéodory metric d associated to X is equivalent to the metric $d_H(p, q) = N(p^{-1}q)$, where

$$N(x, y, t) = \left[(|x|^2 + |y|^2)^2 + t^2 \right]^{\frac{1}{4}}$$

is the natural gauge function on $\mathbb{H}^n$. It thus suffices to consider balls with respect to d_H, $B = B(q, r) = \{p \in \mathbb{H}^n \mid d_H(p, q) < r\}$. The homogeneous dimension attached to the anisotropic dilations $\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$ is $Q = 2n + 2$. One easily sees that $|B(x, r)| \cong r^Q$. The following result holds.

THEOREM 5.1. *There exist positive constants M, R_0, depending on $\mathcal{M}$, such that for every $q \in \mathcal{M}$, and any $0 < r < R_0$, one has*

$$\mu\big(B(q, r)\big) \le M \frac{|B(q, r)|}{r} .$$

We mention that Theorem 5.1 is part of a result, due to Mekias [82], which states that the reverse inequality also holds. Since, the proof in [82] is not available in print and also does not contain full details, given the relevance of the result we provide a detailed proof below.

Taking Theorem 5.1 for granted for a moment, we can use it to implement Theorems 1.4 and 1.9 and obtain trace inequalities on manifolds of codimension one in $\mathbb{H}^n$. If $U \in \mathbb{H}^n$ is an open set such that $\mathcal{M} \subset U$, we thus reach the conclusion that functions in the Folland-Stein Sobolev spaces $\mathcal{L}^{1,1}(U)$, or $\mathcal{L}^{1,p}(U)$, $1 < p < Q$, admit a trace respectively in $L^1(\mathcal{M}, d\mu)$, or in $L^q(\mathcal{M}, d\mu)$, with $q = p\frac{Q-1}{Q-p}$. Moreover, specialized to the present context, (1.6) and (1.11) provide a priori control of the relevant norm of the trace. For instance, from (1.11) we obtain for every $u \in \mathcal{L}^{1,p}(B^*)$

$$\left(\int_B |u - u_{B,\mu}|^q \, d\mu \right)^{\frac{1}{q}} \le C M^{\frac{1}{q}} \left(\int_{B^*} |Xu|^p \, dx \right)^{\frac{1}{p}} ,$$

where M is the constant in Theorem 5.1. The discussion will be complete once we prove Theorem 5.1, to which task we turn next.

Let ρ be the defining function of $\mathcal{M}$. We introduce the quantity

$$\eta(q,r) = \frac{\max\left(w(q), 4|\frac{\partial\rho}{\partial t}(q)|r\right)}{4\|\nabla\rho(q)\|}. \tag{5.2}$$

We start with a preliminary result.

Lemma 5.2. *Suppose that the defining function ρ be at least of class C^2. There exist constants $R_1, C_1, C_2 > 0$ such that for all $q_0 \in \mathcal{M}$ and $r \le R_1$ the following estimate holds*

$$C_1 r^{2n+1}/\eta(q_0,r) \le \sigma(\mathcal{M} \cap B(q_0,r)) \le C_2 r^{2n+1}/\eta(q_0,r). \tag{5.3}$$

Proof. To simplify the notation in the sequel we focus on the three-dimensional Heisenberg group $\mathbb{H} = \mathbb{H}^1$. From the analysis of the latter the general case follows with trivial changes. First, we approximate $\sigma(\mathcal{M}\cap B(q_0,r))$ to first order with $\sigma(T_{q_0}\mathcal{M}\cap B(q_0,r))$. To compute $\sigma(T_{q_0}\mathcal{M}\cap B(q_0,r))$ we use the exponential mapping based at q_0 given in (5.1) and the following famous "Ball-Box" theorem of [83] (see also [59]): *There are strictly positive continuous functions $\overline{C} = \overline{C}(q_0)$ and $R' = R'(q_0)$ such that*

$$\mathrm{Exp}_{q_0}\big(\mathrm{Box}(\overline{C}^{-1}r) \subset B(q_0,r) \subset \mathrm{Exp}_{q_0}(\mathrm{Box}(\overline{C}r)\big)$$

for all $q_0 \in \mathbb{H}^n$ and $r \le R'$ where

$$\mathrm{Box}(r) = \left\{(x,y,t) \mid |x|, |y| \le r, |t| \le r^2\right\}.$$

Consider the tangent plane $T_{q_0}\mathcal{M}$

$$T_{q_0}\mathcal{M} = \left\{(x,y,t)\Big|\frac{\partial\rho}{\partial x}(q_0)(x-x_0) + \frac{\partial\rho}{\partial y}(q_0)(y-y_0) + \frac{\partial\rho}{\partial t}(q_0)(t-t_0) = 0\right\}.$$

We let α, β, γ respectively denote the numbers $\frac{\partial\rho}{\partial x}(q_0)$, $\frac{\partial\rho}{\partial y}(q_0)$, $\frac{\partial\rho}{\partial t}(q_0)$. Choose $A = \alpha + 2y_0\gamma$, $B = \beta - 2x_0\gamma$, $C = -4\gamma$, then from (5.1) one finds that the plane containing $e = (0,0,0)$

$$\mathcal{H} = \left\{(x,y,t) \mid Ax + By + Ct = 0\right\},$$

has the property

$$\mathrm{Exp}_{q_0}(\mathcal{H}) = T_{q_0}\mathcal{M}. \tag{5.4}$$

In view of the fact that Exp_{q_0} is a one to one mapping, we have

$$\mathrm{Exp}_{q_0}\big(\mathcal{H}\cap \mathrm{Box}(\overline{C}^{-1}r)\big) \subset T_{q_0}\mathcal{M}\cap B(q_0,r) \subset \mathrm{Exp}_{q_0}\big(\mathcal{H}\cap\mathrm{Box}(\overline{C}r)\big).$$

Thus, it suffices to consider $\sigma(\mathrm{Exp}_{q_0}(\mathcal{H}\cap \mathrm{Box}(r))$ and make the appropriate adjustment at the end to accomodate the constant $\overline{C}$. We now proceed to determine a coordinate patch for $\mathrm{Exp}_{q_0}(\mathcal{H}\cap \mathrm{Box}(r))$. Consider the cylinder

$$Cyl(r)=\left\{(x,y,t)\,|\,x^2+y^2\le r^2, t\le r^2\right\}.$$

One sees that the quantities $\sigma(\mathcal{H}\cap \mathrm{Box}(r))$ and $\sigma(\mathcal{H}\cap Cyl(r))$ are comparable. Since $\sigma(\mathcal{H}\cap Cyl(r))$ is invariant under rotation around the t-axis, we make a rotation to map the vector (A,B,C) (the normal vector to $\mathcal{H}$) to the vector $(\sqrt{A^2+B^2},0,C)$. We still use $\mathcal{H}$ to denote the rotated plane. It is now easy to see that the range of x for which the (rotated) plane $\mathcal{H}$ lies in $\mathrm{Box}(r)$ is given by

$$\begin{cases} |x|<\frac{|C|}{\sqrt{A^2+B^2}}r^2 & \text{if } r<\frac{\sqrt{A^2+B^2}}{|C|},\\ |x|<r & \text{if } r\ge \frac{\sqrt{A^2+B^2}}{|C|}.\end{cases}$$

One can simply rewrite this condition as $|x|<r^2/\max(\frac{\sqrt{A^2+B^2}}{|C|},r)$. We distinguish two cases. If $C=-4\gamma=-4\frac{\partial\rho}{\partial t}(q_0)\neq 0$. We consider the rectangle

$$\Omega=\left\{(x,y)\ \middle|\ |x|<\frac{r^2}{\max\left(\frac{\sqrt{A^2+B^2}}{|C|},r\right)},\ |y|<r\right\},$$

and the function $\phi:\mathbb{R}^2\to\mathcal{H}$ defined as follows

$$\phi(x,y)=\left(x,y,\left(-\frac{A}{C}x-\frac{B}{C}y\right)\right)=\left(x,y,\left(\frac{\alpha}{4\gamma}+\frac{y_0}{2}\right)x+\left(\frac{\beta}{4\gamma}-\frac{x_0}{2}\right)y\right).$$

(Ω,ϕ) and $(\Omega,\mathrm{Exp}_{q_0}\circ\phi)$ are coordinate patches for $\mathcal{H}\cap\mathrm{Box}(r)$ and $T_{q_0}\mathcal{M}\cap \mathrm{Exp}_{q_0}(\mathrm{Box}(r))$ respectively. Now

$$\sigma\left(\mathrm{Exp}_{q_0}(\mathcal{H}\cap\mathrm{Box}(r))\right)=\sigma\left(\mathrm{Exp}_{q_0}\circ\phi(\Omega)\right)=\int_\Omega\sqrt{1+(\alpha/\gamma)^2+(\beta/\gamma)^2}\,dx\,dy$$
$$=\frac{\sqrt{\alpha^2+\beta^2+\gamma^2}}{|\gamma|}|\Omega|=\frac{\sqrt{\alpha^2+\beta^2+\gamma^2}}{|\gamma|}\frac{r^3}{\max\left(\frac{\sqrt{A^2+B^2}}{|C|},r\right)}.$$

With the above meanings of the quantities $\alpha,\beta,\gamma,A,B,C,$ one easily recognizes that $A=X\rho(q_0)$, $B=Y\rho(q_0)$, and therefore $w(q_0)=\sqrt{A^2+B^2}$.

At this point using the definition (5.1) of the function $\eta(q_0, r)$, one finds from the above chain of equalities

$$\sigma\big(\mathrm{Exp}_{q_0}(\mathcal{H} \cap \mathrm{Box}(r))\big) = \frac{r^3}{\eta(q_0, r)} .$$

If we have $C = -4\gamma = -4\frac{\partial\rho}{\partial t}(q_0) = 0$ instead, we simply let

$$\Omega = \left\{(x, t) \mid |x| < \frac{r}{\max(A/B, 1)}, \ |t| < r^2\right\}$$

and

$$\phi(x, t) = \big(x, -(A/B)x, t\big) .$$

Again, (Ω, ϕ) and $(\Omega, \mathrm{Exp}_{q_0} \circ \phi)$ are coordinate patches for $\mathcal{H} \cap \mathrm{Box}(r)$ and $T_{q_0}\mathcal{M} \cap \mathrm{Exp}_{q_0}(\mathrm{Box}(r))$ respectively. Carrying out the same computations we see that indeed, the expression for $\sigma(\mathrm{Exp}_{q_0}(\mathcal{H} \cap \mathrm{Box}(r))$ also includes the case where $C = -4\gamma = \frac{\partial\rho}{\partial t}(q_0) = 0$.

To finish the proof of the lemma, we simply observe that $\mathcal{M}$ is compact and the constants $\overline{C}(q_0)$ in the "Ball-Box" theorem stated above depend continuously on q_0. Therefore, we can take $C_1 = \min\{\overline{C}^{-1}(q_0) | q_0 \in \mathcal{M}\} > 0$, $C_2 = \max\{\overline{C}^{-1}(q_0) \,|\, q_0 \in \mathcal{M}\} > 0$, $R'' = \min\{R'(q_0) \,|\, q_0 \in \mathcal{M}\} > 0$. Since $\mathcal{M}$ is at least C^2 and compact, we can approximate $\sigma(\mathcal{M} \cap B(q_0, r))$ by $\sigma(T_{q_0}\mathcal{M} \cap B(q_0, r))$ by requesting $r < R_1'$ for some R_1' independent of q_0 so that the error term can be absorbed into the principal term $\sigma(T_{q_0}\mathcal{M} \cap B(q_0, r))$. Letting $R_1 = \min(R'', R_1)$ we then reach the conclusion. □

REMARK 5.3. It is important to observe that if q is a characteristic point, then $\eta(q, r) = r$, and therefore

$$\sigma\big(\mathcal{M} \cap B(q, r)\big) \sim r^{2n} = \frac{|B(q, r)|}{r^2} . \tag{5.5}$$

We now prove that there exist positive constants M, R_0, depending on $\mathcal{M}$, such that for every $q \in \mathcal{M}$ and $0 < r < R_0$ we have

$$\sigma\big(B(q, r) \cap \mathcal{M}\big) \le M\frac{|B(q, r)|}{r^2} . \tag{5.6}$$

PROOF. Since $\|\nabla\rho(q)\| > 0$ for all $q \in \mathcal{M}$ and $w(q) = 0$ for $q \in \Sigma$, $\frac{\partial\rho}{\partial t}(q) \neq 0$ on Σ. By the compactness of Σ, we can find a neighborhood Σ_δ of Σ on which $|\frac{\partial\rho}{\partial t}(q)| \ge C > 0$. Hence,

$$\eta(q, r) \ge \left|\frac{\partial\rho}{\partial t}(q)\right| \ r/\|\nabla\rho(q)\| \ge Cr$$

and

$$\sigma\big(B(q,r)\cap\mathcal{M}\big)\le M\frac{|B(q,r)|}{r^2}$$

for all points $q\in\Sigma_\delta$.

Next, we set $L=\inf\{w(q)\mid q\in\mathcal{M}\setminus\Sigma_\delta\}$. Since Σ is compact and w only vanishes on Σ, we have $L>0$. Taking $R_0<L$ we obiously have

$$\eta(q,r)\ge w(q)/\|\nabla\rho(q)\|\ge CL>CR_0>Cr$$

and hence

$$\sigma\big(\mathcal{M}\cap B(q_0,r)\big)\le M\frac{|B(q,r)|}{r^2}$$

for points $q\in\mathcal{M}\setminus\Sigma_\delta$ and this completes the proof. □

We emphasize that if $p>2$ we can apply Theorem 1.9 with the choice $d\mu=d\sigma$ and conclude that a function in $\mathcal{L}^{1,p}(U)$ has a trace in $L^q(\mathcal{M},d\sigma)$ with $q=p\frac{Q-2}{Q-p}$. In the limiting case $p=2$, we are not able to directly apply Theorem 1.9 to the measure $d\sigma$ by using (5.6), since our general approach would require the stronger assumption

$$\sigma\big(B(q,r)\cap\mathcal{M}\big)\ \le\ M\frac{|B(q,r)|}{r^{2-\epsilon}},\qquad \epsilon>0,$$

which however fails at characteristic points, see (5.5). Nonetheless, interesting sharp results in this direction have been proved by Mekias [82] in some special cases. For instance, this author proved that a function $u\in\mathcal{L}^{1,2}(B)$, where B is the gauge unit ball centered at the identity $e\in\mathbb{H}^n$, admits a trace in $L^2(\mathcal{M},d\sigma)$, where $\mathcal{M}=\{(x,y,t)\in\mathbb{H}^n\mid t=0\}$.

We are ready to give the

Proof of Theorem 5.1. Again, we only consider the case $n=1$ and leave the trivial modification for the case $n>1$ to the reader. Let R_0 and C be as in (5.6). We recall from the proof of (5.6) we have

$$\eta(q_0,r)\ge Cr\quad\text{for all } q_0\in\mathcal{M}\text{ and } 0<r<R_0\,. \tag{5.7}$$

If we use first order Taylor expansion, we obtain easily ([82, Lemma 2 a])

$$w(q)\le C\big(w(q_0)+r\big) \tag{5.8}$$

for some $C>0$ and for all $q\in B(q_0,r)\cap\mathcal{M}$. Now

$$\begin{aligned}
\mu(B(q_0,r)\cap\mathcal{M}) &= \int_{B(q_0,r)\cap\mathcal{M}} w(q)\,d\sigma(q)\\
\text{(by (5.8))}\ &\le C(w(q_0)+r)\sigma(B(q_0,r)\cap\mathcal{M})\\
\text{(by (Lemma 5.2))} &\le CC_2(w(q_0)+r)\frac{r^{2n+1}}{\eta(q_0,r)} \qquad (5.9)\\
&= C\left(\frac{w(q_0)}{\eta(q_0,r)}r^{2n+1}+\frac{r}{\eta(q_0,r)}r^{2n+1}\right)\\
\text{(by (5.7))}\ &\le C\left(\frac{w(q_0)}{\eta(q_0,r)}r^{2n+1}+\frac{r}{Cr}r^{2n+1}\right).
\end{aligned}$$

To continue, we distinguish two cases.

CASE 1.
$$\max\left(w(q_0), 4\left|\frac{\partial\rho}{\partial t}(q_0)\right| r\right) = w(q_0)\,.$$

In this case, recalling (5.2) we obtain from (5.9)

$$\left(\frac{w(q_0)}{\eta(q_0,r)} r^{2n+1} + \frac{r}{Cr} r^{2n+1}\right) \le C\big(\|\nabla\rho(q_0)\| + 1\big) r^{2n+1} \le C(M)\, r^{2n+1}\,. \tag{5.10}$$

CASE 2.
$$\max\left(w(q_0), 4\left|\frac{\partial\rho}{\partial t}(q_0)\right| r\right) = 4\left|\frac{\partial\rho}{\partial t}(q_0)\right| r\,.$$

We now find easily from (5.9)

$$\begin{aligned}\left(\frac{w(q_0)}{\eta(q_0,r)} r^{2n+1} + \frac{r}{Cr} r^{2n+1}\right) &\le \left(\frac{4\left|\frac{\partial\rho}{\partial t}(q_0)\right| r}{4\left|\frac{\partial\rho}{\partial t}(q_0)\right| r}\|\nabla\rho(q_0)\| r^{2n+1} + \frac{r}{Cr} r^{2n+1}\right)\\ &\le C(M)\, r^{2n+1}\,.\end{aligned} \tag{5.11}$$

Combining (5.10) and (5.11) we reach the conclusion. □

6. – Optimal regularity for solutions of quasilinear equations

In this section, we present some interesting applications of Theorem 1.9 to the study of regularity weak solutions of a general class of quasilinear subelliptic equations whose prototypes arise in CR geometry, or in the theory of quasi-conformal mappings between Carnot groups. We consider equations of the type

$$\sum_{j=1}^{m} X_j^* A_j(x, u, Xu) = f(x, u, Xu)\,, \tag{6.1}$$

where $A = (A_1, \dots, A_m) : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$, $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ are measurable functions satisfying the following structural conditions: There exist $1 < p < \infty$, $c_1 \ge 0$ and measurable functions $f_1, f_2, f_3, g_2, g_3, h_3$ on $\mathbb{R}^n$ such that for a.e. $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $\zeta \in \mathbb{R}^m$ we have

$$\begin{cases} |A(x,u,\zeta)| \le c_1|\zeta|^{p-1} + g_2(x)|u|^{p-1} + g_3(x)\\ |f(x,u,\zeta)| \le f_1(x)|\zeta|^{p-1} + f_2(x)|u|^{p-1} + f_3(x)\\ A(x,u,\zeta)\cdot\zeta \ge |\zeta|^p - f_2(x)|u|^p - h_3(x)\,. \end{cases} \tag{6.2}$$

The basic model to keep in mind is the so called subelliptic *p-Laplacian.*

$$\sum_{j=1}^{m} X_j^*\left(|Xu|^{p-2}X_j u\right) = 0\,.$$

Given an open set $\Omega \subset \mathbb{R}^n$, a function $u \in \mathcal{L}^{1,p}_{\mathrm{loc}}(\Omega)$ is said a (weak) solution of (6.1) if for every $\phi \in \overset{\circ}{S}{}^{1,p}(\Omega)$, with compact support in U, one has

$$\sum_{j=1}^{m} \int_\Omega A_j(x,u,Xu)X_j\phi(x)\,dx = \int_\Omega f(x,u,Xu)\phi(x)\,dx\,. \tag{6.3}$$

This type of equation with the above structural conditions was first studied in the Euclidean setting by Serrin in his famous papers [89], [90]. When the system $X = \{X_1, \ldots, X_m\}$ satisfies the finite rank condition [65], then the Harnack inequality and the Hölder continuity (with respect to the Carnot-Carathéodory metric) of solutions of (6.1) were obtained in [14] under structural assumptions which are optimal in the scale of Lebesgue spaces. The local behavior of singular solutions was studied in [17]. In the same framework of [14] earlier non optimal results were obtained by Xu in [99].

The purpose of this section is twofold. On the one hand we show that the three basic assumptions (H.1)-(H.3) suffice to develop a complete regularity theory independently from any specific smoothness and geometric hypothesis on the vector fields (such as, e.g., the finite rank condition). This program is close in spirit to that first developed by Saloff-Coste for linear subelliptic parabolic equations in [88]. On the other hand, by means of Theorem 1.9 we extend the results in [88] to the present setting of a singular Carnot-Carathéodory space. A second novelty consists in the fact that the functions in the structural conditions (6.2) are allowed to belong to Morrey spaces $M_X^{p,\lambda}$ which are optimal. In the Euclidean setting several authors have used D.Adams' trace inequality to study regularity properties of second order elliptic pde's (both linear and nonlinear). For quasilinear equations we refer the reader to [87], [86], [72], [100].

We begin by introducing the relevant Morrey spaces.

DEFINITION 6.1. Given $1 \le p < \infty$, $\lambda > 0$ and a d-bounded open set Ω, a function u is said to belong to the Morrey space $M_X^{p,\lambda}(\Omega)$ associated to the distance d if $u \in L^p_{\mathrm{loc}}(\Omega)$ and

$$\|u\|_{M_X^{p,\lambda}(\Omega)} = \sup_{x\in\Omega,\,0<r<d_0} \left(\frac{r^\lambda}{|\Omega \cap B(x,r)|} \int_{\Omega\cap B(x,r)} |u|^p\,dy \right)^{\frac{1}{p}} < \infty\,,$$

where $d_0 = \min(\mathrm{diam}(\Omega), R_0)$.

We make the following hypothesis on the functions f_i, g_i, h_i in (6.2). If $1 < p < Q$, we assume for some $0 < \epsilon < p$ and $1 < l < \frac{Q}{p-\epsilon}$

$$
\begin{cases} f_1 \in M_X^{p,p-\epsilon}(U), \\ f_2, f_3, h_3 \in M_X^{l,l(p-\epsilon)}(U), \\ g_2, g_3 \in L_{\text{loc}}^{\frac{Q}{p-1}}(U), \end{cases} \tag{6.4}
$$

whereas when $p = Q$ we assume for some $0 < \epsilon < Q$ and $1 < l < \frac{Q}{Q-\epsilon}$

$$
\begin{cases} f_1 \in M_X^{Q,Q-\epsilon}(U), \\ f_2, f_3, h_3 \in M_X^{l,l(Q-\epsilon)}(U), \\ g_2, g_3 \in L_{\text{loc}}^{\frac{Q}{Q-1-\epsilon}}(U). \end{cases} \tag{6.5}
$$

We begin with a basic result about local boundedness of weak solutions.

Theorem 6.2. *Suppose* (H.1)-(H.3) *and* (6.4) *hold. Let $U \subset \mathbb{R}^n$ be a bounded open set with local homogeneous dimension Q and $u \in \mathcal{L}_{\text{loc}}^{1,p}(U)$, with $1 < p < Q$, be a weak solution of* (6.1). *Then there exist $C > 0$, $R_1 > 0$ depending on p, ϵ, the structural assumptions* (6.2) *and* (6.4), *the parameters in* (H.2), (H.3), *and the constants C, β, C_3 in Theorems* 1.9 *and* 3.3 *such that for any $B_R = B(x_0, R)$, for which $B(x_0, 4\beta R) \subset U$ and $R \le R_1$, we have*

$$
\|u\|_{L^\infty(B_R)} \le C \left[\left(\frac{1}{|B_R|} \int_{B_{2R}} |u|^p \, dy \right)^{\frac{1}{p}} + K(R) \right]
$$

where

$$
K(R) = \left(\|g_3\|_{L^{\frac{Q}{p-1}}(B_{\beta R})} + R^\epsilon \|f_3\|_{M_X^{l,l(p-\epsilon)}(B_{\beta R})} \right)^{\frac{1}{p-1}} + \left(R^\epsilon \|h_3\|_{M_X^{l,l(p-\epsilon)}(B_{\beta R})} \right)^{\frac{1}{p}}.
$$

Before giving the proof of Theorem 6.2, we would like to comment about the quantity $K(R)$. In the Euclidean setting, Theorem 6.2 was first established in the scale of L^p spaces in [89] for balls of radius $R = 1$. A rescaling argument then gives the correct powers in the constant $K(R)$. In our setting we have no dilation structure and obtaining the correct $K(R)$ involves some subtler analysis.

Proof of Theorem 6.2. Under the assumptions of Theorems 6.2 and 6.3, the reader can easily verify that (6.3) is well defined. Fix $B^* = B(x_0, 4\beta R) \subset U$ and let u be a solution to (6.1). For fixed $R < R_0$ and $K = K(R)$ we set $\bar{u} = |u| + K$, $\bar{g_2} = |g_2| + K^{1-p}|g_3|$, $\bar{f_2} = |f_2| + K^{1-p}|f_3| + K^{-p}|h_3|$. Then by (II) of Lemma 3.5 in [54] we have $|Xu| = |X\bar{u}|$, and the assumption (6.2) can be rewritten as

$$\text{(6.6)} \qquad \begin{cases} |A(x,u,\zeta)| \le c_1|\zeta|^{p-1} + \bar{g}_2|\bar{u}|^{p-1} \\ |f(x,u,\zeta)| \le f_1|\zeta|^{p-1} + \bar{f}_2|\bar{u}|^{p-1} \\ A(x,u,\zeta)\cdot\zeta \ge |\zeta|^p - \bar{f}_2|\bar{u}|^p . \end{cases}$$

Here, $\bar{f}_2, \bar{g}_2$ satisfy

$$\|\bar{g}_2\|_{L^{\frac{Q}{p-1}}(B^*)} \le \|g_2\|_{L^{\frac{Q}{p-1}}(B^*)} + 1 ,$$
$$\|\bar{f}_2\|_{M_X^{l,l(p-\epsilon)}(B^*)} \le \|f_2\|_{M_X^{l,l(p-\epsilon)}(B^*)} + 2R^{-\epsilon} .$$

We define for $t > K$ and $s \ge 1$

$$F(t) = \begin{cases} t^s & \text{if } K \le t \le h \\ sh^{s-1}t - (s-1)h^s & \text{if } h \le t . \end{cases}$$

With $\gamma = sp - p + 1$ we let

$$G(\tau) = \operatorname{sgn}(\tau)\left[F(|\tau| + K)F'(|\tau| + K)^{p-1} - s^{p-1}K^{\gamma}\right] .$$

By Lemma 4.7 it is easy to verify that $F(\bar{u}), G(u) \in \mathcal{L}^{1,p}(U)$ and the product rule also applies. Since U is (Euclidean) bounded and $B^* \subset U$, then B^* is also a bounded set and hence Theorem 3.3 yields for every $1 \le a < b < \beta$ a function $\eta \in \mathcal{L}^{1,p}(\mathbb{R}^n)$ for every $1 \le p \le \infty$ with $\eta \equiv 1$ on $B(x_0, aR)$ and $\eta \equiv 0$ outside $B(x_0, bR)$ and $|X\eta| \le \frac{C_3}{(b-a)R}$. We then let $\phi = \eta^p G(u)$ and $v = F(\bar{u})$. Proceeding as in [14] we arrive at the following estimate

$$\text{(6.7)} \qquad \begin{aligned} \int_U |\eta Xv|^p\,dy \le c_1 p \int_U |vX\eta||\eta Xv|^{p-1}\,dy + ps^{p-1}\int_U |\bar{g}_2||vX\eta||\eta v|^{p-1}\,dy \\ + \int_U |f_1||\eta v||\eta Xv|^{p-1}\,dy + (1+s)s^{p-1}\int_U |\bar{f}_2||\eta v|^p\,dy . \end{aligned}$$

To control the terms in the right hand side of (6.7) one argues similarly to [14], except for the third and forth integrals which are now estimated by means of Theorem 1.9. We thus omit pointless details and concentrate on the main differences. Fix t so that $max(1, l\frac{p-\epsilon}{p}) < t < l$ and let $\sigma = p(1-t/l) > 0$. Hölder's inequality yields

$$\text{(6.8)} \qquad \int_U |\bar{f}_2||\eta v|^p\,dy \le \left(\int_{B_{\beta R}} |\eta v|^p\,dy\right)^{\frac{\sigma}{p}} \left(\int_{B_{\beta R}} |\bar{f}_2|^{\frac{p}{p-\sigma}}|\eta v|^p\,dy\right)^{\frac{p-\sigma}{p}} .$$

We let $d\mu = |\bar{f}_2|^{\frac{p}{p-\sigma}} dy$ and proceed to verify that μ satisfy the assumption of Theorem 1.9. Since supp $\eta \subset B_{\beta R}$ we can assume that $\bar{f}_2 \equiv 0$ outside $B_{\beta R}$. Fix $x \in U$, $0 < \rho < d_0 = \min(\mathrm{diam}(U), R_0)$. If $x \in B(x_0, 2\beta R)$ then

$$\begin{aligned}
\mu(B(x,\rho)) &= \int_{B(x,\rho)\cap B(x_0,\beta R)} |\bar{f}_2|^{\frac{p}{p-\sigma}} dy \le \left(\int_{B(x,\rho)\cap B(x_0,\beta R)} |\bar{f}_2|^l dy\right)^{\frac{p}{l(p-\sigma)}} \\
&\quad \times |B(x,\rho)|^{1-\frac{p}{l(p-\sigma)}} \\
&= \left(\frac{\rho^{l(p-\epsilon)}}{|B(x,\rho)} \int_{B(x,\rho)\cap B(x_0,\beta R)} |\bar{f}_2|^l dy\right)^{\frac{p}{l(p-\sigma)}} \frac{|B(x,\rho)|}{\rho^{\frac{p(p-\epsilon)}{p-\sigma}}} \\
&\le \|\bar{f}_2\|^{\frac{p}{p-\sigma}}_{M_X^{l,l(p-\epsilon)}(B_{\beta R})} \frac{|B(x,\rho)|}{\rho^{p-\alpha}},
\end{aligned}$$

where $\alpha = \frac{p(\epsilon-\sigma)}{p-\sigma} > 0$ by our choice of t. If $x \notin B(x_0, 2\beta R)$ then either $B(x,\rho) \cap B(x_0, \beta R) = \emptyset$, in this case one has trivially $\mu(B(x,\rho)) = 0 < M|B(x,\rho)|/\rho^{p-\alpha}$, or else there exists $z \in B(x,\rho) \cap B(x_0,\beta R)$, and also it must be $\rho > \beta R$. Thus, $B(x,\rho) \cap B(x_0,\beta R) \subset B(z, 2\rho) \subset B(x, 4\rho)$ and we find

$$\begin{aligned}
\mu(B(x,\rho)) &= \int_{B(x,\rho)\cap B(x_0,\beta R)} |\bar{f}_2|^{\frac{p}{p-\sigma}} dy \le \int_{B(z,2\rho)} |\bar{f}_2|^{\frac{p}{p-\sigma}} dy \\
&\le \left(\frac{\rho^{l(p-\epsilon)}}{|B(x,\rho)} \int_{B(z,2\rho)\cap B(x_0,\beta R)} |\bar{f}_2|^l dy\right)^{\frac{p}{l(p-\sigma)}} \frac{|B(z,2\rho)|}{\rho^{\frac{p(p-\epsilon)}{p-\sigma}}} \\
&\le C\|\bar{f}_2\|^{\frac{p}{p-\sigma}}_{M_X^{l,l(p-\epsilon)}(B_{\beta R})} \frac{|B(x,4\rho)|}{\rho^{p-\alpha}}, \\
\text{(by (H.2))} \quad &\le C\|\bar{f}_2\|^{\frac{p}{p-\sigma}}_{M_X^{l,l(p-\epsilon)}(B_{\beta R})} \frac{|B(x,\rho)|}{\rho^{p-\alpha}}.
\end{aligned}$$

In all cases we conclude

$$\mu\big(B(x,\rho)\big) \le \frac{M|B(x,\rho)|}{\rho^{p-\alpha}},$$

for all $x \in U$ and $0 < \rho \le R_0$ with $M = M(R) = C\|\bar{f}_2\|^{\frac{p}{p-\sigma}}_{M_X^{l,l(p-\epsilon)}(B_{\beta R})}$. An application of Theorem 1.9 gives

$$
\begin{aligned}
\int_{B_{\beta R}} |\bar{f}_2|^{\frac{p}{p-\sigma}} |\eta v|^p \, dy &= \int_{B_{\beta R}} |\eta v|^p \, d\mu \le \mu(B_{\beta R})^{1-p/q} \left(\int_{B_{\beta R}} |\eta v|^q \, d\mu \right)^{\frac{p}{q}} \\
&\le \mu(B_{\beta R})^{1-p/q} C^p M^{\frac{p}{q}} \left(\frac{\beta R}{|B_{\beta R}|^{\frac{1}{Q}}} \right)^{\frac{\alpha Q}{Q-p+\alpha}} \int_{B_{\beta R}} |X(\eta v)|^p \, dy \\
&\le C M R^{\frac{\alpha Q}{Q-p+\alpha} - (p-\alpha)(1-p/q)} |B_{\beta R}|^{1-p/q - \frac{\alpha}{Q-p+\alpha}} \qquad (6.9) \\
&\quad \times \left(\|\eta X v\|^p_{L^p(B_{\beta R})} + \|v X \eta\|^p_{L^p(B_{\beta R})} \right) \\
&= C M R^{\alpha} \left(\|\eta X v\|^p_{L^p(B_{\beta R})} + \|v X \eta\|^p_{L^p(B_{\beta R})} \right).
\end{aligned}
$$

In (6.9) we have used the fact that $q = p \frac{Q-p+\alpha}{Q-p}$, hence $1 - p/q - \frac{\alpha}{Q-p+\alpha} = 0$ and $\frac{\alpha Q}{Q-p+\alpha} - (p-\alpha)(1-p/q) = \alpha$. Using (6.9) in (6.8) we obtain

$$
\begin{aligned}
\int_U |\bar{f}_2| |\eta v|^p \, dy &\le \|\bar{f}_2\|_{M_X^{l,l(p-\epsilon)}(B_{\beta R})} R^{\epsilon - \sigma} \|\eta v\|^{\sigma}_{L^p(B_{\beta R})} \\
&\quad \times \left(\|\eta X v\|^{p-\sigma}_{L^p(B_{\beta R})} + \|v X \eta\|^{p-\sigma}_{L^p(B_{\beta R})} \right).
\end{aligned} \tag{6.10}
$$

Next, we estimate the term containing f_1 in (6.7) using analogous considerations. After applying Hölder inequality, we set $d\mu = |f_1|^p \, dy$ and verify easily that

$$
\mu(B(x,\rho)) \le C \|f_1\|^p_{M_X^{p,p-\epsilon}(B_{\beta R})} |B(x,\rho)| / \rho^{p-\epsilon}.
$$

By Theorem 1.9, having simplified the relevant powers, we obtain

$$
\begin{aligned}
&\int_U |f_1| |\eta v| |\eta X v|^{p-1} \, dy \\
&\le C_p \|f_1\|_{M_X^{p,p-\epsilon}(B_{\beta R})} R^{\frac{\epsilon}{p}} \left(\|\eta X v\|^p_{L^p(B_{\beta R})} + \|\eta X v\|^{p-1}_{L^p(B_{\beta R})} \|v X \eta\|_{L^p(B_{\beta R})} \right).
\end{aligned} \tag{6.11}
$$

We now remark that, contrarily to the case when the coefficients in (6.2) are assumed to belong to the appropriate L^p spaces, the power of $\|\eta X v\|_{L^p(B_{\beta R})}$ in the right hand side of (6.11) is p rather than $1-\epsilon$. Therefore, we must choose R_1 such that for all $R \le R_1$ we have

$$
C_p \|f_1\|_{M_X^{p,p-\epsilon}(B_{\beta R})} R^{\frac{\epsilon}{p}} < 1.
$$

In this way the term

$$
C_p \|f_1\|_{M_X^{p,p-\epsilon}(B_{\beta R})} R^{\frac{\epsilon}{p}} \|\eta X v\|^p_{L^p(B_{\beta R})}
$$

can be absorbed in the left hand side of (6.7). These considerations allow to conclude with $p^* = \frac{pQ}{Q-p}$

$$
\begin{aligned}
\|\eta v\|_{L^{p^*}(B_{\beta R})} \le{}& C s^{\frac{p}{\sigma}} \frac{R}{|B|^{\frac{1}{Q}}} \|v X\eta\|_{L^p(B_{\beta R})} \\
&+ s^{\frac{p}{\sigma}} \left(\frac{R}{|B|^{\frac{1}{Q}}} \right) R^{\frac{\epsilon-\sigma}{\sigma}} \|\bar{f}_2\|^{\frac{1}{\sigma}}_{M_X^{l,l(p-\epsilon)}(B_{\beta R})} \|\eta v\|_{L^p(B_{\beta R})} \\
&+ s^{\frac{p}{\sigma}} \left(\frac{R}{|B|^{\frac{1}{Q}}} \right) R^{\frac{\epsilon-\sigma}{p}} \|\bar{f}_2\|^{\frac{1}{p}}_{M_X^{l,l(p-\epsilon)}(B_{\beta R})} \|\eta v\|^{\frac{\sigma}{p}}_{L^p(B_{\beta R})} \\
&\times \|v X\eta\|^{1-\sigma/p}_{L^p(B_{\beta R})} .
\end{aligned}
\tag{6.12}
$$

We thus come to another point where the analysis departs from that relative to L^p integrability assumptions. We cannot simply use Young's inequality to bound from above the product $\|\eta v\|^{\frac{\sigma}{p}}_{L^p(B_{\beta R})} \|v X\eta\|^{1-\sigma/p}_{L^p(B_{\beta R})}$ in the last term of (6.12) by the sum $\|\eta v\|_{L^p(B_{\beta R})} + \|v X\eta\|_{L^p(B_{\beta R})}$. If we did so, then in view of the estimate of $\|\bar{f}_2\|_{M_X^{l,l(p-\epsilon)}(B^*}$ after (6.6), the term

$$
\left(\frac{R}{|B_R|^{\frac{1}{Q}}} \right) R^{\frac{\epsilon-\sigma}{p}} \|\bar{f}_2\|^{\frac{1}{p}}_{M_X^{l,l(p-\epsilon)}(B_{\beta R})} \|v X\eta\|_{L^p(B_{\beta R})}
$$

would possibly tend to ∞ as $R \to 0$. Instead, we use the estimate $|X\eta| \le \frac{C_3}{(b-a)R}$ in (6.12),coupled with the observation that

$$
\begin{aligned}
R^{\frac{\epsilon}{\sigma}} \|\bar{f}_2\|^{\frac{1}{\sigma}}_{M_X^{l,l(p-\epsilon)}(B_{\beta R})} &\le C R^{\frac{\epsilon}{\sigma}} (\|f_2\|^{\frac{1}{\sigma}}_{M_X^{l,l(p-\epsilon)}(B_{\beta R})} + 2R^{-\epsilon/\sigma}) \\
&\le C (R^{\frac{\epsilon}{\sigma}} \|f_2\|^{\frac{1}{\sigma}}_{M_X^{l,l(p-\epsilon)}(B_{\beta R})} + 2) \\
R^{\frac{\epsilon}{p}} \|\bar{f}_2\|^{\frac{1}{p}}_{M_X^{l,l(p-\epsilon)}(B_{\beta R})} &\le C R^{\frac{\epsilon}{p}} (\|f_2\|^{\frac{1}{p}}_{M_X^{l,l(p-\epsilon)}(B_{\beta R})} + 2R^{-\epsilon/p}) \\
&\le C (R^{\frac{\epsilon}{p}} \|f_2\|^{\frac{1}{p}}_{M_X^{l,l(p-\epsilon)}(B_{\beta R})} + 2) .
\end{aligned}
\tag{6.13}
$$

This leads to the estimate

$$
\|v\|_{L^{p^*}(B_{aR})} \le \frac{C s^{\frac{p}{\sigma}}}{(b-a)|B|^{\frac{1}{Q}}} \|v\|_{L^p(B_{bR})} .
$$

Finally, letting $h \to \infty$ in the definition of F and implementing the Moser iteration scheme (see [14], [89]) we reach the conclusion of the theorem. We mention in closing that (H.2) has been repeatedly used in the above computations without explicit reference.

THEOREM 6.3. *In addition to the hypothesis of Theorem 6.2, suppose also that U be connected. If u is a nonnegative weak solution to (6.1) in U, then there exist $C > 0$, $R_1 > 0$ depending on p, ϵ, the structural assumptions in (6.2) and (6.4), the parameters in (H.2), (H.3) and the constants C, C_3 in Theorems 1.9 and 3.3, such that for any $B_R = B(x_0, R)$, for which $B(x_0, 4\beta R) \subset U$, and $R \leq R_1$, we have*

$$\operatorname*{ess\,sup}_{B_R} u \leq C \left(\operatorname*{ess\,inf}_{B_R} u + K(R) \right) .$$

Here, $K(R)$ is as in Theorem 6.2.

The proof of Theorem 6.3 follows from the ideas already exposed in the proof of Theorem 6.2 and of the Harnack inequality in [14], thereby we omit it. We simply mention that at some point one needs to use the estimates

$$|B(x, R)| \geq \left(R_0^{-Q} \inf_{x \in \bar{U}} |B(x, R_0)| \right) R^Q$$

and

$$\inf_{x \in \bar{U}} |B(x, R_0)| = C_{R_0} > 0 ,$$

which follow from Corollary 2.10 and from (1.4). □

The above results leave out the case $p = Q$. The latter differs from that $1 < p < Q$ in the fact that one cannot use directly Theorem 1.9.

THEOREM 6.4. *Assume (H.1)-(H.3), and that $p = Q$ and (6.5) hold. Let $U \subset \mathbb{R}^n$ be a bounded open set with local homogeneous dimension Q and $u \in \mathcal{L}^{1,Q}_{\mathrm{loc}}(U)$ be a weak solution of (6.1). There exist $C > 0$, $R_1 > 0$, depending on Q, ϵ, the structural conditions (6.2) and (6.5), the parameters in (H.2), (H.3), and constants C, β, C_3 in Theorems 1.9 and 3.3, such that for any $B_R = B(x_0, R)$, for which $B(x_0, 4\beta R) \subset U$ and $R \leq R_1$, one has*

$$\|u\|_{L^\infty(B_R)} \leq C \left[\left(\frac{1}{|B_R|} \int_{B_{2R}} |u|^p \, dy \right)^{\frac{1}{p}} + \tilde{K}(R) \right] ,$$

where

$$\tilde{K}(R) = \left(|B_R|^{\frac{\epsilon}{Q}} \|g_3\|_{L^{\frac{Q}{Q-1-\epsilon}}(B_{\beta R})} + R^\epsilon \|f_3\|_{M_X^{l,l(Q-\epsilon)}(B_{\beta R})} \right)^{\frac{1}{Q-1}}$$
$$+ \left(R^\epsilon \|h_3\|_{M_X^{l,l(Q-\epsilon)}(B_{\beta R})} \right)^{\frac{1}{Q}} .$$

PROOF OF THEOREM 6.4. We note that up to (6.7) the computations are similar to those in the proof of Theorem 6.2. The only difference occurs in the estimates of the terms involving f_1. One has

$$(6.14) \quad \int_U |f_1||\eta v||\eta X v|^{Q-1} dy \leq \left(\int_{B_{\beta R}} |f_1||\eta v|^Q \, dy \right)^{\frac{1}{Q}} \left(\int_{B_{\beta R}} |\eta X v|^Q \, dy \right)^{\frac{Q-1}{Q}}$$

Letting $d\mu = |f_1|^Q dy$ we can verify as in the computations of (6.10) that

$$\mu(B(x,\rho)) \le CM \frac{|B(x,\rho)|}{\rho^{Q-\epsilon}} \tag{6.15}$$

for all $x \in U$ and $0 < \rho \le R_0$. Here $M = M(R) = \|f_1\|^Q_{M_X^{Q,Q-\epsilon}(B_{\beta R})}$. However, in order to apply Theorem 1.9 we need to rewrite (6.15) as follows. We fix γ with $0 < \gamma < \frac{\epsilon Q}{\epsilon + Q}$ and let $\tilde{p} = Q - \gamma < Q$, $\tilde{\epsilon} = \epsilon - \gamma > 0$. Then (6.15) becomes

$$\mu\big(B(x,\rho)\big) \le C\|f_1\|^Q_{M_X^{Q,Q-\epsilon}(B_{\beta R})} \frac{|B(x,\rho)|}{\rho^{\tilde{p}-\tilde{\epsilon}}}. \tag{6.16}$$

Note also that with this choice of γ and if $\tilde{q} = \frac{\tilde{p}(Q-\tilde{p}+\tilde{\epsilon})}{Q-\tilde{p}}$, then $\tilde{q} > Q$. Now with the help of Theorem 1.9 we continue as follows.

$$\left(\int_{B_{\beta R}} |\eta v|^Q |f_1|^Q dy\right)^{\frac{1}{Q}} = \left(\int_{B_{\beta R}} |\eta v|^Q \, d\mu\right)^{\frac{1}{Q}}$$

$$\le \left[\left(\int_{B_{\beta R}} |\eta v|^{\tilde{q}} \, d\mu\right)^{\frac{Q}{\tilde{q}}} \mu(B_{\beta R})^{1-\frac{Q}{\tilde{q}}}\right]^{\frac{1}{Q}}$$

$$\text{(by Theorem 1.9)} \le CM^{\frac{1}{\tilde{q}}} \left(\frac{R}{|B_R|^{\frac{1}{Q}}}\right)^{\frac{\tilde{\epsilon} Q}{\tilde{p}(Q-\tilde{p}+\tilde{\epsilon})}} \left(\int_{B_{\beta R}} |X(\eta v)|^{\tilde{p}} \, dy\right)^{\frac{1}{\tilde{p}}}$$

$$\times CM^{\frac{1}{Q}-\frac{1}{\tilde{q}}} \left(\frac{|B_R|}{R^{\tilde{p}-\tilde{\epsilon}}}\right)^{\frac{1}{Q}-\frac{1}{\tilde{q}}}$$

$$\le C\|f_1\|_{M_X^{Q,Q-\epsilon}(B_{\beta R})} R^{\frac{\epsilon}{Q}} |B_R|^{\frac{-\gamma}{Q(Q-\gamma)}} \|X(\eta v)\|_{L^Q(B_{\beta R})} |B_R|^{\frac{1}{\tilde{p}}-\frac{1}{Q}}$$

$$\le C\|f_1\|_{M_X^{Q,Q-\epsilon}(B_{\beta R})} R^{\frac{\epsilon}{Q}} \left(\|\eta X v\|_{L^Q(B_{\eta R})} + \|v X \eta\|_{L^Q(B_{\beta R})}\right).$$

From this point on, we argue as before to reach the conclusion. □

When $p = Q$ we also have the following Harnack's inequality whose proof we omit altogether.

THEOREM 6.5. *In the same hypothesis of Theorem* 6.4 *let u be a nonnegative solution to* (6.1). *Then, there exist $C > 0$, $R_1 > 0$ depending on p, ϵ, the structural conditions* (6.2) *and* (6.5), *the parameters in* (H.2), (H.3), *and the constants C, β, C_3*

in Theorems 1.9 *and* 3.3, *such that for any* $B_R = B(x_0, R)$, *for which* $B(x_0, 4\beta R) \subset U$, *and* $R \le R_1$, *one has*

$$\operatorname*{ess\,sup}_{B_R} u \le C\left(\operatorname*{ess\,inf}_{B_R} u + \tilde{K}(R)\right).$$

Here, $\tilde{K}(R)$ *is as in Theorem* 6.4

Theorems 6.3 and 6.5 imply that weak solutions are locally Hölder continuous with respect to the Carnot-Carathéodory metric $d(x, y)$.

THEOREM 6.6. *Let* $u \in \mathcal{L}^{1,p}(U)$ *with* $1 < p \le Q$ *be a weak solution to* (6.1), *with* $\operatorname{ess\,sup}_U |u| = L < \infty$. *There exist* $C > 0, \alpha > 0$ *depending on* U, L, *and all parameters involved in the structural and integrability assumptions* (6.2), (6.4), (6.5), *such that*

$$\operatorname*{ess\,sup}_{x,y\in U} \frac{|u(x) - u(y)|}{d(x,y)^\alpha} \le C.$$

The proof of Theorem 6.6 is by now classical, except for one important point. One needs to control the quantities $K(R)$, $\tilde{K}(R)$ in Theorems 6.3 and 6.5, with a power of R. This point is crucial to establish a Hölder modulus of continuity. To, achieve this goal we exploit the following important estimate established in [50].

LEMMA 6.7. *Suppose* (H.1)-(H.2) *hold. There exists* $C^* > 0$ *such that for all* $x \in U$, $0 < R \le R_0$ *and* $0 < t < 1$ *one has*

$$|B(x, tR)| \le C^* t |B(x, R)|.$$

Using Lemma 6.7 we infer for all $x \in U$ and $0 < R < R_0$

$$|B(x, R)| \le \tilde{C} R,$$

where

$$\tilde{C} = \frac{C^*}{R_0} \sup_{x\in\bar{U}} |B(x, R_0)| < \infty$$

in view of Proposition 2.9. We conclude

$$\begin{aligned}
K(R) &\le \left(|B_R|^{\frac{\epsilon}{Q}} \|g_3\|_{L^{\frac{Q}{p-1-\epsilon}}(B_{\beta R})} + R^\epsilon \|f_3\|_{M_X^{l,l(p-\epsilon)}(B_{\beta R})}\right)^{\frac{1}{p-1}} \\
&\quad + \left(R^\epsilon \|h_3\|_{M_X^{l,l(p-\epsilon)}(B_{\beta R})}\right)^{\frac{1}{p}} \\
&\le C R^\gamma \left[R^{\frac{\epsilon}{Q(p-1)} - \gamma} \left(\|g_3\|_{L^{\frac{Q}{p-1-\epsilon}}(B_{\beta R})} + R^{\epsilon - \frac{\epsilon}{Q}} \|f_3\|_{M_X^{l,l(p-\epsilon)}(B_{\beta R})}\right)^{\frac{1}{p-1}} \right. \\
&\quad \left. + R^{\frac{\epsilon}{p} - \gamma} \|h_3\|^{\frac{1}{p}}_{M_X^{l,l(p-\epsilon)}(B_{\beta R})} \right],
\end{aligned}$$

where $\gamma = \min(\frac{\epsilon}{Q(p-1)}, \frac{\epsilon}{p})$. This estimate is the sought for power like behavior needed to complete the proof of Theorem 6.6.

We close this section with the following

REMARK 6.8. The regularity results of this section properly include those in [14]. In the work [28], one of us established a Fefferman-Phong type inequality for Carnot-Carathéodory spaces, and used it to study the regularity of solutions to (6.1). It is interesting to compare the results obtained in [28] with the ones here. It is assumed in [28] that for $1 \leq p < Q$ and for some $0 < \epsilon \leq p$,

$$f_1 \in M_X^{l,l(1-\epsilon)}(U) \quad \text{with} \quad p < l < \frac{Q}{1-\epsilon},$$

$$f_2, f_3, h_3 \in M_X^{l,l(p-\epsilon)}(U) \quad \text{with} \quad 1 < l < \frac{Q}{p-\epsilon},$$

$$g_2, g_3 \in M_X^{l,l(p-1)} \quad \text{with} \quad \frac{p}{p-1} < l < \frac{Q}{p-1}.$$

One easily recognizes that the spaces to which g_2, g_3 are requested to belong to in the work [28] are larger than the ones used in this paper, while the space to which f_1 belongs to in the work [28] is smaller than the one used here. However, combining both results, we obtain the following theorems which simultaneously give the optimal results for the present paper and for [28].

THEOREM 6.9. *Let $U \subset \mathbb{R}^n$ be a bounded open set with local homogeneous dimension Q and let $u \in \mathcal{L}^{1,p}(U)$ with $1 \leq p < Q$ be a weak solution of* (6.1). *Suppose* (H.1)-(H.3) *hold and that for some $0 < \epsilon \leq p$ one has*

$$f_1 \in M_X^{p,p-\epsilon}(U),$$

$$f_2, f_3, h_3 \in M_X^{l,l(p-\epsilon)}(U) \quad \textit{with } 1 < l < \frac{Q}{p-\epsilon},$$

$$g_2, g_3 \in M_X^{l,l(p-1)} \quad \textit{with } \frac{p}{p-1} < l < \frac{Q}{p-1}.$$

There exist $C > 0$, $R_1 > 0$ (depending on p, ϵ, the structural conditions (6.2) *and the above assumptions on $f_1, f_2, f_3, g_2, g_3, h_3$, on the parameters in* (H.2), (H.3) *and on the constants C, C_3 in Theorems* 1.9 *and* 3.3) *such that, for any $B_R = B(x_0, R)$ for which $B(x_0, 4\beta R) \subset U$, and every $R \leq R_1$, we have*

$$\|u\|_{L^\infty(B_R)} \leq C\left[\left(\frac{1}{|B_R|}\int_{B_{2R}} |u|^p\, dy\right)^{\frac{1}{p}} + K(R)\right],$$

where

$$K(R) = \left(\|g_3\|_{M_X^{l,l(p-1)}(B_{\beta R})} + R^\epsilon \|f_3\|_{M_X^{l,l(p-\epsilon)}(B_{\beta R})}\right)^{\frac{1}{p-1}}$$
$$+ \left(R^\epsilon \|h_3\|_{M_X^{l,l(p-\epsilon)}(B_{\beta R})}\right)^{\frac{1}{p}}.$$

THEOREM 6.10. *In addition to the same hypothesis as in Theorem* 6.9 *we also assume that* U *be connected. If* u *is a nonnegative solution to* (6.1), *then as in Theorem* 6.9 *we can find* $C > 0$, $R_1 > 0$ *such that for any* $B_R = B(x_0, R)$ *for which* $B(x_0, 4\beta R) \subset U$ *and* $R \leq R_1$ *we have*

$$\operatorname*{ess\,sup}_{B_R} u \leq C \left(\operatorname*{ess\,inf}_{B_R} u + K(R)\right).$$

REFERENCES

[1] D. ADAMS, *Traces of potentials arising from translation invariant operators*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **25** (1971), 203-217.

[2] D. ADAMS, *A trace inequality for generalized potentials*, Studia Math. **48** (1973), 99-105.

[3] D. ADAMS, *Weighted nonlinear potential theory*, Trans. Amer. Math. Soc. **297** (1986), 73-94.

[4] D. ADAMS – L. HEDBERG, "Function Spaces and Potential Theory", Springer-Verlag, 1996.

[5] M. S. BAOUENDI, *Sur une classe d'opérateurs elliptiques dégénérés*, Bull. Soc. Math France **195** (1967), 45-87.

[6] A. BELLAÏCHE, "Sub-Riemannian Geometry", Birkhäuser, 1996.

[7] M. BIROLI – U. MOSCO, *A Saint-Venant type principle for Dirichlet forms on discontinuous media*, Ann. Mat. Pura Appl. (IV) **169** (1995), 125-181.

[8] M. BIROLI – U. MOSCO, *Sobolev and isoperimetric inequalities for Dirichlet forms on homogeneous spaces*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **6** (1995), 37-44.

[9] M. BIROLI – U. MOSCO, *Sobolev inequalities on homogeneous spaces*, Potential Anal. **4** (1995), 311-324.

[10] J. BOMAN, *L^p estimates for very strongly elliptic systems*, unpublished manuscript.

[11] S. BUCKLEY – P. KOSKELA – G. LU (eds.), *Boman equals John*, In: "Proc. of the 16^{th} Nevanlinna Coll." (Joensuu, 1995), de Gruyter, Berlin, 1996.

[12] H. BUSEMANN, "Recent Synthetic Differential Geometry", Springer-Verlag, 1970.

[13] P. BUSER, *A note on the isoperimetric constant*, Ann. Sci. École Norm. Sup. **4** (1982), 213-230.

[14] L. Capogna – D. Danielli – N. Garofalo, *An embedding theorem and the Harnack inequality for nonlinear subelliptic equations*, Comm. Partial Differential Equations **18** (1993), 1765-1794.

[15] L. Capogna – D. Danielli – N. Garofalo, *An isoperimetric inequality and the geometric Sobolev embedding for vector fields*, Comm. Anal. Geom. **2** (1994), 203-215.

[16] L. Capogna – D. Danielli – N. Garofalo, *Subelliptic mollifiers and a basic pointwise estimate of Poincaré type*, Math. Z. **226** (1997), 147-154.

[17] L. Capogna – D. Danielli – N. Garofalo, *Capacitary estimates and the local behavior of solutions of nonlinear subelliptic equations*, Amer. J. Math. **118** (1997), 1153-1196.

[18] L. Capogna – N. Garofalo, *Boundary behavior of nonegative solutions of subelliptic equations in NTA domains for Carnot-Carathéodory metrics*, J. Fourier Anal. Appl. **4** (1998), to appear.

[19] L. Capogna – N. Garofalo – D. M. Nhieu, *The Dirichlet problem for sub-Laplacians*, preprint (1997).

[20] S. Y. A. Chang – J. M. Wilson – T. H. Wolff, *Some weighted norm inequalities concerning the Schrödinger operators*, Comment. Math. Helv. **60** (1985), 217-246.

[21] S. Chanillo – R. L. Wheeden, *L^p estimates for fractional integrals and Sobolev inequalities with applications to Schrödinger operators*, Comm. Partial Differential Equations **10** (1985), 1077-1166.

[22] F. Chiarenza, *Regularity for solutions of quasilinear elliptic equations under minimal assumptions*, Potential Anal. **4** (1995), 325-334.

[23] F. Chiarenza – M. Frasca, *A remark on a paper by C. Fefferman*, Proc. Amer. Math. Soc. **108** (1990), 407-409.

[24] I. Chavel, "Eigenvalues in Riemannian Geometry", Academic Press, Orlando, 1984.

[25] S. Cohn-Vossen, *Existenz kürzester Wege*, Dokl. Akad. Nauk SSSR **3** (1935), 339-342.

[26] R. Coifman – G. Weiss, "Analyse harmonique non-commutative sur certains espaces homogenes", Springer-Verlag, 1971.

[27] D. Danielli, *Formules de représentation et théoreèmes d'inclusion pour des opérateurs sous-elliptiques*, C.R. Acad. Sci.Paris Sér. I **314** (1992), 987-990.

[28] D. Danielli, *A Fefferman-Phong type inequality and applications to quasilinear subelliptic equations*, Potential Analysis, to appear.

[29] G. David – S. Semmes, *Analysis of and on uniformly rectifiable sets*, Mathematical Surveys and Monographs n. 38, American Mathematical Society, Providence, RI 1993.

[30] G. David – S. Semmes, *Uniform rectifiability and singular sets*, Ann. Inst. H. Poincaré Anal. Non Linéaire **13** (1996), 383-443.

[31] L. C. Evans – R. F. Gariepy, "Measure Theory and Fine Properties of Functions", CRC press, 1992.

[32] H. Federer, "Geometric Measure Theory", Springer-Verlag, 1969.

[33] C. Fefferman, *The uncertainty principle*, Bull. Amer. Math. Soc. **9** (1983), 129-206.

[34] C. Fefferman – D. H. Phong (eds.), *Subelliptic eigenvalue problems*, In: "Proceedings of the Conference in Harmonic Analysis in Honor of A. Zygmund", Wadsworth Math. Ser., Belmont, CA, 1981, pp. 530-606.

[35] C. Fefferman – A. Sanchez-Calle, *Fundamental solutions for second order subelliptic operators*, Ann. of Math. **124** (1986), 247-272.

[36] C. Fefferman – E. Stein, *Some maximal inequalities*, Amer. J. Math. **93** (1971), 107-115.

[37] G. B. FOLLAND, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Mat. **13** (1975), 161-207.

[38] G. B. FOLLAND – E. M. STEIN, "Hardy Spaces on Homogeneous Groups", Princeton Univ. Press., 1982.

[39] B. FRANCHI, *Weighted Sobolev-Poincaré inequalities and pointwise estimates for a class of degenerate elliptic equations*, Trans. Amer. Math. Soc. **327** (1991), 125-158.

[40] B. FRANCHI – S. GALLOT – R. WHEEDEN, *Sobolev and isoperimetric inequalities for degenerate metrics*, Math. Ann. **300** (1994), 557-571.

[41] B. FRANCHI – C. GUTIÉRREZ – R. WHEEDEN, *Weighted Sobolev-Poincaré inequalities for Grushin type operators*, Comm. Partial Differential Equations, 19 **3-4** (1994), 523-604.

[42] B. FRANCHI – E. LANCONELLI, *Une metrique associeé à une classe d'operateurs elliptiques degénérés*, Proceedings of the meeting "Linear Partial and Pseudo Differential Operators", Rend. Sem. Mat. Torino (1984), 105-114.

[43] B. FRANCHI – E. LANCONELLI, *Hölder regularity theorem for a class of linear non uniform elliptic operators with measurable coefficients*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **10** (1983), 523-451.

[44] B. FRANCHI – E. LANCONELLI, *Une condition géométrique pour l'inégalité de Harnack*, J. Math. Pures Appl. **64** (1985), 237-256.

[45] B. FRANCHI – G. LU – R. L. WHEEDEN, *Representation formulas and weighted Poincaré inequalities for Hörmander vector fields*, Ann. Inst. Fourier (Grenoble) **45** (1995), 577-604.

[46] B. FRANCHI – G. LU – R. L. WHEEDEN, *A relationship between Poincaré type inequalities and representation formulas in spaces of homogeneous type*, Internat Math. Res. Notices **1** (1996), 1-14.

[47] B. FRANCHI – R. SERAPIONI, *Pointwise estimates for a class of strongly degenerate elliptic operators*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **14** (1987), 527-568.

[48] B. FRANCHI – R. SERAPIONI – F. SERRA CASSANO, *Meyers-Serrin type theorems and relaxation of variational integrals depending on vector fields*, preprint.

[49] B. FRANCHI – R. SERAPIONI – F. SERRA CASSANO, *Approximation and imbedding theorems for weighted Sobolev spaces associaed with Lipschitz continuous vector fields*, Boll. Un. Mat. Ital. to appear.

[50] B. FRANCHI – R. WHEEDEN, *Some remarks about Poincaré type inequalities and representation formulas in metric spaces of homogeneous type*, preprint.

[51] E. GAGLIARDO, *Caratterizzazione delle tracce sula frontiera relative ad alcune classi di funzioni in n variabili*, Rend. Sem. Mat. Univ. Padova **27** (1957), 284-305.

[52] J. GARCIA CUERVA – J. L. RUBIO DE FRANCIA, "Weighted Norm Inequalities and Related Topics", North-Holland Mat. Stud. n. 116, 1985.

[53] N. GAROFALO, "Recent Developments in the Theory of Subelliptic Equations and Its Geometric Aspects", Birkhäuser, to appear.

[54] N. GAROFALO – D. M. NHIEU, *Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces*, Comm. Pure Appl. Math. **49** (1996), 1081-1144.

[55] N. GAROFALO – D. M. NHIEU, *Lipschitz continuity, global smooth approximations and extension theorems for Sobolev functions in Carnot-Carathéodory spaces*, J. Analyse Math. **74** (1998), 67-97.

[56] M. GIAQUINTA, "Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems", Ann. of Math. Studies 105, Princeton Univ. Press, Princeton, N. J., 1983.

[57] E. GIUSTI, "Minimal Surfaces and Functions of Bounded Variation", Birkhäuser, 1984.

[58] M. GROMOV, "Structures métriques pour les variétés Riemanniennes" (rédigé par J. Lafontaine et P. Pansu), CEDIC ED., Paris, 1981.

[59] M. GROMOV, *Carnot-Carathéodory spaces seen from within*, Inst. Hautes Études Sci. Publ. Math. (1994).

[60] V. V. GRUSHIN, *On a class of hypoelliptic operators*, Math USSR-Sb., 12 **3** (1970), 458-476.

[61] P. HAJLASZ, *Sobolev spaces on an arbitrary metric space*, Potential Anal. **5** (1996), 403-415.

[62] P. HAJLASZ - P. KOSKELA, *Sobolev meets Poincaré*, C. R. Acad. Sci. Paris Sér. I **320** (1995), 1211-1215.

[63] L. HEDBERG, *On certain convolution inequalities*, Proc. Amer. Math. Soc. **36** (1972), 505-510.

[64] L. HEDBERG - T. WOLFF, *Thin sets in nonlinear potential theory*, Ann. Inst. Fourier (Grenoble) **23** (1983), 161-187.

[65] H. HÖRMANDER, *Hypoelliptic second-order differential equations*, Acta Math. **119** (1967), 147-171.

[66] D. JERISON, *The Dirichlet problem for the Kohn Laplacian on the Heisenberg group, II*, J. Funct. Anal. **43** (1981), 224-257.

[67] D. JERISON, *The Poincaré inequality for vector fields satisfying Hörmander's condition*, Duke Math. J. **53** (1986), 503-523.

[68] D. JERISON - C. E. KENIG, *Boundary behavior of harmonic functions in non-tangentially accessible domains*, Adv. Math. **46** (1982), 80-147.

[69] F. JOHN, *Rotation and strain*, Comm. Pure Appl. Math. **14** (1961), 391-413.

[70] R. KERMAN - E. T. SAWYER, *The trace inequality and eigenvalue estimates for Schrödinger operators*, Ann. Inst. Fourier (Grenoble) **36** (1986), 207-228.

[71] A. KUFNER - O. JOHN - S. FUCIK, "Function Spaces", Prague: Academia Pub. House of the Czechoslovak Academy of Sciences, 1977.

[72] G. LIEBERMAN, *Sharp form of estimates for subsolutions and supersolutions of quasilinear elliptic equations involving measures*, Comm. Partial Differential Equations **18** (1993), 1191-1212.

[73] G. LU, *Weighted Poincaré and Sobolev inequalities for vector fields satisfying Hörmander's condition and applications*, Rev. Mat. Iberoamericana **8** (1992), 367-439.

[74] G. LU, *The sharp Poincaré inequality for free vector fields: an endpoint result*, Rev. Mat. Iberoamericana **18** (1994), 453-466.

[75] G. LU, *Embedding theorems on Campanato-Morrey spaces for vector fields and applications*, C. R. Acad. Sci. Paris Sér. I **320** (1995), 429-434.

[76] G. LU, *Embedding theorems on Campanato-Morrey spaces for vector fields of Hörmander type*, Approx. Theory Appl., to appear.

[77] G. LU, *Embedding theorems into Lipschitz and BMO spaces and applications to quasilinear subelliptic differential equations*, Publ. Mat. **40** (1996), 301-329.

[78] P. Maheux – L. Saloff-Coste, *Analyse sur les boules d'un op'erateur sous-elliptique*, Math. Ann. **303** (1995), 713-740.

[79] P. Mattila, "Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability", Cambridg studies in advanced Mathematics n. 44, Cambridge University Press, 1995.

[80] V. G. Mazýa, "Sobolev Spaces", Springer-Verlag, 1985.

[81] V. G. Mazýa – I. E. Verbitsky, *Capacitary inequalities for fractional integrals, with applications to partial differential equations and Sobolev multiplier*, Ark. Mat. **33** (1995), 81-115.

[82] M. Mekias, "Restriction to Hypersurfaces of Non-isotropic Sobolev Spaces", M.I.T. Ph.D Thesis, 1993.

[83] A. Nagel – E. M. Stein – S. Wainger, *Balls and metrics defined by vector fields I: basic properties*, Acta Math. **155** (1985), 103-147.

[84] O. A. Oleinik – E. V. Radkevich, "Second Order Equations with Non-negative Characteristic Form", (Mathematical Analysis 1969), Moscow: Itogi Nauki, 1971 (Russian), English translation: Providence, R.I., Amer. Math. Soc., 1973.

[85] R. S. Phillips – L. Sarason, *Elliptic-parabolic equations of the second order*, J. Math. Mech. **17** (1967/8), 891-917.

[86] J. M. Rakotoson, *Quasilinear equations and spaces of Campanato-Morrey type*, Comm. Partial Differential Equations **16** (1991), 1155-1182.

[87] J. M. Rakotoson – W. P. Ziemer, *Local behavior of solutions of quasilinear elliptic equations with general structure*, Trans. Amer. Math. Soc. **319** (1990), 747-764.

[88] L. Saloff-Coste, *Parabolic Harnack inequality for divergence-form second-order differential operators*, Potential Anal. **4** (1995), 429-467.

[89] J. Serrin, *Local behavior of solutions of quasilinear equations*, Acta Math. **111** (1964), 243-302.

[90] J. Serrin, *Isolated singularities of solutions of quasilinear equations*, Acta Math. **113** (1965), 219-240.

[91] G. Stampacchia, *Problemi al contorno per equazioni di tipo ellittico a derivate parziali e questioni di calcolo delle variazioni connesse*, Ann. Mat. Pura Appl. (4) **33** (1952), 211-238.

[92] E. M. Stein, "Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals", Princeton Univ. Press., 1993.

[93] Robert S. Strichartz, *Sub-Riemannian geometry*, J. Differential Geom. **24** (1986), 221-263.

[94] J. Strömberg – A. Torchinsky, "Weights, Sharp Maximal Functions and Hardy Spaces", Lecture Notes in Mathematics 1381, Springer-Verlag, 1989.

[95] P. Tomter, *Consant mean curvature surfaces in the Heisenberg group*, Proc. Symp. Pure Math. **54** (1993), Part I, 485-495.

[96] N. Th. Varopoulos, *Fonctions harmoniques sure les groupes de Lie*, C.R. Acad. Sci. Paris Sér. I **304** (1987), 519-521.

[97] N. Th. Varopoulos – L. Saloff-Coste – T. Coulhon, *Analysis and geometry on groups*, Cambridge Tracts in Mathematics 100, Cambridge University press, 1992.

[98] S. K. Vodop'yanov, *Weighted L_p-potential theory on homogeneous groups*, Sibirsk., Mat. Zh. **33** (1992), 29-48.

[99] C. J. XU, *Subelliptic variational problems*, Bull. Soc. Math. France **118** (1990), 147-169.

[100] P. ZAMBONI, *Local behavior of solutions of quasilinear elliptic equations with coefficients in Morrey spaces*, Rend. Mat. Appl. **15** (1995), 251-262.

[101] W. P. ZIEMER, "Weakly Differentiable Functions", Springer-Verlag, 1989.

Department of Mathematics
Purdue University
West Lafayette, IN 47907
dxd@math.purdue.edu

Dipartimento di Metodi e Modelli Matematici
Università di Padova, Italy
and Department of Mathematics
Purdue University
West Lafayette, IN 47907
garofalo@ulam.dmsa.unipd.it
garofalo@math.purdue.edu

Institute of Mathematics
Academia Sinica
Nankang, Taipei 11529
Taiwan, R.O.C.
dmnhieu@math.sinica.edu.tw