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## Trace parameters for Teichmüller space of genus 2 surfaces and mapping class group

ABSTRACT. We obtain a representation of the mapping class group of genus 2 surface in terms of a coordinate system of the Teichmüller space defined by trace functions.

**1. Introduction.** We identify  $PSL(2, \mathbb{R})$  with the group of orientation-preserving isometries of the upper half plane  $\mathbb{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$  equipped with the hyperbolic metric  $|dz|/(\text{Im } z)$ .

A Fuchsian subgroup  $G$  of  $PSL(2, \mathbb{R})$  is said to be of type  $(2; -; -; -)$  ([5, p. 38]) if  $\mathbb{H}/G$  is a closed surface of genus 2 and the projection  $\pi : \mathbb{H} \rightarrow \mathbb{H}/G$  is an unbranched covering.  $G$  has a canonical generator system or a *marking*  $E = (A, B, C, D)$  which satisfies

$$[A, B][C, D] = 1,$$

where  $[a, b] = aba^{-1}b^{-1}$  is the commutator of  $a$  and  $b$ , and 1 stands for the unit matrix. We call the pair  $(G, E)$  a *marked Fuchsian group of type*  $(2; -; -; -)$ . Two marked Fuchsian groups  $(G_1, E_1)$  and  $(G_2, E_2)$  are *equivalent* if there exists a matrix  $P \in PSL(2, \mathbb{R})$  such that

$$A_2 = P^{-1}A_1P, \quad B_2 = P^{-1}B_1P, \quad C_2 = P^{-1}C_1P, \quad D_2 = P^{-1}D_1P,$$

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where  $E_j = (A_j, B_j, C_j, D_j)$ ,  $j = 1, 2$ . The *Teichmüller space*  $\mathcal{T}_2$  of type  $(2; -; -; -)$  is the space of all equivalence classes of marked Fuchsian groups of type  $(2; -; -; -)$ . Each marked Fuchsian group  $(G, E)$  can be represented by a tuple  $(A, B, C, D)$  of matrices in  $SL(2, \mathbb{R})$  such that

$$(1.1) \quad \operatorname{tr}A > 0, \operatorname{tr}B > 0, \operatorname{tr}C > 0 \text{ and } \operatorname{tr}D > 0.$$

Therefore, for the rest of this paper, we always assume that  $E = (A, B, C, D)$  consists of matrices satisfying (1.1). In this case  $\operatorname{tr}AB$  and  $\operatorname{tr}CD$  are both positive (this follows from [5, 33.17 (b)]). In [3] we considered the following traces as functions of  $[(G, E = (A, B, C, D))]$  in  $\mathcal{T}_2$ :

$$(1.2) \quad \begin{aligned} a &= \operatorname{tr}A, b = \operatorname{tr}B, z = \operatorname{tr}AB, u = -\operatorname{tr}ACDC^{-1}, \\ v &= -\operatorname{tr}ACD^2, w = -\operatorname{tr}ACD, t = \operatorname{tr}CD. \end{aligned}$$

Since all non trivial elements of  $G$  are hyperbolic, their traces take values in  $\mathbb{R}_{>2} = \{x : x > 2\}$ . It is shown in [3] (see also [4]) that the mapping  $\Phi : \mathcal{T}_2 \rightarrow \mathbb{R}_{>2}^7$  defined by  $\Phi([(G, E)]) = (a, b, z, u, v, w, t)$  is an embedding and  $a, b, z, u, v, w, t$  satisfy the identity

$$(1.3) \quad awt + a^2 + w^2 + t^2 + K^2 + S^2 + 4 - w\sqrt{(K^2 + 4)(S^2 + 4)} = 0,$$

where

$$K = \sqrt{abz - a^2 - b^2 - z^2} \text{ and } S = \sqrt{uvt - u^2 - v^2 - t^2}.$$

The mapping class group  $\mathcal{MC}_2$  is the group of isotopy classes of orientation-preserving homeomorphisms of the orientable closed surface  $S$  of genus 2. It is a subgroup of outer automorphisms of the fundamental group of  $S$  (see [5]).  $\mathcal{MC}_2$  acts on the Teichmüller space  $\mathcal{T}_2$  by changing the marking. The purpose of this paper is to describe a generating system of  $\mathcal{MC}_2$  by using the coordinate-system  $(a, b, z, u, v, w, t)$ . It is an interesting observation that  $\mathcal{MC}_2$  acts on  $\mathcal{T}_2$  as a group of rational transformations.

## 2. Trace identities.

**2.1. Basic trace identities.** The matrices  $A, B$  and  $C$  in  $SL(2, \mathbb{R})$  satisfy the following identities (see [2, §3.4]):

- (I1)  $\operatorname{tr}A = \operatorname{tr}A^{-1}$ ,
- (I2)  $\operatorname{tr}AB + \operatorname{tr}AB^{-1} = \operatorname{tr}A\operatorname{tr}B$ ,
- (I3)  $\operatorname{tr}ABC = \operatorname{tr}A\operatorname{tr}BC + \operatorname{tr}B\operatorname{tr}CA + \operatorname{tr}C\operatorname{tr}AB - \operatorname{tr}A\operatorname{tr}B\operatorname{tr}C - \operatorname{tr}ACB$ .

We shall use repeatedly the following identities, which are consequences of (I1), (I2) and (I3) above:

$$(2.1a) \quad \begin{aligned} \operatorname{tr}[A, B] &= \operatorname{tr}ABA^{-1}B^{-1} \\ &= (\operatorname{tr}A)^2 + (\operatorname{tr}B)^2 + (\operatorname{tr}AB)^2 - \operatorname{tr}A\operatorname{tr}B\operatorname{tr}AB - 2, \end{aligned}$$

$$(2.1b) \quad \operatorname{tr}ABCB = \operatorname{tr}AB\operatorname{tr}BC + \operatorname{tr}AC - \operatorname{tr}A\operatorname{tr}C,$$

$$(2.1c) \quad \operatorname{tr}ABCB^{-1} = \operatorname{tr}A\operatorname{tr}C - \operatorname{tr}AC - \operatorname{tr}AB\operatorname{tr}BC + \operatorname{tr}B\operatorname{tr}ABC.$$

Let  $G$  be a group generated by a finite number of matrices  $A_1, \dots, A_n \in SL(2, \mathbb{R})$  and

$$(2.2) \quad \mathcal{S} = \{\text{tr}(A_{i_1} A_{i_2} \cdots A_{i_r}) : 1 \leq i_1 < i_2 < \cdots < i_r \leq n, 1 \leq r \leq n\}.$$

Then the following fact is well known (see [2, §3.5]).

**Lemma 2.1.** *Let  $g \in G$ . Then  $\text{tr}g$  is an integer polynomial in  $\mathcal{S}$ .*

**2.2. Trace identities for genus 2 surface.** Let  $E = (A, B, C, D)$  be a marking of a Fuchsian group  $G$  of type  $(2; -; -; -)$ . Let  $c = x_1 = \text{tr}C$  and  $d = x_2 = \text{tr}D$ ,  $x_3 = \text{tr}AC$ ,  $x_4 = \text{tr}AD$ ,  $x_5 = \text{tr}BC$ ,  $x_6 = \text{tr}BD$ ,  $x_7 = \text{tr}ABC$ ,  $x_8 = \text{tr}ABD$ ,  $x_9 = \text{tr}BCD$  and  $x_{10} = \text{tr}ABCD$ . Then the set  $\mathcal{S}$  for  $G$  with respect to  $(A, B, C, D)$  is

$$\mathcal{S} = \{a, b, c, d, z, x_3, x_4, x_5, x_6, t, x_7, x_8, x_9, x_{10}\}.$$

The purpose of this section is to find expressions of  $x_1, \dots, x_{10}$  in  $\{a, b, z, u, v, w, t\}$  of (1.2). Then by Lemma 2.1 we can express the trace of any element of  $G$  in  $\{a, b, z, u, v, w, t\}$ . We shall apply this fact to obtain a representation of the mapping class group  $\mathcal{MC}_2$  via rational transformations.

(1) Since  $[A, B] = [C, D]^{-1}$ , we obtain by (2.1a)

$$(2.3) \quad abz - a^2 - b^2 - z^2 = cdt - c^2 - d^2 - t^2.$$

Note that  $\text{tr}[A, B] = a^2 + b^2 + z^2 - abz - 2 < -2$ , since  $G$  is discrete (see, for example [5, 33 D]). In what follows  $K = \sqrt{abz - a^2 - b^2 - z^2}$ .

(2) From  $BAB^{-1} = CDC^{-1}D^{-1}A$  and the basic identity (I3) we obtain

$$a = \text{tr}((ACD) \cdot C^{-1} \cdot D^{-1}) = -wt + cx_3 - ud + wcd - a.$$

and hence

$$(2.4) \quad 2a + wt - cx_3 + ud - wcd = 0.$$

(3) From (I2),  $v = -\text{tr}ACD \cdot D = -(\text{tr}ACD \text{tr}D - \text{tr}AC) = wd + x_3$  and so

$$(2.5) \quad x_3 = v - dw.$$

From this and (2.4) it follows that

$$(2.6) \quad 2a + wt - cv + ud = 0.$$

(4) From (I3),

$$\begin{aligned} -u &= \text{tr}A \cdot CD \cdot C^{-1} = ad + t(\text{tr}AC^{-1}) - wc - atc - x_4 \\ &= ad + t(ac - x_3) - wc - atc - x_4. \end{aligned}$$

It follows from this and (2.5) that

$$(2.7) \quad x_4 = u + ad - tx_3 - wc = u + ad - tv + twd - cw.$$

By substituting  $d = u^{-1}(cv - 2a - wt)$  (see (2.6)) into (2.3) we obtain

$$(uvt - u^2 - v^2)c^2 - (2a + wt)(tu - 2v)c - (K^2 + t^2)u^2 - (2a + tw)^2 = 0.$$

If this identity is regarded as a quadratic equation in  $c$ , it always has a negative root because

$$uvt - u^2 - v^2 = (-\text{tr}[CD^{-1}C^{-1}A^{-1}, ACD^2] - 2) + t^2 > t^2 > 0$$

(see [5, 33 D]) and  $-(K^2 + t^2)u^2 - (2a + tw)^2 < 0$ . Hence the condition  $c = \text{tr}C > 2$  yields

$$(2.8) \quad \begin{aligned} c &= \frac{(2a + tw)(ut - 2v) + u\sqrt{(2a + tw)^2(t^2 - 4) + 4(K^2 + t^2)(S^2 + t^2)}}{2(S^2 + t^2)}, \\ d &= \frac{cv - 2a - wt}{u} \end{aligned}$$

where  $S = \sqrt{uvt - u^2 - v^2 - t^2}$ . By using (1.3) we see that  $(2a + tw)^2(t^2 - 4) + 4(K^2 + t^2)(S^2 + t^2)$  equals

$$\begin{aligned} &\left((t^2 - 4)w + 2\sqrt{(S^2 + 4)(K^2 + 4)}\right)^2 \\ &= \left((t^2 - 4)w + \frac{2(awt + a^2 + t^2 + K^2 + S^2 + 4)}{w}\right)^2. \end{aligned}$$

Now from (2.8) we obtain

$$(2.9) \quad \begin{aligned} c &= \frac{(K^2 + S^2 + t^2 + a^2 + 4)u + w(2atu - 2av - uw + t^2uw - twv)}{w(S^2 + t^2)}, \\ d &= \frac{(K^2 + S^2 + t^2 + a^2 + 4)v + w(2au + twu - vw)}{w(S^2 + t^2)}. \end{aligned}$$

By (2.5), (2.7) and (2.9), we can obtain the expressions of  $x_3 = \text{tr}AC$  and  $x_4 = \text{tr}AD$  in  $(a, b, z, u, v, w, t)$ ,

$$(2.10) \quad \begin{aligned} x_3 &= -\frac{uw(2a + tw) + v(4 + a^2 + K^2 - w^2)}{S^2 + t^2} \\ x_4 &= (ad + u - cw) + t\frac{(4 + a^2 + K^2 - w^2)v + wu(2a + tw)}{S^2 + t^2}. \end{aligned}$$

(5) From (I2) and (2.1c) applied to  $BCDC^{-1}$  we obtain

$$(2.11) \quad \begin{aligned} \text{tr}B^{-1}(CDC^{-1}) &= bd - \text{tr}BCDC^{-1} \\ &= bd - (bd - x_6 - x_5t + cx_9) = x_6 + tx_5 - cx_9. \end{aligned}$$

From (I3),  $\text{tr}B^{-1}CD = bt - x_9$ . Then, from the trace of  $AB^{-1}A^{-1} = B^{-1}CD \cdot C^{-1} \cdot D^{-1}$ , (I2), (I3) and (2.11),

$$\begin{aligned} b &= (\text{tr}B^{-1}CD)t + c\text{tr}B^{-1}C + d\text{tr}(B^{-1}CD \cdot C^{-1}) - (\text{tr}B^{-1}CD)cd - b \\ &= (bt - x_9)(t - cd) + c(bc - x_5) + d(x_6 + tx_5 - cx_9) - b. \end{aligned}$$

Hence

$$(dt - c)x_5 + dx_6 - tx_9 = 2b - bt^2 + bc dt - bc^2.$$

(6) From (I2),  $\text{tr}A^{-1}CD = at + w$ , and from (I2) and (I3),

$$\begin{aligned}\text{tr}B^{-1}A^{-1} \cdot C \cdot D &= zt + c\text{tr}ABD^{-1} + d\text{tr}ABC^{-1} - zcd - \text{tr}B^{-1}A^{-1}DC \\ &= zt + c(zd - x_8) + d(zc - x_7) - zcd - \text{tr}B^{-1}A^{-1}DC \\ &= zt + cdz - dx_7 - cx_8 - \text{tr}B^{-1}A^{-1}DC.\end{aligned}$$

Substituting these into the next equation obtained from  $B^{-1}A^{-1}DC = A^{-1} \cdot B^{-1} \cdot CD$  and (I3),

$$\begin{aligned}\text{tr}B^{-1}A^{-1}DC &= a\text{tr}B^{-1}CD + b\text{tr}A^{-1}CD + zt - abt - \text{tr}B^{-1}A^{-1}CD \\ &= a(bt - x_9) + b(at + w) + zt - abt \\ &\quad - zt - cdz + dx_7 + cx_8 + \text{tr}B^{-1}A^{-1}DC,\end{aligned}$$

we obtain

$$dx_7 + cx_8 - ax_9 = -abt - bw + cdz.$$

(7) From  $B^{-1}CDC^{-1} = \text{tr}AB^{-1}A^{-1}D$ ,  $\text{tr}B^{-1}(CDC^{-1})$  equals

$$\begin{aligned}\text{tr}AB^{-1}A^{-1}D &= \text{tr}B\text{tr}AA^{-1}D - \text{tr}ABA^{-1}D = bd - \text{tr}DABA^{-1} \\ &= bd - (\text{tr}B\text{tr}D - \text{tr}BD - \text{tr}BA\text{tr}AD + \text{tr}A\text{tr}ABD) \\ &= x_6 + zx_4 - ax_8.\end{aligned}$$

Here we have used (I2) and (2.1c). Then from (2.11),

$$tx_5 + ax_8 - cx_9 = zx_4.$$

(8) From  $BA^{-1}B^{-1}C = A^{-1}DCD^{-1}$  and (I2), we have

$$ac - \text{tr}BAB^{-1}C = \text{tr}BA^{-1}B^{-1}C = \text{tr}A^{-1}DCD^{-1} = ac - \text{tr}ADCD^{-1},$$

and hence  $\text{tr}CBAB^{-1} = \text{tr}ADCD^{-1}$ . We have by using (2.1c)

$$\begin{aligned}\text{tr}CBAB^{-1} &= \text{tr}C\text{tr}A - \text{tr}AC - \text{tr}BC\text{tr}AB + \text{tr}B\text{tr}CBA \\ &= ac - x_3 - zx_5 + b(\text{tr}C\text{tr}BA + \text{tr}B\text{tr}CA + \text{tr}A\text{tr}CB \\ &\quad - \text{tr}A\text{tr}B\text{tr}C - \text{tr}ABC) \\ &= ac - x_3 - zx_5 + bcz + b^2x_3 + abx_5 - ab^2c - bx_7\end{aligned}$$

and

$$\begin{aligned}\text{tr}ADCD^{-1} &= \text{tr}A\text{tr}C - \text{tr}AC - \text{tr}AD\text{tr}DC + \text{tr}D\text{tr}ADC \\ &= ac - x_3 - tx_4 + d(\text{tr}A\text{tr}CD + \text{tr}D\text{tr}AC + \text{tr}C\text{tr}AD \\ &\quad - \text{tr}A\text{tr}D\text{tr}C - \text{tr}ACD) \\ &= ac - x_3 - tx_4 + adt + d^2x_3 + cdx_4 - ad^2c + wd.\end{aligned}$$

Thus we obtain

$$(z - ab)x_5 + bx_7 = (b^2 - d^2)x_3 + (t - cd)x_4 + bcz - ab^2c - adt + ad^2c - wd.$$

(9) We use  $C^{-1}BA = \text{tr}DC^{-1}D^{-1}AB$ . Then from (I2) and (I3),

$$\begin{aligned}\text{tr}C^{-1}BA &= zc - \text{tr}CBA \\ &= zc - (cz + bx_3 + ax_5 - abc - x_7) = -bx_3 - ax_5 + abc + x_7.\end{aligned}$$

From (I2) and (2.1c) this equals

$$\begin{aligned}\text{tr}(DC^{-1}D^{-1})AB &= cz - \text{tr}ABDCD^{-1} \\ &= cz - (\text{tr}AB\text{tr}C - \text{tr}ABC - \text{tr}ABD\text{tr}CD \\ &\quad + \text{tr}D\text{tr}(AB \cdot D \cdot C)) \\ &= x_7 + tx_8 - d(zt + dx_7 + cx_8 - zcd - x_{10}).\end{aligned}$$

Hence we obtain

$$-ax_5 + d^2x_7 + (cd - t)x_8 - dx_{10} = -abc + bx_3 - dtz + cd^2z.$$

(10) We use  $D^{-1}C^{-1}B = C^{-1}D^{-1}ABA^{-1}$ . From (I2),  $\text{tr}D^{-1}C^{-1}B = bt - x_9$  and from (I2), (2.1c) and (I3),

$$\begin{aligned}\text{tr}C^{-1}D^{-1}ABA^{-1} &= tb - \text{tr}(DC)ABA^{-1} \\ &= tb - (tb - \text{tr}DCB - \text{tr}DC\text{Atr}AB + \text{tr}A\text{tr}(D \cdot C \cdot AB)) \\ &= (dx_5 + cx_6 + bt - bcd - x_9) + z(dx_3 + cx_4 + at - acd + w) \\ &\quad - a(zt + dx_7 + cx_8 - zcd - x_{10})\end{aligned}$$

we obtain

$$dx_5 + cx_6 - adx_7 - acx_8 + ax_{10} = bcd - zdx_3 - zcx_4 - zw.$$

Let

$$M = \begin{pmatrix} dt - c & d & 0 & 0 & -t & 0 \\ 0 & 0 & d & c & -a & 0 \\ t & 0 & 0 & a & -c & 0 \\ z - ab & 0 & b & 0 & 0 & 0 \\ -a & 0 & d^2 & cd - t & 0 & -d \\ d & c & -ad & -ac & 0 & a \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{pmatrix}$$

and

$$\vec{v} = \begin{pmatrix} 2b - bt^2 + bcdt - bc^2 \\ -abt - bw + cdz \\ zx_4 \\ (b^2 - d^2)x_3 + (t - cd)x_4 + bcz - ab^2c - adt + acd^2 - wd \\ -abc + bx_3 - dzt + cd^2z \\ bcd - dzx_3 - czx_4 - zw \end{pmatrix}.$$

From the results (5)–(10) we obtain  $M\vec{x} = \vec{v}$ . The matrix  $M$  is singular, if  $a = c$ . However, by using (2.4) and (2.7) we can deduce:

$$(2.12) \quad \begin{aligned} x_5 &= \frac{c(2b + a^2b - 2az + bK^2) - tuz + dw(ab + z + zK^2) - v(ab + zK^2)}{K^2 + a^2}, \\ x_6 &= \frac{2(adz - bd) - u(ab + K^2z) + tv(ab + z + K^2z) + (c - dt)w(ab + z + K^2z)}{K^2 + a^2}, \\ x_7 &= \frac{-2cz - btu + avz + wd(b - az)}{K^2 + a^2}, \\ x_8 &= \frac{d(K^2 + a^2 + 2) + auz + vt(b - az) + w(bc - bdt - acz + adtz)}{K^2 + a^2}, \\ x_9 &= \frac{t(2b + a^2b - 2az + bK^2) + dvz + w(ab + K^2z) + u(cz - dtz)}{K^2 + a^2}, \\ x_{10} &= \frac{-2tz + b(c - dt)u + bdv - awz}{K^2 + a^2}. \end{aligned}$$

Expressions for  $x_3$  and  $x_4$  are obtained in (2.10).

**3. Mapping class group.** Let  $G$  be a group of type  $(2; -; -; -)$  and  $E = (A, B, C, D)$  a marking (or a canonical generator system) of  $G$ . We consider the following changes of marking:

$$(3.1) \quad \begin{aligned} \omega_1(E) &= (AB^{-1}, B, C, D), & \omega_2(E) &= (B, BA, C, D), \\ \omega_3(E) &= (B^{-1}CA, B, C, B^{-1}CD), \\ \omega_4(E) &= (A, B, CD^{-1}, D), & \omega_5(E) &= (A, B, C, DC). \end{aligned}$$

Each  $\omega_j$  induces an automorphism of  $G$ , which is also denoted by  $\omega_j$ . The table below shows the images of the elements in the leftmost column under  $\omega_j$ .

	$\omega_1$	$\omega_2$	$\omega_3$	$\omega_4$	$\omega_5$
$A$	$AB^{-1}$	$A$	$B^{-1}CA$	$A$	$A$
$B$	$B$	$BA$	$B$	$B$	$B$
$AB$	$A$	$ABA$	$B^{-1}CAB$	$AB$	$AB$
$ACDC^{-1}$	$AB^{-1}CDC^{-1}$	$ACDC^{-1}$	$B^{-1}CACB^{-1}CDC^{-1}$	$ACDC^{-1}$	$ACD$
$ACD^2$	$AB^{-1}CD^2$	$ACD^2$	$B^{-1}CAC(B^{-1}CD)^2$	$ACD$	$AC(DC)^2$
$ACD$	$AB^{-1}CD$	$ACD$	$B^{-1}CACB^{-1}CD$	$AC$	$ACDC$
$CD$	$CD$	$CD$	$CB^{-1}CD$	$C$	$CDC$

Let  $\omega_{j*} \in \mathcal{MC}_2$  denote the mapping class induced by  $\omega_j$ . Then  $\omega_{1*}, \dots, \omega_{5*}$  generate  $\mathcal{MC}_2$  and satisfy the following relations [1, Theorem 4.8]:

$$\begin{aligned} \omega_{i*}\omega_{j*} &= \omega_{j*}\omega_{i*} \text{ if } |i-j| \geq 2, \ 1 \leq i, j \leq 5, \\ \omega_{j*}\omega_{j+1*}\omega_{j*} &= \omega_{j+1*}\omega_{j*}\omega_{j+1*} \ (j = 1, 2, 3, 4), \\ (\omega_{1*}\omega_{2*}\omega_{3*}\omega_{4*}\omega_{5*})^6 &= 1, \\ \omega_{1*}\omega_{2*}\omega_{3*}\omega_{4*}\omega_{5*}^2\omega_{4*}\omega_{3*}\omega_{2*}\omega_{1*} &= 1. \end{aligned}$$

In this section we represent the action of  $\omega_{j*}$  on  $\mathcal{T}_2$  in the variables  $a, b, z, u, v, w, t$ . More precisely, when  $(A_j, B_j, C_j, D_j) = \omega_j(A, B, C, D)$ , we express

$$\begin{aligned} a_j &= \text{tr}A_j, & b_j &= \text{tr}B_j, & z_j &= \text{tr}A_jB_j, & u_j &= -\text{tr}A_jC_jD_jC_j^{-1}, \\ v_j &= -\text{tr}A_jC_jD_j^2, & w_j &= -\text{tr}A_jC_jD_j, & t_j &= \text{tr}C_jD_j \end{aligned}$$

by using  $a, b, z, u, v, w, t$ . However, for the case of  $\omega_3$  we modify the signs of some traces to obtain positive values.

(Case of  $\omega_{1*}$ ) By using basic trace identities we have  $\text{tr}AB^{-1} = \text{tr}A\text{tr}B - \text{tr}AB = ab - z$ ,

$$w_1 = -\text{tr}AB^{-1}CD = -\text{tr}B\text{tr}ACD + \text{tr}ABCD = bw + x_{10},$$

$$\begin{aligned} u_1 &= -\text{tr}AB^{-1}CDC^{-1} = -\text{tr}B\text{tr}ACDC^{-1} + \text{tr}(AB)CDC^{-1} \ (\because \text{(I2)}) \\ &= bu + (\text{tr}AB\text{tr}D - \text{tr}ABD \\ &\quad - \text{tr}ABC\text{tr}CD + \text{tr}C\text{tr}ABCD) \ (\because \text{(2.1c)}) \\ &= bu + zd - x_8 - tx_7 + cx_{10}, \end{aligned}$$

and

$$\begin{aligned} v_1 &= -\text{tr}AB^{-1}CD^2 = -\text{tr}B\text{tr}ACD^2 + \text{tr}ABCD^2 \ (\because \text{(I2)}) \\ &= bv + (\text{tr}ABCD\text{tr}D - \text{tr}ABC) \ (\because \text{(I2)}) \\ &= bv + dx_{10} - x_7. \end{aligned}$$

Hence

$$\omega_{1*}(a, b, z, u, v, w, t) = (ab - z, b, a, u_1, v_1, w_1, t).$$

(Case of  $\omega_{2*}$ ) Since  $\text{tr}ABA = \text{tr}AB\text{tr}A - \text{tr}B = za - b$ ,

$$\omega_{2*}(a, b, z, u, v, w, t) = (a, z, az - b, u, v, w, t).$$

(Case of  $\omega_{3*}$ ) First we remark that  $\text{tr}B^{-1}CA < 0$  and  $\text{tr}B^{-1}CD < 0$ . To see  $\text{tr}B^{-1}CA < 0$ , for example, note that  $(AB^{-1}, B)$  is a marking for a group of type  $(1; 0; 0; 1)$  and  $\text{tr}A$  and  $\text{tr}B$  are positive. Then we have  $\text{tr}AB^{-1} > 0$ . Then  $(AB^{-1}, C)$  is a marking for a group of type  $(0; 0; 0; 3)$ . Since  $\text{tr}AB^{-1}$  and  $\text{tr}C$  are positive,  $\text{tr}AB^{-1}C < 0$  (see [5, Section 33 A and D]). The calculation for  $\omega_{3*}$  is the most complicated: By using the basic trace identities we have

$$a_3 = \text{tr}B^{-1}CA = \text{tr}B\text{tr}AC - \text{tr}ABC = bx_3 - x_7.$$



$$\begin{aligned}
w_3 &= -\text{tr}(B^{-1}C)(AC)(B^{-1}C)D \\
&= -\text{tr}(AC)(B^{-1}C)D(B^{-1}C) \\
&= -\text{tr}ACB^{-1}C\text{tr}B^{-1}CD - \text{tr}ACD + \text{tr}AC\text{tr}D \\
&= -(\text{tr}B\text{tr}AC^2 - \text{tr}ACBC)(\text{tr}B\text{tr}CD - \text{tr}BCD) + w + dx_3 \\
&= -[b(cx_3 - a) - (x_3x_5 + z - ab)](bt - x_9) + w + dx_3 \\
&= (x_3x_5 + z - bcx_3)(bt - x_9) + w + dx_3,
\end{aligned}$$

$$\begin{aligned}
u_3 &= -\text{tr}(B^{-1}C)(AC)(B^{-1}C)(DC^{-1}) \\
&= -\text{tr}(AC)(B^{-1}C)(DC^{-1})(B^{-1}C) \\
&= -\text{tr}ACB^{-1}C\text{tr}B^{-1}CDC^{-1} - \text{tr}ACDC^{-1} + \text{tr}AC\text{tr}DC^{-1} \\
&= -(\text{tr}AC\text{tr}B^{-1}C - \text{tr}AB)(\text{tr}B\text{tr}D - \text{tr}BCDC^{-1}) + u + x_3(cd - t) \\
&= -(x_3(bc - x_5) - z)[bd - (bd - x_6 - tx_5 + cx_9)] + u + x_3(cd - t) \\
&= (x_3x_5 + z - bcx_3)(x_6 + tx_5 - cx_9) + u + x_3(cd - t),
\end{aligned}$$

$$\begin{aligned}
v_3 &= -\text{tr}B^{-1}CAC(B^{-1}CD)^2 \\
&= -\text{tr}B^{-1}CD\text{tr}B^{-1}CACB^{-1}CD + \text{tr}B^{-1}CAC \\
&= (bt - x_9)[(x_3x_5 + z - bcx_3)(bt - x_9) + w + dx_3] + (bc - x_5)x_3 - z,
\end{aligned}$$

$$t_3 = \text{tr}CB^{-1}CD = \text{tr}CB^{-1}\text{tr}CD - \text{tr}BD = (bc - x_5)t - x_6.$$

In this case  $a_3$ ,  $x_3$ ,  $v_3$  and  $t_3$  are negative. We modify the sign of these parameters and obtain

$$\omega_{3*}(a, b, z, u, v, w, t) = (-a_3, b, -x_3, u_3, -v_3, w_3, -t_3).$$

(Case of  $\omega_{4*}$ ) For the expression of  $\omega_{4*}$  we have easily

$$\omega_{4*}(a, b, z, u, v, w, t) = (a, b, z, u, w, -x_3, c).$$

(Case of  $\omega_{5*}$ ) Since  $-\text{tr}ACDC = -\text{tr}C\text{tr}ACD + \text{tr}ACDC^{-1} = cw - u$ ,

$$\begin{aligned}
v_5 &= -\text{tr}AC(DC)^2 = -\text{tr}CD\text{tr}ACDC + \text{tr}AC \\
&= -t(\text{tr}C\text{tr}ACD - \text{tr}ACDC^{-1}) + x_3 \\
&= cwt - tu + x_3,
\end{aligned}$$

and  $\text{tr}CDC = ct - d$ , we have

$$\omega_{5*}(a, b, z, u, v, w, t) = (a, b, z, w, cwt - tu + x_3, cw - u, ct - d).$$

Now we conclude

**Theorem 3.1.** *The mapping classes  $\omega_{1*}, \omega_{2*}, \omega_{3*}, \omega_{4*}, \omega_{5*}$  are represented by the following rational maps in variables  $a, b, z, u, v, w, t$ :*

(3.2)

$$\begin{aligned}\omega_{1*}(a, b, z, u, v, w, t) &= (ab - z, b, a, u_1, v_1, w_1, t) \\ \omega_{2*}(a, b, z, u, v, w, t) &= (a, z, az - b, u, v, w, t) \\ \omega_{3*}(a, b, z, u, v, w, t) &= (-bx_3 + x_7, b, -x_3, u_3, -v_3, w_3, -bct + x_5t + x_6) \\ \omega_{4*}(a, b, z, u, v, w, t) &= (a, b, z, u, w, -x_3, c) \\ \omega_{5*}(a, b, z, u, v, w, t) &= (a, b, z, w, cwt - tu + x_3, cw - u, ct - d),\end{aligned}$$

where  $c, d, x_3, x_4, x_5, x_6$  and  $x_7$  are given in (2.9) and (2.10) and (2.12).

As it is shown in Section 2,  $x_1 = c, x_2 = d, \dots, x_{10}$  are all rational functions in  $(a, b, z, u, v, w, t)$ . Hence the inverse mappings of  $\omega_{j*}$  ( $j = 1, \dots, 5$ ) are also rational mappings. The expressions in (3.2) in  $(a, b, z, u, v, t)$ , especially the one for  $\omega_{3*}$ , are very complicated.

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