TRACES OF ANISOTROPIC BESOV-LIZORKIN-TRIEBEL SPACES—A COMPLETE TREATMENT OF THE BORDERLINE CASES

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Dedicated to Professor Alois Kufner on the occasion of his 65th birthday

Abstract. Including the previously untreated borderline cases, the trace spaces (in the distributional sense) of the Besov-Lizorkin-Triebel spaces are determined for the anisotropic (or quasi-homogeneous) version of these classes. The ranges of the traces are in all cases shown to be approximation spaces, and these are shown to be different from the usual spaces precisely in the cases previously untreated. To analyse the new spaces, we carry over some real interpolation results as well as the refined Sobolev embeddings of J. Franke and B. Jawerth to the anisotropic scales.

Keywords: Anisotropic Besov and Lizorkin-Triebel spaces, approximation spaces, trace operators, boundary problems, interpolation, atomic decompositions, refined Sobolev embeddings

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1. Introduction

In this paper we present a complete solution of the trace problem for the anisotropic (or rather quasi-homogeneous) Besov and Lizorkin-Triebel spaces, denoted by $B^{s,a}_{p,q}$ and $F^{s,a}_{p,q}$, respectively. The definitions are recalled in Appendix B below.

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Here the trace is the operator γ_0,

$$f(x_1, x_2, \dots, x_{n-1}, x_n) \overset{\gamma_0}{\longmapsto} f(x_1, x_2, \dots, x_{n-1}, 0), \tag{1}$$

which restricts functions on $\mathbb{R}^n$ to the hyperplane $\Gamma = \{x_n = 0\}$ in $\mathbb{R}^n$ $(n \geqslant 2)$—in general this is defined in the obvious way on the subspace $C\big(\mathbb{R}, \mathscr{D}'(\mathbb{R}^{n-1})\big)$ of $\mathscr{D}'(\mathbb{R}^n)$. For the $B^{s,a}_{p,q}(\mathbb{R}^n)$ and $F^{s,a}_{p,q}(\mathbb{R}^n)$ under consideration, this coincides with the extension by continuity from the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, except for $p = \infty$ and $q = \infty$ in which cases $\mathcal{S}$ is not dense; however, because of embeddings, the latter exception is only felt for $F^{a_n,a}_{1,\infty}$, where $s = a_n$ is the lowest possible value, a_n being the modulus of anisotropy associated to x_n. (For simplicity, $\mathbb{R}^n$ is often suppressed in $B^{s,a}_{p,q}$ and $F^{s,a}_{p,q}$ etc. when confusion is unlikely to result.)

The trace problem consists in finding spaces X and Y, as subspaces of $\mathscr{D}'(\mathbb{R}^n)$ and $\mathscr{D}'(\mathbb{R}^{n-1})$, respectively, such that γ_0 yields a continuous, linear *surjection*

$$\gamma_0 \colon X \to Y. \tag{2}$$

In this paper we determine those $B^{s,a}_{p,q}$ and $F^{s,a}_{p,q}$ which allow such a Y to be found, and we moreover determine the optimal Y for these choices of X. (The existence of a right inverse of γ_0 is also discussed.)

One effect of allowing *anisotropic* spaces is that γ_0 is studied on larger domains. However, the main motivation for the anisotropic spaces is that they are indispensable for the fine theory of parabolic boundary problems. For example it is well known that in a treatment of $\partial_t - \Delta$ it is necessary to use $B^{s,a}_{p,q}$ and $F^{s,a}_{p,q}$ for $a = (1, \dots, 1, 2)$ (gathering the moduli of anisotropy to form a vector in the (x,t)-space) and that $F^{s,a}_{p,2}$ (locally) equals the intersection of $L_p\big(\mathbb{R}, H^s_p(\mathbb{R}^{n-1})\big)$ and $H^{s/2}_p\big(\mathbb{R}, L_p(\mathbb{R}^{n-1})\big)$. We refer the reader to works of G. Grubb [15, 16] for a recent treatment, based on L. Boutet de Monvel's pseudo-differential calculus, of parabolic initial boundary problems in anisotropic Besov and Bessel potential spaces.

If desired, the reader may specialise to the isotropic case, which is given by $a = (1, \dots, 1)$.

In the following review of results, comparison with other works is often postponed (for simplicity's sake) to later sections.

For the continuity of γ_0 from $B^{s,a}_{p,q}$ or $F^{s,a}_{p,q}$ to $\mathscr{D}'\big(\mathbb{R}^{n-1}\big)$ two different conditions are necessary for $p \geqslant 1$ and $p < 1$. Introducing $|a| := a_1 + \ldots + a_n$ and similarly $|a'| = a_1 + \ldots + a_{n-1}$ for the spaces over $\mathbb{R}^{n-1}$ (so that in the isotropic case $|a| = n$ and $|a'| = n - 1$), these may be expressed in the following way by letting $t_+ := \max(0, t)$:

$$s \geqslant \max\Big(\frac{a_n}{p}, \frac{|a|}{p} - |a'|\Big) = \frac{a_n}{p} + |a'|\Big(\frac{1}{p} - 1\Big)_+; \tag{3}$$

in other words the correction $|a'|(\frac{1}{p}-1)$ appears for $0<p<1$. In the case of equality it is moreover necessary that $q\leqslant 1$ in the B-case and $p\leqslant 1$ in the F-case; cf. Figure 1.

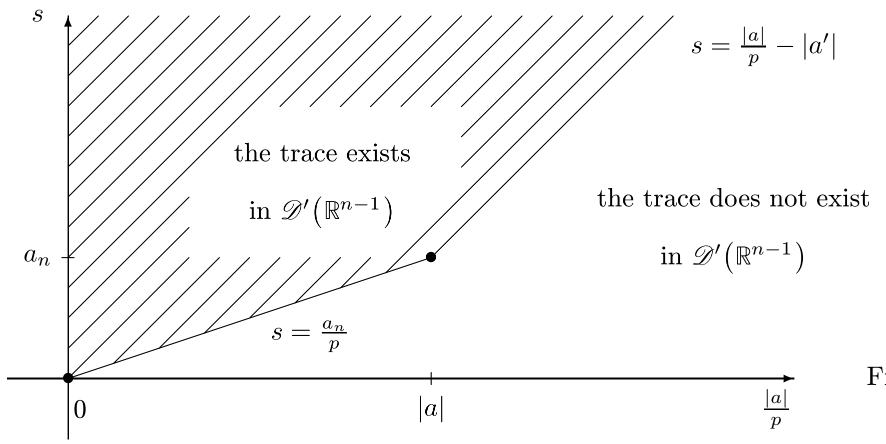

Figure 1

This has earlier been known to specialists (we carry over explicit isotropic counter-examples of the second author [21, Rem. 2.9]). The case $s=\frac{a_n}{p}$ for $p\geqslant 1$ was investigated by V. I. Burenkov and M. L. Gol'dman [5], however only for $q=1$; in the isotropic case $0<q\leqslant 1$ and, for $p<1$, the borderline case $s=\frac{n}{p}-(n-1)$ was treated by J. Johnsen [22], but surjectivity was left open for $p<1$. The present article may therefore be seen as a continuation of [5] and [22]. Emphasis will be on the borderline cases mentioned, since it is known (and comparatively easy) that γ_0 is a bounded, right invertible surjection

$$\gamma_0\colon\ B^{s,a}_{p,q}(\mathbb{R}^n)\to B^{s-\frac{a_n}{p},a'}_{p,q}(\mathbb{R}^{n-1}),\qquad \gamma_0\colon\ F^{s,a}_{p,q}(\mathbb{R}^n)\to B^{s-\frac{a_n}{p},a'}_{p,p}(\mathbb{R}^{n-1}) \tag{4}$$

in any of the *generic* cases (i.e. those with strict inequality in (3)). This is, of course, the well-known loss of $\frac{1}{p}$ in the isotropic case.

For $s=\frac{|a|}{p}-|a'|$ with $p<1$ the optimal Y's are determined below, and it turns out that whenever p, $q\leqslant 1$ they are neither Besov nor Lizorkin-Triebel spaces. So in order to complete the range characterisation, we introduce another scale $A^{s,a}_{p,q}$, which previously has been investigated mainly by Russian specialists in function spaces.

In fact, γ_0 still lowers s by $\frac{a_n}{p}$ and it is a bounded surjection

$$\gamma_0\colon\ B^{\frac{|a|}{p}-|a'|,a}_{p,q}(\mathbb{R}^n)\to A^{|a'|(\frac{1}{p}-1),a'}_{p,q}(\mathbb{R}^{n-1})\quad\text{for}\quad p,q\in\,]0,1], \tag{5}$$

$$\gamma_0\colon\ F^{\frac{|a|}{p}-|a'|,a}_{p,q}(\mathbb{R}^n)\to A^{|a'|(\frac{1}{p}-1),a'}_{p,p}(\mathbb{R}^{n-1})\quad\text{for}\quad 0<p\leqslant 1,\quad 0<q\leqslant\infty. \tag{6}$$

To have instead e.g. a Besov space as the co-domain, one can use an embedding of $A^{|a'|(\frac{1}{p}-1),a'}_{p,q}(\mathbb{R}^{n-1})$ into $B^{|a'|(\frac{1}{r}-r),a'}_{r,\infty}(\mathbb{R}^{n-1})$ for $r = \max(p,q)$ (which is optimal). An investigation of the borderline cases is given in Section 3.2 below.

However, because of the various identifications between the $A^{s,a}_{p,q}$ and the Lebesgue, Besov and Lizorkin-Triebel spaces etc., it is possible to formulate *all* trace results in a concise way in terms of the $A^{s,a}_{p,q}$:

Main Theorem. *For a given anisotropy $a = (a_1, \dots, a_n)$, the following assertions are equivalent:*

(a) *the operator γ_0 is a continuous mapping from $B^{s,a}_{p,q}(\mathbb{R}^n)$ into $\mathscr{D}'(\mathbb{R}^{n-1})$;*

(b) *the operator γ_0 maps $B^{s,a}_{p,q}(\mathbb{R}^n)$ continuously* **onto** $A^{s-\frac{a_n}{p},a'}_{p,q}(\mathbb{R}^{n-1})$;

(c) *the triple (s,p,q) satisfies $s \geqslant \frac{a_n}{p} + |a'|(\frac{1}{p}-1)_+$ and, in case of equality, also $0 < q \leqslant 1$.*

For the Lizorkin-Triebel spaces $F^{s,a}_{p,q}$ the analogous result holds if one replaces $A^{s-\frac{a_n}{p},a'}_{p,q}$ by $A^{s-\frac{a_n}{p},a'}_{p,p}$ in (b) *and replaces '$q \leqslant 1$' by '$p \leqslant 1$' in* (c).

It should be noted that the formal introduction of $A^{s,a}_{p,q}$ in Definition 1 below allows us to give a short, self-contained proof of the main theorem in the $B^{s,a}_{p,q}$-case (thus a unified proof of all underlying borderline cases); the Lizorkin-Triebel case is then deduced from the Besov case by establishing a certain q-independence. This partly follows the work of M. Frazier and B. Jawerth [12], but we point out and correct a flaw in the proof of [12, Th. 11.1], see Remark 3.3 below.

In order to deduce the relations between $A^{|a'|(\frac{1}{p}-1)}_{p,q}(\mathbb{R}^{n-1})$, which enter in (5)–(6) above, and the usual spaces, we need anisotropic versions of the optimal mixed Sobolev embeddings between the two scales $B^{s,a}_{p,q}$ and $F^{s,a}_{p,q}$.

For this purpose we carry over these embeddings (due to B. Jawerth and J. Franke), hence also some necessary real interpolation results for $F^{s,a}_{p,q}$, to the anisotropic setting. See Appendix C below for these results.

In Section 2 below we introduce a working definition (based on a limit) of γ_0 and then present results for the generic cases. The borderline cases are treated in Section 3, in particular the range spaces are presented for the cases with $s = \frac{|a|}{p} - |a'|$ in terms of the approximation spaces $A^{s,a}_{p,q}$, which are formally introduced there for this purpose. The relations between $A^{s,a}_{p,q}$ and the Besov and Lizorkin-Triebel spaces are elucidated in Section 4, and the proofs of the assertions are to be found in Section 5. The appendices collect the notation and the necessary facts about Besov and Lizorkin-Triebel spaces, in particular the extension to the anisotropic case of some well-known facts.

R e m a r k 1.1. We should emphasise that, for $p < 1$, also a different operator has been studied under the label 'trace'. Indeed, when γ_0 is restricted to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, there are extensions by continuity $T\colon B^{s,a}_{p,q} \to Y$ at least if $s > \frac{1}{p}$; while this is effectively weaker than (3), it may only be obtained by taking Y outside of $\mathscr{D}'$, and it was shown in [22] that T is different from γ_0 since

$$\gamma_0(\varphi(x') \otimes \delta_0(x_n)) = \varphi, \qquad T(\varphi(x') \otimes \delta_0(x_n)) = 0 \tag{7}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^{n-1})$.

Moreover, using the sharp result for elliptic boundary operators in Besov and Lizorkin-Triebel spaces obtained in [21], it was also shown in [22] that T is unsuitable for the study of elliptic boundary problems. We shall therefore not consider this other possibility here; it was discussed at length by M. Frazier and B. Jawerth [11, 12] and H. Triebel [40, 41].

R e m a r k 1.2. It is noted once and for all that we consider arbitrary $s \in \mathbb{R}$ and p, $q \in \,]0,\infty]$, although $p < \infty$ is to be understood throughout for the $F^{s,a}_{p,q}$ spaces; and all such admissible parameters are considered unless further restrictions are stated.

2. Generic properties of the trace

To set the scene properly, we introduce the trace in a formal way. The reader should consult Appendices A–B first for the anisotropic spaces and for the corresponding anisotropic distance $|x|_a$ and dilation $t^a x = (t^{a_1}x_1, \dots, t^{a_n}x_n)$, both defined on $\mathbb{R}^n$; here and throughout $a = (a_1, \dots, a_{n-1}, a_n) = (a', a_n)$ will be a given anisotropy.

In addition, $\mathscr{F}$ denotes the Fourier transform and $\mathscr{F}^{-1}$ the inverse, extended from the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ to its dual $\mathcal{S}'(\mathbb{R}^n)$. In different dimensions, say $\mathbb{R}^{n-1}$ and $\mathbb{R}$, Fourier transformation will be indicated by $\mathscr{F}_{n-1}$ and $\mathscr{F}_1$, respectively. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be a function such that

$$\psi(x) = 1 \quad \text{if} \quad |x|_a \leqslant 1 \qquad \text{and} \qquad \psi(x) = 0 \quad \text{if} \quad |x|_a \geqslant 2. \tag{8}$$

For such ψ, we may define a smooth, anisotropic dyadic partition of unity $(\varphi_j)_{j\in\mathbb{N}_0}$ by letting $\varphi_0(x) = \psi(x)$ and

$$\varphi_j(x) = \varphi_0(2^{-ja}x) - \varphi_0(2^{(-j+1)a}x) \quad \text{if} \quad j \in \mathbb{N}. \tag{9}$$

Indeed, since $\psi(2^{Na}\xi) = \sum\limits_{j=0}^{N} \varphi_j(\xi)$, it is clear that

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \quad \text{for} \quad x \in \mathbb{R}^n. \tag{10}$$

2.1. The working definition. Using a fixed ψ of the above type, we have for all $f \in \mathcal{S}'(\mathbb{R}^n)$

$$f = \sum_{j=0}^{\infty} \mathscr{F}^{-1}[\varphi_j \mathscr{F} f] \qquad \text{(convergence in } \mathcal{S}'\text{)}. \tag{11}$$

Since, by the Paley-Wiener-Schwartz Theorem, $\mathscr{F}^{-1}[\varphi_j \mathscr{F} f]$ is continuous, the trace has an immediate meaning for this function. As a temporary working definition we therefore let

$$\gamma_0 f = \lim_{N\to\infty} \sum_{j=0}^{N} \gamma_0\big(\mathscr{F}^{-1}[\varphi_j \mathscr{F} f]\big) \tag{12}$$

whenever this limit exists in $\mathcal{S}'\big(\mathbb{R}^{n-1}\big)$. However, we should make the following remarks to this definition.

On the one hand, it is possible to show the next result (which summarises the most well-known facts on γ_0) by relatively simple arguments:

Theorem 1. *The operator γ_0 maps $B^{s,a}_{p,q}(\mathbb{R}^n)$ (or $F^{s,a}_{p,q}$) continuously into $\mathcal{S}'(\mathbb{R}^{n-1})$ only if (s,p,q) satisfies either*

$$s > \tfrac{a_n}{p} + |a'|\big(\tfrac{1}{p} - 1\big)_+ \tag{13}$$

or, alternatively,

$$s = \tfrac{a_n}{p} + |a'|\big(\tfrac{1}{p} - 1\big)_+ \quad \text{and} \quad 0 < q \leqslant 1 \quad \text{(respectively } p \leqslant 1\text{)}. \tag{14}$$

When (13) holds, then γ_0 is actually a continuous map $B^{s,a}_{p,q}(\mathbb{R}^n) \to B^{s-\frac{a_n}{p},a'}_{p,q}\big(\mathbb{R}^{n-1}\big)$ and $F^{s,a}_{p,q}(\mathbb{R}^n) \to B^{s-\frac{a_n}{p},a'}_{p,p}\big(\mathbb{R}^{n-1}\big)$.

On the other hand, by the same line of thought as in [22], one may, as we show in this paper, deduce from the proof of Theorem 1 that the just defined operator γ_0 coincides with (a restriction of) the map

$$r_0 f := f(0) \quad \text{for } f(t) \text{ in } C\big(\mathbb{R}, \mathscr{D}'(\mathbb{R}^{n-1})\big). \tag{15}$$

To conclude this relation between γ_0 and r_0, we apply the next result where $C_{\mathrm{b}}(\mathbb{R}, X)$ stands for the (supremum normed) space of uniformly continuous, bounded functions valued in the Banach space X:

Proposition 1. *When $B^{s,a}_{p,q}(\mathbb{R}^n)$ and $F^{s,a}_{p,q}(\mathbb{R}^n)$ have parameters s, p and q satisfying (13), or the pertinent version of (14), then*

$$B^{s,a}_{p,q}(\mathbb{R}^n),\ F^{s,a}_{p,q}(\mathbb{R}^n) \hookrightarrow C_{\mathrm{b}}\big(\mathbb{R}, L_{p_1}(\mathbb{R}^{n-1})\big) \tag{16}$$

hold with $p_1 = \max(p,1)$.

Given this result (see Section 5.1.2 for the proof), it follows from the fact that uniform convergence implies pointwise convergence (hence that r_0 is continuous from $C_{\mathrm{b}}(\mathbb{R}, L_{p_1}(\mathbb{R}^{n-1}))$) that, for any f in one of the spaces on the left hand side of (16),

$$r_0 f = \lim_{N\to\infty} r_0 \sum_{j=0}^{N} \mathscr{F}^{-1}\bigl(\varphi_j \mathscr{F} f\bigr) = \sum_{j=0}^{\infty} \mathscr{F}^{-1}\bigl(\varphi_j \mathscr{F} f\bigr)(\cdot, 0) = \gamma_0 f. \tag{17}$$

Indeed, for $q < \infty$ the decomposition in (11) converges in the topology of the $B^{s,a}_{p,q}$ or $F^{s,a}_{p,q}$ space that contains f, hence there is also convergence in $C_{\mathrm{b}}\bigl(\mathbb{R}, L_{p_1}(\mathbb{R}^{n-1})\bigr)$; with one exception it is always possible to reduce to the case with $q < \infty$ by means of embeddings, e.g. a space with $p \leqslant 1$ is a subspace of $F^{a_n,a}_{1,q}$ for some $q < \infty$ according to (14).

The just mentioned exception is the space $F^{a_n,a}_{1,\infty}(\mathbb{R}^n)$ for which (17) requires a sharper argument: for arbitrary $f \in C_{\mathrm{b}}\bigl(\mathbb{R}, L_1(\mathbb{R}^{n-1})\bigr)$ one can take any $\eta \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \eta \,\mathrm{d}x = 1$, let $\eta_k(x) := k^{|a|}\eta(k^a x)$ and then show that

$$\eta_k * f(x', 0) \to f(x', 0) \quad \text{in } L_1\bigl(\mathbb{R}^{n-1}\bigr) \text{ for } k \to \infty. \tag{18}$$

This is sufficient because it applies to any $f \in F^{a_n,a}_{1,\infty}$ by Proposition 1, and while $r_0 f = f(\cdot, 0)$ by definition, $\sum_{j=0}^{N} \mathscr{F}^{-1}\bigl(\varphi_j \mathscr{F} f\bigr)(x) = \eta_k * f(x)$ holds for $\eta = \mathscr{F}^{-1}\psi$ and $k = 2^N$, so that altogether the first equality sign of (17) is justified.

However, (18) may be verified by the usual convolution techniques, for if translation by y' in $\mathbb{R}^{n-1}$ is denoted by $\tau_{y'}$ and $\bigl\| \cdot \bigm| L_1\bigl(\mathbb{R}^{n-1}\bigr)\bigr\|$ is replaced by $\| \cdot \|_1$, the translation invariance gives that

$$\begin{aligned} \bigl\|\eta_k * f(\cdot, x_n) - f(\cdot, x_n)\bigr\|_1 \leqslant \int_{\mathbb{R}^n} &|\eta(z)| \cdot \bigl\|f(\cdot, x_n - k^{-a_n} z_n) - f(\cdot, x_n)\bigr\|_1 \,\mathrm{d}z \\ + \int_{\mathbb{R}^n} &|\eta(z)| \cdot \bigl\|(\tau_{k^{-a'}z'} - I) f(\cdot, x_n)\bigr\|_1 \,\mathrm{d}z. \end{aligned} \tag{19}$$

Setting $x_n = 0$, one may for any $\varepsilon > 0$ take $c > 0$ such that $(\tau_{y'} - I)f(\cdot, 0)$ has L_1-norm less than ε when $|y'|_{a'} < c$ (since $|\cdot|$ and $|\cdot|_{a'}$ define the same topology in $\mathbb{R}^{n-1}$). In the second integral it thus remains to control the region where $|z'|_{a'} \geqslant ck$, but a majorisation by $2\bigl\|f(\cdot, 0) \bigm| L_1\bigr\|$ shows that this contribution is less than ε for all k eventually; by the continuity with respect to x_n, and a similar splitting, the first integral is also $< \varepsilon$ eventually.

Hence (18) holds (the real achievement is the less elementary proof of Proposition 1), and thus (17) is proved for all spaces treated in the present paper. In particular, this means that $\gamma_0 u$ is independent of the choice of ψ (and of the anisotropy a).

Summing up the above discussion, we have proved

Corollary 1. *When the operators* r_0 *and* γ_0 *are defined as in* (15) *and* (12) *above, then*

$$r_0 f = \gamma_0 f \tag{20}$$

holds for all functions f *in the spaces* $B^{s,a}_{p,q}$ *and* $F^{s,a}_{p,q}$ *fulfilling* (13) *or* (14).

Remark 2.1. Our working definition of $\gamma_0 f$ has been used since the late 1970's; cf. [18, 39, 40]. Nevertheless, the consistency with the trace on $C\big(\mathbb{R}, \mathscr{D}'(\mathbb{R}^{n-1})\big)$ was, to our knowledge, first proved for the Besov spaces in [22]. By Proposition 1 and (17) above this consistency holds for all the considered spaces; the consistency extends to the trace defined on the entire Colombeau algebra $\mathscr{G}(\mathbb{R}^n)$, which contains $\mathscr{D}'(\mathbb{R}^n)$, see [29, Prop. 11.1].

2.2. Linear extension. Taking, as we may, η_0 and $\eta \in \mathcal{S}(\mathbb{R})$ such that $\operatorname{supp}\eta_0 \subset\,]-1,1[$ and $\operatorname{supp}\eta \subset\,]1,2[$ and normalised so that

$$\big(\mathscr{F}_1^{-1}\eta_0\big)(0) = \big(\mathscr{F}_1^{-1}\eta\big)(0) = 1, \tag{21}$$

we set

$$\eta_j(t) = \eta(2^{-ja_n}t) \quad \text{for any} \quad t \in \mathbb{R} \quad \text{if} \quad j \geqslant 1. \tag{22}$$

For any $v \in \mathcal{S}'\big(\mathbb{R}^{n-1}\big)$ such that the following series converges, one can now define an extension to $\mathbb{R}^n$ by means of the partition of unity $(\varphi_j)_{j\in\mathbb{N}_0}$ in (9):

$$Kv(x', x_n) = \sum_{j=0}^{\infty} 2^{-ja_n}\,\big(\mathscr{F}_1^{-1}\eta_j\big)(x_n)\,\mathscr{F}_{n-1}^{-1}\big[\varphi_j(\cdot,0)\,\mathscr{F}_{n-1}v\big](x'). \tag{23}$$

Using a homogeneity argument, we have $2^{-ja_n}\big(\mathscr{F}_1^{-1}\eta_j\big)(0) = 1$ for any $j \in \mathbb{N}_0$, so the termwise restriction to $x_n = 0$ gives, with weak convergence in $\mathcal{S}'(\mathbb{R}^{n-1})$,

$$\sum_{j=0}^{N} 2^{-ja_n}\big(\mathscr{F}_1^{-1}\eta_j\big)(0)\,\mathscr{F}_{n-1}^{-1}\big[\varphi_j(\cdot,0)\,\mathscr{F}_{n-1}v\big] \xrightarrow[N\to\infty]{} \sum_{j=0}^{\infty} \mathscr{F}_{n-1}^{-1}\big[\varphi_j(\cdot,0)\,\mathscr{F}_{n-1}v\big] = v. \tag{24}$$

By the working definition of γ_0, this means that $\gamma_0 Kv$ is defined for such v, hence $\gamma_0 K = I$; i.e. K is a linear extension.

When it is understood that the convergence of the series (23) is part of the assertion, one has

Theorem 2. (i) *The operator* K *maps* $B^{s-\frac{a_n}{p},a'}_{p,q}(\mathbb{R}^{n-1})$ *continuously into* $B^{s,a}_{p,q}\big(\mathbb{R}^n\big)$.

(ii) *For* $0 < p < \infty$ *the operator* K *maps* $B^{s-\frac{a_n}{p},a'}_{p,p}\big(\mathbb{R}^{n-1}\big)$ *continuously into* $F^{s,a}_{p,q}(\mathbb{R}^n)$.

Here there are no restrictions in s, that is, the assertions in Theorem 2 hold for all $s \in \mathbb{R}$ (which is to be expected since K is a Poisson operator).

Since the relation $\gamma_0 K = I$ was found above, one has as a consequence the next result.

Theorem 3. *Let* $s > \frac{a_n}{p} + |a'|\bigl(\frac{1}{p} - 1\bigr)_+$. *Then the operator* γ_0 *maps* $B^{s,a}_{p,q}(\mathbb{R}^n)$ *continuously* **onto** $B^{s-\frac{a_n}{p},a'}_{p,q}\bigl(\mathbb{R}^{n-1}\bigr)$, *and for* $0 < p < \infty$ *it maps* $F^{s,a}_{p,q}(\mathbb{R}^n)$ *onto* $B^{s-\frac{a_n}{p},a'}_{p,p}\bigl(\mathbb{R}^{n-1}\bigr)$; *$K$ is in both cases a linear right inverse of* γ_0.

Although Theorems 2–3 above are unsurprising (indeed, well-known in the isotropic case), they deserve to be compared with the borderline results in the next section.

Remark 2.2. The contents of the above theorems are known to a wide extent for the classical parameters $1 \leqslant p,\ q \leqslant \infty$; cf. the works of O. V. Besov, V. P. Ilyin and S. M. Nikol'skij [4], S. M. Nikol'skij [28], V. I. Burenkov and M. L. Gol'dman [5] and G. A. Kalyabin [23]. For the isotropic case we also refer to the works of J. Bergh and J. Löfström [3], M. Frazier and B. Jawerth [11, 12] and H. Triebel [39, 40, 41] as well as to the remarks in [22] and in the present paper. The study of the trace problem for $0 < p < 1$ was initiated by B. Jawerth [18, 19], but the first to find the borderline $s = \frac{n}{p} - (n-1)$ for $0 < p < 1$ seems to be either B. Jawerth or J. Peetre [31, Rem. 2.3]. (Peetre [31, Note 1] actually gives credit to [19] for this, and vice versa in [19, Rem. 2.2].)

Remark 2.3. The borderline $s = \frac{1}{p}$ itself was found by S. M. Nikol'skij in 1951 when he proved the continuity of $\gamma_0\colon B^s_{p,\infty}(\mathbb{R}^n) \to B^{s-\frac{1}{p}+(n-1)(\frac{1}{r}-\frac{1}{p})}_{r,\infty}\bigl(\mathbb{R}^{n-1}\bigr)$ for $s > \frac{1}{p}$ and any $r > p$ (the result was actually anisotropic and valid for the restriction to linear submanifolds of codimension $m \geqslant 1$); cf. [27] and also [28, 6.5]. Traces of Sobolev spaces W^1_p were studied first by N. Aronszajn [2] around 1954. Later, around 1957, E. Gagliardo [13] considered the trace of W^1_1, which constitutes an 'extremal' case; cf. the vertex in Figure 1. However, the first explicit counterexamples for the borderline $s = \frac{1}{p}$ seem to be put forward by G. Grubb [14] (who stated they were due to L. Hörmander); these necessary conditions were then expanded in [21], and in Lemma 1 ff. below these are supplemented to a set of necessary conditions for the anisotropic Besov and Lizorkin-Triebel spaces; in view of this paper the conditions are also *sufficient* for a solution of the distributional trace problem for the spaces considered.

In all remaining cases where γ_0 is a continuous operator into $\mathcal{S}'(\mathbb{R}^{n-1})$ it turns out that γ_0 has properties different from the generic ones in Theorem 1. Recall that it remains to investigate

$$s = \begin{cases} \dfrac{a_n}{p} & \text{if } 1 \leqslant p \leqslant \infty, \\ \dfrac{|a|}{p} - |a'| & \text{if } 0 < p < 1; \end{cases} \tag{25}$$

this amounts to the following five cases, see Figure 1, of which only the subcase $q = 1$ of the first two has been completely covered in the literature hitherto (whilst only the first case and the subcase $q < \infty$ of the fourth have been settled isotropically):

- $B^{\frac{a_n}{p},a}_{p,q}(\mathbb{R}^n)$ with $1 \leqslant p < \infty$ and $0 < q \leqslant 1$;
- $B^{0,a}_{\infty,q}(\mathbb{R}^n)$ for $0 < q \leqslant 1$;
- $B^{\frac{|a|}{p}-|a'|,a}_{p,q}(\mathbb{R}^n)$ for $0 < p < 1$ and $0 < q \leqslant 1$;
- $F^{a_n,a}_{1,q}(\mathbb{R}^n)$ with $0 < q \leqslant \infty$;
- $F^{\frac{|a|}{p}-|a'|,a}_{p,q}(\mathbb{R}^n)$ for $0 < p < 1$ and $0 < q \leqslant \infty$.

However, as a preparation for these cases, some preliminaries are dealt with in the next subsection.

3.1. Approximation spaces and nonlinear extension. To describe the trace classes in the limit situations we introduce another class of spaces, actually a half-scale, related to the approximation by entire analytic functions of exponential type.

Definition 1. Let $p, q \in \,]0, \infty]$ and let (s, p, q) fulfill one of the following two conditions:

$$s > |a|\big(\tfrac{1}{p} - 1\big)_+, \tag{26}$$

$$s = |a|\big(\tfrac{1}{p} - 1\big)_+ \qquad \text{and} \qquad 0 < q \leqslant 1. \tag{27}$$

Then we define the anisotropic *approximation* space $A^{s,a}_{p,q}$ to be the set

(28)

$$A^{s,a}_{p,q}(\mathbb{R}^n) = \Big\{ f \in \mathcal{S}'(\mathbb{R}^n);\ \forall j \in \mathbb{N}_0,\ \exists h_j \in L_p(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n),$$
$$\operatorname{supp} \mathscr{F} h_j \subset \big\{\xi;\ |\xi|_a \leqslant 2^j\big\},\ f \underset{\mathcal{S}'}{=} \sum_{j=0}^{\infty} h_j,\ \Big(\sum_{j=0}^{\infty} 2^{jsq} \big\| h_j \mid L_p(\mathbb{R}^n) \big\|^q \Big)^{1/q} < \infty \Big\};$$

this is equipped with the quasi-norm

$$\big\| f \mid A^{s,a}_{p,q}(\mathbb{R}^n) \big\| = \inf \left(\sum_{j=0}^{\infty} 2^{jsq} \big\| h_j \mid L_p(\mathbb{R}^n) \big\|^q \right)^{1/q}, \tag{29}$$

where the infimum is taken over all admissible representations $f = \sum h_j$. (If $q = \infty$, the ℓ_q-norm should be replaced by the supremum over j in both instances above.)

S. M. Nikol'skij and O. V. Besov have (in connection with Besov spaces for $1 \leqslant p \leqslant \infty$) defined such spaces $A^{s,a}_{p,q}$. For a comprehensive treatment and additional references, see the monograph of S. M. Nikol'skij [28, 3.3 and 5.6]; cf. also H. Triebel [40, 2.5.3].

The restrictions on s make the $A^{s,a}_{p,q}(\mathbb{R}^n)$ continuously embedded into $\mathcal{S}'(\mathbb{R}^n)$, cf. the following proposition and its corollary on the identifications between the $A^{s,a}_{p,q}$ and other classes of functions. Here and throughout $C_{\mathrm{b}}(\mathbb{R}^n)$ denotes the set of all uniformly continuous, bounded, complex-valued functions on $\mathbb{R}^n$ equipped with the supremum norm.

Proposition 2. (i) $A^{s,a}_{p,q}(\mathbb{R}^n) = B^{s,a}_{p,q}(\mathbb{R}^n)$ *if, and only if,* $s > |a| \left(\frac{1}{p} - 1\right)_+$.
(ii) *Let* $1 \leqslant p < \infty$ *and* $0 < q \leqslant 1$. *Then* $A^{0,a}_{p,q}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$.
(iii) *Let* $0 < q \leqslant 1$. *Then* $A^{0,a}_{\infty,q}(\mathbb{R}^n) = C_{\mathrm{b}}(\mathbb{R}^n)$.

In the affirmative cases the quasi-norms are equivalent.

Remark 3.1. Parts (ii) and (iii) are contained in Burenkov and Gol'dman's work [5], at least implicitly. An isotropic counterpart is stated in Oswald [30], albeit with the spaces based on approximation by splines. Furthermore, the "if"-part in (i) has been known before, cf. Nikol'skij [28, 3.3 and 5.6], Triebel [40, 2.5]; recently also Netrusov [26] considered this issue.

In virtue of Proposition 2 the continuity of $A^{s,a}_{p,q} \hookrightarrow \mathcal{S}'$ is clear for $p \geqslant 1$; when $0 < p < 1$ the anisotropic Nikol'skij-Plancherel-Polya inequality (cf. [43, 2.13]) and the restrictions (26)–(27) immediately give an embedding into L_1, so one has

Corollary 2. *The classes* $A^{s,a}_{p,q}(\mathbb{R}^n)$ *are continuously embedded into* $L_{\max(1,p)}(\mathbb{R}^n)$ *(which for any* $p \in \,]0, \infty]$ *is a subspace of* $\mathcal{S}'$*).*

For a more detailed comparison of the classes $A^{s,a}_{p,q}(\mathbb{R}^n)$ in the remaining limit situations with spaces of Besov-Lizorkin-Triebel type we refer to Section 4.

To make extensions to $\mathbb{R}^n$ of suitable f in $\mathcal{S}'(\mathbb{R}^{n-1})$, we consider any f such that $f = \sum\limits_{j=0}^{\infty} h_j$ holds in $\mathcal{S}'(\mathbb{R}^{n-1})$ with $\operatorname{supp} \mathscr{F}_{n-1} h_j \subset \left\{ \xi' ;\ |\xi'|_{a'} \leqslant 2^j \right\}$. Analogously to K, the extension Ef is defined (whenever the following series converges in $\mathcal{S}'$) as

$$Ef(x',x_n)=\sum_{j=0}^{\infty}2^{-ja_n}\mathscr{F}_1^{-1}[\eta_j](x_n)\ h_j(x')\,. \tag{30}$$

Observe that this is not a mapping—despite the notation—since each f equals many sums like $\sum h_j$.

By the homogeneity properties of the Fourier transform,

$$\lim_{N\to\infty}\sum_{j=0}^{N}2^{-ja_n}\mathscr{F}_1^{-1}[\eta_j](0)\ h_j(x')=\sum_{j=0}^{\infty}h_j(x')=f(x')\,. \tag{31}$$

Hence $\gamma_0\,Ef = f$ is valid for such f, in particular when f belongs to some $A^{s,a'}_{p,q}(\mathbb{R}^{n-1})$. For simplicity's sake, a particular extension Ef is said to depend *boundedly* on f (although Ef is not a map) when (32) or (33) below holds:

Theorem 4. *There exists a constant c such that the inequalities*

$$\big\|Ef\mid B^{s,a}_{p,q}(\mathbb{R}^n)\big\|\leqslant c\,\big\|f\mid A^{s-\frac{a_n}{p},a'}_{p,q}(\mathbb{R}^{n-1})\big\|, \tag{32}$$

$$\big\|Eg\mid F^{s,a}_{p,q}(\mathbb{R}^n)\big\|\leqslant c\,\big\|g\mid A^{s-\frac{a_n}{p},a'}_{p,p}(\mathbb{R}^{n-1})\big\| \tag{33}$$

hold for all $f\in A^{s-\frac{a_n}{p},a'}_{p,q}(\mathbb{R}^{n-1})$ and all $g\in A^{s-\frac{a_n}{p},a'}_{p,p}(\mathbb{R}^{n-1})$, respectively.

Remark 3.2. Approximation spaces like $A^{s,a}_{p,q}$ have been used for the trace problem earlier, e.g. directly by P. Oswald [30] and in the proofs of V. I. Burenkov and M. L. Gol'dman [5]. Traces of the approximation spaces themselves have been investigated by Yu. V. Netrusov [26].

3.2. Consequences for the trace. Using the tools from the previous subsection, one can now prove

Theorem 5. *The assertions of the introduction's main theorem are valid, and for any $v(x')$ in $A^{s-\frac{a_n}{p},a'}_{p,q}(\mathbb{R}^{n-1})$ or $A^{s-\frac{a_n}{p},a'}_{p,p}(\mathbb{R}^{n-1})$ there is an extension Ev in $B^{s,a}_{p,q}(\mathbb{R}^n)$ or $F^{s,a}_{p,q}(\mathbb{R}^n)$, respectively, depending boundedly on such v; in the generic cases Ev may be defined by means of a bounded operator Kv.*

It may be beneficial to list the consequences for the various borderline cases, so we will formulate separate results with detailed references. For brevity it is in the following understood that the extensions Ef depend boundedly on f, when f is viewed as an element of the pertinent range space for γ_0.

Corollary 3. *Let $1 \leqslant p < \infty$ and $0 < q \leqslant 1$. Then the trace operator γ_0 maps $B^{\frac{a_n}{p},a}_{p,q}(\mathbb{R}^n)$ continuously onto $L_p(\mathbb{R}^{n-1})$ with extensions $Ef \in B^{\frac{a_n}{p},a}_{p,q}(\mathbb{R}^n)$ depending boundedly on f.*

The first contribution to this was made by S. Agmon and L. Hörmander [1], who dealt with $p = 2$, $q = 1$ and $a = (1, \dots, 1)$. For $1 \leqslant p < \infty$, $q = 1$ and $a = (1, \dots, 1)$, the first part of the the result was stated by J. Peetre [31]. The anisotropic variant was proved by V. I. Burenkov and M. L. Gol'dman [5] for $q = 1$. Later M. Frazier and B. Jawerth [11] and J. Johnsen [22] gave proofs (except for the extensions) by other methods for all $q \leqslant 1$ in the isotropic situation.

Corollary 4. *When $0 < q \leqslant 1$, then the trace γ_0 maps $B^{0,a}_{\infty,q}(\mathbb{R}^n)$ continuously onto $C_{\mathrm{b}}\big(\mathbb{R}^{n-1}\big)$ with extensions $Ef \in B^{0,a}_{\infty,q}(\mathbb{R}^n)$ depending boundedly on f.*

For $q = 1$ the first proof of this result was given by V. I. Burenkov and M. L. Gol'dman [5].

Corollary 5. *When $p,\ q \in\ ]0, 1]$, then γ_0 maps $B^{\frac{|a|}{p}-|a'|,a}_{p,q}$ continuously onto the space $A^{|a'|(\frac{1}{p}-1),a'}_{p,q}\big(\mathbb{R}^{n-1}\big)$ with extensions $Ef \in B^{\frac{|a|}{p}-|a'|,a}_{p,q}\big(\mathbb{R}^n\big)$ depending boundedly on f.*

Corollary 5 should be a novelty; the determination of the trace space as an approximation space does not, to our knowledge, have any forerunners.

Corollary 6. *The trace operator γ_0 maps $F^{a_n,a}_{1,q}(\mathbb{R}^n)$ continuously onto $L_1(\mathbb{R}^{n-1})$, and there are extensions $Ef \in F^{a_n,a}_{1,q}(\mathbb{R}^n)$ depending boundedly on f.*

In [13], Gagliardo proved $\gamma_0\big(W^1_1(\mathbb{R}^n)\big) = L_1\big(\mathbb{R}^{n-1}\big)$, which is closely related to Corollary 6 because of the embedding $F^1_{1,2}(\mathbb{R}^n) \hookrightarrow W^1_1(\mathbb{R}^n) \hookrightarrow B^1_{1,\infty}(\mathbb{R}^n)$. M. Frazier and B. Jawerth [12] were the first to attempt a proof of the corollary, but their argument seems rather flawed; cf. Remark 3.3 below; H. Triebel obtained the first part of Corollary 6 by another approach based on atomic decompositions [41, 4.4.3], but without making it clear that $F^{a_n,a}_{1,\infty}(\mathbb{R}^n)$ is a subspace of $C\big(\mathbb{R}, \mathscr{D}'(\mathbb{R}^{n-1})\big)$.

The next result is an analogue of Corollary 5.

Corollary 7. *When $0 < p < 1$, then γ_0 maps $F^{\frac{|a|}{p}-|a'|,a}_{p,q}(\mathbb{R}^n)$ continuously onto the space $A^{|a'|(\frac{1}{p}-1),a'}_{p,p}\big(\mathbb{R}^{n-1}\big)$ with extensions $Ef \in F^{\frac{|a|}{p}-|a'|,a}_{p,q}(\mathbb{R}^n)$ depending boundedly on f.*

A final remark to the existence of a linear right inverse of γ_0: J. Peetre [31] ($a=(1,\dots,1)$) and V. I. Burenkov and M. L. Gol'dman [5] (general anisotropic case) have shown that if $1\leqslant p<\infty$ and $s=a_n/p$, then there exists no linear extension operator mapping $L_p(\mathbb{R}^{n-1})$ boundedly to $B^{\frac{a_n}{p},a}_{p,1}(\mathbb{R}^n)$; whether this remains true in other borderline cases seems to be unknown.

R e m a r k 3.3. Directly below [12, Th. 11.1] the authors write (in our notation): "We will show directly that $\gamma_0(F^s_{p,q})$ is independent of q. Given this, we have $\gamma_0(F^s_{p,q})=\gamma_0(F^s_{p,p})$ and all conclusions follow from the [Besov space] results in [11, Sect. 5], since $B^s_{p,p}=F^s_{p,p}$." However, it is evident from their proof that they tacitly assume γ_0 to be defined on $F^s_{p,q}$ for *two* arbitrary sum-exponents (q and r in $]0,\infty]$), and not just for $q=p$, but they never support this implicit claim by arguments.

Although it is true (and trivial to verify for the generic cases as well as for $s=\frac{a_n}{p}-|a'|$ when $p<1$, in view of the Sobolev embedding into $B^{a_n,a}_{1,1}(\mathbb{R}^n)$), it does require a proof that γ_0 is well defined on $F^{a_n,a}_{1,q}(\mathbb{R}^n)$, since if $1<q\leqslant\infty$ this space is strictly larger than any Besov space on which γ_0 is defined; cf. the vertex in Figure 1. The authors claim to have covered such F-spaces as a novelty, but in view of the described flaw it should be appropriate that we prove that $F^{a_n,a}_{1,\infty}(\mathbb{R}^n)\hookrightarrow C_{\mathrm{b}}\big(\mathbb{R},L_1(\mathbb{R}^{n-1})\big)$ and that $\gamma_0(F^{a_n,a}_{1,q})$ is independent of $q\in\,]0,\infty]$; see Proposition 1 ff. and Proposition 8 below.

4. A sharper comparison of the Besov-Lizorkin-Triebel classes and the approximation spaces

In Proposition 2 we have identified $A^{s;a}_{p,q}$ with standard function spaces in the generic cases. We now investigate the remaining borderline case $s=|a|\big(\frac{1}{p}-1\big)$ for $p<1$; the analysis involves the continuity properties of γ_0 proved in Theorem 5 above.

Concerning the borderline with $s=|a|\big(\frac{1}{p}-1\big)$, it is noteworthy that the two cases $q\leqslant p<1$ and $p<q\leqslant 1$ give quite different results:

Theorem 6. *Let $0<p<1$ and $0\leqslant q\leqslant 1$. Then*

$$A^{|a|(\frac{1}{p}-1),a}_{p,q}(\mathbb{R}^n)\hookrightarrow B^{|a|(\frac{1}{r}-1),a}_{r,u}(\mathbb{R}^n) \tag{34}$$

holds if, and only if,

$$\max(p,q)\leqslant r\leqslant\infty \qquad \text{and} \qquad u=\infty\,, \tag{35}$$

and whenever $0<r<\infty$ and $0<u\leqslant\infty$, then

$$A^{|a|(\frac{1}{p}-1),a}_{p,q}(\mathbb{R}^n)\not\subset F^{|a|(\frac{1}{r}-1),a}_{r,u}(\mathbb{R}^n). \tag{36}$$

Conversely,

$$B^{|a|(\frac{1}{r}-1),a}_{r,u}(\mathbb{R}^n) \hookrightarrow A^{|a|(\frac{1}{p}-1),a}_{p,q}(\mathbb{R}^n) \tag{37}$$

holds if

$$0 < r \leqslant p \qquad \text{and} \qquad 0 < u \leqslant q\,, \tag{38}$$

while

$$F^{|a|(\frac{1}{r}-1),a}_{r,u}(\mathbb{R}^n) \hookrightarrow A^{|a|(\frac{1}{p}-1),a}_{p,q}(\mathbb{R}^n) \tag{39}$$

holds if one of the following conditions does so:

$$\big(0 < r < p \quad \text{and} \quad r \leqslant q\big) \qquad \text{or} \qquad \big(r = p \leqslant q \quad \text{and} \quad 0 < u \leqslant q\big). \tag{40}$$

The necessity of the conditions (38) and (40) has been obtained for the parts concerning r and p, but not for the sum-exponents; cf. Remark 5.1 below.

By application of the above results, it is clear that, for $p < 1$, one has embeddings

$$B^{|a|(\frac{1}{p}-1),a}_{p,q}(\mathbb{R}^n) \hookrightarrow A^{|a|(\frac{1}{p}-1),a}_{p,q}(\mathbb{R}^n) \hookrightarrow B^{|a|(\frac{1}{r}-1),a}_{r,\infty}(\mathbb{R}^n), \qquad r = \max(p,q). \tag{41}$$

The latter is *optimal* in the sense that any space $B^{s,a}_{p,q}$ or $F^{s,a}_{p,q}$ (with $s = |a|(\frac{1}{p}-1)$), which $A^{s,a}_{p,q}$ is embedded into, actually also has the Besov space on the right hand side as an embedded subspace. This follows from (35)–(36) and the usual embeddings.

However, a somewhat sharper argument yields the following result:

Proposition 3. *For $0 < p < 1$ and $0 < q \leqslant 1$ the space $A^{|a|(\frac{1}{p}-1),a}_{p,q}(\mathbb{R}^n)$ is neither a Besov space $B^{s',a}_{p',q'}(\mathbb{R}^n)$ nor a Lizorkin-Triebel space $F^{s',a}_{p',q'}(\mathbb{R}^n)$ for any admissible (s',p',q').*

From this proposition, from Theorem 5, and the fact that also L_1 is neither a Besov nor a Lizorkin-Triebel space, we obtain

Corollary 8. *For $0 < p \leqslant 1$ the ranges of the trace operator γ_0,*

$$\gamma_0\left(B^{\frac{|a|}{p}-|a'|,a}_{p,q}(\mathbb{R}^n)\right), \quad 0 < q \leqslant 1, \qquad \text{and} \qquad \gamma_0\left(F^{\frac{|a|}{p}-|a'|,a}_{p,q}(\mathbb{R}^n)\right), \quad 0 < q \leqslant \infty,$$

do not belong to the scales of anisotropic Besov-Lizorkin-Triebel spaces on $\mathbb{R}^{n-1}$.

The embedding of the trace spaces $\gamma_0\big(B^{\frac{|a|}{p}-|a'|,a}_{p,q}(\mathbb{R}^n)\big)$ into $B^{|a'|(\frac{1}{r}-1),a'}_{r,\infty}\big(\mathbb{R}^{n-1}\big)$ for $r = \max(p,q)$, cf. (34), was first proved by Johnsen [22] (isotropic situation). By the above discussion, the present results are sharper and optimal.

A final remark on the borderline cases. For fixed p one may ask for the largest spaces on which the trace exists. One should clearly minimise s and maximise q (which is done throughout this paper), but the dependence on the anisotropy a may also be considered. For the isotropic spaces (indicated without the a) the following embeddings hold, for $p < \infty$:

$$B^{\frac{1}{p}+(n-1)(\frac{1}{p}-1)_+}_{p,q}(\mathbb{R}^n) \hookrightarrow B^{\frac{a_n}{p}+|a'|(\frac{1}{p}-1)_+,a}_{p,q}(\mathbb{R}^n), \tag{42}$$

$$F^{\frac{1}{p}+(n-1)(\frac{1}{p}-1)_+}_{p,q}(\mathbb{R}^n) \hookrightarrow F^{\frac{a_n}{p}+|a'|(\frac{1}{p}-1)_+,a}_{p,q}(\mathbb{R}^n); \tag{43}$$

i.e. the anisotropic spaces are larger than the corresponding isotropic ones. Furthermore, if $p < 1$, then the Besov spaces $B^{\frac{a_n}{p}+|a'|(\frac{1}{p}-1)_+,a}_{p,1}(\mathbb{R}^n)$ and the Lizorkin-Triebel spaces $F^{\frac{a_n}{p}+|a'|(\frac{1}{p}-1)_+,a}_{p,\infty}(\mathbb{R}^n)$ are the largest possible, however, all of them are contained in $F^{a_n;a}_{1,\infty}(\mathbb{R}^n)$ which is the largest possible for $p = 1$. If $1 < p < \infty$, the only candidate is the Besov space $B^{\frac{a_n}{p},a}_{p,1}(\mathbb{R}^n)$. Returning to the anisotropic classes with different anisotropies, it is not hard to see that the spaces depend increasingly on each component a_j of a. So, although a_n is bounded by $a_n \leqslant s \cdot p$, within the anisotropic spaces there are no maximal spaces on which the trace makes sense. For $p = \infty$ there is a largest space having a continuous trace, namely $C_{\mathrm{b}}(\mathbb{R}^n)$. But, as mentioned in the introduction, all these anisotropic spaces are subclasses of $C\big(\mathbb{R}, \mathscr{D}'(\mathbb{R}^{n-1})\big)$ with its natural notion of a trace.

5. Remaining proofs

Some proofs below are essentially just anisotropic variants of known techniques (scattered in the journals and main references like [35, 40, 41]), but even so the most important ones are presented (or sketched) here for the reader's convenience. It would lead too far to do this consistently, so in the remaining cases we shall have to make do with indications of the necessary changes, however.

5.1. The assertions in Section 2.

5.1.1. Proof of Theorem 1. By means of elementary embeddings (cf. around (115) below), the 'only if' part is, for $p < \infty$, a consequence of the following lemma, which is carried over from [21, Lem. 2.8].

Lemma 1. *Let $0 < p < \infty$. For any $u \in \mathcal{S}(\mathbb{R}^{n-1})$ there exists $u_k \in \mathcal{S}(\mathbb{R}^n)$ such that*

$$\gamma_0 u_k(x') = u(x') \quad \text{for any} \quad k \in \mathbb{N}, \tag{44}$$

$$\lim_{k\to\infty} u_k = 0 \quad \text{in} \quad B^{\frac{a_n}{p},a}_{p,q}(\mathbb{R}^n) \quad \text{if} \quad 1 < q \leqslant \infty, \tag{45}$$

$$\lim_{k\to\infty} u_k = 0 \quad \text{in} \quad F^{\frac{a_n}{p},a}_{p,q}(\mathbb{R}^n) \quad \text{if} \quad 1 < p < \infty. \tag{46}$$

If $0 < p \leqslant 1$, there exists $v_k \in \mathcal{S}(\mathbb{R}^n)$ such that $\lim_{k\to\infty} \gamma_0 v_k = \delta_0$ in $\mathcal{S}'(\mathbb{R}^{n-1})$ while

$$\lim_{k\to\infty} v_k = 0 \quad \text{in} \quad B^{\frac{|a|}{p}-|a'|,a}_{p,q}(\mathbb{R}^n) \quad \text{for} \quad 1 < q \leqslant \infty; \tag{47}$$

here δ_0 stands for the Dirac measure at $x' = 0$.

To prove Lemma 1 one can set $u_k(x) = u(x')w_k(x_n)$ with

$$w_k(x_n) = \frac{1}{k}\sum_{l=1}^{k} w\big(2^{la_n}x_n\big) \tag{48}$$

where $w \in \mathcal{S}(\mathbb{R})$ with $\operatorname{supp}\widehat{w} \subset \big\{\xi_n \in \mathbb{R};\, \frac{3}{4} \leqslant |\xi_n|^{1/a_n} \leqslant 1\big\}$ and $w(0) = 1$. Moreover,

$$v_k(x', x_n) = \frac{1}{k}\sum_{l=k+1}^{2k} 2^{l|a'|}\, f\big(2^{la'}x'\big)\, g\big(2^{la_n}x_n\big) \tag{49}$$

has the claimed properties at least when $f \in \mathcal{S}(\mathbb{R}^{n-1})$ and $g \in \mathcal{S}(\mathbb{R})$ satisfy

$$\operatorname{supp}\mathscr{F}_{n-1}f \subset \big\{\xi' \in \mathbb{R}^{n-1};\ |\xi'|_{a'} \leqslant \tfrac{1}{2}\big\} \quad \text{and} \quad \int_{\mathbb{R}^{n-1}} f(x')\,\mathrm{d}x' = 1, \tag{50}$$

$$\operatorname{supp}\mathscr{F}g \subset \big\{\xi_n \in \mathbb{R};\ |\xi_n|^{1/a_n} \leqslant \tfrac{1}{2}\big\} \quad \text{and} \quad g(0) = 1. \tag{51}$$

Indeed, it is not hard to check that the norms are $\mathscr{O}\big(k^{\frac{1}{q}-1}\big)$, respectively $\mathscr{O}\big(k^{\frac{1}{p}-1+\varepsilon}\big)$ for the $F^{s,a}_{p,q}$-norm, see [21, Lem. 2.8], for one may use the fact that if $s > 0$ (it is here the restriction $p < \infty$ is needed), then there exists a constant $c > 0$ such that

$$\big\|f \otimes g \mid B^{s,a}_{p,q}(\mathbb{R}^n)\big\| \leqslant c\,\big\|f \mid B^{s,a'}_{p,q}(\mathbb{R}^{n-1})\big\|\,\big\|g \mid B^{s/a_n}_{p,q}(\mathbb{R})\big\|, \tag{52}$$

$$\big\|f \otimes g \mid F^{s,a}_{p,q}(\mathbb{R}^n)\big\| \leqslant c\,\big\|f \mid F^{s,a'}_{p,q}(\mathbb{R}^{n-1})\big\|\,\big\|g \mid F^{s/a_n}_{p,q}(\mathbb{R})\big\|. \tag{53}$$

The isotropic version of (52)–(53) is due to J. Franke, see [10, Lem. 1]. For the anisotropic version one may use a paramultiplicative decomposition (in Yamazaki's sense [43, 44]) of the direct product $f \otimes g$ and apply the estimates in [43] (cf. the case $f = \delta_0$ treated in [21]).

For the space $B^{0,a}_{\infty,q}(\mathbb{R}^n)$ with $1 < q \leqslant \infty$, one can modify the proof of the lemma by taking $u \in \mathcal{S}$ to have a sufficiently small (non-empty) spectrum such as the ball $\{\xi' ;\ |\xi'|_{a'} \leqslant \frac{1}{2}\}$. Then the norm of u_k is seen to be $\mathscr{O}(k^{\frac{1}{q}-1})$ if, instead of (52), one calculates directly by means of an anisotropic Lizorkin representation (in the language of [40, 2.6]) with a *smooth* partition of unity. (This means that the φ_j entering the norm of $B^{s,a}_{p,q}$ should be replaced by some η_{jk}, with k in a finite j-independent set, such that $\bigcup_k \operatorname{supp} \eta_{jk}$ equals a 'corridor' (the complement of an n-dimensional rectangle inside a dilation by 2^a of itself) and such that each η_{jk} is a product $\theta_k(2^{ja'}\xi')\,\omega_k(2^{ja_n}\xi_n)$; this is well known to give an equivalent quasi-norm for $B^{s,a}_{p,q}$ by Lemma 3; an isotropic version may be found in [22].)

It remains to prove the stated continuity of γ_0. For the Lizorkin-Triebel case one may show that the operator γ_0 is defined on $F^{s,a}_{p,q}$ for every (s,p,q) considered in the theorem, see Proposition 1, and that $\gamma_0(F^{s,a}_{p,q})$ is independent of q; this last fact is obtained in Appendix C.4 below. Then the identification $B^{s,a}_{p,p} = F^{s,a}_{p,p}$ reduces the question to the Besov case; for the boundedness of γ_0 one may use the inequality (134) in Appendix C.4 below.

For the treatment of $B^{s,a}_{p,q}$, we use the short argument of [22, Sect. 3]; the idea is to combine the Nikol'skij-Plancherel-Polya inequality with the Paley-Wiener-Schwartz Theorem to deduce the crucial mixed-norm estimate (57).

First we remark that if $h \in \mathcal{S}'(\mathbb{R}^n)$ satisfies $\operatorname{supp} \mathscr{F} h \subset \{\xi \in \mathbb{R}^n ;\ |\xi|_a \leqslant A\}$ for some $A > 0$, then the restriction $h(x', \cdot)$, obtained by freezing x', fulfils

$$\operatorname{supp} \mathscr{F}_{x_n \mapsto \xi_n} h(x', \xi_n) \subset \{\xi_n \in \mathbb{R};\ |\xi_n| \leqslant A^{a_n}\}. \tag{54}$$

This is a consequence of (91) below and the Paley-Wiener-Schwartz Theorem (cf. [17, 7.3.1]), for these give that $g := h(x', \cdot)$ is analytic and satisfies

$$g(x_n + \mathrm{i} y_n) \leqslant C(x')\big(1 + |x_n|\big)^N \mathrm{e}^{A^{a_n}|y_n|}. \tag{55}$$

Now, let $f \in B^{s,a}_{p,q}(\mathbb{R}^n)$. Applying (54) with $A = 2^j$ to $\mathscr{F}^{-1}[\varphi_j \mathscr{F} f](x', \cdot)$, the Nikol'skij-Plancherel-Polya inequality, see for example [40, 1.3.2], yields

$$\sup_{x_n \in \mathbb{R}} \big|\mathscr{F}^{-1}[\varphi_j \mathscr{F} f](x', x_n)\big| \leqslant c\, 2^{j a_n/p} \Big(\int_{\mathbb{R}} \big|\mathscr{F}^{-1}[\varphi_j \mathscr{F} f](x', x_n)\big|^p \,\mathrm{d}x_n \Big)^{1/p} \tag{56}$$

where the constant c does not depend on x', f and j; hence x'-integration gives

$$\big\| \sup_{x_n} \mathscr{F}^{-1}[\varphi_j \mathscr{F} f](\cdot, x_n) \ \big|\ L_p(\mathbb{R}^{n-1}) \big\| \leqslant c\, 2^{j a_n/p} \big\| \mathscr{F}^{-1}[\varphi_j \mathscr{F} f] \,\big|\, L_p(\mathbb{R}^n) \big\|. \tag{57}$$

Estimating the supremum from below by the value for $x_n = 0$ and arguing as for (54), we obtain

$$\operatorname{supp} \mathscr{F}_{x' \mapsto \xi'} \big(\mathscr{F}^{-1}[\varphi_j \mathscr{F} f](x', 0)\big) \subset \big\{\xi' \in \mathbb{R}^{n-1};\ |\xi'|_{a'} \leqslant 2^j\big\}. \tag{58}$$

So under the restriction $s > \frac{a_n}{p} + |a'|\big(\frac{1}{p} - 1\big)_+$, Lemma 4 of Appendix D below yields the convergence of the series $\sum_{j=0}^{\infty} \mathscr{F}^{-1}[\varphi_j \mathscr{F} f](\cdot, 0)$ in $\mathcal{S}'(\mathbb{R}^{n-1})$ as well as the boundedness of $\gamma_0 \colon B^{s,a}_{p,q}(\mathbb{R}^n) \to B^{s - \frac{a_n}{p}, a'}_{p,q}(\mathbb{R}^{n-1})$.

5.1.2. Proof of Proposition 1. The particular value $x_n = 0$ is unimportant in (58), so when the application of the Nikol'skij-Plancherel-Polya inequality above is repeated, a slightly stronger conclusion is reached: for $p_1 := \max(1, p)$ and $f_j := \mathscr{F}^{-1}[\varphi_j \mathscr{F} f]$,

$$\sup_{x_n \in \mathbb{R}} \big\| f_j(\cdot, x_n) \mid L_{p_1}(\mathbb{R}^{n-1}) \big\| \leqslant c\, 2^{j\left(\frac{a_n}{p} + |a'|(\frac{1}{p} - 1)_+\right)} \, \| f_j \mid L_p \|. \tag{59}$$

If C_{b} temporarily stands for bounded, continuous functions, the left hand side is the norm of f_j in $C_{\mathrm{b}}\big(\mathbb{R}, L_{p_1}(\mathbb{R}^{n-1})\big)$, and this is in $\ell_1(\mathbb{N}_0)$ with respect to j because the right hand side is so. Consequently, the series $\sum f_j$ converges in $\mathcal{D}'(\mathbb{R}^n)$ to f and in $C_{\mathrm{b}}\big(\mathbb{R}, L_{p_1}(\mathbb{R}^{n-1})\big)$, and because the latter space is continuously embedded into the former (a reference to this folklore could be Prop. 3.5 and (5.4) in [22]), this shows that $f \in C_{\mathrm{b}}\big(\mathbb{R}, L_{p_1}(\mathbb{R}^{n-1})\big)$. Moreover, the triangle inequality applied to $f = \sum f_j$ yields continuity of

$$B^{s,a}_{p,q}(\mathbb{R}^n) \hookrightarrow C_{\mathrm{b}}\big(\mathbb{R}, L_{p_1}(\mathbb{R}^{n-1})\big) \qquad \text{whenever } s \geqslant \tfrac{a_n}{p} + |a'|\big(\tfrac{1}{p} - 1\big)_+, \text{ provided } q \leqslant 1 \text{ in case of equality.} \tag{60}$$

Finally, the uniform continuity with respect to x_n should be verified. However, if translation by $h \in \mathbb{R}$ is denoted by τ_h, that is $\tau_h f(x) = f(x', x_n - h)$, then the boundedness above yields the following, say with $s = \frac{a_n}{p} + |a'|\big(\frac{1}{p} - 1\big)_+$ for simplicity:

$$\sup_{x_n \in \mathbb{R}} \big\| f(\cdot, x_n) - f(\cdot, x_n - h) \mid L_{p_1}(\mathbb{R}^{n-1}) \big\| \leqslant c \Big(\sum_{j=0}^{\infty} 2^{sjq} \big\| (1 - \tau_h) f_j \mid L_{p_1} \big\|^q \Big)^{\frac{1}{q}}. \tag{61}$$

Indeed, this is clear since $1 - \tau_h$ commutes with $\mathscr{F}^{-1} \varphi_j \mathscr{F}$, and so it remains to note that the right hand side tends to 0 for $h \to 0$ by majorised convergence.

Since $F^{s,a}_{p,q} \hookrightarrow B^{s,a}_{p,\infty}$ may be used for the generic Lizorkin-Triebel cases, it suffices to consider $F^{s,a}_{p,q}$ in the borderline cases with $s = \frac{|a|}{p} - |a'|$ and $p \leqslant 1$; however, by the Sobolev embeddings it is enough to treat $p = 1$, hence to show that

$$F^{a_n, a}_{1,\infty}(\mathbb{R}^n) \hookrightarrow C_{\mathrm{b}}\big(\mathbb{R}, L_1(\mathbb{R}^{n-1})\big). \tag{62}$$

To do so, we replace the above use of the Nikol'skij-Plancherel-Polya inequality by an application of the Jawerth embedding $F^1_{1,\infty}(\mathbb{R}) \hookrightarrow B^0_{\infty,1}(\mathbb{R})$. As above, for $f \in F^{a_n,a}_{1,\infty}(\mathbb{R}^n)$,

$$f(x) = \sum_{j=0}^{\infty} f_j(x). \tag{63}$$

Note, as a preparation, that this series converges pointwise for $x \notin N$, where N is a Borel set in $\mathbb{R}^n$ with $\operatorname{meas}(N) = 0$; indeed, this follows since $\sum \|f_j \mid L_1\| < \infty$ must hold for any f in $F^{a_n,a}_{1,\infty}(\mathbb{R}^n)$.

By Fubini's theorem, there is also a null set $M \subset \mathbb{R}^{n-1}$ such that $x' \notin M$ implies

$$\int_{\mathbb{R}} \sup_j \bigl|2^{ja_n} f_j(x', x_n)\bigr| \, \mathrm{d}x_n < \infty. \tag{64}$$

Invoking Lemma 4 of Appendix D one therefore obtains a function $x_n \mapsto g(x', x_n)$ in $F^{a_n}_{1,\infty}(\mathbb{R})$ for which $g(x', \cdot) = \sum f_j(x', \cdot)$; using $a_n \geqslant 1$ to apply the Jawerth embedding, we have

$$\begin{aligned} |g(x', x_n)| &\leqslant \bigl\|g(x', \cdot) \mid B^0_{\infty,1}(\mathbb{R})\bigr\| \\ &\leqslant c \Bigl\|\sum f_j(x', \cdot) \mid F^{a_n}_{1,\infty}(\mathbb{R})\Bigr\| \leqslant c' \int_{\mathbb{R}} \sup_j \bigl|2^{ja_n} f_j(x', x_n)\bigr| \, \mathrm{d}x_n < \infty. \end{aligned} \tag{65}$$

The x'-dependence of g is not arbitrary as it seems to be, for there is another null set $M' \subset \mathbb{R}^{n-1}$ such that $M \subset M'$ and when $x' \notin M'$, then

$$g(x', x_n) = f(x', x_n) \quad \text{for } x_n \text{ a.e. in } \mathbb{R}. \tag{66}$$

Indeed, the section $N_{x'} = \bigl\{x_n ;\ (x', x_n) \in N\bigr\}$ is a Borel set and the relation $0 = \operatorname{meas}(N) = \int_{\mathbb{R}^{n-1}} \operatorname{meas}(N_{x'}) \, \mathrm{d}x'$ gives that $\operatorname{meas}(N_{x'}) = 0$ for x' outside a null set M', which may be assumed to contain M. So by (63), $f(x', \cdot) = \sum f_j(x', \cdot)$ holds outside $N_{x'}$ whenever $x' \notin M'$. But, since the norm series $\sum \bigl\|f_j(x', \cdot) \mid L_1(\mathbb{R})\bigr\|$ is estimated by the integral in (64), hence is finite, the series $\sum f_j(x', \cdot)$ converges to $g(x', \cdot)$ in $L_1(\mathbb{R})$. So, by the fact that a pointwise limit coincides a.e. with a limit in mean, (66) is obtained.

It is thus justified to integrate both sides of (65) with respect to x', and therefore

$$\sup_{x_n \in \mathbb{R}} \bigl\|f(\cdot, x_n) \mid L_1(\mathbb{R}^{n-1})\bigr\| \leqslant c \bigl\|f \mid F^{a_n,a}_{1,\infty}\bigr\|. \tag{67}$$

Now the uniform continuity in x_n of any f in $F^{a_n,a}_{1,\infty}$ may be shown by an argument analogous to the Besov case above. This proves (62) and thus the proposition.

5.1.3. Proof of Theorem 2. For part (i), the streamlined method of [22, Sect. 3] may be adopted as follows. Writing

$$Kv(x',x_n)=\sum_{j=0}^{\infty}u_j(x',x_n), \tag{68}$$

where for any $j\in\mathbb{N}_0$

$$u_j(x',x_n)=2^{-ja_n}\left(\mathscr{F}_1^{-1}\eta_j\right)(x_n)\,\mathscr{F}_{n-1}^{-1}\left[\varphi_j(\cdot,0)\,\mathscr{F}_{n-1}v\right](x'), \tag{69}$$

it is straightforward to see that u_j has a compact spectrum: using the triangle inequality one can see that $|\xi|_a\sim|\xi'|_{a'}+|\xi_n|^{1/a_n}$ and then find a constant $A>0$ such that

$$\operatorname{supp}\mathscr{F}u_j\subset\bigl\{\xi;\ 2^{ja_n}\leqslant|\xi_n|\leqslant 2^{(j+1)a_n},\ |\xi'|_{a'}\leqslant c\,2^j\bigr\}\subset\bigl\{\xi;\ \tfrac{2^j}{A}\leqslant|\xi|_a\leqslant A\,2^j\bigr\}. \tag{70}$$

For any $j\in\mathbb{N}_0$ we have, when $\check{\eta}:=\mathscr{F}_1^{-1}\eta$ and $\hat{v}:=\mathscr{F}_{n-1}v$,

$$2^{sj}\bigl\|u_j\mid L_p(\mathbb{R}^n)\bigr\|=2^{j(s-\frac{a_n}{p})}\bigl\|\check{\eta}\mid L_p(\mathbb{R})\bigr\|\cdot\bigl\|\mathscr{F}_{n-1}^{-1}[\varphi_j(\cdot,0)\hat{v}]\mid L_p(\mathbb{R}^{n-1})\bigr\| \tag{71}$$

where the right hand side is in ℓ_q provided $v\in B^{s-\frac{a_n}{p},a'}_{p,q}\bigl(\mathbb{R}^{n-1}\bigr)$. By Lemma 3, this implies

$$\bigl\|Kv\mid B^{s,a}_{p,q}(\mathbb{R}^n)\bigr\|\leqslant c\Bigl(\sum_{j=0}^{\infty}2^{jsq}\bigl\|u_j\mid L_p(\mathbb{R}^n)\bigr\|^q\Bigr)^{\frac{1}{q}}\leqslant c'\bigl\|v\mid B^{s-\frac{a_n}{p},a'}_{p,q}\bigr\| \tag{72}$$

and the proof of (i) is complete.

When $v\in B^{s,a'}_{p,p}(\mathbb{R}^{n-1})$ one can modify the corresponding proof of [40, 2.7.2] in the following way: by the $F^{s,a}_{p,q}$-part of Lemma 3, the series for Kv converges to an element of $F^{s+\frac{a_n}{p},a}_{p,q}(\mathbb{R}^n)$ if we can show a certain estimate; this is done as in [40]. The main thing is to get the correct auxiliary inequality which is

$$\bigl|2^{-ja_n}\mathscr{F}_1^{-1}\eta_j(x_n)\bigr|\leqslant c\bigl(1+2^{ja_n}|x_n|\bigr)^{-\delta} \tag{73}$$

for a sufficiently large positive δ; in addition it is convenient to split the integration there over the subintervals $I_l=\bigl]-2^{-la_n},-2^{-(l+1)a_n}\bigr]\cup\bigl[2^{-(l+1)a_n},2^{-la_n}\bigr[$.

5.2. Assertions in Section 3.

5.2.1. Proof of Proposition 2. For part (i) the definitions and line (93) below yield $B^{s,a}_{p,q}(\mathbb{R}^n)\hookrightarrow A^{s,a}_{p,q}(\mathbb{R}^n)$. However, if $s>|a|\bigl(\frac{1}{p}-1\bigr)_+$, then Lemma 4, cf. the appendix, implies that the two spaces are equal.

To prove parts (ii) and (iii), let $f \in L_p(\mathbb{R}^n)$ for some $p \in [1,\infty[$ (the case $f \in C_{\mathrm{b}}(\mathbb{R}^n)$ for $p=\infty$ is treated similarly). Selecting a subsequence j_k of $\mathbb{N}_0$ such that $\lambda_k := 2^{j_k}$ (with $\lambda_0 = 1$) satisfies

$$\big\|\mathscr{F}^{-1}[\psi(\lambda_k^a\cdot)\,\mathscr{F}f] - f \mid L_p(\mathbb{R}^n)\big\| < 2^{-k-1}\,\big\|f \mid L_p(\mathbb{R}^n)\big\| \qquad \text{for } k \in \mathbb{N}_0 \tag{74}$$

(which is possible as seen from the usual convolution estimates), one clearly has that the series

$$f(x) = \mathscr{F}^{-1}[\psi\mathscr{F}f](x) + \sum_{k=1}^{\infty} \big(\mathscr{F}^{-1}[\psi(\lambda_k^a\cdot)\mathscr{F}f](x) - \mathscr{F}^{-1}[\psi(\lambda_{k-1}^a\cdot)\mathscr{F}f](x)\big) \tag{75}$$

converges in $\big\|\cdot \mid L_p(\mathbb{R}^n)\big\|$. Setting h_j equal to the kth summand in (75) for $j = j_k$ and $h_j = 0$ for all other j, it is found by the definition of the $A^{s;a}_{p,q}$ that $f \in A^{0;a}_{p,q}(\mathbb{R}^n)$ for all q and

$$\big\|f \mid A^{0;a}_{p,q}(\mathbb{R}^n)\big\| \leqslant c\,\big\|f \mid L_p(\mathbb{R}^n)\big\|. \tag{76}$$

The converse inclusion may be shown for $0 < q \leqslant 1$, for when $f \in A^{0;a}_{p,q}(\mathbb{R}^n)$ is written as $f = \sum h_j$ according to the definition, then the embedding $l_q \hookrightarrow l_1$ yields

$$\sum_{j=0}^{\infty} \big\|h_j \mid L_p(\mathbb{R}^n)\big\| \leqslant \Big(\sum_{j=0}^{\infty} 2^{j\cdot 0\cdot q}\big\|h_j \mid L_p(\mathbb{R}^n)\big\|^q\Big)^{1/q}, \tag{77}$$

so that the completeness of L_p gives $\big\|f \mid L_p(\mathbb{R}^n)\big\| \leqslant \big\|f \mid A^{0;a}_{p,q}(\mathbb{R}^n)\big\|$. The proof is complete.

5.2.2. Proof of Theorem 4. Here one can use the same strategy as for Theorem 2, except that a given v in the approximation space should be written as $v = \sum h_j$ and h_j should then replace $\mathscr{F}^{-1}_{n-1}\varphi_j(\cdot,0)\mathscr{F}_{n-1}v$; this works because also h_j has its spectrum in the ball $\{\xi';\ |\xi'|_{a'} \leqslant 2^j\}$ and because the representation $\sum h_j$ can be chosen such that its relevant norm is less than $2\,\big\|v \mid A^{s-\frac{a_n}{p},a'}_{p,q}(\mathbb{R}^{n-1})\big\|$.

5.2.3. Proof of Theorem 5 (and of the main theorem). Obviously (b) entails (a) in the main theorem; cf. Corollary 2. The fact that (a) implies (c) is proved in connection with Theorem 1, and the extensions depending boundedly on v were established in Theorem 4 and, for K, in Theorem 2. So it remains to prove (c) $\Longrightarrow$ (b).

When f is in $B^{s;a}_{p,q}$ one derives (57) as before; since $s \geqslant \frac{a_n}{p} + |a'|\big(\frac{1}{p} - 1\big)_+$ it is straightforward to see from (57) that $\sum \mathscr{F}^{-1}[\varphi_j\mathscr{F}f](\cdot,0)$ converges in L_p if $p \geqslant 1$ or, by the Nikol'skij-Plancherel-Polya inequality, in L_1 if $p < 1$. So we may write

$\gamma_0 f = \sum \mathscr{F}^{-1}[\varphi_j \mathscr{F} f](\cdot,0)$, and (57), (58) and the definition of the approximation spaces thereafter show the boundedness of γ_0 from $B^{s,a}_{p,q}$ into $A^{s-\frac{a_n}{p},a'}_{p,q}$. The Lizorkin-Triebel case follows from the Besov case as before; cf. Propositions 1 and 8 below.

5.3. The assertions in Section 4.

Proof of Theorem 6. *Step 1.* To deduce all the embeddings, note that when $s = |a|\big(\frac{1}{p}-1\big)$ and $r = \max(p,q)$, then it is easy to establish that

$$B^{s,a}_{p,q}(\mathbb{R}^n) \hookrightarrow A^{s,a}_{p,q}(\mathbb{R}^n) \hookrightarrow B^{|a|(\frac{1}{r}-1),a}_{r,\infty}(\mathbb{R}^n); \tag{78}$$

in fact, the first inclusion is obvious from the definitions, and the second follows if Lemma 5 below is invoked in addition. Then the sufficiency of (35) and (38) is clear in view of the Sobolev embeddings, and from (115) or the anisotropic Jawerth embedding in Proposition 7 below it is seen analogously that (40) implies (39).

Step 2. To show that (34) implies the u-part of (35) in Theorem 6, we shall use the already proved boundedness of γ_0; it is therefore convenient to replace (34) by the assumption that $A^{|a'|(\frac{1}{p}-1),a'}_{p,q}(\mathbb{R}^{n-1}) \hookrightarrow B^{|a'|(\frac{1}{r}-1),a'}_{r,u}(\mathbb{R}^{n-1})$ for some $n \geqslant 2$.

It suffices to find Schwartz functions g_k such that

$$\sup_{k\in\mathbb{N}} \big\| g_k \mid B^{\frac{|a|}{p}-|a'|,a}_{p,q}(\mathbb{R}^n)\big\| < \infty, \tag{79}$$

$$\big\|\gamma_0 g_k \mid B^{|a'|(\frac{1}{r}-1),a'}_{ru}(\mathbb{R}^{n-1})\big\| \geqslant c k^{1/u} \quad \text{for any} \quad k \geqslant 2. \tag{80}$$

Indeed, it then follows from (34) and the boundedness of γ_0 from $B^{\frac{|a|}{p}-|a'|,a}_{p,q}$ that

$$ck^{1/u} \leqslant c' \big\| \gamma_0 g_k \,|A^{|a'|(\frac{1}{p}-1),a'}_{p,q}(\mathbb{R}^{n-1})\big\| \leqslant c'' \big\| g_k \,|B^{\frac{|a|}{p}-|a'|,a}_{p,q}(\mathbb{R}^n)\big\|, \tag{81}$$

which contradicts (79) unless $u = \infty$.

To prove the existence of such g_k, note first that it is possible to take the partition of unity $1 = \sum \varphi_j$ such that, say, $\varphi_0(\xi) = 1$ for $|\xi|_a \leqslant \frac{11}{10}$ and $\varphi_0(\xi) = 0$ for $|\xi|_a \geqslant \frac{13}{10}$ (this choice is consistent with the conventions in [20], which will be convenient later). Similarly the η of Section 2.2 should fulfill $\operatorname{supp} \eta \subset \,]1, \frac{21}{20}[$.

With some k_0 to be determined, we set

$$g_k(x',x_n) = \big(\mathscr{F}_1^{-1}\eta\big)\big(2^{(k+k_0)a_n}x_n\big) \sum_{j=0}^{k} \mathscr{F}_{n-1}^{-1}[\varphi_j(\cdot,0)](x'). \tag{82}$$

Then $\operatorname{supp}\mathscr{F}g_k$ is contained in the set where both $|\xi'|_{a'} < \frac{13}{10}\cdot 2^k$ and $2^{(k+k_0)a_n} < |\xi_n| < \frac{21}{20}\cdot 2^{(k+k_0)a_n}$ hold, and consequently (since $|\xi_n|^{1/a_n} \leqslant |\xi|_a \leqslant |\xi'|_{a'} + |\xi_n|^{1/a_n}$) the number k_0 may be taken so large that for all k,

$$\text{(83)} \qquad \operatorname{supp}\mathscr{F}g_k \subset \big\{\xi\in\mathbb{R}^n\,;\ 2^{k+k_0} \leqslant |\xi|_a \leqslant \tfrac{11}{10}\,2^{k+k_0}\big\} \subset \{\xi\,;\ \varphi_{k+k_0}(\xi)=1\}.$$

By the definition, the $B^{s,a}_{p,q}$-norm of g_k is therefore equal to

$$2^{(k+k_0)\left(\frac{|a|}{p}-|a'|\right)} \times \big\| g_k \mid L_p(\mathbb{R}^n)\big\|,$$

so since $\sum\limits_{j=0}^{k}\varphi_j = \psi(2^{-ka}\cdot)$, we conclude

$$\text{(84)} \quad \big\|g_k \mid L_p\big\| \leqslant c2^{-k\frac{a_n}{p}}\big\|\mathscr{F}_1^{-1}\eta \mid L_p(\mathbb{R})\big\|\cdot 2^{k\left(|a'|-\frac{|a'|}{p}\right)}\big\|\mathscr{F}^{-1}_{n-1}[\psi(\cdot,0)] \mid L_p(\mathbb{R}^{n-1})\big\|$$

and the claim in (79) follows immediately.

To prove (80), note that the definition of the Besov norm implies

$$\text{(85)} \qquad \begin{aligned} &\big\| g_k(\cdot,0) \mid B^{|a'|(\frac{1}{r}-1),a'}_{r,u}(\mathbb{R}^{n-1})\big\| \\ &\qquad \geqslant \Big(\sum_{j=0}^{k-1} 2^{j|a'|(\frac{1}{r}-1)u}\big\|\mathscr{F}^{-1}_{n-1}(\varphi_j(\cdot,0)) \mid L_r(\mathbb{R}^{n-1})\big\|^u\Big)^{1/u} \\ &\qquad \geqslant ck^{1/u}. \end{aligned}$$

If one assumes (34), now in dimension n again, it then follows that $B^{|a|(\frac{1}{p}-1),a}_{p,q} \hookrightarrow A^{|a|(\frac{1}{p}-1),a}_{p,q} \hookrightarrow B^{|a|(\frac{1}{r}-1),a}_{r,\infty}$ and this embedding between the Besov spaces implies $r \geqslant p$ (as one may show by considering the functions ϱ_k in [20, Lem. 4.1] for $-k\in\mathbb{N}_0$; cf. Remark 4.6 there). That the embedding (34) implies $r\geqslant q$ may be proved by a standard technique; it is e.g. easy to adapt the proof of the similar, isotropic statement in [22, Prop. 3.2].

Step 3. If there is an embedding as in (36), then a Sobolev embedding gives for some finite $t>r$ that $A^{|a|(\frac{1}{p}-1),a}_{p,q} \hookrightarrow B^{|a|(\frac{1}{t}-1),a}_{t,t}$, and this would contradict the u-part of (35).

Proof of Proposition 3. Suppose for $s := |a|\big(\frac{1}{p}-1\big)$ that $A^{s,a}_{p,q}(\mathbb{R}^n) = X$, where X denotes either a Besov or a Lizorkin-Triebel space with some parameter (τ,ϱ,ω).

Since $B^{s,a}_{p,q} \hookrightarrow A^{s,a}_{p,q} \hookrightarrow X$ it follows from the necessity of the parameter restrictions of the usual embeddings in the Besov and Lizorkin-Triebel scales that

$$\text{(86)} \qquad \tau\leqslant s, \qquad \varrho\geqslant p, \qquad \tau-\tfrac{|a|}{\varrho} \leqslant s-\tfrac{|a|}{p}.$$

Moreover, since $X \hookrightarrow B^{|a|(\frac{1}{r}-1),a}_{r,\infty}$ for $r = \max(p,q)$, it follows in the same way, since $B^{|a|(\frac{1}{r}-1),a}_{r,\infty}$ has differential dimension $-|a|$, that

$$\tau - \tfrac{|a|}{\varrho} \geqslant -|a| = s - \tfrac{|a|}{p}. \tag{87}$$

Hence $\tau - \frac{|a|}{\varrho} = s - \frac{|a|}{p}$, and therefore X is a Besov space according to (36).

Using these conclusions, we obtain from the necessity of (35) that $\varrho \geqslant r$, and because

$$B^{|a|(\frac{1}{\varrho}-1),a}_{\varrho,\omega} = X \hookrightarrow B^{|a|(\frac{1}{r}-1),a}_{r,\infty}, \tag{88}$$

we find that $\varrho = r$. Then we see from this and (34)–(35) that $\omega = \infty$.

Finally we conclude that $A^{s,a}_{p,q} = X = B^{|a|(\frac{1}{r}-1),a}_{r,\infty}$ must hold under the assumption made at the beginning of the proof; but this conclusion is absurd since the Dirac measure δ_0 belongs to $B^{|a|(\frac{1}{r}-1),a}_{r,\infty}$, which then contradicts the inclusion $A^{s,a}_{p,q} \subset L_1$ that one obtains from Corollary 2.

Remark 5.1. The necessity of the first inequality in (38) may be obtained by means of the special Schwartz functions ϱ_k constructed in [20, Lem. 4.1]; these may be inserted for $-k \in \mathbb{N}$ into the inequality expressing the boundedness of the embeddings $B^{|a|(\frac{1}{r}-1),a}_{r,u} \hookrightarrow A^{|a|(\frac{1}{p}-1),a}_{p,q} \hookrightarrow L_p$. However, the second part of (38) is not so easy to handle, for when one analogously inserts the $\varrho^{(s-\frac{|a|}{p})}_{N,l}$, the $A^{s,a}_{p,q}$-norms of these functions are troublesome to calculate.

To prove the necessity of (40), given (39), it may be used for any $t < r$ that

$$B^{|a|(\frac{1}{t}-1),a}_{t,t} \hookrightarrow F^{|a|(\frac{1}{r}-1),a}_{r,u} \hookrightarrow A^{|a|(\frac{1}{p}-1),a}_{p,q}; \tag{89}$$

from (the established necessity of) (38) it follows that $t \leqslant p$—hence by taking the supremum over such t that $r \leqslant p$, which is the r-part of (40).

Clearly this would give $r \leqslant \min(p,q)$ if (38) could be shown to be necessary in its entirety. So, if $r < p$ we would have deduced that $r \leqslant q$. Moreover, for $r = p$ the conclusion $r \leqslant \min(p,q)$ would reduce to the inequality $p \leqslant q$, and so it would remain to be proved that $u \leqslant q$. Here one could try to calculate the norms of the $\theta^{(s)}_N$ of [20, Lem. 4.1], but then the same difficulties would occur as above for (38).

Appendix A. Notation

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing C^∞-functions on $\mathbb{R}^n$, equipped with the usual topology; $\mathcal{S}'(\mathbb{R}^n)$ denotes the topological dual, the space of all tempered distributions on $\mathbb{R}^n$. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then $\widehat{\varphi} = \mathscr{F}\varphi$ and $\check{\varphi} = \mathscr{F}^{-1}\varphi$ are respectively the Fourier and the inverse Fourier transform of φ, extended in the usual way from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$.

The space of uniformly continuous, bounded functions on $\mathbb{R}^n$, valued in a Banach space X, is denoted by $C_{\mathrm{b}}(\mathbb{R}^n, X)$; for $X = \mathbb{C}$ the Banach space is suppressed.

For a normed or quasi-normed space X we denote by $\|x \mid X\|$ the norm of the vector x. Recall that X is quasi-normed when the triangle inequality is weakened to $\|x + y \mid X\| \leqslant c\big(\|x \mid X\| + \|y \mid X\|\big)$ for some $c \geqslant 1$ independent of x and y.

All unimportant positive constants are denoted by c, occasionally with additional subscripts within the same formulas. The equivalence "$\mathrm{term}_1 \sim \mathrm{term}_2$" means that there exist two constants $c_1, c_2 > 0$ independent of the variables in the two terms such that $c_1\,\mathrm{term}_1 \leqslant \mathrm{term}_2 \leqslant c_2\,\mathrm{term}_1$.

Appendix B. Anisotropic function spaces

The conventions we adopt here are, by and large, those of [43, 44]. For each coordinate x_i in $\mathbb{R}^n$ a weight a_i is given such that $\min(a_1, \dots, a_n) = 1$. The vector $a = (a_1, \dots, a_n)$ is called an n-dimensional anisotropy, and $|a| := a_1 + \ldots + a_n$. If $a = (1, \dots, 1)$ we have the *isotropic* case.

For a given $a = (a_1, \dots, a_n)$, the action of $t \in [0, \infty)$ on $x \in \mathbb{R}^n$ is defined by the formula

$$t^a x = \big(t^{a_1} x_1, \dots, t^{a_n} x_n\big). \tag{90}$$

For $t > 0$ and $s \in \mathbb{R}$ we set $t^{sa}x = (t^s)^a x$. In particular, $t^{-a}x = (t^{-1})^a x$ and $2^{-ja}x = (2^{-j})^a x$.

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $x \neq 0$, let $|x|_a$ be the unique positive number t such that

$$\frac{x_1^2}{t^{2a_1}} + \ldots + \frac{x_n^2}{t^{2a_n}} = 1 \tag{91}$$

and let $|0|_a = 0$ for $x = 0$.

By [43, 1.4], [43, 3.8] the map $|\cdot|_a$ is an anisotropic distance function, which is C^∞ and coincides with $|\cdot|$ in the isotropic case. (Anisotropic distance functions are continuous maps $u\colon \mathbb{R}^n \to \mathbb{R}$ fulfilling $u(x) > 0$ if $x \neq 0$ and $u(t^a x) = t u(x)$ for all $t > 0$ and all $x \in \mathbb{R}^n$; any two such functions u and u' are equivalent in the sense that

$u(x) \sim u'(x)$, see [36] and [6, 1.2.3].) Moreover, setting $a_{\max} = \max\{a_i\,;\ 1 \leqslant i \leqslant n\}$ one has, cf. [43, 1.4], for any $x \in \mathbb{R}^n$ that

$$\min\bigl\{|x|, |x|^{1/a_{\max}}\bigr\} \leqslant |x|_a \leqslant \max\bigl\{|x|, |x|^{1/a_{\max}}\bigr\}. \tag{92}$$

If $(\varphi_j)_{j\in\mathbb{N}_0}$ is the anisotropic partition of unity from Section 2, then for any $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$f = \sum_{j=0}^{\infty} \mathscr{F}^{-1}(\varphi_j \mathscr{F} f) \qquad \text{with convergence in} \quad \mathcal{S}'(\mathbb{R}^n). \tag{93}$$

Let $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, $s \in \mathbb{R}$. The anisotropic Besov space $B^{s,a}_{p,q}(\mathbb{R}^n)$ and, provided $p < \infty$, the anisotropic Lizorkin-Triebel space $F^{s,a}_{p,q}(\mathbb{R}^n)$ are defined to consist of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the following quasi-norms are finite:

$$\bigl\|f \mid B^{s,a}_{p,q}(\mathbb{R}^n)\bigr\| = \Bigl(\sum_{j=0}^{\infty} 2^{jsq} \bigl\|\mathscr{F}^{-1}(\varphi_j \mathscr{F} f) \mid L_p(\mathbb{R}^n)\bigr\|^q \Bigr)^{1/q}, \tag{94}$$

$$\bigl\|f \mid F^{s,a}_{p,q}(\mathbb{R}^n)\bigr\| = \Bigl\|\Bigl(\sum_{j=0}^{\infty} 2^{jsq} \bigl|\mathscr{F}^{-1}(\varphi_j \mathscr{F} f)(\cdot)\bigr|^q \Bigr)^{1/q} \Bigm| L_p(\mathbb{R}^n)\Bigr\|, \tag{95}$$

respectively (with the usual modification if $q = \infty$).

Both $B^{s,a}_{p,q}(\mathbb{R}^n)$ and $F^{s,a}_{p,q}(\mathbb{R}^n)$ are quasi-Banach spaces (Banach spaces if $p \geqslant 1$ and $q \geqslant 1$) which are independent of the choice of $(\varphi_j)_{j\in\mathbb{N}_0}$.

The embeddings $\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^{s,a}_{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n) \hookrightarrow F^{s,a}_{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ hold true for all admissible values of p, q, s. Furthermore, if both $p < \infty$ and $q < \infty$, the ranges of these inclusions are all dense; cf. [43, 3.5] and [6, 1.2.10]. The results on embeddings are reviewed (and extended) in Appendix C.3 below; let us conclude with a few identifications.

If $1 < p < \infty$ and $s \in \mathbb{R}$ then $F^{s,a}_{p,2}(\mathbb{R}^n) = H^{s,a}_p(\mathbb{R}^n)$ with equivalent quasi-norms; hereby

$$H^{s,a}_p(\mathbb{R}^n) = \Bigl\{ f \in \mathcal{S}'(\mathbb{R}^n)\,;\ \Bigl\| \Bigl(\sum_{k=1}^{n} \bigl(1 + \xi_k^2\bigr)^{s/(2a_k)} \widehat{f}\Bigr)^{\vee} \Bigm| L_p(\mathbb{R}^n) \Bigr\| < \infty \Bigr\} \tag{96}$$

is the anisotropic Bessel potential space; cf. [37, Rem. 11], [38, 2.5.2], and [43, 3.11].

Furthermore, if $1 < p < \infty$, $s \in \mathbb{R}$ and if $s_1 = \frac{s}{a_1} \in \mathbb{N}$, $\ldots$, $s_n = \frac{s}{a_n} \in \mathbb{N}$ then $F^{s,a}_{p,2}(\mathbb{R}^n) = W^{s,a}_p(\mathbb{R}^n)$ (with equivalent quasi-norms), where

$$W^{s,a}_p(\mathbb{R}^n) = \Bigl\{ f \in \mathcal{S}'(\mathbb{R}^n); \bigl\|f \mid L_p(\mathbb{R}^n)\bigr\| + \sum_{k=1}^{n} \Bigl\| \frac{\partial^{s_k} f}{\partial x_k^{s_k}} \Bigm| L_p(\mathbb{R}^n) \Bigr\| < \infty \Bigr\} \tag{97}$$

is the classical anisotropic Sobolev space on $\mathbb{R}^n$. If $s > 0$ and $\frac{s}{a_k} \notin \mathbb{N}$ for $k = 1, \ldots, n$, then $B^{s,a}_{\infty,\infty}(\mathbb{R}^n) = C^{s,a}(\mathbb{R}^n)$ are the anisotropic Hölder spaces.

Anisotropic spaces have been intensively studied by S. M. Nikol'skij [28], and by O. V. Besov, V. P. Il'in and S. M. Nikol'skij [4]. See also works of M. Yamazaki [43, 44], H.-J. Schmeisser and H. Triebel [33, 4.2], A. Seeger [34], P. Dintelmann [6, 7] etc.

Appendix C. Properties of the anisotropic spaces

The point of this section is to sketch how Propositions 7 and 8 may be proved in the present anisotropic set-up. Along the way, we also prove some interpolation formulas that are of interest in their own right. The main tool will be anisotropic versions of the atomic decompositions of [11, 12], see W. Farkas [8].

Appendix C.1. Atomic decompositions. As a preparation, we recall some basic notions of atomic decompositions in an anisotropic setting.

Consider the lattice $\mathbb{Z}^n$ as a subset of $\mathbb{R}^n$. If $\nu \in \mathbb{N}_0$ and $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, we denote by $Q^a_{\nu m}$ the rectangle in $\mathbb{R}^n$ centred at $2^{-\nu a} m = (2^{-\nu a_1} m_1, \ldots, 2^{-\nu a_n} m_n)$, which has sides parallel to the axes and side lengths $2^{-\nu a_1}, \ldots, 2^{-\nu a_n}$, respectively ($Q^a_{0m}$ is a cube with side length 1). If $Q^a_{\nu m}$ is such a rectangle in $\mathbb{R}^n$ and $c > 0$, then $cQ^a_{\nu m}$ denotes the concentric rectangle with side lengths $c2^{-\nu a_1}, \ldots, c2^{-\nu a_n}$. If $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n_0$ is a multi-index and if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then $x^\beta := x_1^{\beta_1} \ldots x_n^{\beta_n}$, and we write $a\beta = a_1\beta_1 + \ldots + a_n\beta_n$. If $E \subset \mathbb{R}^n$ is Lebesgue measurable, then $|E|$ denotes its Lebesgue measure. Now, anisotropic atoms are defined as follows.

Definition 2. Let $s \in \mathbb{R}$, $0 < p \leqslant \infty$ and $K, L \in \mathbb{R}$. A function $\varrho\colon \mathbb{R}^n \to \mathbb{C}$ for which $D^\beta \varrho$ exists when $a\beta \leqslant K$ (or ϱ is continuous if $K \leqslant 0$) is called an anisotropic $(s,p)_{K,L}$-atom, if

$$\operatorname{supp} \varrho \subset cQ^a_{\nu m} \quad \text{for some } \nu \in \mathbb{N},\ m \in \mathbb{Z}^n \text{ and } c > 1, \tag{98}$$

$$\left|D^\beta \varrho(x)\right|^{|a|} \leqslant \left|Q^a_{\nu m}\right|^{s - a\beta - \frac{|a|}{p}} \quad \text{if } a\beta \leqslant K, \tag{99}$$

$$\int_{\mathbb{R}^n} x^\beta \varrho(x)\, \mathrm{d}x = 0 \quad \text{if } a\beta \leqslant L. \tag{100}$$

If conditions (98) and (99) are satisfied for $\nu = 0$, then ϱ is called an anisotropic 1_K-atom.

If the atom ϱ is located at $Q^a_{\nu m}$ (i.e. $\operatorname{supp} \varrho \subset cQ^a_{\nu m}$ with $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, $c > 1$) we denote it by $\varrho^a_{\nu m}$. The value of the number $c > 1$ in (98) is unimportant; it allows, at any level ν, a controlled overlap of the supports of different $\varrho^a_{\nu m}$.

The main advantage of the atomic approach is that one can often reduce a problem given in $B^{s,a}_{p,q}$ or $F^{s,a}_{p,q}$ to some corresponding sequence spaces; these are here denoted by $b_{p,q}$ and $f^a_{p,q}$. If $Q^a_{\nu m}$ is a rectangle as above, let $\chi_{\nu m}$ be the characteristic function of $Q^a_{\nu m}$; then

$$2^{\nu|a|/p}\chi_{\nu m} \tag{101}$$

is the $L_p(\mathbb{R}^n)$-normalised characteristic function of $Q^a_{\nu m}$ whenever $0 < p \leqslant \infty$.

If $0 < p, q \leqslant \infty$, then $b_{p,q}$ is the collection of all $\lambda = \{\lambda_{\nu m} \in \mathbb{C};\ \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ such that

$$\|\lambda \mid b_{p,q}\| = \Bigl(\sum_{\nu=0}^{\infty}\Bigl(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^p\Bigr)^{q/p}\Bigr)^{1/q} \tag{102}$$

is finite (usual modification if $p = \infty$ and/or $q = \infty$). Furthermore, $f^a_{p,q}$ is the collection of all such sequences λ for which

$$\|\lambda \mid f^a_{p,q}\| = \Bigl\|\Bigl(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}2^{\nu|a|/p}\chi_{\nu m}(\cdot)|^q\Bigr)^{1/q}\Bigm| L_p(\mathbb{R}^n)\Bigr\| \tag{103}$$

(usual modification if $p = \infty$ and/or $q = \infty$) is finite.

For $0 < p \leqslant \infty$ and $0 < q \leqslant \infty$ we will use the abbreviations

$$\sigma_p = |a|\bigl(\tfrac{1}{p} - 1\bigr)_+ \quad \text{and} \quad \sigma_{p,q} = |a|\bigl(\tfrac{1}{\min(p,q)} - 1\bigr)_+. \tag{104}$$

Proposition 4. (i) *Let $0 < p < \infty$, $0 < q \leqslant \infty$ and $s \in \mathbb{R}$, and let K, $L \in \mathbb{R}$ fulfill*

$$K \geqslant a_{\max} + s \quad \text{if } s \geqslant 0, \tag{105}$$

$$L \geqslant \sigma_{p,q} - s. \tag{106}$$

Then $g \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $F^{s,a}_{p,q}(\mathbb{R}^n)$ if, and only if, for some $\lambda = (\lambda_{\nu m})$ in $f^a_{p,q}$,

$$g = \sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\varrho^a_{\nu m}, \text{ with convergence in } \mathcal{S}'(\mathbb{R}^n), \tag{107}$$

where $\varrho^a_{\nu m}$ are anisotropic 1_K-atoms ($\nu = 0$) or anisotropic $(s,p)_{K,L}$-atoms ($\nu \in \mathbb{N}$).

Furthermore, $\inf \|\,\lambda\,|f^a_{p,q}\|$ with the infimum taken over all admissible representations (107) is an equivalent quasi-norm in $F^{s,a}_{p,q}(\mathbb{R}^n)$.

(ii) *The analogous statements are valid for the Besov spaces $B^{s,a}_{p,q}(\mathbb{R}^n)$ for $0 < p \leqslant \infty$ provided $\sigma_{p,q}$ and $f^a_{p,q}$ (together with its norm $\|\cdot|f^a_{p,q}\|$) are replaced by σ_p and $b_{p,q}$, respectively.*

P r o o f. The proposition is a slightly different version of the atomic decomposition theorem proved in [8]; the modifications needed are immaterial, so we omit details. □

Appendix C.2. Real Interpolation. Our aim is to prove a refined Sobolev embedding due to B. Jawerth [18] and J. Franke [10] in the isotropic context. As usual $(\cdot,\cdot)_{\theta,q}$ denotes the real interpolation.

Lemma 2. *Let $s_0 \neq s_1$ and $s=(1-\theta)s_0+\theta s_1$, where $0<\theta<1$. If $0<p\leqslant\infty$ then*

$$\big(B^{s_0,a}_{p,q_0}(\mathbb{R}^n), B^{s_1,a}_{p,q_1}(\mathbb{R}^n)\big)_{\theta,q} = B^{s,a}_{p,q}(\mathbb{R}^n), \tag{108}$$

$$\big(F^{s_0,a}_{p,q_0}(\mathbb{R}^n), F^{s_1,a}_{p,q_1}(\mathbb{R}^n)\big)_{\theta,q} = B^{s,a}_{p,q}(\mathbb{R}^n), \tag{109}$$

provided $0<p<\infty$ in the last formula.

Formulas (108) and (109) can be proved using the same arguments as in [40, 2.4.2], and we refrain from doing this here. In the case of constant s, there is another result:

Proposition 5. *Let $0<\theta<1$ and $0<p_0<p<p_1<\infty$. When $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$, then*

$$\big(F^{s,a}_{p_0,q}(\mathbb{R}^n), F^{s,a}_{p_1,q}(\mathbb{R}^n)\big)_{\theta,p} = F^{s,a}_{p,q}(\mathbb{R}^n). \tag{110}$$

Proof. Replacing (if necessary) $|\cdot|_a$ by an equivalent anisotropic distance function we may assume

$$\big\{x\in\mathbb{R}^n\,;\ |x|_a\leqslant 2\big\}\subset[-\pi,\pi]^n. \tag{111}$$

Let $f^a_{p,q}$ as in (103). By immaterial modifications of the proof of [12, (6.10)] we have

$$\big(f^a_{p_0,q}, f^a_{p_1,q}\big)_{\theta,p} = f^a_{p,q} \quad \text{for} \quad \tfrac{1}{p}=\tfrac{1-\theta}{p_0}+\tfrac{\theta}{p_1}. \tag{112}$$

Then (110) is a consequence of (112) and the following anisotropic φ-transform. □

Proposition 6. *Let $(\varphi_j)_{j\in\mathbb{N}_0}$ be a partition of unity as in Section 2, and let $\varrho\in\mathcal{S}(\mathbb{R}^n)$ fulfill $\varrho(x)=1$ if $|x|_a\leqslant 2$ and $\operatorname{supp}\varrho\subset[-\pi,\pi]^n$. The operators $U_\varphi\colon F^{s,a}_{p,q}(\mathbb{R}^n)\to f^a_{p,q}$ and $T_\varrho\colon f^a_{p,q}\to F^{s,a}_{p,q}(\mathbb{R}^n)$ defined by*

$$U_\varphi(g)=\big\{(2\pi)^{-n/2}\,2^{\nu\left(s-\frac{n}{p}\right)}(\varphi_\nu\widehat{g})^\vee(2^{-\nu a}m)\ \big|\ \nu\in\mathbb{N}_0\,,m\in\mathbb{Z}^n\big\} \tag{113}$$

for $g\in F^{s,a}_{p,q}(\mathbb{R}^n)$ and by

$$T_\varrho(\lambda)=\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\,2^{-\nu\left(s-\frac{n}{p}\right)}\check{\varrho}(2^{\nu a}\cdot -m) \tag{114}$$

for $\lambda=\{\lambda_{\nu m}\,;\ \nu\in\mathbb{N}_0\,,m\in\mathbb{Z}^n\}$ belonging to $f^a_{p,q}$, respectively, are bounded.

Furthermore, $(T_\varrho\circ U_\varphi)(g)=g$ for any $g\in F^{s,a}_{p,q}(\mathbb{R}^n)$ and $\|U_\varphi(\cdot)\,|\,f^a_{p,q}\|$ is an equivalent quasi-norm on $F^{s,a}_{p,q}(\mathbb{R}^n)$.

The corresponding results hold for the $B^{s,a}_{p,q}(\mathbb{R}^n)$ spaces with $b_{p,q}$ in place of $f^a_{p,q}$.

P r o o f. For $p, q < \infty$ a proof may be found in [7], where density of $\mathcal{S}(\mathbb{R}^n)$ in $B^{s,a}_{p,q}(\mathbb{R}^n)$ and $F^{s,a}_{p,q}(\mathbb{R}^n)$ was used. In the remaining cases one may proceed, for example, as in [42, 14.15]. $\square$

The proposition, due to P. Dintelmann [7, Theorem 1], represents the anisotropic version of a theorem originally proved by M. Frazier and B. Jawerth, see [12].

Appendix C.3. Embeddings. In addition to elementary embeddings (monotonicity in s and q) we have

$$B^{s,a}_{p,\min(p,q)}(\mathbb{R}^n) \hookrightarrow F^{s,a}_{p,q}(\mathbb{R}^n) \hookrightarrow B^{s,a}_{p,\max(p,q)}(\mathbb{R}^n); \tag{115}$$

hence $B^{s,a}_{p,p} = F^{s,a}_{p,p}$ whenever $0 < p < \infty$. There are, moreover, the Sobolev embeddings

$$B^{s,a}_{p,q}(\mathbb{R}^n) \hookrightarrow B^{t,a}_{r,q}(\mathbb{R}^n) \qquad \text{and} \qquad F^{s,a}_{p,q}(\mathbb{R}^n) \hookrightarrow F^{t,a}_{r,u}(\mathbb{R}^n) \tag{116}$$

provided that

$$p < r \qquad \text{and} \qquad s - \tfrac{|a|}{p} = t - \tfrac{|a|}{r}; \tag{117}$$

q and u are independent of each other. These assertions may conveniently be found in [43]. There is also an anisotropic version of the Jawerth-Franke embedding (which has features in common with both of the above types):

Proposition 7. *Let $0 < p_0 < p < p_1 \leqslant \infty$, $s_1 < s < s_0$ and $0 < q \leqslant \infty$. Then*

$$B^{s_0,a}_{p_0,p}(\mathbb{R}^n) \hookrightarrow F^{s,a}_{p,q}(\mathbb{R}^n) \hookrightarrow B^{s_1,a}_{p_1,p}(\mathbb{R}^n) \tag{118}$$

provided $s_0 - \frac{|a|}{p_0} = s - \frac{|a|}{p} = s_1 - \frac{|a|}{p_1}$.

P r o o f. Let $0 < p_0 < p' < p < p'' < \infty$ and let

$$s' = s + \tfrac{|a|}{p_0} - \tfrac{|a|}{p'} \quad \text{and} \quad s'' = s + \tfrac{|a|}{p_0} - \tfrac{|a|}{p''}. \tag{119}$$

As a consequence of (116) we obtain

$$F^{s',a}_{p_0,q}(\mathbb{R}^n) \hookrightarrow F^{s,a}_{p',q}(\mathbb{R}^n) \quad \text{and} \quad F^{s'',a}_{p_0,q}(\mathbb{R}^n) \hookrightarrow F^{s,a}_{p'',q}(\mathbb{R}^n). \tag{120}$$

There exists $\theta \in \,]0,1[$ such that $\frac{1}{p} = \frac{1-\theta}{p'} + \frac{\theta}{p''}$. Then $s_0 = (1-\theta)s' + \theta s''$.

Using (109), elementary properties of the interpolation, and (110) we have

(121)
$$B^{s_0,a}_{p_0,p}(\mathbb{R}^n) = \big(F^{s',a}_{p_0,q}(\mathbb{R}^n), F^{s'',a}_{p_0,q}(\mathbb{R}^n)\big)_{\theta,p} \hookrightarrow \big(F^{s,a}_{p',q}(\mathbb{R}^n), F^{s,a}_{p'',q}(\mathbb{R}^n)\big)_{\theta,p} = F^{s,a}_{p,q}(\mathbb{R}^n)$$

and this gives the first embedding in (118).

To prove the second, let now $p' < p < p'' < p_1 \leqslant \infty$ and $s^{(j)} = s - \frac{|a|}{p^{(j)}} + \frac{|a|}{p_1}$ for $j = 1, 2$. Then by (116) and (115) we have $F^{s,a}_{p^{(j)},q}(\mathbb{R}^n) \hookrightarrow B^{s^{(j)},a}_{p_1,\infty}(\mathbb{R}^n)$, and $s_1 = (1-\theta)s' + \theta s''$ when $\frac{1-\theta}{p'} + \frac{\theta}{p''} = \frac{1}{p}$, so applying (110) and (108) we conclude that

(122)
$$F^{s,a}_{p,q}(\mathbb{R}^n) = \big(F^{s,a}_{p',q}(\mathbb{R}^n), F^{s,a}_{p'',q}(\mathbb{R}^n)\big)_{\theta,p} \hookrightarrow \big(B^{s',a}_{p_1,\infty}(\mathbb{R}^n), B^{s'',a}_{p_1,\infty}(\mathbb{R}^n)\big)_{\theta,p} = B^{s_1,a}_{p_1,p}(\mathbb{R}^n).$$

□

Remark 8.1. The second and first part of (118) was proved by B. Jawerth [18] and by J. Franke [10], respectively, in the isotropic case. The present extension to $a \neq (1,\dots,1)$ would have been beneficial for e.g. the fine continuity properties of the pointwise product as investigated in [20], where it was necessary to distinguish between the isotropic and anisotropic cases in numerous places.

Appendix C.4. The q-independence of the range space $\gamma_0\big(F^{s,a}_{p,q}(\mathbb{R}^n)\big)$.

Proposition 8. *When γ_0 is defined on both $F^{s,a}_{p,q}(\mathbb{R}^n)$ and $F^{s,a}_{p,t}(\mathbb{R}^n)$, then*

$$\gamma_0\big(F^{s,a}_{p,q}(\mathbb{R}^n)\big) = \gamma_0\big(F^{s,a}_{p,t}(\mathbb{R}^n)\big). \tag{123}$$

Proof. We proceed as in [12, 11.1]. If $q < t$, the elementary embedding $F^{s,a}_{p,q}(\mathbb{R}^n) \hookrightarrow F^{s,a}_{p,t}(\mathbb{R}^n)$ implies $\gamma_0\big(F^{s,a}_{p,q}(\mathbb{R}^n)\big) \subset \gamma_0\big(F^{s,a}_{p,t}(\mathbb{R}^n)\big)$. To prove the converse inclusion, let K and L fulfill (105)–(106) and let us write $g \in F^{s,a}_{p,t}(\mathbb{R}^n)$ as

$$g = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\varrho^{a,t}_{\nu m}, \quad \text{convergence in } \mathcal{S}'(\mathbb{R}^n), \tag{124}$$

with $\|\lambda \mid f^a_{p,t}\| \leqslant c\,\|g \mid F^{s,a}_{p,t}(\mathbb{R}^n)\|$; cf. (107). We claim there exists a function $\widetilde{g} \in F^{s,a}_{p,q}(\mathbb{R}^n)$ with $\gamma_0(g) = \gamma_0(\widetilde{g})$. Only such rectangles $Q^a_{\nu m}$ for which the relevant $cQ^a_{\nu m}$ intersects $\Gamma = \{x;\ x_n = 0\}$, are important. Let

$$A = \big\{(\nu, m) \in \mathbb{N}_0 \times \mathbb{Z}^n;\ cQ^a_{\nu m} \cap \Gamma \neq \emptyset\big\}. \tag{125}$$

We define $\widetilde{\lambda}_{\nu m} = \lambda_{\nu m}$ if $(\nu, m) \in A$ and otherwise $\widetilde{\lambda}_{\nu m} = 0$, and put $\widetilde{\lambda} = \{\widetilde{\lambda}_{\nu m};\ \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$. Let now $\psi \in S(\mathbb{R})$ be such that $\operatorname{supp}\psi \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $\psi(0) = 1$ and, moreover,

$$\int_{\mathbb{R}} z^{\beta_n} \psi(z)\, \mathrm{d}z = 0 \quad \text{for all} \quad \beta_n \in \mathbb{N}_0 \quad \text{such that} \quad a_n\beta_n \leqslant L. \tag{126}$$

Using this we define

$$\widetilde{\varrho}^{a,q}_{\nu m}(x', x_n) = \varrho^{a,t}_{\nu m}(x', 0)\, \psi(2^{\nu a_n} x_n) \tag{127}$$

and remark that $\widetilde{\varrho}^{a,q}_{\nu m}$ is supported in a rectangle $c\widetilde{Q}^a_{\nu m}$ where $\widetilde{Q}^a_{\nu m}$ has sides parallel to the axes, is centred at $(2^{-\nu a_1} m_1, \ldots, 2^{-\nu a_{n-1}} m_{n-1}, 0)$ and its side lengths are respectively $2^{-\nu a_1}, \ldots, 2^{-\nu a_{n-1}}, 2^{-\nu a_n}$. Furthermore, if $\beta = (\beta', \beta_n) \in \mathbb{N}^n_0$ is such that $a\beta \leqslant K$ and if $\nu \in \mathbb{N}$ then

$$\left|D^\beta \widetilde{\varrho}^{a,q}_{\nu m}(x)\right| \leqslant c \left|D^{\beta'} \varrho^{a,t}_{\nu m}(x', 0)\right| \cdot 2^{\nu a_n \beta_n} \leqslant c'\, 2^{-\nu\left(s - \frac{|a|}{p}\right)}\, 2^{\nu a\beta}. \tag{128}$$

It follows that each $\widetilde{\varrho}^{a,q}_{\nu m}$, up to an unimportant constant, is an anisotropic 1_K-atom for $\nu = 0$ or an anisotropic $(s,p)_{K,L}$-atom (due to its product structure and to the assumptions on the function ψ there are no problems in checking the moment conditions).

Defining

$$\widetilde{g} = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \widetilde{\lambda}_{\nu m} \widetilde{\varrho}^{a,q}_{\nu m} = \sum_{(\nu,m)\in A} \lambda_{\nu m} \widetilde{\varrho}^{a,q}_{\nu m} \tag{129}$$

we have $\gamma_0(g) = \gamma_0(\widetilde{g})$. For $(\nu, m) \in A$, let

$$\widetilde{E}^a_{\nu m} = \left\{(x_1, \ldots, x_n) \in \widetilde{Q}^a_{\nu m};\ 2^{-(\nu+1)a_n - 1} < x_n \leqslant 2^{-\nu a_n - 1}\right\}. \tag{130}$$

Obviously,

$$\frac{\left|\widetilde{E}^a_{\nu m}\right|}{\left|\widetilde{Q}^a_{\nu m}\right|} = \frac{1 - 2^{-a_n}}{2} > 0. \tag{131}$$

Then (with the usual modification for $q = \infty$)

$$\left\|\widetilde{\lambda} \mid f^a_{p,q}\right\| \sim \left\| \Big(\sum_{(\nu,m)\in A} |\lambda_{\nu m}\, \widetilde{\chi}^{(p)}_{\nu m}(\cdot)|^q \Big)^{1/q} \Big| L_p\big(\mathbb{R}^n\big) \right\| \tag{132}$$

where $\widetilde{\chi}^{(p)}_{\nu m}$ denotes the $L_p(\mathbb{R}^n)$-normalised characteristic function of the rectangle $\widetilde{E}^a_{\nu m}$. Using an inequality of Fefferman-Stein type, see [9] for the anisotropic Hardy-Littlewood maximal function, the proof is a simple anisotropic counterpart of [12, 2.7].

For $(\nu, m) \in A$ the sets $\widetilde{E}^a_{\nu m}$ are pairwise disjoint and so at most one term in the sum on the right-hand side is nonzero. Hence q and $1/q$ cancel in (132) and may therefore be replaced by t and $1/t$. So, with the usual modification if $t = \infty$,

$$\|\widetilde{\lambda} \mid f^a_{p,q}\| \sim \Big\| \Big(\sum_{(\nu,m)\in A} |\lambda_{\nu m}\, \widetilde{\chi}^{(p)}_{\nu m}(\cdot)|^t \Big)^{1/t} \Big| L_p(\mathbb{R}^n) \Big\| \leqslant c\, \|\lambda \mid f^a_{p,t}\|. \tag{133}$$

The last relation together with (129) prove the fact that $\widetilde{g} \in F^{s,a}_{p,q}(\mathbb{R}^n)$ and

$$\|\widetilde{g} \mid F^{s,a}_{p,q}(\mathbb{R}^n)\| \leqslant c\, \|\widetilde{\lambda} \mid f^a_{p,q}\| \leqslant c'\, \|\lambda \mid f^a_{p,t}\| \leqslant c''\, \|g \mid F^{s,a}_{p,t}(\mathbb{R}^n)\|. \tag{134}$$

This verifies our claim. □

Remark 8.2. The q-independence of the traces of Lizorkin-Triebel spaces has also been treated by Yu. V. Netrusov [25].

Appendix C. Series of entire functions with compact spectra

First a few well-known results on convergence of certain series are recalled:

Lemma 3. *Let $0 < A < \infty$ and let $\{f_k\}_{k=0}^\infty$ be a sequence of functions on $\mathbb{R}^n$ such that*

$$\begin{cases} \operatorname{supp} \mathscr{F} f_k \subset \big\{\xi;\ \tfrac{1}{A} 2^k \leqslant |\xi|_a \leqslant A\, 2^k\big\}, \quad k = 1, \dots, \\ \operatorname{supp} \mathscr{F} f_0 \subset \big\{\xi;\ |\xi|_a \leqslant A\big\}. \end{cases} \tag{135}$$

Then one has, for all $s \in \mathbb{R}$ and $0 < p \leqslant \infty$, that

$$\Big\| \sum_{k=0}^\infty f_k \Big| B^{s,a}_{p,q}(\mathbb{R}^n) \Big\| \leqslant c \Big(\sum_{k=0}^\infty 2^{skq} \| f_k \,| L_p(\mathbb{R}^n)\|^q \Big)^{1/q}. \tag{136}$$

More precisely, if the right-hand side is finite, then $\sum_{k=0}^\infty f_k$ converges in $\mathcal{S}'$ to a distribution satisfying this inequality (where c depends on a, A, p, s, n but not on $\{f_k\}$).

For $p < \infty$ an analogous result holds for $F^{s,a}_{p,q}$, provided that (on the right hand side of (136)) the ℓ_q-norm is calculated pointwise at each $x \in \mathbb{R}^n$ before the L_p-norm is taken.

Lemma 4. *Let* $0 < A < \infty$ *and let* $\{f_k\}_{k=0}^{\infty}$ *be a sequence of functions on* $\mathbb{R}^n$ *such that*

$$\operatorname{supp} \mathscr{F} f_k \subset \{\xi;\ |\xi|_a \leqslant A\, 2^k\}, \quad k = 0,\, 1,\, \dots, \tag{137}$$

and suppose that

$$s > |a|\big(\tfrac{1}{p} - 1\big)_+. \tag{138}$$

Then the statements in (136) *ff. hold true. Moreover, for* $p < \infty$ *one has the analogous result for the space* $F^{s,a}_{p,q}$ *provided* (138) *is replaced by the condition*

$$s > |a| \max\big(\tfrac{1}{p} - 1, \tfrac{1}{q} - 1, 0\big). \tag{139}$$

These lemmas are proved in [43], but see also [20], [24] or [32, 2.3.2]. For the borderline case with equality in (138) we refer to [22, Th. 3.1]; it is straightforward to get anisotropic variants of this result, so without proof we state what is needed above for the case $p < 1$:

Lemma 5. *Let* $\{f_k\}_{k=0}^{\infty}$ *be a sequence of functions such that* (137) *is satisfied for some* $A < \infty$. *Let* $0 < p < 1$, $0 < q \leqslant 1$ *and suppose that* $s = |a|\big(\frac{1}{p} - 1\big)$. *Then we have*

$$\Big\| \sum_{k=0}^{\infty} f_k \,\Big|\, L_1(\mathbb{R}^n) \Big\| \leqslant c \Big(\sum_{k=0}^{\infty} 2^{k|a|(\frac{1}{p}-1)q} \| f_k \mid L_p(\mathbb{R}^n) \|^q \Big)^{1/q}. \tag{140}$$

More precisely, if the right-hand side is finite, then $\sum\limits_{k=0}^{\infty} f_k$ *converges in* $L_1(\mathbb{R}^n)$ *to a distribution* f *satisfying this inequality (where* c *depends on* A, p, s, n *but not on* f_k, $k = 0, 1, \dots$*). Moreover, this limit* f *also satisfies*

$$\Big\| \sum_{k=0}^{\infty} f_k \,\Big|\, B^{|a|(\frac{1}{r}-1),a}_{r,\infty}(\mathbb{R}^n) \Big\| \leqslant c \Big(\sum_{k=0}^{\infty} 2^{k|a|(\frac{1}{p}-1)q} \, \| f_k \,| L_p(\mathbb{R}^n) \|^q \Big)^{1/q} \tag{141}$$

for $r = \max(p, q)$.

It is of course a stronger fact that the sum f belongs to the space $A^{s,a}_{p,q}$, but the lemma is needed for the proof of (41) above.

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