PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 3, November 1981

TRACES OF *BMO*-SOBOLEV SPACES

ROBERT S. STRICHARTZ[1]

ABSTRACT. The trace operator $RF(x) = F(x, 0)$ where $F(x, t)$ is a function of $x \in \mathbf{R}^n$ and $t \in \mathbf{R}^1$ maps $I_\alpha(BMO)$, the *BMO*-Sobolev space of Riesz potentials of order α of functions of bounded mean oscillation on $\mathbf{R}^{n+1}$, onto the homogeneous Besov space $\Lambda_\alpha^0(\infty, \infty)$ on $\mathbf{R}^n$, for $\alpha > 0$. A right inverse is given by the extension operator $Ef(x, t) = \mathcal{F}^{-1}(e^{-t^2|\xi|^2}\hat{f}(\xi))$.

1. Introduction. The space $I_\alpha(BMO)$ of Riesz potentials of order α of functions of bounded mean oscillation has been studied in [4], [8], and [9] as a substitute for the L^p-Sobolev spaces when $p = \infty$. One point left open in [8] is the characterization of the traces of functions in $I_\alpha(BMO)$ for $\alpha > 0$. Since $I_\alpha(BMO) \subseteq \Lambda_\alpha^0(\infty, \infty)$ (this was proved in [8] and was essentially known earlier) and the trace of functions in $\Lambda_\alpha^0(\infty, \infty)$ must obviously also be in $\Lambda_\alpha^0(\infty, \infty)$ for $\alpha > 0$, the trace operator $RF(x) = F(x, 0)$ maps $I_\alpha(BMO)$ into $\Lambda_\alpha^0(\infty, \infty)$. In this paper we will prove that the mapping is onto by showing that the extension operator

$$Ef(x, t) = \mathcal{F}^{-1}\left(e^{-t^2|\xi|^2}\hat{f}(\xi)\right)$$
$$= (4\pi)^{-n/2}t^{-n}\int f(x - y)e^{-|y|^2/4t^2}\,dy$$

maps $\Lambda_\alpha^0(\infty, \infty)$ into $I_\alpha(BMO)$ for all $\alpha \geqslant 0$. This is an exact analogue of the well-known theorem of Gagliardo [3] and Stein [6] that R maps $I_\alpha(L^p)$ onto $\Lambda_{\alpha-1/p}^0(p, p)$ for $1 < p < \infty$ and $\alpha > 1/p$, with the same extension operator. The trace theorem can be routinely transplanted to the context of compact manifolds and submanifolds; in particular the L^p estimates for elliptic boundary value problems [1] are valid for *BMO*-Sobolev spaces on the manifold and Besov spaces on the boundary.

2. Preliminaries. A general reference for all unexplained notation is Stein [7]. A locally integrable (real or complex valued) function defined on $\mathbf{R}^n$ is said to be of *bounded mean oscillation* if the mean oscillation of f on any cube Q

$$MO(f, Q) = \frac{1}{|Q|}\int_Q |f(x) - M(f, Q)|\,dx$$

is uniformly bounded, where $M(f, Q)$ denotes the mean of f on Q,

$$M(f, Q) = \frac{1}{|Q|}\int_Q f(x)\,dx$$

Received by the editors July 18, 1980 and, in revised form, December 19, 1980.
1980 *Mathematics Subject Classification*. Primary 46E35.
Key words and phrases. Bounded mean oscillation, Sobolev space, Besov space, trace theorem.
[1] Research supported in part by NSF grant MCS-8002771.

0002-9939/81/0000-0515/$02.25

and $|Q|$ denotes the Lebesgue measure of Q. The Banach space BMO consists of functions of bounded mean oscillation modulo constants equipped with the norm $\|f\|_{BMO} = \sup_Q MO(f, Q)$.

The Riesz potentials are defined on the space of tempered distributions modulo polynomials by $I_\alpha f = \mathcal{F}^{-1}(|\xi|^{-\alpha}\hat{f}(\xi))$, where $\alpha \in \mathbf{R}$, so we may define the BMO-Sobolev spaces $I_\alpha(BMO)$ as the image of BMO under I_α.

We will be interested only in the case $\alpha > 0$ when $I_\alpha(BMO)$ consists of locally integrable functions modulo polynomials (with a little care one can define $I_\alpha(BMO)$ as a space of functions modulo polynomials of degree $< \alpha$). In the important special case when α is an integer we can describe $I_\alpha(BMO)$ more succinctly as the space of functions whose derivatives of order exactly α are BMO.

The homogeneous Besov space $\Lambda^0_\alpha(\infty, \infty)$ is for noninteger $\alpha > 0$ simply the usual Hölder class of order α, with Zygmund's modification of using higher difference for integer α. Thus for $0 < \alpha < 1, f \in \Lambda^0_\alpha(\infty, \infty)$ if and only if $|f(x + y) - f(x)| < M|y|^\alpha$ with the least M the $\Lambda^0_\alpha(\infty, \infty)$-norm, identifying functions that differ by constants, while for $\alpha = 1$ the condition is $|f(x + 2y) - 2f(x + y) + f(x)| < M|y|$. There are myriad equivalent characterizations of these spaces; our preference is for one due to Peetre [5] that treats all values of α simultaneously: $f \in \Lambda^0_\alpha(\infty, \infty)$ if and only if $\|\sigma_s * f\|_\infty < Ms^\alpha$ for all $s > 0$ where $\sigma_s(x) = s^{-n}\sigma(x/s)$ are the dilations of a fixed test function σ (note $\hat{\sigma}_s(\xi) = \hat{\sigma}(s\xi)$) which satisfies the conditions

(1) $\hat{\sigma} \in \mathfrak{D}$ with support in an annular ring, say $\frac{1}{4} < |\xi| < 4$.

(2) $\hat{\sigma} = 1$ in a smaller annular ring, say $\frac{1}{2} < |\xi| < 2$.

(3) $\hat{\sigma}$ is radial and nonnegative.

The least M is equivalent to the $\Lambda^0_\alpha(\infty, \infty)$-norm. The exact choice of σ does not change the class $\Lambda^0_\alpha(\infty, \infty)$, and conditions (2) and (3) can be considerably weakened. We note in passing that $\int_0^\infty \hat{\sigma}(s\xi)^2(ds/s)$ is a positive radial function homogeneous of degree zero, hence a nonzero constant for $\xi \neq 0$, from which we can conclude

$$f = c\int_0^\infty \sigma_s * \sigma_s * f \, \frac{ds}{s} \tag{2.1}$$

for certain constant c provided we identify functions modulo polynomials. This is perhaps the key identity in the entire theory of Besov spaces.

Now let $x \in \mathbf{R}^n$ and $t \in \mathbf{R}^1$. We will use lower case letters such as f to denote functions of x and upper case letters such as F to denote functions of x and t. The trace operator $RF(x) = F(x, 0)$ is not always well defined for locally integrable F. However if $F \in I_\alpha(BMO)$ for $\alpha > 0$ then $F \in \Lambda^0_\alpha(\infty, \infty)$ (see [8, Theorem 3.4]) and so is continuous. From the Hölder-Zygmund description of $\Lambda^0_\alpha(\infty, \infty)$ it follows that R maps $\Lambda^0_\alpha(\infty, \infty)$ of $\mathbf{R}^n$ to $\Lambda^0_\alpha(\infty, \infty)$ of $\mathbf{R}^{n-1}$ for $\alpha > 0$.

The extension operator

$$Ef(x, t) = \mathcal{F}^{-1}\left(e^{-t^2|\xi|^2}\hat{f}(\xi)\right)$$

is well defined for any tempered distribution f, and obviously is right-inverse to R, $REf = f$.

THEOREM. *E is a bounded operator from $\Lambda^0_\alpha(\infty, \infty)$ to $I_\alpha(BMO)$ for all $\alpha > 0$.*

3. Proof of Theorem. We will prove the theorem in the special case when $\alpha = k$, an integer; the result in general then follows by interpolation since both scales of spaces are preserved by the complex method of interpolation (see [2] for Besov spaces and [8] for BMO-Sobolev spaces). Thus suppose $f \in \Lambda^0_k(\infty, \infty)$, so

$$\|\sigma_s * f\|_\infty \leq Ms^k \tag{3.1}$$

where $M = \|f\|_{\Lambda^0_k(\infty,\infty)}$. To prove the theorem for $\alpha = k$ we need to show that $Ef \in I_k(BMO)$, or equivalently $(\partial/\partial x)^\beta(\partial/\partial t)^j Ef \in BMO$ for all integers j and multi-indices β with $j + |\beta| = k$. Writing $(\partial/\partial x)^\beta(\partial/\partial t)^j Ef = G$ we need to show

$$MO(G, Q) \leq cM \tag{3.2}$$

for every cube Q in $\mathbf{R}^{n+1}$ with sides parallel to the axes, where M is the constant in (3.1) and c is independent of f (the value of c may vary from equation to equation). Any such cube Q is of the form $Q_r \times I_r$ where Q_r is a cube in $\mathbf{R}^n$ of side length r and I_r is an interval in $\mathbf{R}^1$ of length r.

Now from the identity (2.1) we have

$$G = \int_0^\infty \left(\frac{\partial}{\partial x}\right)^\beta \left(\frac{\partial}{\partial t}\right)^j E(\sigma_s * \sigma_s * f)\frac{ds}{s}, \tag{3.3}$$

and given the cube Q and hence r we can split G into two parts $G = G_1 + G_2$, where we take the integral from 0 to r in (3.3) for G_1, and from r to ∞ for G_2. To establish (3.2) we will show

$$\frac{1}{r}\int_{I_r} \sup_x |G_1(x, t)|\, dt \leq cM \tag{3.4}$$

and

$$\frac{1}{r}\int_{I_r} \sup_{x \in Q} |G_2(x, t) - G_2(x_0, t_0)|\, dt \leq cM \tag{3.5}$$

where (x_0, t_0) is the center of Q, for then

$$\begin{aligned} MO(G, Q) &\leq 2M(|G - G_2(x_0, t_0)|, Q) \\ &\leq 2M(|G_1|, Q) + 2M(|G_2 - G_2(x_0, t_0)|Q) \leq cM. \end{aligned}$$

To establish (3.4) and (3.5) we need to examine the form of G. Since $Ef = \mathcal{F}^{-1}(e^{-t^2|\xi|^2}\hat{f}(\xi))$ we have

$$G = \left(\frac{\partial}{\partial x}\right)^\beta \left(\frac{\partial}{\partial t}\right)^j Ef = \mathcal{F}^{-1}\left(p(t\xi)e^{-t^2|\xi|^2}q_k(\xi)\hat{f}(\xi)\right)$$

where p and q_k are polynomials in $\mathbf{R}^n$ and q_k is homogeneous of degree k. If we write

$$h_t(x) = \mathcal{F}^{-1}\left(p(t\xi)e^{-t^2|\xi|^2}q_k(\xi)\right) \tag{3.6}$$

then

$$G_1 = \int_0^r \sigma_s * h_t * \sigma_s * f\frac{ds}{s}$$

hence

$$\sup_x |G_1(x, t)| \leq \int_0^r \|\sigma_s * h_t\|_1 \|\sigma_s * f\|_\infty \frac{ds}{s}$$
$$\leq M \int_0^r s^{k-1} \|\sigma_s * h_t\|_1 \, ds.$$

LEMMA. $\|\sigma_s * h_t\|_1 \leq g(t/s)e^{-ct^2/s^2}s^{-k}$ *for some polynomial* g.

PROOF. We have

$$\sigma_s * h_t = \mathcal{F}^{-1}\big(\hat{\sigma}(s\xi)p(t\xi)e^{-t^2|\xi|^2}q_k(\xi)\big).$$

The L^1-norm is unchanged by the change of variable $\xi \to s^{-1}\xi$ on the Fourier transform side, so

$$\|\sigma_s * h_t\|_1 = s^{-k} \|\mathcal{F}^{-1}\big(\hat{\sigma}(\xi)p(ts^{-1}\xi)e^{-t^2|\xi|^2/s^2}q_k(\xi)\big)\|_1.$$

Thus it suffices to show

$$\|\sigma * h_t\|_1 \leq g(t)e^{-ct^2}.$$

But this follows easily from the well-known estimate

$$\|\sigma * h_t\|_1 \leq c \sum_{|\beta| \leq n+1} \left\| \left(\frac{\partial}{\partial \xi}\right)^\beta (\sigma * h_t)\hat{}(\xi) \right\|_1$$

since $\hat{\sigma}$ has support in an annular ring. Q.E.D.

Returning to the proof of the theorem, we apply the lemma to obtain

$$\frac{1}{r}\int_{I_r} \sup_x |G_1(x, t)| \, dt \leq M\frac{1}{r}\int_{I_r}\int_0^r g(t/s)e^{-ct^2/s^2}s^{-1} \, ds \, dt$$
$$\leq M\frac{1}{r}\int_0^r \int_0^\infty g(t/s)e^{-ct^2/s^2} \, dt \, s^{-1} \, ds$$
$$= M\frac{1}{r}\int_0^r \int_0^\infty g(t)e^{-ct^2} \, dt \, ds = cM$$

which proves (3.4).

Turning to (3.5), we first write

$$G_2(x, t) - G_2(x_0, t_0) = (G_2(x, t) - G_2(x_0, t)) + (G_2(x_0, t) - G_2(x_0, t_0))$$

and estimate the two differences separately. For the first we have

$$\left| \int_r^\infty \int (\sigma_s * h_t(x - y) - \sigma_s * h_t(x_0 - y))\sigma_s * f(y) \, dy \frac{ds}{s} \right|$$
$$\leq M \int_r^\infty \int |\sigma_s * h_t(x - y) - \sigma_s * h_t(x_0 - y)| \, dy \, s^{k-1} \, ds.$$

But by the fundamental theorem of the calculus

$$\int |\sigma_s * h_t(x - y) - \sigma_s * h_t(x_0 - y)| \, dy$$
$$\leq |x - x_0| \int_0^1 \int |\nabla_x \sigma_s * h_t(x_0 - y + \lambda(x - x_0))| \, dy \, d\lambda$$
$$\leq |x - x_0| \, \|\nabla_x \sigma_s * h_t\|_1.$$

For $x \in Q_r$ we have $|x - x_0| \leqslant cr$ so

$$\sup_{x \in Q} |G_2(x, t) - G(x_0, t)| \leqslant Mcr \int_r^\infty \|\nabla_x \sigma_s * h_t\|_1 s^{k-1}\, ds.$$

The second difference can be estimated, for $t \in I_r$, using the mean value theorem, by

$$\begin{aligned} |G_2(x_0, t) - G_2(x_0, t_0)| &\leqslant \int_r^\infty \left\|(t - t_0)\frac{\partial}{\partial t}\sigma_s * h_{t_1}\right\|_1 \|\sigma_s * f\|_\infty \frac{ds}{s} \\ &\leqslant Mr \int_r^\infty \left\|\frac{\partial}{\partial t}\sigma_s * h_{t_1}\right\|_1 s^{k-1}\, ds \end{aligned}$$

where $t_1 \in I_r$. But now all x and t first derivatives of $\sigma_s * h_t$ are of the same form with k increased by 1, so we may apply the Lemma to estimate both $\|\nabla_x \sigma_s * h_t\|_1$ and $\|(\partial/\partial t)(\sigma_s * h_{t_1})\|_1$ by cs^{-k-1} since $g(t)e^{-t^2}$ is bounded. Thus

$$\frac{1}{r}\int_{I_r} \sup_{x \in Q} |G_2(x, t) - G_2(x_0, t_0)|\, dt \leqslant cM \int_{I_r}\int_r^\infty s^{-2}\, ds\, dt = cM$$

which establishes (3.5) and completes the proof of the theorem.

References

1. S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions*. I, Comm. Pure Appl. Math. **12** (1959), 623–727.

2. A. P. Calderon, *Intermediate spaces and interpolation, the complex method*, Studia Math. **24** (1964), 113–190.

3. E. Gagliardo, *Caratterizzazioni delle trace sulla frontiera relative ad alcune classi di funzioni in n variabili*, Rend. Sem. Mat. Padua **27** (1957), 284–305.

4. U. Neri, *Fractional integration in the space H^1 and its dual*, Studia Math. **53** (1975), 175–189.

5. J. Peetre, *Sur les espaces de Besov*, C. R. Acad. Sci. Paris Ser. A-B **264** (1967), A281–A283.

6. E. M. Stein, *The characterization of functions arising as potentials*. II, Bull. Amer. Math. Soc. **68** (1962), 577–582.

7. ______, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N.J., 1970.

8. R. Strichartz, *Bounded mean oscillation and Sobolev spaces*, Indiana Univ. Math. J. **29** (1980), 539–558.

9. A. Torchinsky and R. Wheeden, *On the Marcinkiewicz integral and functions of bounded mean oscillation*, Indiana Univ. Math. J. **29** (1980), 479–494.

Department of Mathematics, Cornell University, Ithaca, New York 14853