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TRACKING CONTROL OF A VIBRATING STRING WITH AN INTERIOR MASS VIEWED AS DELAY SYSTEM

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Abstract. A vibrating string, modelled by the wave equation, with an interior mass is considered. It is viewed as a linear delay system. A trajectory tracking problem is solved using a new type of controllability.

1. Introduction

The "hybrid system" modelling a vibrating string with an interior point mass is analyzed in a remarkable recent paper by Hansen and Zuazua [6]. Using Hilbert's uniqueness method [7, 8], on the one hand, and nonharmonic Fourier series [13], on the other hand, they give a detailed description of its exact controllability and stabilization by boundary feedback.

The present study, which is a companion paper of [3], aims to solve another natural control problem, namely tracking a trajectory of the mass position. We exploit the well known relation [1] between the undamped wave equation and linear delay systems.

We freely use [3], where relations to several classical structural properties of delay systems are established. The present case study illustrates in particular the importance of *π-freeness* [3] for tracking control of a delay system, in a similar spirit as the "flatness based control" of nonlinear (finite dimensional) systems [2]. See also [11] for other examples.

In the next Section the delay system model is derived from the hybrid one. Its π-freeness is established in Section 3. As in [6], two cases are distinguished: position control on both boundaries or on either one, in which case the other end is fixed. The tracking control in the single control case is treated and illustrated by simulations in Section 4. Related examples may

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be found in [4, 12]. In [12] the control of an Euler-Bernoulli flexible beam is sketched, which uses some of the tools developed in [3].

2. Hybrid system and delay system model

The device under study can be modelled as a hybrid system: a (one-dimensional) wave equation for each interval between the boundary and a point mass together with a second order linear ordinary differential equation describing the motion of the mass;

$$\rho_1 \partial_t^2 p = \tau_1 \partial_x^2 p \qquad x \in [-L_1, 0], t \in \mathbb{R}^+, \tag{2.1a}$$

$$\rho_2 \partial_t^2 q = \tau_2 \partial_x^2 q \qquad x \in [0, L_2], t \in \mathbb{R}^+. \tag{2.1b}$$

Here $p(x,t)$ and $q(x,t)$ represent the deformation at abscissa x and time t of the respective parts of the string, supposed homogeneous, occupying $[-L_1, 0]$ and $[0, L_2]$. The physical parameters are their densities ρ_1 and ρ_2 and the tensions τ_1, τ_2. The position $z(t)$ of the mass M, attached to the string at $x = 0$, satisfies

$$z(t) = p(0,t) = q(0,t) \qquad t \in \mathbb{R}^+, \tag{2.1c}$$

$$M\partial_t^2 z(t) + \tau_1 \partial_x p(0,t) - \tau_2 \partial_x q(0,t) = 0 \qquad t \in \mathbb{R}^+. \tag{2.1d}$$

Applying position controls $u(t)$ and $v(t)$ at the ends leads to Dirichlet boundary conditions

$$p(-L_1, t) = u(t) \qquad t \in \mathbb{R}^+, \tag{2.1e}$$

$$q(L_2, t) = v(t) \qquad t \in \mathbb{R}^+. \tag{2.1f}$$

The (compatible) initial conditions are

$$p(x,0) = p^0(x), \quad \partial_t p(x,0) = p^1(x), \qquad x \in [-L_1, 0], \tag{2.1g}$$

$$q(x,0) = q^0(x), \quad \partial_t q(x,0) = q^1(x), \qquad x \in [0, L_2], \tag{2.1h}$$

$$z(0) = z^0 = p^0(0) = q^0(0), \tag{2.1i}$$

$$\partial_t z(0) = z^1 = p^1(0) = q^1(0). \tag{2.1j}$$

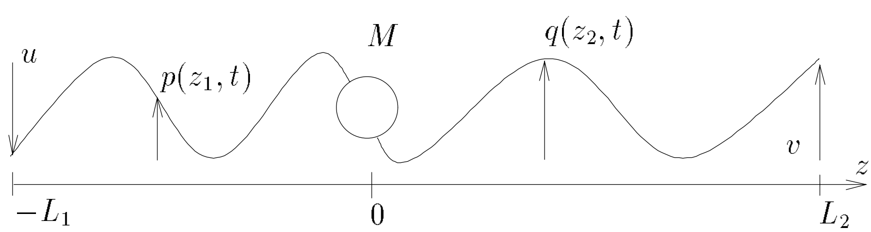

Figure 1. The vibrating string with an interior point mass

The general solution of (2.1) in the convolution ring $\mathcal{D}'(\mathbb{R})$ of distributions is well known to be

$$p(x,t) = (\delta_{\mu_1 x} * \phi_1)(t) + (\delta_{-\mu_1 x} * \psi_1)(t), \tag{2.2}$$

$$q(x,t) = (\delta_{\mu_2 x} * \phi_2)(t) + (\delta_{-\mu_2 x} * \psi_2)(t), \tag{2.3}$$

with $\phi_1, \phi_2, \psi_1, \psi_2 \in \mathcal{D}'(\mathbb{R})$, δ_h denoting the Dirac distribution concentrated at h, and

$$\mu_1 = \sqrt{\rho_1/\tau_1}, \qquad \mu_2 = \sqrt{\rho_2/\tau_2}.$$

The boundary conditions (2.1c), (2.1e), and (2.1f) then read

$$\phi_1(t) + \psi_1(t) - \phi_2(t) - \psi_2(t) = 0 \tag{2.4a}$$

$$M(\delta'' * (\phi_1 + \psi_1)(t)) + \tau_1\mu_1(\delta' * (\phi_1 - \psi_1)(t)) - \tau_2\mu_2(\delta' * (\phi_2 - \psi_2(t)) \ = 0 \tag{2.4b}$$

$$(\delta_{-\mu_1 L_1} * \phi_1)(t) + (\delta_{\mu_1 L_1} * \psi_1)(t) = u(t) \tag{2.4c}$$

$$(\delta_{\mu_2 L_2} * \phi_2)(t) + (\delta_{-\mu_2 L_2} * \psi_2)(t) = v(t) \tag{2.4d}$$

where δ' and δ'' denote the derivatives of the Dirac distribution concentrated at 0. Since $\bar{\phi}_1(t) = \phi_1(t) + \alpha(t)$ and $\bar{\psi}_1(t) = \psi_1(t) - \alpha(t)$ give rise to the same solution for all $\alpha(t)$, we can choose to have

$$M(\delta' * (\phi_1 + \psi_1))(t) + \tau_1\mu_1(\phi_1 - \psi_1)(t) - \tau_2\mu_2(\phi_2 - \psi_2)(t) = 0. \tag{2.5}$$

3. Structural properties

Defining

$$\eta_1 = \tau_1\mu_1/M, \qquad \eta_2 = \tau_2\mu_2/M$$

and the localized delay operators σ_1 and σ_2 of respective amplitudes $\mu_1 L_1$ and $\mu_2 L_2$, the preceding equations can be resumed as

$$\phi_1 + \psi_1 = \phi_2 + \psi_2 \tag{3.1a}$$

$$\frac{d}{dt}(\phi_1 + \psi_1) + \eta_1(\phi_1 - \psi_1) - \eta_2(\phi_2 - \psi_2) = 0 \tag{3.1b}$$

$$\sigma_1\phi_1 + \sigma_1^{-1}\psi_1 = u \tag{3.1c}$$

$$\sigma_2^{-1}\phi_2 + \sigma_2\psi_2 = v. \tag{3.1d}$$

Remark 3.1. Here we use the notation σ for the delay operators rather than δ as in [3] in order to avoid any confusion with the Dirac distribution.

3.1. Case of two controls

The delay system model is defined as the $\mathbb{R}[\frac{d}{dt}, \sigma_1, \sigma_2]$-module $\Lambda^{u,v}$ generated by $\{\phi_1, \psi_1, \phi_2, \psi_2, u, v\}$ with the relations (3.1).

We study the structure of the delay system $\Lambda^{u,v}$ with two controls first.

Theorem 3.2. *The system $\Lambda^{u,v}$ is torsion free, but not free. It is σ_2-free with basis $(\sigma_1\phi_1 - u,\ \phi_1 + \psi_1)$.*

Proof. The presentation matrix of $\Lambda^{u,v}$ associated with (3.1) is

$$P_{\Lambda^{u,v}}(\tfrac{d}{dt}, \sigma_1, \sigma_2) = \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ \dfrac{d}{dt} + \eta_1 & \dfrac{d}{dt} - \eta_1 & -\eta_2 & \eta_2 & 0 & 0 \\ \sigma_1^2 & 1 & 0 & 0 & -\sigma_1 & 0 \\ 0 & 0 & 1 & \sigma_2^2 & 0 & -\sigma_2 \end{bmatrix}.$$

Calculating the minors which define the variety $\mathcal{V}_{\Lambda^{u,v}}$ associated to $\Lambda^{u,v}$, one realizes that $P_{\Lambda^{u,v}}(s_1, s_2, s_3)$ has a loss of rank for $s_2 = s_3 = 0$ and

$s_1 = \eta_2 - \eta_1$ only. This proves that $\Lambda^{u,v}$ is torsion free but not free (*see* [9] and [3, Proposition 3.1 and Theorem 3.2]).

Introduce $b_1 = \sigma_1\phi_1 - u$ and $b_2 = \phi_1 + \psi_1$. The independence of b_1 and b_2 over the quotient field $\mathbb{R}(\frac{d}{dt}, \sigma_1, \sigma_2)$ can be checked on (3.1) using elementary linear algebra. This readily implies their independence over $\mathbb{R}[\frac{d}{dt}, \sigma_1, \sigma_2]$. Let us check that b_1 and b_2 are generators of $\Lambda^{u,v}$. One has $\sigma_1 b_1 = \sigma_1^2\phi_1 - \sigma_1 u = -\psi_1$, whence $\psi_1 = -\sigma_1 b_1$, $\phi_1 = b_2 + \sigma_1 b_1$, $u = \sigma_1 b_2 - (1-\sigma_1^2)b_1$. Then, using $\frac{d}{dt}b_2 + \eta_1(b_2 + 2\sigma_1 b_1) = \eta_2(2\phi_2 - b_2)$ and $\psi_2 = b_2 - \phi_2$

$$\begin{aligned}\phi_2 &= \frac{1}{2\eta_2}\frac{d}{dt}b_2 + \frac{\eta_1}{\eta_2}\sigma_1 b_1 + \frac{\eta_1+\eta_2}{2\eta_2}b_2,\\ \psi_2 &= -\frac{1}{2\eta_2}\frac{d}{dt}b_2 - \frac{\eta_1}{\eta_2}\sigma_1 b_1 + \frac{\eta_2-\eta_1}{2\eta_2}b_2.\end{aligned}$$

Finally, $\sigma_2 v = \phi_2 + \sigma_2^2\psi_2$ yields

$$v = \frac{(\sigma_2^{-1} - \sigma_2)}{2\eta_2}\frac{d}{dt}b_2 + \frac{\eta_1\sigma_1(\sigma_2 - \sigma_2^{-1})}{\eta_2}b_1 + \frac{(\eta_1+\eta_2)\sigma_2^{-1} + (\eta_2-\eta_1)\sigma_2}{2\eta_2}b_2.$$

The $\mathbb{R}[\frac{d}{dt}, \sigma_1, \sigma_2]$-linear independence of b_1 and b_2 is a direct consequence of $\mathrm{rk}_{\mathbb{R}[d/dt,\sigma_1,\sigma_2]}\Lambda^{u,v} = 2$. □

3.2. Case of a single control

Let the system be controlled on one boundary only. Therefore, consider $v = 0$ in (3.1d), as in [6]. Equations (3.1) then yield

$$\phi_1 + \psi_1 = (1-\sigma_2^2)\psi_2 \tag{3.2a}$$

$$\sigma_1^2\phi_1 + \psi_1 = \sigma_1 u \tag{3.2b}$$

$$\frac{d}{dt}(\phi_1+\psi_1) + \eta_1(\phi_1 - \psi_1) + \eta_2(1+\sigma_2^2)\psi_2 = 0. \tag{3.2c}$$

Denote as Λ^u the $\mathbb{R}[\frac{d}{dt}, \sigma_1, \sigma_2]$-module generated by $\{\phi_1,\ \psi_1,\ \phi_2,\ \psi_2,\ u\}$ with the relations (3.2).

PROPOSITION 3.3. *The system Λ^u is torsion free but not free. It is σ_1-free, with basis ψ_2.*

Proof. The presentation matrix of Λ^u associated with (3.2) is

$$P_{\Lambda^u}(\tfrac{d}{dt}, \sigma_1, \sigma_2) = \begin{bmatrix} 1 & 1 & \sigma_2^2 - 1 & 0\\ \sigma_1^2 & 1 & 0 & -\sigma_1\\ \frac{d}{dt} + \eta_1 & \frac{d}{dt} - \eta_1 & \eta_2(1+\sigma_2^2) & 0\end{bmatrix}.$$

The associated variety of zeros $\mathcal{V}_{\Lambda^u}$ is the curve in the s_1, s_2, s_3 space given by

$$s_2 = 0, \quad s_1 = \eta_1 - \eta_2\frac{1+s_3^2}{1-s_3^2}.$$

It follows that Λ^u is torsion free, but not free (*see* [3], Theorems 12 and 13).

We now show that Λ^u is σ_1-free, with basis ψ_2. The relation $v = 0$ in (3.1d) implies $\phi_2 = -\sigma_2^2\psi_2$. Using this together with (3.2a) in (3.2c) yields

$$\phi_1 = -\left[(1-\sigma_2^2)\frac{d}{dt} + (\eta_1+\eta_2)\sigma_2^2 + (\eta_2-\eta_1)\right]\frac{\psi_2}{2\eta_1} \tag{3.3a}$$

$$\psi_1 = \left[(1-\sigma_2^2)\frac{d}{dt} + (\eta_2-\eta_1)\sigma_2^2 + (\eta_1+\eta_2)\right]\frac{\psi_2}{2\eta_1} \tag{3.3b}$$

$$u = \Big[(\sigma_1^{-1}-\sigma_1)\big((1-\sigma_2^2)\frac{d}{dt} + \eta_2\sigma_2^2 - \eta_1\big) + (\sigma_1^{-1}+\sigma_1)(\eta_2-\eta_1\sigma_2^2)\Big]\frac{\psi_2}{2\eta_1} \tag{3.3c}$$

and ψ_2 generates $\mathbb{R}[\frac{d}{dt}, \sigma_1, \sigma_2, \sigma_1^{-1}] \otimes_{\mathbb{R}[d/dt,\sigma_1,\sigma_2]} \Lambda^u$; it is then a basis for this latter module. □

4. Mass tracking

From Proposition 3.3 and equations (2.1c) and (2.2), we get

$$z = (1-\sigma_2^2)\psi_2. \tag{4.1}$$

THEOREM 4.1. *The system Λ^u is $\sigma_1(1-\sigma_2^2)$-free, with basis z.*

Thus, the control law allowing to track a desired trajectory z_d of z, following directly from (3.3c), involves an advance of $\mu_1 L_1$. Additionally, we have from (2.2)

$$p(x,t) = (\delta_{\mu_1 x} * \phi_1)(t) + (\delta_{-\mu_1 x} * \psi_1)(t), \qquad x \in [-L_1, 0]$$
$$q(x,t) = (\delta_{\mu_2 x} * \phi_2)(t) + (\delta_{-\mu_2 x} * \psi_2)(t), \qquad x \in [0, L_2]$$

and ϕ_1 and ψ_1 are given in terms of z through (3.3a), (3.3b), and (4.1). These formulae allow to compute the explicit solution of (2.1) and the control law $u_d(t)$ yielding a desired trajectory $z_d(t)$ of the mass position, as the one shown in Figure 2. The corresponding control function $u_d(t)$ is depicted in Figure 3 and the displacements along the string are shown on Figure 4; they correspond to $\mu_1 = 1$, $\mu_2 = \sqrt{2}$, $\eta_1 = 1$, $\eta_2 = \sqrt{2}$, and $L_1 = 1$, $L_2 = 2$.

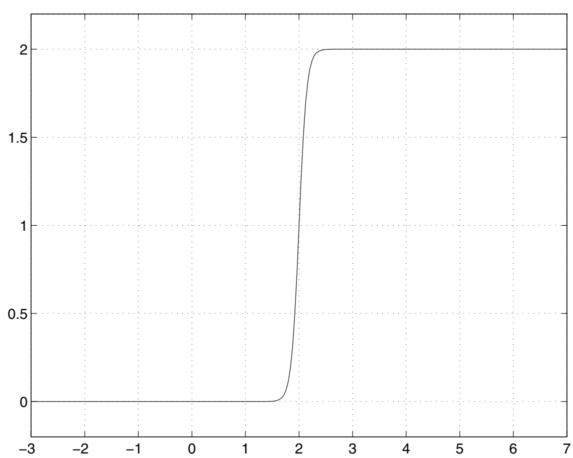

FIGURE 2. *The desired output $z_d(t)$*

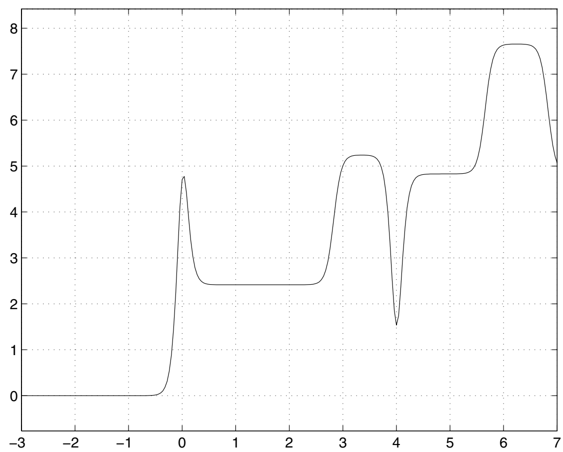

FIGURE 3. *The control law* $u_d(t)$

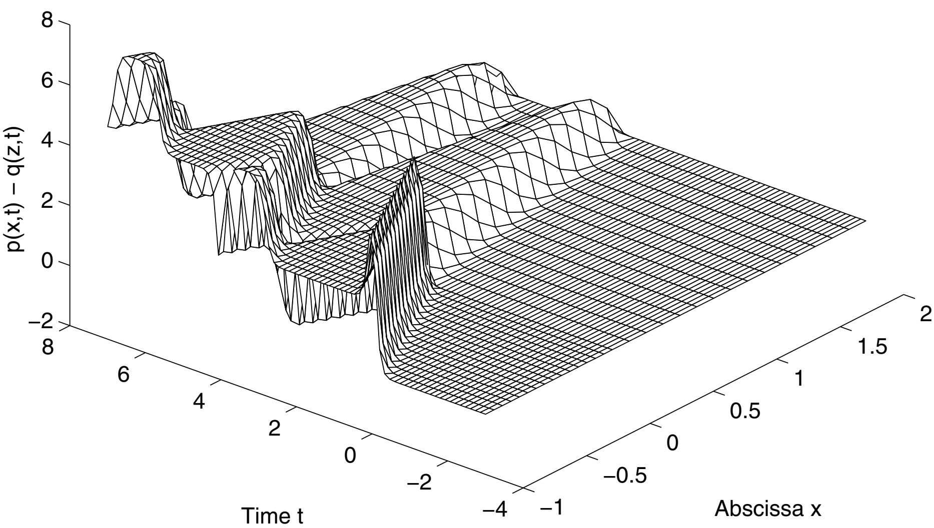

FIGURE 4. *Graph of* $p(x,t)$ $(x \in [-1,0])$ *and* $q(x,t)$ $(x \in [0,2])$

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