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Tracking control of mechanical systems with impacts

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Abstract— **In this paper controllers are designed such that the state trajectories of mechanical systems with impacts converge to a reference trajectory that contains impacts. The impact times of the plant will typically not coincide with those of the reference, such that the Euclidean tracking error intrinsically behaves in an unstable manner. Therefore, an alternative approach is needed and we propose to study the convergence of a non-Euclidean tracking error measure that corresponds to the intuitive notion of tracking: impact times of the plant converge to those of the reference, and the plant follows the reference away from the impacts. Sufficient conditions for asymptotic stability in terms of this tracking error are presented, and the results are illustrated with a bouncing ball example.**

I. Introduction

In hybrid systems, such as robotic systems with impacts, digitally controlled physical systems, and electrical circuits with switches, the continuous-time dynamics and discrete dynamics are intertwined. Due to this ambivalent nature, hybrid systems can show more complex behaviour than can occur in ordinary differential equations (ODEs) or discrete systems, and conventional control approaches are not directly applicable.

Most existing results in the literature of hybrid control systems deal with the stability of time-independent sets (especially with equilibrium points), such that the stability can be analysed using Lyapunov functions, see e.g. [1]–[8]. Essentially, such a set is asymptotically stable when a Lyapunov function decreases both during flow and jumps (i.e. discrete events), cf. [2], [3], [5]. Extensions of these results allow for Lyapunov functions that increase during jumps, as long as this increase is compensated by a larger decrease during flow, or vice versa. Using these stability results, several control strategies have been developed for the stabilisation of a time-independent set, see e.g. [5], [8].

Few results exist where controllers are designed to make a system track a given, *time-varying*, reference trajectory, that contains both continuous-time behaviour and discrete events. When the jump times of the plant trajectories can be guaranteed to coincide with jumps of the reference, then stable behaviour of the Euclidean tracking error is possible and several tracking problems have been solved in this setting, see, e.g., [9]–[11], [11], [12]. In [13], observer problems are considered for a class of hybrid systems where a similar condition is exploited, namely, that the jumps of the plant and the observer coincide. Requiring the jump times of plant and reference (or plant and observer) to coincide is a strong condition that limits the number of problems that can be solved in this manner.

In hybrid systems with state-triggered jumps, such as mechanical systems with unilateral constraints and impacts, the jump times of the plant and jump times of the reference are in general not coinciding. To illustrate this behaviour, we consider the bouncing trajectories of a ball with unit mass on a table as an example. The continuous-time solutions are described by the following ODE, where x_1 represents the position and x_2 the velocity with respect to the table surface:

$$\dot{x} = \begin{bmatrix} x_2 \\ -G + u + \lambda(x_1, x_2) \end{bmatrix}, \quad x_1 \geq 0, \tag{1a}$$

where G is the gravitational acceleration, u is a force that can be applied to the system, and the contact force λ between the ball satisfies the following set-valued force law, cf. [14],

$$\lambda(x_1, x_2) \in \begin{cases} 0, & (x_1\ x_2) \neq (0\ 0), \\ [0, \infty), & (x_1\ x_2) = (0\ 0), \end{cases} \tag{1b}$$

and avoids penetration of the table by the ball. Motion according to (1a) is only possible when the distance x_1 between the table and the ball is non-negative. If the ball arrives at the surface $x_1 = 0$, then a Newton-type impact law with restitution coefficient equal to one is assumed, modelled as

$$x^+ = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}, \quad x_1 = 0 \text{ and } x_2 < 0.$$

We consider the following reference trajectory:

$$r = \begin{bmatrix} \tau + \frac{G}{2}\tau^2 \\ 1 - G\tau \end{bmatrix}, \quad \tau = t \bmod \tfrac{2}{G}.$$

Suppose that a control signal u is designed such that a plant trajectory x tracks the reference r (in fact, such a controller will be constructed in Section IV), then we expect behaviour as given in Fig. 1, where the positions x_1 and r_1 converge to each other and the jump times show a vanishing mismatch. During the time interval caused by this jump-time mismatch, the large velocity error $|x_2 - r_2|$ implies that the Euclidean error $|x-r|$ is large, as shown in Fig. 1c. Since this behaviour also occurs when the initial error $|x - r|$ is arbitrarily small, the Euclidean error displays unstable behaviour in the sense of Lyapunov. This "peaking behaviour" was observed in [7], [9]–[11], [13], [15], and is expected to occur in all hybrid systems with state-triggered jumps when considering tracking or observer design problems. However, although the Euclidean error behaves in an unstable fashion, from

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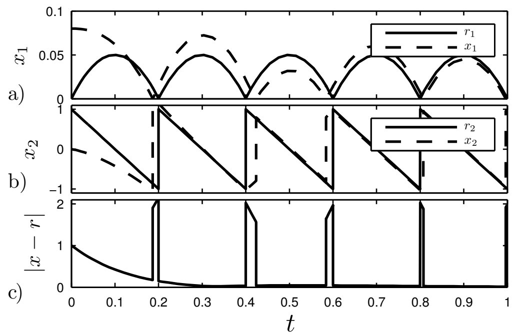

Fig. 1. a),b) Exemplary trajectories of (1). c) Euclidean tracking error $|x-r|$.

a control engineering point of view, the trajectories shown in Fig. 1 display desirable behaviour. For this reason, we formulate a tracking problem that considers the behaviour shown in Fig. 1 as a solution, since the jump times of the plant converge to the jump times of the reference and the distance between the plant and reference trajectories converges to zero during time intervals where no jumps occur. This tracking problem is less restrictive compared to problems where stability of the Euclidean error is required (see [9], [10], [12]), such that the class of tracking problems that can be considered is widened significantly.

Several solutions have already been presented to formalise tracking problems that consider behaviour as shown in Fig 1 as desired. However, these approaches are either tailored to specific systems, or it is not clear how to formulate conditions under which such tracking problems are solved. Most notably, in [15], [16], the tracking of a billiard system is considered, where the position of the ball is always required to be close to the reference, but the error in velocity is not studied for a small time interval near the jump instances. In addition, the convergence of jump times is required.

In order to study tracking problems with non-matching jump times, we propose an alternative approach using a non-Euclidean distance between the plant and reference states, where stable behaviour of this distance corresponds to the desired and intuitive notion of tracking. Since this distance incorporates information on the velocity during the time interval near jumps, the tracking problem can be formulated based on this error only. This step is instrumental in our approach as it allows us to derive sufficient conditions under which the tracking problem is solved that are based on the instantaneous state, the time-derivatives of the states, and the possible jumps that can occur. Since this behaviour is given directly in the description of the hybrid system, our approach is advantageous to the approach in [15], [16], where a return map is required to prove convergence of jump times. In addition to this novel formulation of the tracking problem for systems with jumps, we indeed present such sufficient conditions for the stability of the tracking error defined using the novel distance. Finally, tracking controllers are designed for mechanical systems with unilateral constraints.

For ease of exposition, we focus in this paper on mechanical systems with one degree of freedom and impacts, but we advocate that our general philosophy is also applicable for other classes of hybrid systems.

The outline of this paper is as follows. In Section II, we introduce the class of systems under study. In Section III, the new perspective on tracking is formulated by requiring asymptotic stability with respect to a non-Euclidean tracking error measure, and sufficient conditions for asymptotic stability are formulated. Controllers solving the tracking problem are designed in Section IV, and are illustrated in Section V with an example. Section VI presents the conclusions.

II. MECHANICAL SYSTEMS WITH IMPACTS

In this paper, we consider mechanical systems with one degree of freedom and a single unilateral constraint with impact, see Fig. 2a). Such systems can be modelled with

$$\dot{x} = \begin{bmatrix} x_2 \\ f(t,x) + u + \lambda(x_1, x_2) \end{bmatrix}, \quad x \in C := [0,\infty) \times \mathbb{R} \tag{2a}$$

$$x^+ = g(x) = \begin{bmatrix} x_1 \\ -\epsilon x_2 \end{bmatrix}, \quad x \in D := \{0\} \times (-\infty, 0). \tag{2b}$$

Here, (2a) describes the continuous-time flow of trajectories with control signal u, λ the contact force in the constraint, and $f(t,x)$ representing other forces. Equation (2b) models the impact such that the velocity changes sign, where parameter $\epsilon \in (0,1]$ takes into account the energy dissipation during impact. The unilateral constraint force $\lambda(x_1, x_2)$ satisfies (1b) and ensures that $x \geq 0$ is not violated.

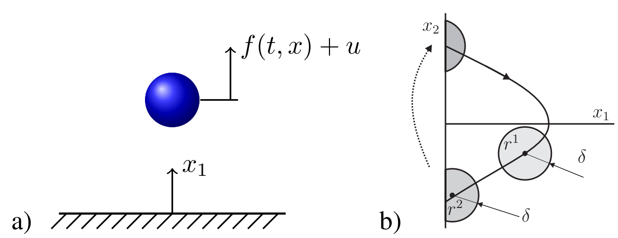

Fig. 2. a) Example of a mechanical system described by (2). b) Neighbourhoods $\{x \in C \cup D : d(x, r^i) < \delta\}$ for two distinct points r^i, $i = 1, 2$, where d is given in (4).

In this paper, we will present controllers that solve a local tracking problem for a reference trajectory r that is non-Zeno, and bounded away from the origin. For all trajectories near this reference the contact force λ vanishes, such that the trajectories are described with the simpler hybrid system

$$\dot{x} = F(t,x,u) := \begin{bmatrix} x_2 \\ f(t,x) + u \end{bmatrix}, \ x \in C := [0,\infty) \times \mathbb{R} \tag{3a}$$

$$x^+ = g(x) := \begin{bmatrix} x_1 \\ -\epsilon x_2 \end{bmatrix}, \quad x \in D := \{0\} \times (-\infty, 0). \tag{3b}$$

Throughout this paper, we assume that u is bounded and f is continuous in x and locally essentially bounded in t. Solutions of (3) are considered in the sense of [5].

III. Tracking control

A. Formulation of the tracking problem

For certain classes of hybrid systems with state-triggered jumps, such as the hybrid system given in (3), the exact properties of the jumps can be used to compare a reference trajectory with a plant trajectory, when one of them just experienced a jump, and the other did not. A distance function between two trajectories that allows such a comparison incorporates the structure of the jumps, as described in (3b). In this paper we employ such distance functions to formulate and solve the tracking problem. Let us first illustrate the main idea by constructing an appropriate distance function for systems of the form (3) in the case of fully elastic impacts, i.e., $\epsilon = 1$, for example the system (1) presented in the introduction. We use the property that the velocity x_2 changes sign at impacts, and the position x_1 is zero, see (3b). Hence, if we want to compare a reference r with plant state x when one of them just experienced a jump, then the distance $|x + r|$ is appropriate. Away from jump instances, typically, the conventional distance $|x - r|$ can be used. To distinguish when the distance $|x - r|$ or $|x + r|$ should be considered, we use the minimum of both, such that the novel distance is given by

$$d(r, x) = \min(|x - r|, |x + r|). \tag{4}$$

In Fig. 2b), the neighbourhoods $\{x \in C \cup D : d(x, r^i) < \delta\}$ of two different points $r^i,\ i = 1, 2$, are shown. Essentially, the tracking error measure d allows to compare a reference trajectory with a plant trajectory, "as if" both of them already jumped, cf. the points with positive x_2 in Fig. 2b), where the measure d corresponds to the error $|x - g(r^2)|$.

Now, to account for a non-ideal impact, i.e., $\epsilon \neq 1$, we reshape the space $C \cup D$ using a function $M_e : C \cup D \to C \cup D$ as follows. Essentially, we aim to employ the distance function (4), but, since trajectories jump from $(0, x_2)$ to $(0, -\epsilon x_2)$ when $x_2 < 0$, we will "stretch" the positive x_2−axis, such that $M_e((0, x_2)) = (0, \frac{1}{\epsilon} x_2)$ for $x_2 > 0$, and $M_e((0, x_2)) = (0, x_2)$ for $x_2 \leq 0$. Using an interpolation which is linear with respect to $\arctan \frac{x_2}{x_1}$, we define a continuous function

$$M_\epsilon(x) := \left(\tfrac{1}{2} + \tfrac{1}{\pi}\text{atan}(x_2, x_1)\right) \tfrac{1}{\epsilon} x + \left(\tfrac{1}{2} - \tfrac{1}{\pi}\text{atan}(x_2, x_1)\right) x,$$

for all $x \in C \cup D$, with $\text{atan}(a, b) = \arctan(\frac{a}{b})$, $b \neq 0$ and $\text{atan}(a, 0) = \frac{\pi}{2}\text{sign}(a)$. Using M_ϵ, we define the tracking error for the system (3) as

$$d(r, x) := \min(|M_\epsilon(x) - M_\epsilon(r)|, |M_\epsilon(x) + M_\epsilon(r)|). \tag{5}$$

For $\epsilon = 1$ this distance is equal to (4) since $M_1(x) = x$. Note that the distance (5) is not equivalent with the Euclidean distance $|x - r|$, such that asymptotically stable behaviour of d does not imply asymptotically stable behaviour of the Euclidean distance. Analogous to the common approach in tracking control for ODEs, we consider reference trajectories r that are solutions to (3) for a given feedforward signal $u = u_{\text{ff}}(t)$ and design a state-dependent control law $u = u_d(t, r, x)$.

In order to study the stability of the closed-loop system, we compare the dynamics of the reference with the dynamics of the plant and evaluate $d(r, x)$ along trajectories. For this purpose, we create an extended hybrid system with state $q = (r^\top, x^\top)^\top$. The dynamics of this hybrid system is given by

$$\dot{q} = F_e(t, q), \quad q \in C^2 \tag{6a}$$

$$q^+ = \text{col}(q_1, -\epsilon q_2, q_3, q_4), \qquad q \in D \times (C \cup D) \tag{6b}$$

$$q^+ = \text{col}(q_1, q_2, q_3, -\epsilon q_4), \qquad q \in (C \cup D) \times D, \tag{6c}$$

where $\text{col}(a, b) = (a^\top\ b^\top)^\top$, and

$$F_e(t, q) := \text{col}(q_2, f(t, r) + u_{\text{ff}}(t), q_4, f(t, x) + u_d(t, r, x)). \tag{7}$$

The main advantage of considering this extended hybrid system is that a joint hybrid time domain is created, such that hybrid times $(t, j) \in \text{dom}\ q$ denote the continuous-time t lapsed, and j gives the total number of jumps that occurred in both x and r. Hence, we can evaluate $\bar{r}(t, j) := (q_1, q_2)^\top(t, j)$ and $\bar{x}(t, j) := (q_3, q_4)(t, j)$ at any time $(t, j) \in \text{dom}\ q$ to evaluate $d(\bar{r}(t, j), \bar{x}(t, j))$. If two distinct trajectories x and r from (3) would be considered, then $\text{dom}\ r \neq \text{dom}\ x$. In this case, if one defines a time-dependent tracking error at time $(t, j) \in \text{dom}\ x$, then it is not clear what time $(t_r, j_r) \in \text{dom}\ r$ is appropriate to compare x with r. Such problems are avoided by studying the extended dynamics in (6).

We will formulate a tracking problem by requiring the design of a state-dependent tracking control law $u = u_d(t, r, x)$ such that the reference r has the asymptotic stability property as formalised as follows.

Definition 1 (Stability with respect to distance d) *Let d be given in* (5). *A reference trajectory $r(t, j)$ of* (3) *is*

- stable with respect to d *if for all $t_0, j_0 \geq 0$ and $\varepsilon > 0$ there exists a $\delta(t_0, j_0, \varepsilon) > 0$ such that $\forall t \geq t_0, \forall j \geq j_0$:*

$$d(\bar{r}(t_0, j_0), \bar{x}(t_0, j_0)) < \delta(t_0, j_0, \varepsilon) \Rightarrow d(\bar{r}(t, j), \bar{x}(t, j)) < \varepsilon; \tag{8}$$

- asymptotically stable with respect to d *if it is stable with respect to d and δ in* (8) *can be chosen such that:*

$$d(\bar{r}(t_0, j_0), \bar{x}(t_0, j_0)) < \delta(t_0, j_0, \varepsilon) \Rightarrow \lim_{t+j\to\infty} d(\bar{r}(t, j), \bar{x}(t, j)) \to 0.$$

We formalise the tracking problem as follows.

Definition 2 (Tracking problem) *Given a hybrid system* (3) *with reference trajectory r, design a control law $u_d(t, r, x)$ such that the trajectory r is asymptotically stable with respect to d given in* (5).

We note that asymptotic stability of the reference with respect to the distance d, as given in Definition 1, directly implies that the difference between the jump times of the reference and plant vanishes when trajectories of (3a) arrive at D in a non-tangential fashion. Additionally, the function $d(r, x)$ is designed such that, when $d(r, x)$ is small and r is away from $D \cup g(D)$, then $|x - r|$ is small. Hence, a controller solving the tracking problem in Definition 2 induces the intuitive notion of tracking: jump times of reference and plant converge to each other, and, away from the jump times, the Euclidean tracking error converges to zero.

The tracking problem in Definition 2 does allow the occurrence of the peaking phenomenon of the Euclidean error, as depicted in Fig 1c. The reason of this fact is found in the novel definition of the tracking error as in (5) and the related stability notion in Definition 1. Namely, it embeds a more intuitive and less restrictive notion of the closeness of jumping trajectories. Moreover, the tracking problem defined here can be solved with a state-dependent controller that does not induce jumps directly.

The distance d depends only on the current state of the plant and the reference, and not on the complete trajectories. For this reason, the evaluation of Lyapunov-like functions along trajectories can be used to study the behaviour of d and to analyse whether a given control law $u_d(t, r, x)$ solves the tracking problem formulated in Definition 2.

B. Sufficient conditions for stability of hybrid trajectories

In this section, we formulate sufficient conditions that guarantee that trajectories of the closed-loop dynamics are defined for $t \to \infty$ and the hybrid tracking problem is solved for a given control law $u_d(t, r, x)$. Hereto, Lyapunov functions are employed that are non-increasing over jumps and converge to zero during flow when $t \to \infty$.

In order to guarantee that trajectories of (6) have hybrid time domains that are unbounded in $t-$direction, we require that r is non-Zeno and bounded:

Assumption 1 *The reference trajectory* $r = \begin{bmatrix} r_1 & r_2 \end{bmatrix}^\top$ *is non-Zeno, bounded, and the unique solution of* (3) *with feedforward signal* $u_{\text{ff}}(t)$ *and initial condition* $r(0, 0)$.

The required uniqueness of the reference trajectory implies that $(q_1, q_2)^\top = \bar{r}$ when the initial condition $(q_1, q_2)^\top = \bar{r}(0, 0)$ is chosen. This implies that if the set $\{q \in (C \cup D)^2 : d(\bar{r}, \bar{x}) = 0\}$ is asymptotically stable with respect to d under the dynamics (6), then the tracking problem in Definition 2 is solved, since the initial condition can be chosen such that $\bar{r}(0, 0)$ is appropriately initialised. In the following theorem, we present sufficient conditions under which the tracking problem is solved, and in addition, an estimate of the basin of attraction of r is given. We note that this theorem is proven by evaluation of d along trajectories $q(t, j)$ of (6) for $t \to \infty$. For this reason, x should not display Zeno behaviour, which, for the estimate of the basin of attraction, is ensured in statement *(ii)* of Theorem 1 by a technical condition. Note, that the time-dependency in the theorem originates from the reference signal $r(t, j)$, which is known a priori in the tracking problem.

Theorem 1 *Consider a hybrid system* (3)*, distance d given in* (5)*, reference trajectory r, and feedforward signal u_{ff} satisfying Assumption 1. Let the control law $u_d(t, r, x)$ be given. If we let $F_e(t, r, x)$ be defined in* (7)*, then the following statements hold:*

(i) *The reference r is asymptotically stable with respect to d for the system* (6) *if there exist functions $\alpha_{1,2} \in \mathcal{K}_\infty$, a continuously differentiable function $V(r, x)$ and scalars $c, \delta > 0$ such that*

$$\alpha_1(d(r(t,j), x)) \leq V(r(t,j), x) \leq \alpha_2(d(r(t,j), x)) \quad \text{(9a)}$$

holds for all $x \in C \cup D$, $(t, j) \in \operatorname{dom} r$, and

$$V(g(r(t,j)), x) \leq V(r(t,j), x), \quad \text{for } r(t,j) \in D \quad \text{(9b)}$$

$$V(r(t,j), g(x)) \leq V(r(t,j), x), \quad \text{for } x \in D \quad \text{(9c)}$$

$$\langle \nabla V, F_e(t, (r(t,j)^\top, x^\top)^\top) \rangle < -cV(r(t,j), x), \quad \text{for } x, r(t,j) \in C \quad \text{(9d)}$$

hold for all $(t, j) \in \operatorname{dom} r$ and all $x \in C \cup D$ such that $d(x, r(t, j)) < \delta$.

(ii) *If* (9) *holds for all $(t, j) \in \operatorname{dom} r$, all $x \in A := \{x \in C \cup D : V(r(t, j), x) \leq K\}$ and if the origin is not contained in the set A for any $(t, j) \in \operatorname{dom} r$, then the domain $\{x \in C \cup D : V(r(0, 0), x) \leq K\}$ is contained in the basin of attraction of r.*

Proof: First, we prove stability by evaluating $V(\bar{r}, \bar{x})$ along trajectories of (6). Let $\varepsilon \geq 0$, as given in Definition 1, be chosen arbitrarily. In the following argument we select initial conditions such that along closed-loop trajectories, firstly, the tracking error $d(\bar{x}, \bar{r})$ satisfies $d(\bar{x}, \bar{r}) < \delta$, such that (9b)-(9d) hold, and secondly, $d(\bar{x}, \bar{r}) < \varepsilon$, proving stability.

Consider a trajectory $\bar{x}$ with initial conditions $\bar{x}(t_0, j_0)$ satisfying $d(\bar{x}(t_0, j_0), \bar{r}(t_0, j_0)) \leq \delta^*$, where $\delta^* < \alpha_2^{-1}\alpha_1(\min(\delta, \varepsilon))$. For the sake of contradiction, assume that for such initial conditions, there exists times $(t^\dagger, j^\dagger) \in \operatorname{dom} q$, $(t^\dagger, j^\dagger) > (t_0, j_0)$ such that

$$d(\bar{r}(t^\dagger, j^\dagger), \bar{x}(t^\dagger, j^\dagger)) \geq \min(\delta, \varepsilon). \quad (10)$$

From the structure of (5) it follows that d, when evaluated along trajectories of (6), changes continuously with respect to continuous-time t and remains constant over jumps. Hence, (10) implies that there exists a $(t^*, j^*) \in \operatorname{dom} q$, with $t^* \leq t^\dagger$ and $j^* \leq j^\dagger$, such that

$$d(\bar{r}(t^*, j^*), \bar{x}(t^*, j^*)) = \min(\delta, \varepsilon) \quad (11)$$

and $d(\bar{r}(t, j), \bar{x}(t, j)) < \min(\delta, \varepsilon)$, $\forall (t, j) \in (\operatorname{dom} q \cap [t_0, t^*) \times [j_0, j^*])$. In the time interval $(\operatorname{dom} q \cap [t_0, t^*) \times [j_0, j^*])$, the function $V(\bar{r}, \bar{x})$, evaluated over trajectories of (6), is non-increasing both over jumps, as given in (9b)-(9c), and over flow, as given in (9d), such that we conclude $V(\bar{r}(t^*, j^*), \bar{x}(t^*, j^*)) \leq V(\bar{r}(t_0, j_0), \bar{x}(t_0, j_0))$. Using (9a), this implies $d(\bar{r}(t^*, j^*), \bar{x}(t^*, j^*)) \leq \alpha_1^{-1}\alpha_2(d(\bar{r}(t_0, j_0), \bar{x}(t_0, j_0)))$, and by design of δ^*, we obtain $d(\bar{r}(t^*, j^*), \bar{x}(t^*, j^*)) < \min(\delta, \varepsilon)$, which contradicts (11). Hence, we have proven that $d(\bar{r}(t, j), \bar{x}(t, j)) < \min(\delta, \varepsilon)$, $\forall (t, j) \in \operatorname{dom} q$, such that, first, statement *(i)* implies that the local conditions (9b)-(9d) hold along trajectories from initial conditions satisfying

$d(\bar{x}(t_0, j_0), \bar{r}(t_0, j_0)) \leq \delta^*$, and second, stability as given in Definition 1 is proven with $\delta(t_0, j_0, \varepsilon) = \delta^*$.

Now, we prove that the trajectory q has a time domain which is unbounded in t-direction when $\bar{x}(0,0)$ is sufficiently close to $\bar{r}(0,0)$. By Assumption 1, the trajectory $\bar{r}$ is unique and non-Zeno. This directly implies $\bar{r}(t,j)$ is bounded away from the origin, i.e., there exists a $d_0 > 0$ such that

$$|\bar{r}(t,j)| > d_0, \quad \forall (t,j) \in \operatorname{dom} q. \tag{12}$$

For initial conditions $\bar{x}(0,0)$ with $d(\bar{x}(0,0), \bar{r}(0,0)) \leq \delta^\dagger$, with strictly positive $\delta^\dagger < \min(\delta^*, \alpha_2^{-1}(\alpha_1(d_0)))$, it follows from the same reasoning as used to prove stability that $d(\bar{r}(t,j), \bar{x}(t,j)) \leq d_0, \ \forall (t,j) \in \operatorname{dom} q$, such that the definition of d yields $|\bar{x}(t,j)| \geq |\bar{r}(t,j)| - d_0$. Combination with (12) implies that $\bar{x}(t,j) \neq 0, \forall (t,j) \in \operatorname{dom} q$, such that $\bar{x}$ is non-Zeno. Boundedness of $\bar{x}$ follows from stability and implies that, at any time instant, the time-evolution of (3a) can be extended for increasing t unless $\bar{x}$ leaves C. Finally, trajectories x of (3) can not leave $C \cup D$, such that the time domain $\operatorname{dom} x$ is unbounded in t-direction. Since both x and r have a time domain which is unbounded in t-direction, $\operatorname{dom} q$ is unbounded in t-direction.

Now, we prove asymptotic stability. From (9b)-(9d) it follows that $V(\bar{r}(t,j), \bar{x}(t,j)) \leq e^{-ct} V(\bar{r}(0,0), \bar{x}(0,0)), \quad \forall (t,j) \in \operatorname{dom} q$. Hence, (9a) implies that $d(\bar{r}(t,j), \bar{x}(t,j)) \leq \alpha_1^{-1}(e^{-ct}\alpha_2(d(\bar{r}(0,0), \bar{x}(0,0))))$, proving *(i)*.

Consider an arbitrary initial condition $\bar{x}(0,0) \in \{x \in C \cup D : V(r(0,0), x) \leq K\}$. When (9) holds for all x such that $V(\bar{r}, \bar{x}) \leq K$, then evaluation of V along trajectories of (6) proves that $V(\bar{r}(t,j), \bar{x}(t,j)) \leq V(\bar{r}(0,0), \bar{x}(0,0)), \quad \forall (t,j) \in \operatorname{dom} q$. The assumption that the origin is not contained in the set $\{x \in C \cup D : V(r(t,j), x) \leq K\}$ for any $(t,j) \in \operatorname{dom} r$, as posed in *(ii)*, directly implies $\bar{x}$ is non-Zeno, such that $\operatorname{dom} q$ is unbounded in t-direction. From (9b)-(9d) it follows that $V(\bar{r}(t,j), \bar{x}(t,j)) \leq e^{-ct} V(\bar{r}(0,0), \bar{x}(0,0)), \ \forall (t,j) \in \operatorname{dom} q$, which proves asymptotic convergence of d to zero using (9a). Since $\bar{x}(0,0) \in \{x \in C \cup D : V(r(0,0), x) \leq K\}$ is chosen arbitrarily, all trajectories from these initial conditions converge asymptotically to r, proving *(ii)*. ■

IV. Controller design

In this section, we design a tracking control law $u_d(t,r,x)$ for system (3), based on the definition of the tracking error measure d in (5), for the case of fully elastic impacts ($\epsilon = 1$). When the trajectory x is sufficiently close to r and neither of them experiences a jump in the near future or past, then the tracking error $d(r,x)$ given in (5) is given by $|x - r|$. Along flowing solutions, this error could accurately be controlled using a controller with PD-type feedback, given by:

$$u = u_{\text{ff}}(t) - (f(t,x) - f(t,r)) - [k_p \, k_d](x - r), \tag{13}$$

where $k_p, k_d > 0$. Implementation of this controller yields the error dynamics $\ddot{x}_1 - \ddot{r}_1 = -k_p(x_1 - r_1) - k_d(\dot{x}_1 - \dot{r}_1)$, such that $\begin{bmatrix} x_1 - r_1 & \dot{x}_1 - \dot{r}_1 \end{bmatrix}^\top = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top$ is an asymptotically stable equilibrium point of this error dynamics.

However, if either the reference or the plant just experienced a jump, $d(r,x)$ as in (5) is given by $|x + r|$. The dynamics of $x + r$ is stable during flow when this error dynamics satisfies $\ddot{x}_1 + \ddot{r}_1 = -k_p(x_1 + r_1) - k_d(\dot{x}_1 + \dot{r}_1)$, which is obtained by the controller design:

$$u = -u_{\text{ff}}(t) - f(t,x) - f(t,r) - [k_p \, k_d](x + r). \tag{14}$$

Based on these insights, we propose a controller that switches between (13) and (14). To choose the partitioning of the state space where either (13) or (14) are applied, the following candidate Lyapunov function $V(r,x)$ is considered:

$$V_d(r,x) = \tfrac{1}{2}(x-r)^\top P (x-r); \ \ V_m(r,x) = \tfrac{1}{2}(x+r)^\top P (x+r)$$
$$V(r,x) = \min(V_d(r,x), V_m(r,x)), \tag{15}$$

where a positive definite matrix P is chosen such that

$$A^\top P + PA \prec -cP, \tag{16}$$

with $c > 0$ and $A = \begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix}$. For strictly positive k_p, k_d, the matrix A is Hurwitz, which implies that such P and c exist, cf. [17, Theorem 4.6]. The function V remains constant during jumps, such that (9b) and (9c) in Theorem 1 are satisfied. The lower and upper bounds (9a) are satisfied with $\alpha_1(d(x,r)) = \frac{1}{2}\lambda_{\min}(P) d(x,r)^2$ and $\alpha_2(d(x,r)) = \frac{1}{2}\lambda_{\max}(P) d(x,r)^2$, where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the minimum and maximum eigenvalue of P, respectively.

Based on the Lyapunov function candidate V, a control law $u = u_d(t,r,x)$ is designed, such that V decreases when $V_d(r,x) \neq V_m(r,x)$:

$$u = \begin{cases} u_{\text{ff}}(t) + f(t,r) - f(t,x) - [k_p \, k_d](x-r), & V_d \leq V_m \\ -u_{\text{ff}}(t) - f(t,r) - f(t,x) - [k_p \, k_d](x+r), & V_d > V_m. \end{cases} \tag{17}$$

Using the Lyapunov function (15), the following theorem is derived, that shows that the control law (17) solves the tracking problem.

Theorem 2 *Consider a hybrid system* (3)*, tracking error* d *given in* (5)*, reference trajectory* r*, and feedforward signal* u_{ff} *satisfying Assumption 1. Application of the control law* $u_d(t,r,x)$ *as defined in* (15) *and* (17)*, with* $k_p, k_d > 0$*, to the hybrid system* (3) *makes the reference trajectory* r *asymptotically stable with respect to* d*. In addition, the set* $\{x \in C \cup D : V(r(0,0), x) \leq K\}$ *is contained in the basin of attraction of* r*, where* K *is chosen to satisfy*

$$K < \min_{(t,j) \in \operatorname{dom} r} V(r(t,j), 0). \tag{18}$$

Proof: This proof can be found in [18] and is omitted here due to length restrictions. ■

V. Illustrative example

Consider system (3) with $\epsilon = 1$, $f(t,x) = -G$, where $G = 9.81$. Let the reference r be given by

$$r = \begin{cases} \begin{bmatrix} \frac{3}{4}G(\tau - \tau^2) \\ \frac{3}{4}G(1 - 2\tau) \end{bmatrix}, & \tau = t \bmod 4 \in [0,1] \\ \begin{bmatrix} \frac{3G}{4}(\tau - 1) - \frac{G}{4}(\tau - 1)^2 \\ \frac{3G}{4} - \frac{G}{2}(\tau - 1) \end{bmatrix}, & \tau = t \bmod 4 \in [1,4], \end{cases}$$

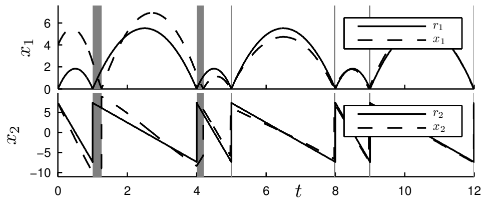

Fig. 3. Reference r and plant trajectory x of (3), where $f(t,x) = -9.81$, $u_{\rm ff}$ given in (19), and the control law (17) is applied with $[k_p\ k_d] = [1\ 0.5]$.

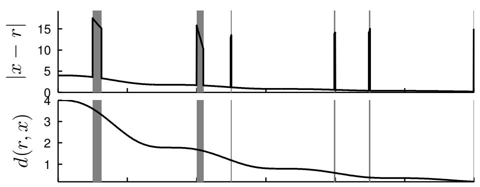

Fig. 4. Tracking error expressed in Euclidean distance $|x-r|$ and distance function $d(r,x)$ given in (4) for system (3), where $f(t,x) = -9.81$, $u_{\rm ff}$ given in (19), and control law (17) is applied with $[k_p\ k_d] = [1\ 0.5]$.

which is shown with a solid line in Fig. 3 and satisfies Assumption 1 for the initial condition $r(0,0) = \begin{bmatrix} 0 & \frac{3G}{4} \end{bmatrix}^\top$ and feedforward signal u:

$$u_{\rm ff}(t) = \begin{cases} -\frac{G}{2}, & \tau = t \bmod 4 \in [0,1] \\ \frac{G}{2}, & \tau = t \bmod 4 \in [1,4]. \end{cases} \quad (19)$$

The control law $u_d(t,r,x)$ given in (17) with $[k_p\ k_d] = [1\ \ 0.5]$ and $P = \begin{bmatrix} 2.25 & 0.5 \\ 0.5 & 2 \end{bmatrix}$ is applied to system (3) with initial condition $x(0,0) = \begin{bmatrix} 4 & \frac{3G}{4} \end{bmatrix}^\top$. This trajectory is shown with the dashed line in Fig. 3. Observe that the hybrid trajectory x converges to r during flow, and the jump instances of r and x converge to each other.

Both the Euclidean distance $|x-r|$ and the distance $d(r,x)$ between both trajectories are shown in Fig. 4. Clearly, the Euclidean distance displays the unstable "peaking" behaviour.

During the time period where $V_d(r,x) < V_m(r,x)$, the first case of controller (17) is active, which has a conventional feedback action of PD-type. However, during the time intervals that are depicted gray in Fig. 3, the reference experienced an impact and the plant not, or vise versa, such that $V_m(r,x) < V_d(r,x)$, and consequently, the second case of (17) is active. Hence, the feedback action is not disturbed by the peaking phenomenon of the Euclidean tracking error, and the controller (17) solves the tracking problem as formulated in Definition 2.

VI. Conclusion

In this paper, tracking problems were considered for a class of hybrid systems, which model mechanical systems with a unilateral constraint. In these systems, plant trajectories generically exhibit state-triggered jumps that do not coincide with the jumps of the reference trajectory. As a consequence, the Euclidean tracking error behaves in an unstable manner, even though the trajectories behave desirably. Therefore, we provided a new and appropriate formulation for the tracking problem in terms of a novel distance between the reference and the plant state. Sufficient conditions were formulated that guarantee local asymptotic stability of the reference trajectory with respect to this tracking error. In addition, a controller design procedure was proposed that solves the tracking problem for the class of systems with elastic impacts. Finally, the results were illustrated through simulations.

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