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Trajectory Tracking of Underactuated Skid-Steering Robot

Dariusz Pazderski and Krzysztof Kozłowski

Abstract— **This paper considers problem of approximation of admissible trajectory for skid-steering mobile robot at kinematic level. Nonholonomic constraints at kinematic and dynamic level are taken into account. The trajectory tracking control problem is solved using practical stabilizer using tunable oscillator with novel method of tuning. The stability result is proved using Lyapunov analysis and takes into account uncertainty of kinematics. In order to ensure stable motion of the robot the scaling method is used. Theoretical considerations are illustrated by simulation results.**

I. INTRODUCTION

In robotics applications wheeled vehicles play crucial role. The most popular group of mobile robots takes advantage of differential drive mechanism mainly because of simplicity and good mobility. In general two main categories of such construction can be distinguished, i.e. vehicles for which non-slip and pure-rolling conditions may be assumed [2] and vehicles for which skid phenomena is used for proper operation. Although skidding effect between wheels and surface may be observed for all vehicles, only for the second group known as skid-steering vehicles it is necessary to change their heading.

Skid-steering structure is commonly used in robotics that is due to its mechanical robustness. In particular skid-steering mobile robots (SSMRs) are quite similar to robots equipped with two-wheeled driving system (i.e. unicycle-like robots). However, there is an important difference between them, namely for SSMR ground-wheels interaction and skidding effect play an important role within high range of velocities and accelerations. Since ground reaction forces are very difficult to calculate and measure the model of SSMR dynamics is not accurate. Moreover, in spite of that fact that skidding allows to change robot's orientation, extensive skidding causes the motion to be unstable – hence it is necessary to limit velocity of the vehicle.

Pure rolling of wheel without slip is one of the main source of the first order nonholonomic constraints in mechanical systems. As a result the set of admissible velocities becomes constrained but the dimension of configuration space is not reduced. For SSMR because of skidding one can ask if it can be regarded as nonholonomic system similar to the two-wheeled robot taking into account that from a mechanical point of view it is an underactuated system with nonintegrable dynamics (i.e. with acceleration constraints). According to work done by Lewis [6] SSMR is strictly dynamic system since it cannot be reduced to smooth kinematic system without lack of knowledge concerning admissible trajectories. However, in [8] it was shown how overconstrained dynamic system may be reduced to kinematic system with changed structure.

In this paper we formally show that kinematics of SSMR can be approximated by kinematics of unicycle-like robot. According to authors' knowledge such problem have not been properly investigated in the robotics literature. Previously, in some papers (see [3] and [9]) for control purposes authors assumed an ideal nonholonomic constraint and used dynamic model of SSMR with Lagrange multipliers which cannot be justified taking into account physical properties of the system.

Here we show that it is possible to consider SSMR at kinematic level assuming non-stationary nonholonomic constraint. In the case of slow motion one can obtain simple approximation of admissible trajectories.

In order to present usefulness of this approach we formulate kinematic control law based on tunable oscillator [4] and transverse functions [7] which is robust to bounded lateral skidding. Taking into account the part of dynamics we give a condition of stable motion with respect to position of instantaneous center of rotation.

Next, we consider control problem assuming that linear and angular velocity can be treated as control input and neglect the task of enforcing this velocities by actuators. Such approach is used for dividing control tasks onto two levels, i.e. kinematic and dynamic which can be relatively easy realized in real applications.

The paper is organized as follows. In Section II kinematic and dynamic model of SSMR is presented and approximation of admissible trajectories of SSMR at kinematic level is discussed. In the next section the control law using tunable oscillator is developed with respect to limited lateral skidding velocity. In Section IV simulation results are presented. Concluding remarks are given in Section V.

II. SSMR MODEL

A. Kinematics

In this paper we consider an example of SSMR equipped with Four Wheel Drive (4-WD) and moving on the plane (Fig. 1) with respect to the inertial frame X_gY_g. A local frame x_ly_l is attached to its center of mass (COM). Let $\boldsymbol{q} \triangleq [\theta\ X\ Y]^T \in \mathbb{S}^1 \times \mathbb{R}^2$ denote generalized coordinates describing robot's position, X and Y, in the inertial frame

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D. Pazderski is with Faculty of Computer Science and Management, Chair of Control and Systems Engineering, Poznan University of Technology, ul. Piotrowo 3a, 60-965 Poznan, Poland Dariusz.Pazderski@put.poznan.pl

K. Kozłowski is with Faculty of Computer Science and Management, Chair of Control and Systems Engineering, Poznan University of Technology, ul. Piotrowo 3a, 60-965 Poznan, Poland Krzysztof.Kozlowski@put.poznan.pl

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and orientation, θ, of the local frame with respect to the inertial one.

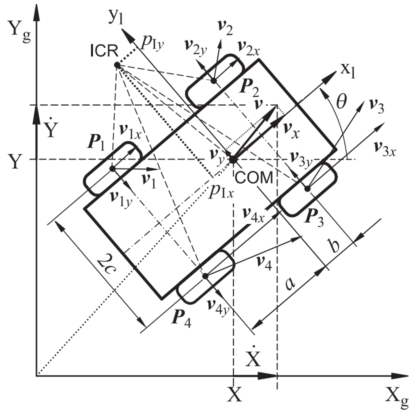

Fig. 1. Kinematics of SSMR

In the local frame one can describe robot motion using vector $\boldsymbol{\eta} \triangleq [\omega \ v_x \ v_y]^T \in \mathbb{R}^3$, where ω, v_x and v_y denotes angular, longitudinal and lateral velocities of the robot, respectively. From Fig. 1 one can easily find the following map

$$\dot{\boldsymbol{q}} = \boldsymbol{\Theta}(\boldsymbol{q})\,\boldsymbol{\eta}, \tag{1}$$

where

$$\boldsymbol{\Theta}(\boldsymbol{q}) \triangleq \boldsymbol{\Theta}(q_1) \triangleq \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{R}(\theta) \end{bmatrix} \tag{2}$$

and $\boldsymbol{R}(\theta) \in \mathrm{SO}(2)$. Taking into account SSMR kinematics (1) one can rewrite it in the form similar to unicycle kinematics

$$\dot{\boldsymbol{q}} = \boldsymbol{S}(\boldsymbol{q})\,\boldsymbol{\eta}^* + \boldsymbol{d}(\boldsymbol{q}), \tag{3}$$

with

$$\boldsymbol{S}(\boldsymbol{q}) \triangleq \begin{bmatrix} 1 & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix}, \tag{4}$$

$\boldsymbol{\eta}^* \triangleq \begin{bmatrix} \omega & v_x \end{bmatrix}^T$ and $\boldsymbol{d}(\boldsymbol{q}) \triangleq \begin{bmatrix} 0 & -\sin\theta & \cos\theta \end{bmatrix}^T v_y$. The term $\boldsymbol{d}$ can be considered as disturbance which is dependent on lateral velocity v_y resulting from skidding.

Taking into account position of instantaneous center of rotation (ICR) one can find the following first order constraint

$$\boldsymbol{A}(\boldsymbol{q}, p_{Ix})\,\dot{\boldsymbol{q}} = 0, \tag{5}$$

where $\boldsymbol{A}(\boldsymbol{q}, p_{Ix}) \triangleq \begin{bmatrix} p_{Ix} & -\sin\theta & \cos\theta \end{bmatrix}$ is a constraint matrix dependent on current value of ICR x-coordinate expressed in the local frame. Equation (5) is not integrable, hence it describes first order but non-stationary nonholonomic constraint in the case when $|p_{Ix}| \in \mathcal{L}_\infty$. Indeed evolution of p_{Ix} cannot be derived from kinematics equation since non-skidding condition between wheels and surface is generally violated. As a result such system cannot be accurately reduced to smooth kinematic system [6].

B. Dynamics

The dynamics of SSMR for plane motion can be modeled using the following equation

$$\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}} = \boldsymbol{Q}_A + \boldsymbol{Q}_R, \tag{6}$$

where $\boldsymbol{M} \in \mathbb{R}^{3\times3}$ denotes positive definite inertia matrix, $\boldsymbol{C} \in \mathbb{R}^{3\times3}$ is used to describe centrifugal and Coriolis inertia forces, $\boldsymbol{Q}_A \in \mathbb{R}^3$ is a vector of active forces (produces by actuators) and $\boldsymbol{Q}_R \in \mathbb{R}^3$ is a vector of resistive forces which mainly result from wheels-ground interaction. Taking into account geometry of considered robot one calculate active forces as follows

$$\boldsymbol{Q}_A \triangleq \boldsymbol{B}(\boldsymbol{q})\,\boldsymbol{\tau} \tag{7}$$

where $\boldsymbol{B}(\boldsymbol{q}) \in \mathbb{R}^{3\times2}$ denotes input matrix and $\boldsymbol{\tau} = [\tau_L \ \tau_R]^T \in \mathbb{R}^2$ is an input vector determining torques produced by pairs of wheels on the left and right side of the vehicle, respectively (see Fig. 2).

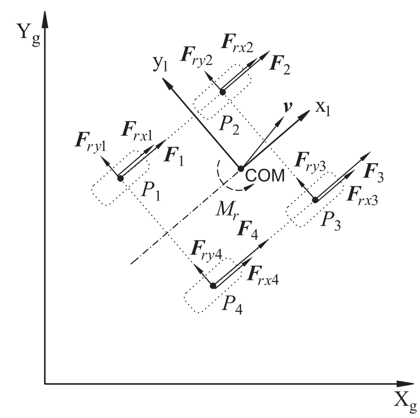

Fig. 2. Active and resistive forces

For simplicity of analysis we assume that mass distribution of the vehicle is homogeneous and constant as well as the origin of local frame is placed at COM – hence, inertia matrix takes the following form: $\boldsymbol{M} = diag\{I, m, m\}$, while m, I represents the mass and inertia, respectively and $\boldsymbol{C} \equiv \mathbf{0}$. Then using kinematics (1) and taking into account (7) one can rewrite equation (6) in the following form

$$\bar{\boldsymbol{M}}\dot{\boldsymbol{\eta}} + \bar{\boldsymbol{C}}\boldsymbol{\eta} = \bar{\boldsymbol{B}}\boldsymbol{\tau} + \bar{\boldsymbol{Q}}_R, \tag{8}$$

where

$$\begin{aligned} \bar{\boldsymbol{M}} &= \boldsymbol{M}, & \bar{\boldsymbol{Q}}_R &= \begin{bmatrix} M_r & F_{rx} & F_{ry} \end{bmatrix}^T, \\ \bar{\boldsymbol{C}} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -m \\ 0 & m & 0 \end{bmatrix}\omega, & \bar{\boldsymbol{B}} &= \tfrac{1}{r}\begin{bmatrix} -c & c \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \end{aligned} \tag{9}$$

with

$$\begin{aligned} F_{rx} &= \textstyle\sum_{i=1}^4 F_{rxi}, \\ F_{ry} &= \textstyle\sum_{i=1}^4 F_{ryi}, \\ M_r &= c\left(-\textstyle\sum_{i=1,2} F_{rxi} + \sum_{i=3,4} F_{rxi}\right) + \\ &\quad -a\textstyle\sum_{i=1,4} F_{ryi} + b\sum_{i=2,3} F_{ryi} \end{aligned}$$

and $r = r_i$ (it is supposed that radius of each wheel is the same). Here F_{rx} is used in order to describe resultant resistive force in longitudinal direction including rolling resistant of wheels, motors and gears. The term F_{ry} denotes the resultant constraint force in lateral direction as a result of wheel-ground interactions and it is hard to model accurately (in general tyre-ground model may be considered – see for example [11]). However, for simplicity in references [5], [11] it is often assumed that in the case of skid-steering vehicles lateral force F_{ryi} for i-th wheel can be described using Coulomb friction model as follows

$$F_{ryi} \triangleq -\mu_i N_i \, \mathrm{sgn} v_{yi}, \tag{10}$$

where μ_i is a friction coefficient, N_i is the wheel ground contact force which result from gravity and v_{yi} is the lateral velocity of wheel (as indicated in Fig. 1).

C. Nonintegrable dynamics

Dynamic equations (8) show that considered system is underactuated, since $\dim \boldsymbol{q} > \dim \boldsymbol{\tau}$. In opposite to simple nonholonomic wheeled robots which can be considered as kinematic systems in this case kinematic reduction is not possible. Taking into account (8) in the more detailed form one can distinguished the following two subsystems

$$\begin{bmatrix} I\dot{\omega} \\ m\dot{v}_x \end{bmatrix} + \begin{bmatrix} 0 \\ -mv_y\omega \end{bmatrix} + \begin{bmatrix} -M_r \\ -F_{rx} \end{bmatrix} = \begin{bmatrix} \frac{c}{r}(-\tau_L + \tau_R) \\ \frac{1}{r}(\tau_L + \tau_R) \end{bmatrix} \tag{11}$$

and

$$m\dot{v}_y + mv_x\omega = F_{ry}. \tag{12}$$

The first subsystem (11) is fully actuated. The second one, i.e. lateral dynamics (12), describes acceleration constraint that is nonintegrable. As a result it can be regarded as second order nonholonomic constraint. For SSMR significant lateral skidding is undesirable during normal operation. Therefore the velocity v_y should be limited. According to [10] we may consider the following proposition

Proposition 1: Assuming that linear and angular velocities of the vehicle satisfy

$$|\omega v_x| \leq g \sum_{i=1}^{4} \gamma_i \mu_i \tag{13}$$

where g is the value of gravity and

$$\gamma_i \triangleq \frac{1}{a+b} \begin{cases} b & \text{for } i = 1,4 \\ a & \text{for } i = 2,3 \end{cases}, \tag{14}$$

the motion of the vehicle is stable in the sense that x-coordinate of ICR is bounded as

$$-a \leq p_{Ix} \leq b. \tag{15}$$

Therefore if the robot moves relatively slow skidding effect is reduced significantly.

D. Approximation of admissible trajectories

From a practical point of view it is hard to model or measure interaction forces. Therefore it is almost impossible to generate trajectory off-line which is feasible for SSMR. Instead of it one can consider its approximation based on kinematics of unicycle-like robot. Moreover for low velocities (i.e. limited value of product $|v_x\omega|$) value of v_y is highly reduced. As a result one may introduce the following definition:

Definition 1: The trajectory $\boldsymbol{q}$ which is a solution to the following kinematic equation

$$\dot{\boldsymbol{q}} = \boldsymbol{S}(\boldsymbol{q})\,\boldsymbol{\eta}^* \tag{16}$$

is called *almost admissible trajectory* for the system (3) if value of $|v_x\omega|$ is small enough.

III. CONTROL LAW

A. Tracking error definition

In this paper we use so called left-invariant operation [1], [7] which takes into account symmetry of the control system (3) described on SE(2) Lie group. The operation $\circ$ can be defined as follows

$$\boldsymbol{a} \circ \boldsymbol{b} \triangleq \boldsymbol{a} + \boldsymbol{\Theta}(a_1)\,\boldsymbol{b}, \tag{17}$$

where $\boldsymbol{a} \triangleq [a_1\ a_2\ a_3]^T \in \mathbb{S}^1 \times \mathbb{R}^2$, $\boldsymbol{b} \triangleq [b_1\ b_2\ b_3]^T \in \mathbb{S}^1 \times \mathbb{R}^2$ are elements of Lie group and $\boldsymbol{\Theta}(\cdot)$ is defined by (2). According to [7] we can define so-called transformed tracking error with respect to the moving frame as

$$\tilde{\boldsymbol{q}} \triangleq \boldsymbol{q}_r^{-1} \circ \boldsymbol{q} = \boldsymbol{\Theta}^T(\theta_r)(\boldsymbol{q}_r - \boldsymbol{q}), \tag{18}$$

where $\boldsymbol{q}_r \triangleq [\theta_r\ X_r\ Y_r]^T$ denotes reference orientation and position. Next, taking the time derivative of (18) and using (3) one can obtain

$$\dot{\tilde{\boldsymbol{q}}} = \boldsymbol{S}(\tilde{\boldsymbol{q}})\,\boldsymbol{\eta}^* + \boldsymbol{f}_d(\tilde{\boldsymbol{q}}, \boldsymbol{q}_r, \dot{\boldsymbol{q}}_r) + \boldsymbol{d}(\tilde{\boldsymbol{q}}), \tag{19}$$

where

$$\boldsymbol{f}_d(\tilde{\boldsymbol{q}}, \boldsymbol{q}_r, \dot{\boldsymbol{q}}_r) = \begin{bmatrix} -\omega_r \\ -\boldsymbol{J}\begin{bmatrix}\tilde{q}_2 \\ \tilde{q}_3\end{bmatrix}\omega_r - \boldsymbol{R}^T(\theta_r)\begin{bmatrix}\dot{X}_r \\ \dot{Y}_r\end{bmatrix} \end{bmatrix} \tag{20}$$

is the drift term dependent on reference trajectory with $\boldsymbol{J} \triangleq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\omega_r \triangleq \dot{\theta}_r$.

B. Control law development

Here we use the concept of practical stabilization originally introduced by Dixon and coauthors [4] and next developed and generalized by Morin and Samson [7] (they utilize frequency method of control). The control task at kinematic level can be formulated as follows:

Definition 2: Find bounded controls $v_x(t), \omega(t)$ for kinematics (1) such, that for initial condition $\tilde{\boldsymbol{q}}(0)$ the Euclidean norm of the error $\tilde{\boldsymbol{q}}(t)$ is bounded as:

$$\lim_{t\to\infty} \|\tilde{\boldsymbol{q}}(t)\| \leq \varepsilon, \tag{21}$$

where ε is an assumed error envelope, which can be made arbitrary small.

In order to facilitate the control solution we define auxiliary error taking into account left-invariant operation (17) as follows

$$\boldsymbol{z} \triangleq \tilde{\boldsymbol{q}} \circ \boldsymbol{x}_d^{-1} = \tilde{\boldsymbol{q}} - \boldsymbol{\Theta}\left(\tilde{q}_1 - x_{d1}\right) \boldsymbol{x}_d \quad (22)$$

where $\boldsymbol{x}_d \triangleq \left[x_{d1}\ x_{d2}\ x_{d3}\right]^T \in \mathbb{R}^3$ is a vector containing harmonic-like signals (according to terminology used by Morin and Samson $\boldsymbol{x}_d$ is a transverse function). Frequency of these signals can be seen as a new virtual input which allows to render desired trajectory at each direction in the state space (i.e. approximates it with desired accuracy).

Taking the time derivative of (22) and using (19) one has

$$\begin{aligned}\dot{\boldsymbol{z}} = \boldsymbol{S}\left(\tilde{\boldsymbol{q}}\right) \boldsymbol{\eta}^* - \boldsymbol{\Theta}\left(z_1\right) \dot{\boldsymbol{x}}_d - d\boldsymbol{\Theta}\left(z_1\right) \boldsymbol{x}_d \left(\omega - \dot{x}_{d1}\right) + \quad (23)\\ +\boldsymbol{f}_r\left(\tilde{\boldsymbol{q}}, \boldsymbol{q}_r, \dot{\boldsymbol{q}}_r, \boldsymbol{x}_d\right) + \boldsymbol{d}\left(\tilde{\boldsymbol{q}}\right)\end{aligned}$$

where $\boldsymbol{f}_r\left(\tilde{\boldsymbol{q}}, \boldsymbol{q}_r, \dot{\boldsymbol{q}}_r, \boldsymbol{x}_d\right) \triangleq \boldsymbol{f}_d\left(\tilde{\boldsymbol{q}}, \boldsymbol{q}_r, \dot{\boldsymbol{q}}_r\right) + d\boldsymbol{\Theta}\left(z_1\right) \boldsymbol{x}_d \omega_r$ with

$$d\boldsymbol{\Theta}\left(\theta\right) = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{R}\left(\theta\right) \boldsymbol{J} \end{bmatrix}. \quad (24)$$

Now based on [4] and [7] we assume that $\boldsymbol{x}_d$ is generated using tunable linear oscillator as follows

$$\boldsymbol{x}_d \triangleq \begin{bmatrix} \boldsymbol{\Phi}_1 \\ \frac{1}{2}\boldsymbol{\xi}^T \boldsymbol{\Phi}_2 \end{bmatrix} \boldsymbol{\xi}, \quad (25)$$

where $\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_2 \in \mathbb{R}^{2\times 2}$ are matrices with scaling functions and $\boldsymbol{\xi} \in \mathbb{R}^2$ is the solution of the following differential equation

$$\dot{\boldsymbol{\xi}} \triangleq \boldsymbol{J}\boldsymbol{\xi}\Omega \quad (26)$$

with Ω denoting instantaneous frequency and initial condition $\boldsymbol{\xi}\left(0\right)^T \boldsymbol{\xi}\left(0\right) = 1$. Assuming that

$$\boldsymbol{\Phi}_i\left(t\right) \triangleq \begin{bmatrix} {}^i\varphi_{11}\left(t\right) & {}^i\varphi_{12}\left(t\right) \\ {}^i\varphi_{21}\left(t\right) & {}^i\varphi_{22}\left(t\right) \end{bmatrix}, \quad (27)$$

taking the time derivative of (25), substituting the result to (23) and making some algebraic manipulations one can finally write

$$\begin{aligned}\dot{\boldsymbol{z}} = \boldsymbol{\Sigma}\left(\boldsymbol{z}, \boldsymbol{x}_d\right) \boldsymbol{H}\left(\boldsymbol{x}_d\right) \begin{bmatrix} \boldsymbol{\eta}^* \\ \Omega \end{bmatrix} + \boldsymbol{f}_\Phi\left(\tilde{\boldsymbol{q}}, \boldsymbol{\xi}\right) + \quad (28)\\ +\boldsymbol{f}_r\left(\tilde{\boldsymbol{q}}, \boldsymbol{q}_r, \dot{\boldsymbol{q}}_r, \boldsymbol{x}_d\right) + \boldsymbol{d}\left(\tilde{\boldsymbol{q}}\right),\end{aligned}$$

where

$$\boldsymbol{\Sigma}\left(\boldsymbol{z}, \boldsymbol{x}_d\right) \triangleq \begin{bmatrix} 1 & 0 \\ -\boldsymbol{R}\left(z_1\right) \boldsymbol{J} \begin{bmatrix} x_{d2} \\ x_{d3} \end{bmatrix} & \boldsymbol{R}\left(z_1\right) \end{bmatrix} \in \mathbb{R}^{3\times 3}, \quad (29)$$

$$\boldsymbol{H}\left(\boldsymbol{x}_d\right) \triangleq \begin{bmatrix} \boldsymbol{S}\left(\boldsymbol{x}_d\right) & -\begin{bmatrix} \boldsymbol{\Phi}_1 \\ \boldsymbol{\xi}^T \boldsymbol{\Phi}_2 \end{bmatrix} \boldsymbol{J}\boldsymbol{\xi} \end{bmatrix} \in \mathbb{R}^{3\times 3}, \quad (30)$$

and

$$\boldsymbol{f}_\Phi\left(\tilde{\boldsymbol{q}}, \boldsymbol{\xi}\right) \triangleq \begin{bmatrix} -\left[1\ 0\right] \dot{\boldsymbol{\Phi}}_1 \\ \boldsymbol{R}\left(z_1\right) \left(\boldsymbol{J} \begin{bmatrix} x_{d2} \\ x_{d3} \end{bmatrix} \left[1\ 0\right] \dot{\boldsymbol{\Phi}}_1 - \begin{bmatrix} \left[1\ 0\right] \dot{\boldsymbol{\Phi}}_1 \\ \frac{1}{2}\boldsymbol{\xi}^T \dot{\boldsymbol{\Phi}}_2 \end{bmatrix}\right) \end{bmatrix} \boldsymbol{\xi}. \quad (31)$$

In order to calculate control signal $\boldsymbol{\eta}^*$ matrices $\boldsymbol{H}$ and $\boldsymbol{\Sigma}$ must be invertible. For matrix $\boldsymbol{\Sigma}$ one can show that such property is always satisfied since $\det \boldsymbol{\Sigma} \equiv 1$. However, invertibility of $\boldsymbol{H}$ is guaranteed only for properly chosen set of scaling functions ${}^i\varphi_{jk}$. Calculating determinant of $\boldsymbol{H}$ one can find the following limitations

$$\sqrt{\left({}^1\varphi_{11}\right)^2 + \left({}^1\varphi_{12}\right)^2} < \frac{\pi}{2},\ {}^1\varphi_{21} \geq 0,\ {}^1\varphi_{22} > 0, \quad (32)$$

$${}^2\varphi_{11} = 0,\ {}^2\varphi_{12} = {}^2\varphi_{21} < {}^1\varphi_{11}{}^1\varphi_{22},\ {}^2\varphi_{22} \geq 0. \quad (33)$$

Next we can formulate the control solution as follows.

Proposition 2: The smooth control law given as

$$\begin{aligned}\begin{bmatrix} \boldsymbol{\eta}^* \\ \Omega \end{bmatrix} \triangleq \left(\boldsymbol{\Sigma}\left(\boldsymbol{z}, \boldsymbol{x}_d\right) \boldsymbol{H}\left(\boldsymbol{z}\right)\right)^{-1} \left(-\boldsymbol{K}\boldsymbol{z} + \right. \quad (34)\\ -\boldsymbol{f}_\Phi\left(\tilde{\boldsymbol{q}}, \boldsymbol{\xi}\right) - \boldsymbol{f}_r\left(\tilde{\boldsymbol{q}}, \boldsymbol{q}_r, \dot{\boldsymbol{q}}_r, \boldsymbol{x}_d\right) + \\ \left. -\rho \bar{\boldsymbol{d}}\left(\tilde{\boldsymbol{q}}\right) f_s^a\left(\bar{\boldsymbol{d}}^T \tilde{\boldsymbol{q}}\right)\right),\end{aligned}$$

where $-\boldsymbol{K} \in \mathbb{R}^{3\times 3}$ is Hurwitz-stable matrix, ρ is a scalar function which satisfies

$$\rho > \left|v_y\right|, \quad (35)$$

$$f_s^a\left(y\right) \triangleq \frac{\rho y}{\rho \left|y\right| + \varepsilon_s} \quad (36)$$

is an approximation of non smooth $\operatorname{sgn}\left(\cdot\right)$ function with constant $\varepsilon_s > 0$ and $\bar{\boldsymbol{d}}\left(\tilde{\boldsymbol{q}}\right) \triangleq \left[0\ \ -\sin \tilde{q}_1\ \ \cos \tilde{q}_1\right]^T$, $\left\|\dot{\boldsymbol{q}}_r\right\| \in \mathcal{L}_\infty$, $\left\|\dot{\boldsymbol{\Phi}}_i\right\| \in \mathcal{L}_\infty$ ensures practical stabilization in the sense given by (21).

Proof: Firstly, we define Lyapunov function candidate as

$$V \triangleq \frac{1}{2}\boldsymbol{z}^T \boldsymbol{z}. \quad (37)$$

Next, taking the time derivative of (37), using (23) with control (34) one has

$$\begin{aligned}\dot{V} = -\boldsymbol{z}^T \boldsymbol{K}\boldsymbol{z} + \boldsymbol{z}^T \boldsymbol{d}\left(\tilde{\boldsymbol{q}}\right) - \rho \boldsymbol{z}^T \bar{\boldsymbol{d}}\left(\tilde{\boldsymbol{q}}\right) f_s^a\left(\bar{\boldsymbol{d}}^T\left(\tilde{\boldsymbol{q}}\right) \boldsymbol{z}\right) = \quad (38)\\ = -\boldsymbol{z}^T \boldsymbol{K}\boldsymbol{z} + \boldsymbol{z}^T \bar{\boldsymbol{d}}\left(\tilde{\boldsymbol{q}}\right) \left(v_y - \rho f_s^a\left(\bar{\boldsymbol{d}}^T\left(\tilde{\boldsymbol{q}}\right) \boldsymbol{z}\right)\right). \quad (39)\end{aligned}$$

Taking into account definition (36) and condition (35) one can find the following inequality

$$\dot{V} \leq -\boldsymbol{z}^T \boldsymbol{K}\boldsymbol{z} + \frac{\rho \left|\bar{\boldsymbol{d}}^T\left(\tilde{\boldsymbol{q}}\right) \boldsymbol{z}\right| \varepsilon_s}{\rho \left|\bar{\boldsymbol{d}}^T\left(\tilde{\boldsymbol{q}}\right) \boldsymbol{z}\right| + \varepsilon_s} \leq -\lambda \left\|\boldsymbol{z}\right\|^2 + \varepsilon_s \quad (40)$$

with $\lambda > 0$. Then solving inequality (40) and using (37) one has

$$\left\|\boldsymbol{z}\left(t\right)\right\| \leq \sqrt{\left\|\boldsymbol{z}\left(0\right)\right\| \exp\left(-2\lambda t\right) + \frac{\varepsilon_s}{2\lambda}\left(1 - \exp\left(-2\lambda t\right)\right)}. \quad (41)$$

Hence, auxiliary error in the steady-state is bounded as follows

$$\lim_{t\to\infty} \left\|\boldsymbol{z}\left(t\right)\right\| \leq \varepsilon_2, \quad (42)$$

where $\varepsilon_2 = \sqrt{\frac{\varepsilon_s}{2\lambda}}$. The steady-state error in the configuration space can be estimated as

$$\lim_{t\to\infty} \left\|\tilde{\boldsymbol{q}}\left(t\right)\right| \leq \varepsilon_1 + \varepsilon_2, \quad (43)$$

where ε_1 is dependent on scaling functions ${}^{i}\varphi_{jk}$ according to the following relationship

$$\varepsilon_1^2 = \lim_{t\to\infty}\left(\sum_{i=1}^{2}\sum_{j=1}^{2}\left({}^{1}\varphi_{ij}\left(t\right)\right)^2 + \left({}^{2}\varphi_{12}\left(t\right)\right)^2 + \right. \quad (44)$$

$$\left. + \frac{1}{4}\left({}^{2}\varphi_{22}\left(t\right)\right)^2 + {}^{2}\varphi_{22}\left(t\right){}^{2}\varphi_{12}\left(t\right)\right).$$

■

C. Controller tuning

In spite of stability result proved in previous subsection a good performance of the controller is related to proper selection of tuning matrices $\mathbf{\Phi}_i$ which influence both transient and steady-state. In this paper we use novel method based on adaptive scaling taking into account current error $\boldsymbol{z}$. In order to do that we introduce the filtered error z_s which a solution of the following linear second order differential equation

$$T^2\ddot{z}_s + 2T\dot{z}_s + z_s = \sqrt{z_2^2 + z_3^2 + \epsilon^2}, \quad (45)$$

with initial condition $z_s\left(0\right) > 0$ and $\dot{z}_s\left(0\right) = 0$, $T > 0$ and $\epsilon > 0$. Taking into account (42) it is easy to show that in the steady state

$$\lim_{t\to\infty} z_s\left(t\right) \leq \varepsilon_2 + \epsilon. \quad (46)$$

Filtered error is then used to scale ${}^{i}\varphi_{jk}$. For example in particular case one can use the following set of functions

$${}^{1}\varphi_{11} = \frac{\pi}{2}\tanh\left(c_{11a}z_s + c_{11b}z_s^2\right),\ {}^{1}\varphi_{22} = c_{22}z_s, \quad (47)$$

$${}^{1}\varphi_{12} = {}^{1}\varphi_{21} = 0,\ {}^{2}\varphi_{12} = \frac{1}{2}{}^{1}\varphi_{11}{}^{1}\varphi_{22},\ {}^{2}\varphi_{22} = 0, \quad (48)$$

with $c_{11a}, c_{11b}, c_{22} > 0$, which satisfy the conditions (32) and (33). Such tuning method allows to limit oscillatory behavior of the control systems as well as it allows to obtain good accuracy in the steady state (determined by (46 and (43))).

D. Velocity limitation

To ensure stable motion of SSMR the robot should move relatively slow, i.e. product $|v_x\omega|$ should be limited. However the control law formulated in proposition 2 does not take into account this limitation. Therefore we propose to use scaling kinematic signal and time similar to method given in [10]. It should be assumed that reference trajectory is chosen in such a way that it can be approximated with desired accuracy using feasible value of velocities, i.e. that condition given in proposition 2 can be satisfied.

E. Extension to the dynamic level

In this paper we concentrate on control problem at kinematic level only. Dynamic description used here is only necessary to prove that within range of low velocities kinematic approximation of admissible velocity is justified. For simplicity we assume that desired velocities ω and v_x generated by the controller can be realized with high accuracy. In order to consider problem of control at dynamic level one can use backstepping technique taking into account that velocity signals are time-differentiable.

IV. SIMULATION RESULTS

In order to show effectiveness of the considered controller numerical simulations in Matlab/Simulink environment were conducted. The model of SSMR used here is based on kinematic equation (1) and takes into account the lateral dynamics (12). The parameters of the SSMR model were chosen to correspond to the parameters of the real experimental mobile robot 4WD SSMR MMS (Modular Mobile System) built in our laboratory as follows:

$$a = b = 0.075\ [m],\ m = 14\ [kg],\ g = 9.81\ [m/s^2].$$

It was assumed that friction coefficient μ_i is time varying and satisfies $0.4 < \mu_i < 1$. The reference eight-like shaped trajectory $\boldsymbol{q}_r$ was calculated based on unicycle model (16) and parametrized as follows

$$\begin{bmatrix} X_r\left(t\right) \\ Y_r\left(t\right) \end{bmatrix} = \begin{bmatrix} 0.7\sin\left(0.4t\right) \\ 0.7\cos\left(0.2t\right) \end{bmatrix}. \quad (49)$$

In order to include motor limitation and to ensure motion stability scaling input signals algorithm [10] was used. The parameters of the controller and tuning functions were selected as: $\boldsymbol{K} \triangleq diag\left\{1,1,1\right\}$, $\varepsilon_s = 0.05$, $\rho = 0.2$, $T = 1.5$, $z_s\left(0\right) = 1$, $\epsilon = 0.08$, $c_{11a} = 0.75$, $c_{11b} = 6$, $c_{22} = 1$, $\boldsymbol{\xi}\left(0\right) = \left[-1\ 0\right]^T$. The initial configuration was chosen as $\boldsymbol{q}\left(0\right) = \left[0\ 1\ -1\right]^T$.

The results of simulations for SSMR are illustrated in Figs. 3-6. According to Fig. 3 one can see that tracking errors are bounded to the nonzero values. The most difficult is to decrease orientation error when the curvature of the path is high. The difficulty in achieving lower tracking error lies in the fact that almost admissible reference trajectory can be only approximated. Hence when we take into account velocity limitation it is clear that in practice accuracy of tracking is restricted. From Fig. 4 one can conclude that used reference trajectory is hard to be tracked – robot motion is not smooth since linear velocity changes its direction in order to keep tracking error to be within desired set. Lateral velocity v_y does not achieve higher values and it satisfies $|v_y| \leq \rho$. Fig. 5 illustrates that auxiliary errors converges quite fast to the small neighborhood of zero. In Fig. 6 one can conclude that the robot motion is stable – x-coordinate of ICR remains in the desired range. Filtered error z_s used for controller tuning tends to assumed value based on current auxiliary error – therefore oscillatory behavior during transient states is avoided.

For comparison purpose the similar simulation was conducted for unicycle model (i.e. assuming $v_y = 0$). The controller was not changed. From Figs. 7 and 8 one can see that results of stabilization are better than for SSMR since tracked reference trajectory in this case becomes admissible.

V. CONCLUDING REMARKS

In this paper we formally show that control of SSMR is possible based on unicycle kinematics. Additionally we propose to use control law which ensures practical stabilization. Such approach seems to be justified when we consider structural and parametric uncertainties in SSMR

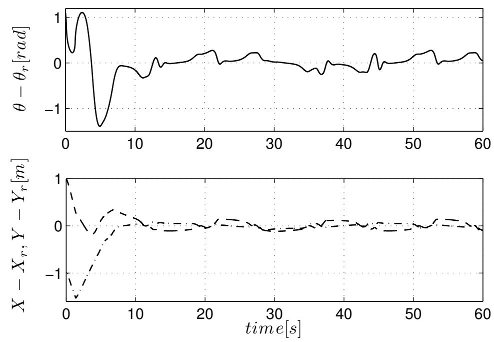

Fig. 3. SSMR – tracking errors: $\theta - \theta_r[rad]$ (–), $X - X_r[m]$ (– –), $Y - Y_r[m]$ (–.–)

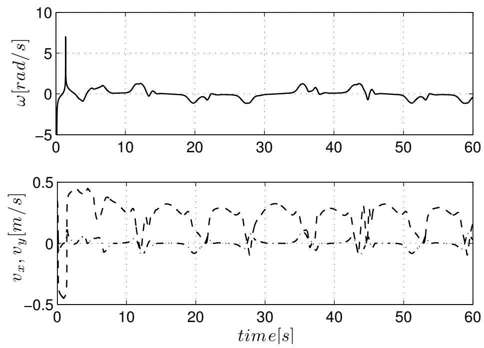

Fig. 4. SSMR – velocities: $\omega[rad/s]$ (–), $v_x[m/s]$ (– –), $v_y[m/s]$ (–.–)

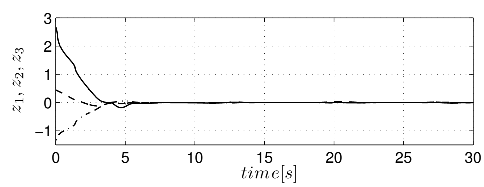

Fig. 5. SSMR – auxiliary errors: z_1 (–), z_2 (– –), z_3 (–.–)

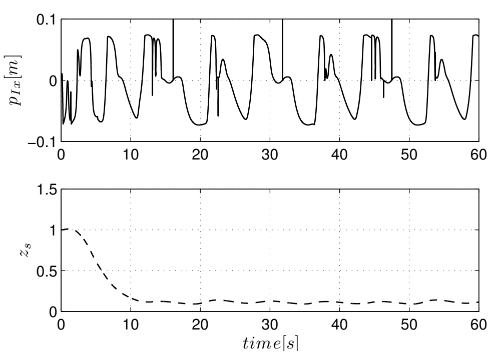

Fig. 6. SSMR – coordinate of ICR and filtered error: $p_{Ix}[m]$ (–), z_s (– –)

model. Hence for such class of system one should not expect high accuracy of control. In the future control scheme taking into account robot dynamics will be considered thoroughly and the results will be illustrated experimentally.

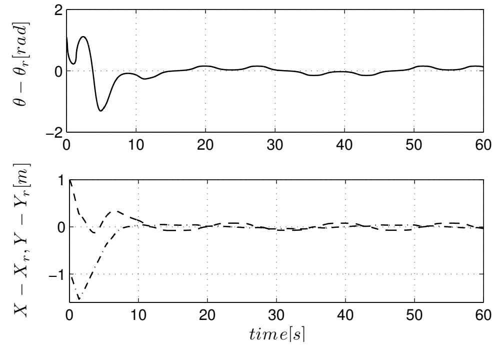

Fig. 7. Unicycle – tracking errors: $\theta - \theta_r[rad]$ (–), $X - X_r[m]$ (– –), $Y - Y_r[m]$ (–.–)

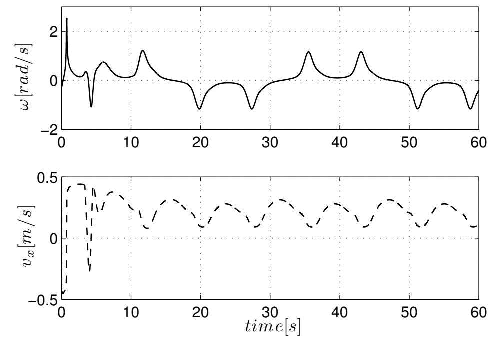

Fig. 8. Unicycle – velocities: $\omega[rad/s]$ (–), $v_x[m/s]$ (– –)

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