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Trans-Sasakian 3-Manifolds with Reeb Flow Invariant Ricci Operator

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Abstract: Let M be a three-dimensional trans-Sasakian manifold of type (α, β). In this paper, we obtain that the Ricci operator of M is invariant along Reeb flow if and only if M is an α-Sasakian manifold, cosymplectic manifold or a space of constant sectional curvature. Applying this, we give a new characterization of proper trans-Sasakian 3-manifolds.

Keywords: trans-Sasakian 3-manifold; Reeb flow symmetry; Ricci operator

1. Introduction

A trans-Sasakian manifold is usually denoted by $(M, \phi, \xi, \eta, g, \alpha, \beta)$, where both α and β are smooth functions and (ϕ, ξ, η, g) is an almost contact metric structure. M is said to be proper if either $\alpha = 0$ or $\beta = 0$. When $\beta = 0$, α is a constant if $\dim M \geq 5$ (see [1]) and in this case M becomes an α-Sasakian manifold if $\alpha \in \mathbb{R}^*$ or a cosymplectic manifold if $\alpha = 0$. This conclusion is not necessarily true for dimension three. However, unlike the above case, when $\alpha = 0$, β is not necessarily a constant even if $\dim M \geq 5$ or M is compact for dimension three (see [2]). The set of all trans-Sasakian manifolds of type $(0, \beta)$ coincides with that of all f-cosymplectic manifolds (see [3]) or f-Kenmotsu manifolds (see [4–6]). A trans-Sasakian manifold of dimension ≥ 5 must be proper (see [1]). In the geometry of trans-Sasakian 3-manifolds, there exists a basic interesting problem, that is:

Under what condition is a trans-Sasakian 3-manifold proper?

De [7–12], Deshmukh [13–15], Wang and Liu [16] and Wang [2,17] answered this question from various points of view. In this paper, we study this question under a new geometric condition. Before stating our main results, we recall some results related with such a condition.

On an almost contact metric manifold (M, ϕ, ξ, η, g), the Ricci operator of M is said to be Reeb flow invariant if it satisfies

$$\mathcal{L}_\xi Q = 0, \tag{1}$$

where $\mathcal{L}$, ξ and Q are the Lie derivative, Reeb vector field and the Ricci operator, respectively. Cho in [18] proved that a contact metric 3-manifold satisfies Equation (1) if and only if it is Sasakian or locally isometric to $SU(2)$ (or $SO(3)$), $SL(2, R)$ (or $O(1, 2)$), the group $E(2)$ of rigid motions of Euclidean 2-plane. Cho in [19] proved that an almost cosymplectic 3-manifold satisfies (1) if and only if it is either cosymplectic or locally isometric to the group $E(1, 1)$ of rigid motions of Minkowski 2-space. In addition, Cho and Kimura in [20] proved that an almost Kenmotsu 3-manifold satisfies (1) if and only if it is of constant sectional curvature -1 or a non-unimodular Lie group. Reeb flow invariant Ricci operators were also investigated on the unit tangent sphere bundle of a Riemannian manifold (see [21]), even on real hypersurfaces in complex two-plane Grassmannians (see [22]). In this paper, we obtain a new characterization of proper trans-Sasakian 3-manifolds by employing (1) and proving

Theorem 1. *The Ricci operator of a trans-Sasakian* 3*-manifold is invariant along Reeb flow if and only if the manifold is an* α*-Sasakian manifold, cosymplectic manifold or a space of constant sectional curvature.*

According to calculations shown in Section 3, we observe that Ricci parallelism with respect to the Levi–Civita connection (i.e., $\nabla Q = 0$) is stronger than a Reeb flow invariant Ricci operator. Thus, we have

Remark 1. *Theorem 1 is an extension of Wang and Liu [16] (Theorem 3.12).*

Some corollaries induced from Theorem 1 are also given in the last section.

2. Trans-Sasakian Manifolds

On a smooth Riemannian manifold (M, g) of dimension $2n+1$, we assume that ϕ, ξ and η are $(1,1)$-type, $(1,0)$-type and $(0,1)$-type tensor fields, respectively. According to [23], M is called an almost contact metric manifold if

$$\begin{aligned} \phi^2 X = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \eta(\phi X) = 0, \\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\ \eta(X) = g(X, \xi) \end{aligned} \tag{2}$$

for any vector fields X and Y. An almost contact metric manifold is said to be normal if $[\phi, \phi] = -2d\eta \otimes \xi$, where $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ.

A normal almost contact metric manifold is called a *trans-Sasakian manifold* (see [1]) if

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{3}$$

for any vector fields X, Y and two smooth functions α, β. In particular, a three-dimensional almost contact metric manifold is trans-Sasakian if and only if it is normal (see [24,25]).

A normal almost contact metric manifold is called an α*-Sasakian manifold* if $d\eta = \alpha\Phi$ and $d\Phi = 0$, where $\alpha \in \mathbb{R}^*$ (see [26]). An α-Sasakian manifold reduces to a Sasakian manifold (see [23]) when $\alpha = 1$. A normal almost contact metric manifold is called a β*-Kenmotsu manifold* if it satisfies $d\eta = 0$ and $d\Phi = 2\beta\eta \wedge \Phi$, where $\beta \in \mathbb{R}^*$ (see [26]). A β-Kenmotsu manifold becomes a Kenmotsu manifold when $\beta = 1$. A normal almost contact metric manifold is called a *cosymplectic manifold* if it satisfies $d\eta = 0$ and $d\Phi = 0$.

Putting $Y = \xi$ into (3) and using (2), we have

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi) \tag{4}$$

for any vector field X. In this paper, all manifolds are assumed to be connected.

3. Reeb Flow Invariant Ricci Operator on Trans-Sasakian 3-Manifolds

In this section, we give a proof of our main result Theorem 1. First, we introduce the following two important lemmas (see [12]) which are useful for our proof.

Lemma 1. *On a trans-Sasakian* 3*-manifold of type* (α, β) *we have*

$$\xi(\alpha) + 2\alpha\beta = 0. \tag{5}$$

Lemma 2. *On a trans-Sasakian 3-manifold of type (α,β), the Ricci operator is given by*

$$\begin{aligned} Q=&\left(\frac{r}{2}+\xi(\beta)-\alpha^2+\beta^2\right)\mathrm{id}-\left(\frac{r}{2}+\xi(\beta)-3\alpha^2+3\beta^2\right)\eta\otimes\xi\\ &+\eta\otimes(\phi(\nabla\alpha)-\nabla\beta)+g(\phi(\nabla\alpha)-\nabla\beta,\cdot)\otimes\xi, \end{aligned} \tag{6}$$

where by ∇f we mean the gradient of a function f.

We also need the following lemma (see [17])

Lemma 3. *On a trans-Sasakian 3-manifold of type (α,β), the following three conditions are equivalent:*

(1) *The Reeb vector field is minimal or harmonic.*
(2) *The following equation holds: $\phi\nabla\alpha-\nabla\beta+\xi(\beta)\xi=0$ ($\Leftrightarrow\nabla\alpha+\phi\nabla\beta+2\alpha\beta\xi=0$).*
(3) *The Reeb vector field is an eigenvector field of the Ricci operator.*

Lemma 4. *The Ricci operator on a cosymplectic 3-manifold is invariant along the Reeb flow.*

The above lemma can be seen in [19]

Lemma 5. *The Ricci operator on an α-Sasakian 3-manifold is invariant along the Reeb flow.*

Proof. According to Lemma 2 and the definition of an α-Sasakian 3-manifold, the Ricci operator is given by

$$QX=\left(\frac{r}{2}-\alpha^2\right)X-\left(\frac{r}{2}-3\alpha^2\right)\eta(X)\xi, \tag{7}$$

for any vector field X and certain nonzero constant α. Moreover, according to [16] (Corollary 3.10), we observe that the scalar curvature r is invariant along the Reeb vector field ξ, i.e., $\xi(r)=0$. In fact, such an equation can be deduced directly by using the formula $\mathrm{div}Q=\frac{1}{2}\nabla r$ and (7). Applying $\xi(r)=0$, it follows directly from (7) that $\mathcal{L}_\xi Q=0$. □

Proof of Theorem 1. Let M be a trans-Sasakian 3-manifold and e be a unit vector field orthogonal to ξ. Then, $\{\xi,e,\phi e\}$ forms a local orthonormal basis on the tangent space for each point of M. The Levi–Civita connection ∇ on M can be written as the following (see [12])

$$\begin{aligned} \nabla_\xi\xi=&0,\ \nabla_\xi e=\lambda\phi e,\ \nabla_\xi\phi e=-\lambda e,\\ \nabla_e\xi=&\beta e-\alpha\phi e,\ \nabla_e e=-\beta\xi+\gamma\phi e,\ \nabla_e\phi e=\alpha\xi-\gamma e,\\ \nabla_{\phi e}\xi=&\alpha e+\beta\phi e,\ \nabla_{\phi e}e=-\alpha\xi-\delta\phi e,\ \nabla_{\phi e}=-\beta\xi+\delta e, \end{aligned} \tag{8}$$

where λ, γ and δ are smooth functions on some open subset of the manifold. We assume that the Ricci operator is invariant along the Reeb flow. From (1) and (4), we have

$$0=(\mathcal{L}_\xi Q)X=(\nabla_\xi Q)X+\alpha\phi QX-\alpha Q\phi X+\beta\eta(QX)\xi-\beta\eta(X)Q\xi \tag{9}$$

for any vector field X.

By using the local basis $\{\xi,e,\phi e\}$ and Lemma 2, the Ricci operator can be rewritten as the following:

$$\begin{aligned} Q\xi=&\phi\nabla\alpha-\nabla\beta+(2\alpha^2-2\beta^2-\xi(\beta))\xi,\\ Qe=&\left(\frac{r}{2}+\xi(\beta)-\alpha^2+\beta^2\right)e-(\phi e(\alpha)+e(\beta))\xi,\\ Q\phi e=&\left(\frac{r}{2}+\xi(\beta)-\alpha^2+\beta^2\right)\phi e+(e(\alpha)-\phi e(\beta))\xi. \end{aligned} \tag{10}$$

Replacing X in (9) by ξ, we obtain

$$\begin{aligned}\nabla_\xi(\phi\nabla\alpha-\nabla\beta)+\xi(2\alpha^2-2\beta^2-\xi(\beta))\xi+\alpha(-\nabla\alpha+\xi(\alpha)\xi-\phi\nabla\beta)\\+2\beta(\alpha^2-\beta^2-\xi(\beta))\xi-\beta(\phi\nabla\alpha-\nabla\beta)-\beta(2\alpha^2-2\beta^2-\xi(\beta))\xi=0.\end{aligned}\tag{11}$$

Taking the inner product of the above equation with ξ, e and ϕe, respectively, we obtain

$$\begin{aligned}\xi(\xi(\beta))+2\beta\xi(\beta)+4\alpha^2\beta&=0,\\ \alpha e(\alpha)-\beta\phi e(\alpha)-\beta e(\beta)-\alpha\phi e(\beta)&=0,\\ \beta e(\alpha)+\alpha\phi e(\alpha)+\alpha e(\beta)-\beta\phi e(\beta)&=0,\end{aligned}\tag{12}$$

where we have employed Lemma 1. The addition of the second term of (12) multiplied by α to the third term of (12) multiplied by β gives

$$(\alpha^2+\beta^2)(e(\alpha)-\phi e(\beta))=0.\tag{13}$$

Following (13), we consider the following several cases.

Case i: $\alpha^2+\beta^2=0$, or equivalently, $\alpha=\beta=0$. In this case, the manifold becomes a cosymplectic 3-manifold. The proof for this case is completed because of Lemma 4.

Case ii: $\alpha^2+\beta^2\neq 0$. It follows immediately from (13) that $e(\alpha)-\phi e(\beta)=0$, or equivalently, $g(\nabla\alpha+\phi\nabla\beta,e)=0$. Because e is assumed to be an arbitrary vector field, it follows that $\nabla\alpha+\phi\nabla\beta=\eta(\nabla\alpha+\phi\nabla\beta)\xi$, i.e.,

$$\nabla\alpha+\phi\nabla\beta+2\alpha\beta\xi=0,\tag{14}$$

or equivalently, $\phi\nabla\alpha-\nabla\beta+\xi(\beta)\xi=0$, where we have used Lemma 1. When $\beta=0$, it follows from (14) that α is a nonzero constant. Thus, the proof can be done by applying Lemma 5. In what follows, we consider the last case.

Case iii: $\alpha^2+\beta^2\neq 0$ and $\beta\neq 0$. In this context, (10) becomes

$$\begin{aligned}Q\xi&=2(\alpha^2-\beta^2-\xi(\beta))\xi,\\ Qe&=\left(\frac{r}{2}+\xi(\beta)-\alpha^2+\beta^2\right)e,\\ Q\phi e&=\left(\frac{r}{2}+\xi(\beta)-\alpha^2+\beta^2\right)\phi e.\end{aligned}\tag{15}$$

Replacing X by e in (9) and using (8), (15), we acquire

$$0=(\mathcal{L}_\xi Q)e=\xi\left(\frac{r}{2}+\xi(\beta)-\alpha^2+\beta^2\right)e.$$

With the aid of Lemma 1 and the first term of (12), from the previous relation, we have

$$\xi(r)=0.\tag{16}$$

From (15), we calculate the derivative of the Ricci operator as the following:

$$\begin{aligned}(\nabla_\xi Q)\xi&=0,\\ (\nabla_e Q)e&=e(A)e-\beta A\xi+2\beta(\alpha^2-\beta^2-\xi(\beta))\xi,\\ (\nabla_{\phi e}Q)\phi e&=\phi e(A)\phi e-\beta A\xi+2\beta(\alpha^2-\beta^2-\xi(\beta))\xi,\end{aligned}\tag{17}$$

where we have used the first term of (8) and (12) and, for simplicity, we put

$$A=\frac{r}{2}+\xi(\beta)-\alpha^2+\beta^2.\tag{18}$$

On a Riemannian manifold, we have $\mathrm{div}Q = \frac{1}{2}\nabla r$. In this context, it is equivalent to

$$g((\nabla_\xi Q)\xi + (\nabla_e Q)e + (\nabla_{\phi e} Q)\phi e, X) = \frac{1}{2}X(r) \tag{19}$$

for any vector field X. Replacing X in (19) by ξ and recalling (16) and the first term of (12), we obtain $2\beta(A - 2\alpha^2 + 2\beta^2 + 2\xi(\beta)) = 0$, or equivalently,

$$\xi(\beta) - \alpha^2 + \beta^2 = -\frac{r}{6}, \tag{20}$$

where we have used the assumption $\beta \neq 0$ and (18). According to (15), it is clear to see that the manifold is Einstein, i.e, $Q = \frac{r}{3}\mathrm{id}$. Because the manifold is of dimension three, then it must be of constant sectional curvature. □

A Riemannian manifold is said to be locally symmetric if $\nabla R = 0$ and this is equivalent to $\nabla Q = 0$ for dimension three. Wang and Liu in [16] proved that a trans-Sasakian 3-manifold is locally symmetric if and only if it is locally isometric to the sphere space $\mathbb{S}^3(c^2)$, the hyperbolic space $\mathbb{H}^3(-c^2)$, the Euclidean space $\mathbb{R}^3$, product space $\mathbb{R} \times \mathbb{S}^2(c^2)$ or $\mathbb{R} \times \mathbb{H}^2(-c^2)$, where c is a nonzero constant. According to [16], on a locally symmetric trans-Sasakian 3-manifold, the Reeb vector field is an eigenvector field of the Ricci operator. Thus, following Lemma 3 and relations (9) and (10), we observe that Ricci parallelism is stronger than the Reeb flow invariant Ricci operator. Hence, our main result in this paper extends [16] (Theorem 3.12).

From Theorem 1, we obtain a new characterization of proper trans-Sasakian 3-manifolds.

Theorem 2. *A compact trans-Sasakian 3-manifold with Reeb flow invariant Ricci operator is homothetic to either a Sasakian manifold or a cosymplectic manifold.*

Proof. As seen in the proof of Theorem 1, a trans-Sasakian 3-manifold with Reeb flow invariant Ricci operator is a α-Sasakian manifold, a cosymplectic manifold or a space of constant sectional curvature. It is well known that an α-Sasakian manifold is homothetic to a Sasakian manifold. Moreover, there do exist compact Sasakian and cosymplectic manifolds. To complete the proof, we need only to prove that *Case iii* in the proof of Theorem 1 cannot occur.

Let M be a trans-Sasakian 3-manifold satisfying *Case iii*. According to (14) and Lemma 5, we know that the Reeb vector field is minimal or harmonic. It has been proved in [17] (Lemma 5.1) that when ξ of a compact trans-Sasakian 3-manifold is minimal or harmonic, then α is a constant. Because the manifold is of constant sectional curvature, then the scalar curvature r is also a constant. Therefore, the differentiation of (20) along ξ gives

$$\xi(\xi(\beta)) + 2\beta\xi(\beta) = 0. \tag{21}$$

Adding the above equation to the first term of (12) implies that $\alpha = 0$ because of $\beta \neq 0$. Using this in (14), we have $\nabla\beta = \xi(\beta)\xi$. The following proof follows directly from [2]. For sake of completeness, we present the detailed proof.

Applying $\nabla\beta = \xi(\beta)\xi$ and (7), we obtain

$$\nabla_X \nabla\beta = X(\xi(\beta))\xi + \xi(\beta)(\beta X - \beta\eta(X)\xi) = 0$$

for any vector field X. Contracting X in the previous relation and using (21), we obtain $\Delta\beta = \xi(\xi(\beta)) + 2\beta\xi(\beta) = 0$. Because the manifold is assumed to be compact, the application of the divergence theorem gives that β is a non-zero constant. Next, we show that this is impossible. In fact, the application of (4) gives that $\mathrm{div}\xi = 2\beta$. Since the manifold is assumed to be compact, it follows that $\beta = 0$, a contradiction. This completes the proof. □

Theorem 2 can also be written as follows.

Theorem 3. *A compact trans-Sasakian* 3*-manifold with Reeb flow invariant Ricci operator is proper.*

The curvature tensor R of a trans-Sasakian 3-manifold is given by (see [10,27])

$$\begin{aligned} & R(X,Y)Z \\ =& B(g(Y,Z)X - g(X,Z)Y) - Cg(Y,Z)\eta(X)\xi \\ & + g(Y,Z)(\eta(X)(\phi\nabla\alpha - \nabla\beta) - g(\nabla\beta - \phi\nabla\alpha, X)\xi) \\ & + Cg(X,Z)\eta(Y)\xi - g(X,Z)(\eta(Y)(\phi\nabla\alpha - \nabla\beta) - g(\nabla\beta - \phi\nabla\alpha, Y)\xi) \\ & - (g(\nabla\beta - \phi\nabla\alpha, Z)\eta(Y) + g(\nabla\beta - \phi\nabla\alpha, Y)\eta(Z))X - C\eta(Y)\eta(Z)X \\ & + (g(\nabla\beta - \phi\nabla\alpha, Z)\eta(X) + g(\nabla\beta - \phi\nabla\alpha, X)\eta(Z))X + C\eta(X)\eta(Z)Y \end{aligned} \tag{22}$$

for any vector fields X, Y, Z, where, for simplicity, we set

$$B = \frac{r}{2} + 2\xi(\beta) - 2\alpha^2 + 2\beta^2, \; C = \frac{r}{2} + \xi(\beta) - 3\alpha^2 + 3\beta^2. \tag{23}$$

Substituting (14) and (20) into (22), with the aid of (23), we get

$$R(X,Y)Z = \frac{r}{6}(g(Y,Z)X - g(X,Z)Y)$$

for any vector fields X, Y, Z. This implies that, on a trans-Sasakian 3-manifold satisfying *Case iii* in the proof of Theorem 1, we do not know whether $\alpha = 0$ or not. In view of this, we introduce an interesting question:

Problem 1. *Is there a non-proper and non-compact trans-Sasakian* 3*-manifold of constant sectional curvature?*

Remark 2. *According to De and Sarkar [10] (Theorem 5.1), we observe that a compact trans-Sasakian* 3*-manifold of constant sectional curvature is either α-Sasakian or β-Kenmotsu.*

Remark 3. *Given a trans-Sasakian* 3*-manifold, following proof of Theorem 1, we still do not know whether β is a constant or not even when $\alpha = 0$ and the manifold is compact (see [2]).*

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